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FUKUSHIMA TYPE DECOMPOSITION FOR SEMI-DIRICHLET FORMS

ZHI-MING MA, WEI SUN AND LI-FEI WANG

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Abstract. We present a Fukushima type decomposition in the setting of general quasi-regular semi-Dirichlet forms. The decomposition is then employed to give a transformation formula for martingale additive functionals. Applications of the results to some concrete examples of semi-Dirichlet forms are given at the end of the paper. We discuss also the uniqueness question about the Doob-Meyer decomposition on optional sets of interval type.

Introduction. The celebrated Fukushima's decomposition and related transformation rules play the roles of the Doob-Meyer decomposition and Itô's formula in the framework of Dirichlet forms. They have been used to investigate the properties of a large class of stochastic processes that are not semi-martingales such as additive functionals of Brownian motion which are not necessarily of bounded variation (cf. e.g. [25], [5] and references therein). Fukushima's decomposition was originally established for regular symmetric Dirichlet forms (cf. [7] and [8, Theorem 5.2.2]) and then extended to the non-symmetric and quasi-regular cases (cf. [20, Theorem 5.1.3] and [18, Theorem VI.2.5]). Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E; m)$ with associated Markov process $((X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ (we refer the reader to [8, 18, 17] for notations and terminologies of this paper). If $u \in D(\mathcal{E})$, then Fukushima's decomposition tells us that there exist a unique martingale additive functional (MAF in short) $M^{[u]}$ of finite energy and a unique continuous additive functional $N^{[u]}$ of zero energy such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]} . \tag{1}$$

Hereafter $\tilde{u}$ denotes an $\mathcal{E}$-quasi-continuous m-version of u.

Compared with Dirichlet form, semi-Dirichlet form is a more general framework arising from various applications. In the viewpoint of applications, and also by the interests of the theory its own, it is natural to ask if we can extend Fukushima's decomposition from the setting of Dirichlet forms to that of semi-Dirichlet forms. For example, do we have Fukushima's decomposition for the following simple local semi-Dirichlet form?

$$\mathcal{E}(u, v) = \int_0^1 u'v'dx + \int_0^1 \sqrt{x}u'vdx , \quad u, v \in D(\mathcal{E}) := H_0^{1,2}(0, 1) .$$

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Note that the assumption of the existence of dual Markov process plays a crucial role in Fukushima's decomposition for Dirichlet forms. In fact, without that assumption, the usual definition of energy of AFs is questionable. Here we would like to point out that although Fukushima's decomposition was even considered for generalized Dirichlet forms (cf. [26] and [24]), which is a more general framework than semi-Dirichlet forms (see [23]), up to now Fukushima's decomposition for generalized Dirichlet forms has only been given under the additional assumption that their dual forms are also sub-Markovian. For a quasi-regular semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, we may use the semi-$h$ transform method to associate $(\mathcal{E}, D(\mathcal{E}))$ with a sub-Markovian dual form (cf. [10]). However, without imposing further assumptions, we cannot expect to obtain Fukushima's decomposition for general $u \in D(\mathcal{E})$; we can only expect to obtain the decomposition (1) for functions u in the domain of the generator of $(\mathcal{E}, D(\mathcal{E}))$, which is just the classical Doob-Meyer decomposition.

To our knowledge, the paper [16] appears to be the first publication on the Fukushima type decomposition in the setting of semi-Dirichlet forms without assuming that the dual form is sub-Markovian. In that paper the authors introduced a condition of local control (cf. Condition 1.5 below) and under the condition they obtained the Fukushima type decomposition for $u \in D(\mathcal{E})_{loc}$ where $(\mathcal{E}, D(\mathcal{E}))$ is a local semi-Dirichlet form. The main method employed in [16] is the localization and pasting technique. For a non-local semi-Dirichlet form, the jump part of $M^{[u]}$ is in general not locally consistent, which causes some extra difficulty in implementing the localization and pasting technique. Afterwards, one of the authors of the present paper investigated further in [28] the Fukushima type decomposition for general quasi-regular semi-Dirichlet forms. Motivated by some idea of Kuwae [15] and employing also the localization and pasting technique, he obtained the Fukushima type decomposition for $u \in D(\mathcal{E})_{loc}$ under a suitable condition (S) (see Theorem 1.4 below). Meanwhile Professor Oshima sent us a manuscript of his new book [21], in which he proved Fukushima's decomposition for $u \in D(\mathcal{E})_b$ in the setting of regular semi-Dirichlet forms satisfying his condition $(\mathcal{E}.5)$. The main techniques employed by Oshima in developing Fukushima's decomposition are the weak sense energy and his ingenious auxiliary bilinear form, different from the localization and pasting technique employed in [16] and [28].

In this paper we shall report and develop further the Fukushima type decomposition based on [28], and discuss some related topics. Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form which is not necessarily local. We show that under a suitable assumption (i.e. Assumption 1.3 below), a function $u \in D(\mathcal{E})_{loc}$ admits a Fukushima type decomposition if and only if it satisfies Condition (S), and the decomposition is unique. Roughly speaking, here u admits a Fukushima type decomposition means that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]},$$

where $M^{[u]}$ is a locally square integrable MAF on the set $I(\zeta) := [\![0, \zeta[\![\cup [\![\zeta_i]\!]$, with ζ being the lifetime of X and ζ_i the totally inaccessible part of ζ; and $N^{[u]}$ is a local AF which is continuous and has zero quadratic variation on $I(\zeta)$. For details see Theorem 1.4 below. It is worth to point out that Assumption 1.3 mentioned above is weaker than the condition of local control in [16] and the condition ($\mathcal{E}$.5) in [21]. We are very grateful to Professor Oshima for sending us his new book [21]. The condition ($\mathcal{E}$.5) in [21] stimulated us to formulate Assumption 1.3.

The reader might notice that in the above description we used $I(\zeta)$ instead of $[\![0, \zeta[\![$, the latter is customarily used in the literature. The reason of this variation is that we discovered that the decomposition on $I(\zeta)$ is unique, but it may fail to be unique on $[\![0, \zeta[\![$. This difference is essentially due to the fact that $I(\zeta)$ is a predictable set of interval type while $[\![0, \zeta[\![$ is not necessarily predictable. This discovery exposes not only an oversight in the previous paper [16], but also similar oversights in the literature e.g. [15] and [2] (however, see Remark 2.5 below). The oversight may be traced back even to Theorem 8.26 of the book [11], which exposes a question about the uniqueness of the Doob-Meyer decomposition on optional sets of interval type. We shall discuss this question in detail in Section 2 below.

The rest of the paper is organized as follows. In Section 1, we present a general Fukushima type decomposition for semi-Dirichlet forms. We divide it into three subsections. In Subsection 1.1, we present basic settings and statement of the theorem, and provide some discussions and remarks about the theorem. In Subsection 1.2, we give the proof of the theorem. In Subsection 1.3, we study the local energies of $M^{[u]}$ and $N^{[u]}$. In Section 2, we discuss in detail the question about the uniqueness of the Doob-Meyer decomposition on optional sets of interval type. In Section 3, we give a transformation formula for MAFs based on the Fukushima type decomposition. In Section 4, we apply our results to two concrete examples of semi-Dirichlet forms appearing in recent papers.

1. Fukushima type decomposition.

1.1. Statement of the theorem and discussions. The basic setting of this paper is the same as that in [16] with some necessary modifications, e.g., $(\mathcal{E}, D(\mathcal{E}))$ in this paper is not assumed to be local. To fix the notations and also for the convenience of the reader, below we restate our setting of which some contents are taken from [16]. Let E be a metrizable Lusin space and m a σ-finite positive measure on its Borel σ-algebra $\mathcal{B}(E)$. We consider a quasi-regular semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E; m)$. Hereafter for notations and terminologies related to quasi-regular semi-Dirichlet forms we refer to [17]. Denote by $(T_t)_{t\geq 0}$ and $(G_\alpha)_{\alpha\geq 0}$ (resp. $(\hat{T}_t)_{t\geq 0}$ and $(\hat{G}_\alpha)_{\alpha\geq 0}$) the semigroup and resolvent (resp. co-semigroup and co-resolvent) associated with $(\mathcal{E}, D(\mathcal{E}))$. Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be an m-tight special standard process which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$ in the sense that $P_t f$ is an $\mathcal{E}$-quasi-continuous m-version of $T_t f$ for all $f \in \mathcal{B}_b(E) \cap L^2(E; m)$ and all $t > 0$, where $(P_t)_{t\geq 0}$ denotes the semigroup associated with $\mathbf{M}$ (cf. [17, Theorem 3.8]).

Similar to the symmetric case, in the setting of semi-Dirichlet forms there is also a one-to-one correspondence between the family of all equivalent classes of positive continuous additive functionals and the family S of smooth measures. The contents below concerning positive continuous additive functionals and S are taken from [16]. We remark that the reader can now find more detailed descriptions and discussions in [21] on the potential theory of semi-Dirichlet forms including the correspondence between positive continuous additive functionals and smooth measures.

Recall that a positive measure μ on $(E, \mathcal{B}(E))$ is called *smooth* (w.r.t. $(\mathcal{E}, D(\mathcal{E}))$), denoted by $\mu \in S$, if $\mu(N) = 0$ for each $\mathcal{E}$-exceptional set $N \in \mathcal{B}(E)$ and there exists an $\mathcal{E}$-nest $\{F_k\}$ of compact subsets of E such that

$$\mu(F_k) < \infty \quad \text{for all } k \in \mathbb{N}.$$

A family $(A_t)_{t\geq 0}$ of functions on Ω is called an *additive functional* (AF in short) of $\mathbf{M}$ if:

(i) A_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a defining set $\Lambda \in \mathcal{F}$ and an exceptional set $N \subset E$ which is $\mathcal{E}$-exceptional such that $P_x[\Lambda] = 1$ for all $x \in E\backslash N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $(0, \infty)$ and has left limits on $(0, \zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$, and

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega), \quad \forall\, s, t \geq 0. \tag{2}$$

Hereafter ζ denotes the lifetime of $X := (X_t)_{t\geq 0}$.

Two AFs $A = (A_t)_{t\geq 0}$ and $B = (B_t)_{t\geq 0}$ are said to be equivalent, denoted by $A = B$, if they have a common defining set Λ and a common exceptional set N such that $A_t(\omega) = B_t(\omega)$ for all $\omega \in \Lambda$ and $t \geq 0$. An AF $A = (A_t)_{t\geq 0}$ is called a continuous AF (CAF in short) if $t \mapsto A_t(\omega)$ is continuous on $(0, \infty)$. It is called a positive CAF (PCAF in short) if $A_t(\omega) \geq 0$ for all $t \geq 0$, $\omega \in \Lambda$.

LEMMA 1.1 (cf. [16, Theorem A.8], see also [21, Section 4.1]). *Let A be a PCAF. Then there exists a unique $\mu \in S$, which is referred to as the Revuz measure of A and is denoted by μ_A, such that*:

For any γ-co-excessive function g ($\gamma \geq 0$) in $D(\mathcal{E})$ and $f \in \mathcal{B}^+(E)$,

$$\lim_{t\downarrow 0} \frac{1}{t} E_{g\cdot m}((fA)_t) = \langle f \cdot \mu, \tilde{g}\rangle.$$

Conversely, let $\mu \in S$, then there exists a unique (up to the equivalence) PCAF A such that $\mu = \mu_A$.

Throughout this paper, we fix a function $\phi \in L^1(E; m)$ with $0 < \phi \leq 1$ m-a.e. and set $h = G_1\phi$, $\hat{h} = \hat{G}_1\phi$. Denote $\tau_B := \inf\{t > 0 \,|\, X_t \notin B\}$ for $B \subset E$. Let V be a quasi-open subset of E. We denote by $X^V = (X_t^V)_{t\geq 0}$ the part process of X on V and denote by $(\mathcal{E}^V, D(\mathcal{E})_V)$ the part form of $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(V; m)$. It is known that X^V is a standard process and $(\mathcal{E}^V, D(\mathcal{E})_V)$ is a quasi-regular semi-Dirichlet form (cf. [14]). Denote by $(T_t^V)_{t\geq 0}$, $(\hat{T}_t^V)_{t\geq 0}$, $(G_\alpha^V)_{\alpha\geq 0}$ and $(\hat{G}_\alpha^V)_{\alpha\geq 0}$ the semigroup, co-semigroup, resolvent and co-resolvent associated with $(\mathcal{E}^V, D(\mathcal{E})_V)$, respectively. Define

$$\bar{h}^V := \hat{G}_1^V\phi. \tag{3}$$

Then $\bar{h}^V \in D(\mathcal{E})_V$ and $\bar{h}^V$ is 1-co-excessive. Denote $D(\mathcal{E})_{V,b} := \mathcal{B}_b(E) \cap D(\mathcal{E})_V$.

For an AF $A = (A_t)_{t\geq 0}$ of X^V, we define

$$e^V(A) := \lim_{t\downarrow 0} \frac{1}{2t} E_{\bar{h}^V\cdot m}(A_t^2)$$

whenever the limit exists in $[0,\infty]$. Define

$$\begin{aligned}\dot{\mathcal{M}}^V := \{M \mid M \text{ is an AF of } X^V,\ E_x(M_t^2) < \infty, E_x(M_t) = 0 \\ \text{for all } t \geq 0 \text{ and } \mathcal{E}\text{-q.e. } x \in V, e^V(M) < \infty\}\,,\end{aligned}$$

$$\begin{aligned}\mathcal{N}_c^V := \{N \mid N \text{ is a CAF of } X^V, E_x(|N_t|) < \infty \text{ for all } t \geq 0 \\ \text{and } \mathcal{E}\text{-q.e. } x \in V, e^V(N) = 0\}\,,\end{aligned}$$

$$\begin{aligned}\Theta := \Bigg\{ \{V_n\} \mid V_n \text{ is } \mathcal{E}\text{-quasi-open},\ V_n \subset V_{n+1}\ \mathcal{E}\text{-q.e.} \\ \forall\, n \in \mathbb{N},\ \text{and } E = \bigcup_{n=1}^{\infty} V_n\ \mathcal{E}\text{-q.e.} \Bigg\}\,,\end{aligned}$$

and

$$\begin{aligned}D(\mathcal{E})_{loc} := \{u \mid \exists\, \{V_n\} \in \Theta \text{ and } \{u_n\} \subset D(\mathcal{E}) \\ \text{such that } u = u_n\ m\text{-a.e. on } V_n,\ \forall\, n \in \mathbb{N}\}\,.\end{aligned}$$

In what follows we shall employ the notion of local AFs introduced in [8] as follows.

DEFINITION 1.2 (cf. [8, page 271]). A family $A = (A_t)_{t\geq 0}$ of functions on Ω is called a *local AF* of $\mathbf{M}$, if A satisfies all the requirements for an AF as stated in above (i) and (ii), except that the additivity property (2) is required only for $s, t \geq 0$ with $t + s < \zeta(\omega)$.

Two local AFs $A^{(1)}$, $A^{(2)}$ are said to be equivalent if for $\mathcal{E}$-q.e. $x \in E$, it holds that

$$P_x(A_t^{(1)} = A_t^{(2)}; t < \zeta) = P_x(t < \zeta)\,, \quad \forall\, t \geq 0\,.$$

We now define

$$\begin{aligned}\dot{\mathcal{M}}_{loc} := \{M \mid M \text{ is a local AF of } \mathbf{M},\ \exists\, \{V_n\}, \{E_n\} \in \Theta \text{ and } \{M^n \mid M^n \in \dot{\mathcal{M}}^{V_n}\} \\ \text{such that } E_n \subset V_n,\ M_{t\wedge\tau_{E_n}} = M^n_{t\wedge\tau_{E_n}},\ t \geq 0,\ n \in \mathbb{N}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_c := \{N \mid N \text{ is a local AF of } \mathbf{M}\,,\ \exists\, \{E_n\} \in \Theta \text{ such that } t \mapsto N_{t\wedge\tau_{E_n}} \\ \text{is continuous and of zero quadratic variation},\ n \in \mathbb{N}\}\,.\end{aligned}$$

In the above definition, $\{N_{t\wedge\tau_{E_n}}\}$ is said to be of zero quadratic variation if its quadratic variation vanishes in P_m-measure, more precisely, if it satisfies

$$\sum_{k=0}^{[T/\varepsilon_l]} (N_{\{(k+1)\varepsilon_l\}\wedge\tau_{E_n}} - N_{\{k\varepsilon_l\}\wedge\tau_{E_n}})^2 \to 0 \quad \text{as } l \to \infty \text{ in } P_m\text{-measure}\,,$$

for any $T > 0$ and any sequence $\{\varepsilon_l\}_{l\in\mathbb{N}}$ converging to 0.

We use ζ_i to denote the totally inaccessible part of ζ, by which we mean that ζ_i is an $\{\mathcal{F}_t\}$-stopping time and is the totally inaccessible part of ζ w.r.t. P_x for $\mathcal{E}$-q.e. $x \in E$. In Section 2 below we shall give a proof for the existence and uniqueness of such ζ_i, where the uniqueness is in the sense of P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Write $I(\zeta) := [\![0, \zeta[\![\cup [\![\zeta_i]\!]$. We can show that there exists a $\{V_n\} \in \Theta$ such that for any $\{U_n\} \in \Theta$, $I(\zeta) = \bigcup_n [\![0, \tau_{V_n \cap U_n}]\!]$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$ (see Proposition 2.4 below). Therefore $I(\zeta)$ is a predictable set of interval type (cf. [11, Theorem 8.18]). In this paper a local AF M is called a locally square integrable MAF on $I(\zeta)$, denoted by $M \in \mathcal{M}_{loc}^{I(\zeta)}$, if $M \in (\mathcal{M}_{loc}^2)^{I(\zeta)}$ in the sense of [11, Definition 8.19].

Denote by $J(dx, dy)$ and $K(dx)$ the jumping and killing measures of $(\mathcal{E}, D(\mathcal{E}))$, respectively (cf. [12]). Let $(N(x, dy), H_s)$ be a Lévy system of X and μ_H the Revuz measure of the PCAF H. Then we have $J(dy, dx) = \frac{1}{2}N(x, dy)\mu_H(dx)$ and $K(dx) = N(x, \Delta)\mu_H(dx)$.

We put the following assumption:

ASSUMPTION 1.3. There exist $\{V_n\} \in \Theta$ and locally bounded functions $\{C_n\}$ on $\mathbb{R}$ such that for each $n \in \mathbb{N}$, if $u, v \in D(\mathcal{E})_{V_n,b}$ then $uv \in D(\mathcal{E})$ and

$$\mathcal{E}(uv, uv) \leq C_n(\|u\|_\infty + \|v\|_\infty)(\mathcal{E}_1(u, u) + \mathcal{E}_1(v, v)) .$$

Now we can state the main theorem of this section.

THEOREM 1.4. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption* 1.3. *Then for $u \in D(\mathcal{E})_{loc}$ the following two assertions are equivalent to each other.*

(i) *u admits a Fukushima type decomposition. That is, there exist $M^{[u]} \in \mathcal{M}_{loc}^{I(\zeta)}$ and $N^{[u]} \in \mathcal{L}_c$ such that*

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0, \quad P_x\text{-}a.s. \quad for\ \mathcal{E}\text{-}q.e.\ x \in E. \tag{4}$$

(ii) *u satisfies Condition (S) specified below.*

$$(S): \quad \mu_u(dx) := \int_E (\tilde{u}(x) - \tilde{u}(y))^2 J(dy, dx) \text{ is a smooth measure.}$$

Moreover, if u satisfies Condition (S), then the decomposition (4) *is unique up to the equivalence of local AFs, and the continuous part of $M^{[u]}$ belongs to $\dot{\mathcal{M}}_{loc}$.*

The proof of Theorem 1.4 will be given in the next subsection. In the remainder of this subsection we provide some remarks and discussions about the theorem.

In [16], the authors obtained a Fukushima type decomposition for $u \in D(\mathcal{E})_{loc}$ where $(\mathcal{E}, D(\mathcal{E}))$ is a local quasi-regular Dirichlet form satisfying the condition of local control as stated below.

CONDITION 1.5. *There exists $\{V_n\} \in \Theta$ such that for each $n \in \mathbb{N}$ there exist a Dirichlet form $(\eta^{(n)}, D(\eta^{(n)}))$ on $L^2(V_n; m)$ and a constant $C_n > 1$ satisfying $D(\eta^{(n)}) = D(\mathcal{E})_{V_n}$ and for any $u \in D(\mathcal{E})_{V_n}$,*

$$\frac{1}{C_n}\eta_1^{(n)}(u, u) \leq \mathcal{E}_1(u, u) \leq C_n \eta_1^{(n)}(u, u) .$$

It is clear that Assumption 1.3 is more general than Condition 1.5. Hence we have the following remark.

REMARK 1.6. Theorem 1.4 extends the corresponding result of [16].

In [21], Oshima discussed various topics of regular semi-Dirichlet forms under his condition $(\mathcal{E}.5)$. In particular, he proved in Theorem 5.1.5 a weak sense of Fukushima's decomposition for $u \in D(\mathcal{E})_b$. Below is the condition $(\mathcal{E}.5)$ of [21] stated in our context.

Condition $(\mathcal{E}.5)$**.** If $u, v \in D(\mathcal{E})$ and $w \in L^2(E; m)$ satisfy $|w(x) - w(y)| \leq |u(x) - u(y)| + |v(x) - v(y)|$ and $|w(x)| \leq |u(x)| + |v(x)|$ for any $x, y \in E$, then $w \in D(\mathcal{E})$ and $|\mathcal{E}(w, w)| \leq K(\mathcal{E}_1(u, u) + \mathcal{E}_1(v, v))$ for some K depending on $\|u\|_\infty$ and $\|v\|_\infty$.

It is easy to see that Condition $(\mathcal{E}.5)$ implies the following condition.

CONDITION 1.7. *There exists a locally bounded function C on $\mathbb{R}$ such that if $u, v \in D(\mathcal{E})_b$, then $uv \in D(\mathcal{E})$ and*

$$\mathcal{E}(uv, uv) \leq C(\|u\|_\infty + \|v\|_\infty)(\mathcal{E}_1(u, u) + \mathcal{E}_1(v, v)). \tag{5}$$

PROPOSITION 1.8. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ satisfies Condition 1.7, then any $u \in D(\mathcal{E})_b$ satisfies Condition (S), and hence admits a Fukushima type decomposition.*

PROOF. Since Condition 1.7 is a special case of Assumption 1.3, hence by Theorem 1.4 we need only to check that any $u \in D(\mathcal{E})_b$ satisfies Condition (S). By the quasi-homeomorphism method (cf. [6] or [12, Theorem 3.8]), without loss of generality below we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form. Let $\{E_n\}$ be a sequence of relatively compact open sets such that $E = \bigcup_n E_n$ and $\{v_n\} \subset D(\mathcal{E}) \cap C_0(E)$ satisfying $v_n = 1$ on E_n for each $n \in \mathbb{N}$. We choose a sequence of relatively compact open sets $G_l \uparrow E$ and a sequence of numbers $\delta_l \downarrow 0$ such that the set $\Gamma_l := \{(x, y) \in G_l \times G_l \mid |\rho(x, y) \geq \delta_l\}$ is a continuous set w.r.t. J for every $l \in \mathbb{N}$, where ρ is the metric of E. For $\beta > 0$, let σ_β be the unique positive Radon measures on $E \times E$ satisfying

$$(\beta G_\beta f, g) = \int_{E \times E} f(x)g(y)\sigma_\beta(dx, dy)\,, \quad \forall f, g \in D(\mathcal{E}) \cap C_0(E)\,.$$

Let $u \in D(\mathcal{E}) \cap C_0(E)$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\frac{1}{2}\int_{E_n}\int_E (u(x) - u(y))^2 N(x, dy)\mu_H(dx) \\
&\quad \leq \frac{1}{2}\int_E\int_E v_n(x)(u(x) - u(y))^2 N(x, dy)\mu_H(dx) \\
&\quad \leq \lim_{l\to\infty}\int\int_{\Gamma_l}(u(x) - u(y))^2 v_n(y)J(dx, dy) \\
&\quad = \lim_{l\to\infty}\lim_{\beta\to\infty}\frac{\beta}{2}\int\int_{\Gamma_l}(u(x) - u(y))^2 v_n(y)\sigma_\beta(dx, dy) \\
&\quad \leq \lim_{\beta\to\infty}\frac{\beta}{2}\int_E\int_E (u(x) - u(y))^2 v_n(y)\sigma_\beta(dx, dy)
\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{\beta\to\infty} \frac{\beta}{2}\{(\beta G_\beta 1_E, u^2 v_n) - 2(\beta G_\beta u, u v_n) + (\beta G_\beta u^2, v_n)\}\\
&\leq \lim_{\beta\to\infty} \{\beta(u - \beta G_\beta u, u v_n) - \frac{\beta}{2}(u^2 - \beta G_\beta u^2, v_n)\}\\
&= \mathcal{E}(u, u v_n) - \frac{1}{2}\mathcal{E}(u^2, v_n)\,, \qquad (6)
\end{aligned}
$$

which implies that u satisfies Condition (S).

For general $u \in D(\mathcal{E})_b$, we may select a sequence of functions $\{u_k\} \subset D(\mathcal{E}) \cap C_0(E)$ such that $u_k \to u$ w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm as $k \to \infty$ and $\|u_k\|_\infty \leq \|u\|_\infty$ for $k \in \mathbb{N}$. Then by (5), (6) and Fatou's lemma, we can show that $\int_{E_n}\int_E (\tilde{u}(x) - \tilde{u}(y))^2 N(x, dy)\mu_H(dx) < \infty$. Hence u satisfies Condition (S), which completes the proof. □

REMARK 1.9. Proposition 1.8 shows that Theorem 1.4 is an extension of [21, Theorem 5.1.5].

We would like to point out that the methods of [21] in developing Fukushima's decomposition are different from ours. In the next subsection we shall see that Theorem 1.4 is proved by the localization and pasting technique. The main techniques employed by Oshima in developing his Theorem 5.1.5 are the weak sense energy and the ingenious auxiliary bilinear form invented in [21]. We take this opportunity to thank Professor Oshima for sending us his manuscript [21]. The condition ($\mathcal{E}$.5) in [21] stimulated us to formulate Assumption 1.3.

REMARK 1.10. Theorem 1.4 extends the corresponding results of [8, Theorem 5.5.1] and [15, Theorem 4.2] from the symmetric case to the semi-Dirichlet form case.

Note that for a symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, Assumption 1.3 is satisfied automatically. Also, $u \in D(\mathcal{E})_{loc}$ satisfies Condition (S) trivially if $(\mathcal{E}, D(\mathcal{E}))$ is local. When $(\mathcal{E}, D(\mathcal{E}))$ is non-local, Condition (S) is necessary even in the symmetric case. In developing stochastic analysis with Nakao's integral, Kuwae obtained in [15] a generalized Fukushima decomposition in the symmetric case for a subclass of $D(\mathcal{E})_{loc}$, which is equivalent to impose Condition (S) for $u \in D(\mathcal{E})_{loc}$. In this paper when dealing with purely discontinuous part of $M^{[u]}$, we adopted some idea from [15] without making use of Nakao's integral. One of the authors of this paper has joint work with others extending Nakao's integral to non-symmetric Dirichlet forms (cf. [1]). We feel that Nakao's integral can also be extended to semi-Dirichlet forms.

REMARK 1.11. In Theorem 1.4 if we use $\mathcal{M}_{loc}^{[\![0,\zeta[\![}$ instead of $\mathcal{M}_{loc}^{I(\zeta)}$, then the uniqueness of the decomposition may fail to be true.

We shall discuss the above remark and related topics in detail in Section 2 below.

1.2. Proof of the theorem. Before proving Theorem 1.4, we prepare some lemmas.

We fix a $\{V_n\} \in \Theta$ satisfying Assumption 1.3. Without loss of generality, we assume that $\widetilde{\hat{h}}$ is bounded on each V_n, otherwise we may replace V_n by $V_n \cap \{\widetilde{\hat{h}} < n\}$. Since $\bar{h}^{V_n} =$

$\hat{G}_1^{V_n}\phi \le \hat{G}_1\phi = \hat{h}$, $\bar{h}^{V_n}$ is bounded on V_n. To simplify notations, we write

$$\bar{h}_n := \bar{h}^{V_n}\,.$$

LEMMA 1.12 ([16, Lemma 2.6]). *Let $u \in D(\mathcal{E})_{V_n,b}$. Then there exist unique $M^{n,[u]} \in \dot{\mathcal{M}}^{V_n}$ and $N^{n,[u]} \in \mathcal{N}_c^{V_n}$ such that* for *$\mathcal{E}$-q.e. $x \in V_n$,*

$$\tilde{u}(X_t^{V_n}) - \tilde{u}(X_0^{V_n}) = M_t^{n,[u]} + N_t^{n,[u]}\,, \quad t \ge 0\,, \quad P_x\text{-}a.s.$$

Lemma 1.12 has been given in [16] under Assumption 1.5 and the additional assumption that $(\mathcal{E}, D(\mathcal{E}))$ is local; however, it can be easily extended to general semi-Dirichlet forms under Assumption 1.3 with the similar proof.

We now fix a $u \in D(\mathcal{E})_{loc}$ satisfying Condition (S). Then there exist $\{V_n^1\} \in \Theta$ and $\{u_n\} \subset D(\mathcal{E})$ such that $u = u_n$ m-a.e. on V_n^1. By [17, Proposition 3.6], we may assume without loss of generality that each u_n is $\mathcal{E}$-quasi-continuous. By [17, Proposition 2.16], there exists an $\mathcal{E}$-nest $\{F_n^2\}$ of compact subsets of E such that $\{u_n\} \subset C\{F_n^2\}$. Denote by V_n^2 the fine interior of F_n^2. Then $\{V_n^2\} \in \Theta$. Since u satisfies Condition (S), there exists an $\mathcal{E}$-nest $\{F_n^3\}$ of compact subsets of E such that $\mu_u(F_n^3) < \infty$. Denote by V_n^3 the fine interior of F_n^3. Since the killing measure $K(dx)$ is a smooth measure, there exists an $\mathcal{E}$-nest $\{F_n^4\}$ of compact subsets of E such that $K(F_n^4) < \infty$. Denote by V_n^4 the fine interior of F_n^4. Define $V_n' = V_n \cap V_n^1 \cap V_n^2 \cap V_n^3 \cap V_n^4$. Then $\{V_n'\} \in \Theta$, each u_n is bounded on V_n', and

$$\begin{aligned}
&\int_{V_n'}\int_{E_\Delta} (\tilde{u}(x) - \tilde{u}(y))^2 N(x, dy)\mu_H(dx) \\
&\qquad = \int_{V_n'}\int_E (\tilde{u}(x) - \tilde{u}(y))^2 J(dy, dx) + \int_{V_n'} \tilde{u}^2(x)K(dx) \\
&\qquad < \infty\,.
\end{aligned}$$

For $n \in \mathbb{N}$, we define $E_n = \{x \in E \mid \widetilde{h_n}(x) > \frac{1}{n}\}$, where $h_n := G_1^{V_n}\phi$. Then $\{E_n\} \in \Theta$ satisfying $\overline{E}_n^{\mathcal{E}} \subset E_{n+1}$ $\mathcal{E}$-q.e. and $E_n \subset V_n$ $\mathcal{E}$-q.e. for each $n \in \mathbb{N}$ (cf. [14, Lemma 3.8]). Here $\overline{E}_n^{\mathcal{E}}$ denotes the $\mathcal{E}$-quasi-closure of E_n. Define $f_n = n\widetilde{h_n} \wedge 1$. Then $f_n = 1$ on E_n and $f_n = 0$ on V_n^c. Since f_n is a 1-excessive function of $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ and $f_n \le n\widetilde{h_n} \in D(\mathcal{E})_{V_n}$, hence $f_n \in D(\mathcal{E})_{V_n}$ by [19, Remark 3.4(ii)]. Denote by Q_n the bound of $|u_n|$ on V_n'. By [14, (2.1)] and Assumption 1.3, we find that $[(-Q_n f_n) \vee u_n \wedge (Q_n f_n)]f_n \in D(\mathcal{E})_{V_n,b}$. To simplify notation, below we use still u_n to denote $[(-Q_n f_n) \vee u_n \wedge (Q_n f_n)]$ and use still E_n to denote $E_n \cap V_n'$. Then we have $\{E_n\} \in \Theta$, $E_n \subset V_n$, $u_n, u_n f_n \in D(\mathcal{E})_{V_n,b}$, and $u = u_n = u_n f_n$ on E_n for $n \in \mathbb{N}$. Denote by $\{\mathcal{F}_t^n\}$ the minimum completed admissible filtration of X^{V_n}. For $n < l$, we have $\mathcal{F}_t^n \subset \mathcal{F}_t^l \subset \mathcal{F}_t$. Since $E_n \subset V_n$, τ_{E_n} is an $\{\mathcal{F}_t^n\}$-stopping time.

LEMMA 1.13 ([13, Lemma 25.3]). *For any optional time T and predictable process Y, the random variable $Y_T 1_{(T<\infty)} \in \mathcal{F}_{T-}$.*

Hereafter for a martingale M, we denote by M^c and M^d its continuous part and purely discontinuous part, respectively.

LEMMA 1.14. *For $n < l$, we have $M^{n,[u_n f_n],c}_{t\wedge\tau_{E_n}} = M^{l,[u_l f_l],c}_{t\wedge\tau_{E_n}}$, $t \geq 0$, P_x-a.s.* for *$\mathcal{E}$-q.e. $x \in V_n$.*

PROOF. Let $n < l$. Since $M^{n,[u_n f_n]} \in \dot{\mathcal{M}}^{V_n}$, $M^{n,[u_n f_n]}$ is an $\{\mathcal{F}^n_t\}$-martingale by the Markov property. Since τ_{E_n} is an $\{\mathcal{F}^n_t\}$-stopping time, $\{M^{n,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ is an $\{\mathcal{F}^n_{t\wedge\tau_{E_n}}\}$-martingale. Denote $\Upsilon^n_t = \sigma\{X^{V_n}_{s\wedge\tau_{E_n}} \,|\, 0 \leq s \leq t\}$. Then $\{M^{n,[u_n f_n],c}_{t\wedge\tau_{E_n}}\}$ is a $\{\Upsilon^n_t\}$-martingale. Denote $\Upsilon^{n,l}_t = \sigma\{X^{V_l}_{s\wedge\tau_{E_n}} \,|\, 0 \leq s \leq t\}$. Similarly, we can show that $\{M^{l,[u_n f_n],c}_{t\wedge\tau_{E_n}}\}$ is a $\{\Upsilon^{n,l}_t\}$-martingale. Since

$$X^{V_l}_s = X_s = X^{V_n}_s, \quad s < \tau_{E_n}, \quad P_x\text{-a.s.} \quad \text{for } \mathcal{E}\text{-q.e. } x \in V_n, \tag{7}$$

we find that $\Upsilon^n_{(t\wedge\tau_{E_n})-} = \Upsilon^{n,l}_{(t\wedge\tau_{E_n})-}$. Hence $\{M^{l,[u_n f_n],c}_{t\wedge\tau_{E_n}}\} \in \Upsilon^{n,l}_{(t\wedge\tau_{E_n})-}$ by Lemma 1.13 and therefore $\{M^{l,[u_n f_n],c}_{t\wedge\tau_{E_n}}\}$ is a $\{\Upsilon^n_t\}$-martingale. Moreover, $N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}} \in \Upsilon^{n,l}_{(t\wedge\tau_{E_n})-} = \Upsilon^n_{(t\wedge\tau_{E_n})-} \subset \mathcal{F}^n_{t\wedge\tau_{E_n}}$.

Let $N \in \mathcal{N}^{V_j}_c$ for some $j \in \mathbb{N}$. Then, for any $T > 0$,

$$\begin{aligned}
\sum_{k=1}^{[rT]} E_{\bar{h}_j\cdot m}\big[(N_{\frac{k+1}{r}} - N_{\frac{k}{r}})^2\big] &\leq \sum_{k=1}^{[rT]} e^T\big(E_\cdot(N^2_{\frac{1}{r}}), e^{-\frac{k}{r}}\hat{T}^{V_j}_{\frac{k}{r}}\bar{h}_j\big) \\
&\leq \sum_{k=1}^{[rT]} e^T\big(E_\cdot(N^2_{\frac{1}{r}}), \bar{h}_j\big) \\
&\leq rTe^T E_{\bar{h}_j\cdot m}(N^2_{\frac{1}{r}}) \to 0 \quad \text{as } r \to \infty.
\end{aligned}\tag{8}$$

Hence

$$\sum_{k=1}^{[rT]}(N_{\frac{k+1}{r}} - N_{\frac{k}{r}})^2 \to 0 \text{ as } r \to \infty \text{ in } P_{\bar{h}_j\cdot m}\text{-measure}.$$

Therefore, the quadratic variations of $\{N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ and $\{N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ vanish in $P_{\bar{h}_l\cdot m}$-measure and $P_{\bar{h}_n\cdot m}$-measure, respectively.

By (7), we find that for $\mathcal{E}$-q.e. $x \in V_n$,

$$\begin{aligned}
&M^{n,[u_n f_n],c}_{t\wedge\tau_{E_n}} + M^{n,[u_n f_n],d}_{t\wedge\tau_{E_n}} + N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} \\
&\quad = \widetilde{u_n f_n}(X^{V_n}_{t\wedge\tau_{E_n}}) - \widetilde{u_n f_n}(X^{V_n}_0) \\
&\quad = \widetilde{u_n f_n}(X^{V_l}_{t\wedge\tau_{E_n}}) - \widetilde{u_n f_n}(X^{V_l}_0) \\
&\quad = M^{l,[u_n f_n],c}_{t\wedge\tau_{E_n}} + M^{l,[u_n f_n],d}_{t\wedge\tau_{E_n}} + N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}, \quad P_x\text{-a.s.}
\end{aligned}$$

Then $\{M^{n,[u_n f_n],d}_{t\wedge\tau_{E_n}}\} \in \Upsilon^n_t$, and $\{M^{n,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ and $\{M^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ are $\{\Upsilon^n_t\}$-martingales. Hence $M^{n,[u_n f_n],c}_{t\wedge\tau_{E_n}} = M^{l,[u_n f_n],c}_{t\wedge\tau_{E_n}}$ and $N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} = N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}$, P_x-a.s. for m-a.e. $x \in V_n$. This implies that $E_m(\langle M^{n,[u_n f_n],c}_{\cdot\wedge\tau_{E_n}} - M^{l,[u_n f_n],c}_{\cdot\wedge\tau_{E_n}}\rangle_t) = 0$, $\forall t \geq 0$. By Lemma 1.1, we find that $M^{n,[u_n f_n],c}_{t\wedge\tau_{E_n}} = M^{l,[u_n f_n],c}_{t\wedge\tau_{E_n}}$, $\forall t \geq 0$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_n$.

Since $u_n f_n = u_l f_l = u$ on E_n, similar to [15, Lemma 2.4], we can show that $M_t^{l,[u_n f_n],c} = M_t^{l,[u_l f_l],c}$ when $t < \tau_{E_n}$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_l$. Then $M_{t\wedge\tau_{E_n}}^{l,[u_n f_n],c} = M_{t\wedge\tau_{E_n}}^{l,[u_l f_l],c}$, $t \geq 0$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_l$. Therefore $M_{t\wedge\tau_{E_n}}^{n,[u_n f_n],c} = M_{t\wedge\tau_{E_n}}^{l,[u_l f_l],c}$, $t \geq 0$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_n$. □

PROOF OF THEOREM 1.4. (a) Suppose that u satisfies Condition (S). We shall show that u admits the Fukushima type decomposition (4).

We define $M_{t\wedge\tau_{E_n}}^{[u],c} := \lim_{l\to\infty} M_{t\wedge\tau_{E_n}}^{l,[u_l f_l],c}$ and $M_t^{[u],c} := 0$ for $t > \zeta$ if there exists some n such that $\tau_{E_n} = \zeta$ and $\zeta < \infty$; or $M_t^{[u],c} := 0$ for $t \geq \zeta$, otherwise. By Lemma 1.14, $M^{[u],c}$ is well defined and $M_{t\wedge\tau_{E_n}}^{[u],c} = M_{t\wedge\tau_{E_n}}^{n,[u_n f_n],c}$ for $t \geq 0$ and $n \in \mathbb{N}$. Hence $M^{[u],c} \in \dot{\mathcal{M}}_{loc}$. Define $M_t^n := M_{t\wedge\tau_{E_n}}^{n+1,[u_{n+1} f_{n+1}],c}$ for $t \geq 0$ and $n \in \mathbb{N}$. Then $M_{t\wedge\tau_{E_n}}^{[u],c} = M_{t\wedge\tau_{E_n}}^n$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_{n+1}$ by Lemma 1.14. Since $\overline{E}_n^{\mathcal{E}} \subset E_{n+1} \subset V_{n+1}$ $\mathcal{E}$-q.e. implies that $P_x(\tau_{E_n} = 0) = 1$ for $x \notin V_{n+1}$, $M_{t\wedge\tau_{E_n}}^{[u],c} = M_{t\wedge\tau_{E_n}}^n$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$.

Next we show that M^n is an $\{\mathcal{F}_t\}$-martingale. In fact, by the fact that τ_{E_n} is an $\{\mathcal{F}_t^{n+1}\}$-stopping time, we find that $1_{\{\tau_{E_n}\leq s\}}$ is $\mathcal{F}_{s\wedge\tau_{E_n}}^{n+1}$-measurable for any $s \geq 0$. Let $0 \leq s_1 < \cdots < s_k \leq s < t$ and $g \in \mathcal{B}_b(\mathbb{R}^k)$. Then, we obtain by the fact $M^{n+1,[u_{n+1} f_{n+1}],c} \in \dot{\mathcal{M}}^{V_{n+1}}$ that for $\mathcal{E}$-q.e. $x \in V_{n+1}$,

$$\begin{aligned}
&\int_\Omega M_t^n g(X_{s_1}, \ldots, X_{s_k}) dP_x \\
&= \int_{\{\tau_{E_n}\leq s\}} M_t^n g(X_{s_1}, \ldots, X_{s_k}) dP_x + \int_{\{\tau_{E_n}> s\}} M_t^n g(X_{s_1}, \ldots, X_{s_k}) dP_x \\
&= \int_{\{\tau_{E_n}\leq s\}} M_s^n g(X_{s_1}, \ldots, X_{s_k}) dP_x \\
&\quad + \int_\Omega M_{t\wedge\tau_{E_n}}^{n+1,[u_{n+1} f_{n+1}],c} g(X_{s_1\wedge\tau_{E_n}}^{V_{n+1}}, \ldots, X_{s_k\wedge\tau_{E_n}}^{V_{n+1}}) 1_{\{\tau_{E_n}>s\}} dP_x \\
&= \int_{\{\tau_{E_n}\leq s\}} M_s^n g(X_{s_1}, \ldots, X_{s_k}) dP_x \\
&\quad + \int_\Omega M_{s\wedge\tau_{E_n}}^{n+1,[u_{n+1} f_{n+1}],c} g(X_{s_1\wedge\tau_{E_n}}^{V_{n+1}}, \ldots, X_{s_k\wedge\tau_{E_n}}^{V_{n+1}}) 1_{\{\tau_{E_n}>s\}} dP_x \\
&= \int_{\{\tau_{E_n}\leq s\}} M_s^n g(X_{s_1}, \ldots, X_{s_k}) dP_x + \int_{\{\tau_{E_n}> s\}} M_s^n g(X_{s_1}, \ldots, X_{s_k}) dP_x \\
&= \int_\Omega M_s^n g(X_{s_1}, \ldots, X_{s_k}) dP_x .
\end{aligned}$$

Obviously, the equality holds for $x \notin V_{n+1}$. Hence M^n is an $\{\mathcal{F}_t\}$-martingale. By Proposition 2.4 below, $\bigcup_n [\![0, \tau_{E_n}]\!] \supseteq I(\zeta)$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Therefore $M^{[u],c} \in \mathcal{M}_{loc}^{I(\zeta)}$.

We define $\phi(x, y) = \tilde{u}(y) - \tilde{u}(x)$, $\phi_l(x, y) = (\tilde{u}(y) - \tilde{u}(x)) 1_{\{|\tilde{u}(x)-\tilde{u}(y)|>\frac{1}{l}\}}$, and

$$M_t^l := \sum_{0<s\leq t} \phi_l(X_{s-}, X_s) - \int_0^t \int_{E_\Delta} \phi_l(X_s, y) N(X_s, dy) dH_s$$

for $l \in \mathbb{N}$. Denote $T_r^l := \inf\{t > 0 \mid |M_t^l| \geq r\}$ for $r \in \mathbb{N}$. Then, $\{T_r^l\}$ is an $\{\mathcal{F}_t\}$-stopping time and

$$\begin{aligned}|M^l_{t\wedge T^l_r\wedge\tau_{E_n}}| &\leq |M^l_{t\wedge T^l_r\wedge\tau_{E_n}-}| + |\phi(X_{t\wedge T^l_r\wedge\tau_{E_n}-}, X_{t\wedge T^l_r\wedge\tau_{E_n}})| \\ &\leq r + |\phi(X_{t\wedge T^l_r\wedge\tau_{E_n}-}, X_{t\wedge T^l_r\wedge\tau_{E_n}})|\,.\end{aligned}$$

We define (cf. [16, Theorem A.3])

$$\hat{S}^*_{00} := \{\mu \in S_0 \mid \hat{U}_1\mu \leq c\hat{G}_1\phi \text{ for some constant } c > 0\}. \tag{9}$$

Let $\nu \in S^*_{00}$ satisfying $\nu(E) < \infty$. Then, by [16, Lemma A.9], we get

$$\begin{aligned}E_\nu\big[(M^l_{t\wedge T^l_r\wedge\tau_{E_n}})^2\big] &\leq 2r^2\nu(E) + 2E_\nu\Big[\sum_{0<s\leq t\wedge\tau_{E_n}}\phi^2(X_{s-}, X_s)\Big] \\ &= 2r^2\nu(E) + 2E_\nu\Big[\int_0^{t\wedge\tau_{E_n}}\int_{E_\Delta}\phi^2(X_s, y)N(X_s, dy)dH_s\Big] \\ &\leq 2r^2\nu(E) + 2C_\nu(1+t)\int_{E_n}\widetilde{\hat{h}}\int_{E_\Delta}\phi^2(x, y)N(x, dy)\mu_H(dx) \\ &< \infty\,,\end{aligned}$$

where C_ν is a positive constant. Hence, for fixed n and r, $\{M^l_{t\wedge T^l_r\wedge\tau_{E_n}}\}$ is a square integrable purely discontinuous P_ν-martingale. By [8, Corollary A.3.1], we find that

$$(M^l_{t\wedge T^l_r\wedge\tau_{E_n}})^2 - \sum_{s\leq t}(\Delta M^l_{s\wedge T^l_r\wedge\tau_{E_n}})^2 = (M^l_{t\wedge T^l_r\wedge\tau_{E_n}})^2 - \sum_{s\leq t\wedge T^l_r\wedge\tau_{E_n}}\phi_l^2(X_{s-}, X_s)$$

is a P_ν-martingale, which implies that

$$\begin{aligned}E_\nu\big[(M^l_{t\wedge\tau_{E_n}})^2\big] &\leq \liminf_{r\to\infty} E_\nu\big[(M^l_{t\wedge T^l_r\wedge\tau_{E_n}})^2\big] \\ &= \liminf_{r\to\infty} E_\nu\Big[\sum_{0<s\leq t\wedge T^l_r\wedge\tau_{E_n}}\phi_l^2(X_{s-}, X_s)\Big] \\ &= E_\nu\Big[\sum_{s\leq t\wedge\tau_{E_n}}\phi_l^2(X_{s-}, X_s)\Big] \\ &\leq E_\nu\left[\int_0^{t\wedge\tau_{E_n}}\int_{E_\Delta}\phi^2(X_s, y)N(X_s, dy)dH_s\right] \\ &\leq C_\nu(1+t)\int_{E_n}\widetilde{\hat{h}}(x)\int_{E_\Delta}\phi^2(x, y)N(x, dy)\mu_H(dx) \\ &< \infty\,.\end{aligned}$$

Thus $\{M^l_{t\wedge\tau_{E_n}}\}$ is a P_ν-square-integrable martingale. Since $\{M^l_{t\wedge T^l_r\wedge\tau_{E_n}}\}_{r=1}^\infty$ is $L^2(P_\nu)$-bounded, by virtue of Banach-Saks theorem, we obtain that

$$E_\nu\big[(M^l_{t\wedge\tau_{E_n}})^2\big] = E_\nu\left[\int_0^{t\wedge\tau_{E_n}}\int_{E_\Delta}\phi_l^2(X_s, y)N(X_s, dy)dH_s\right].$$

By Doob's maximum inequality, we obtain that for any $\alpha > 0$ and l, k,

$$P_\nu\left(\sup_{0\le s\le T}|M^l_{s\wedge\tau_{E_n}} - M^k_{s\wedge\tau_{E_n}}| > \alpha\right) \le \frac{4C_\nu(1+T)}{\alpha^2}\int_{E_n}\tilde{\hat{h}}(x)\int_{E_\Delta}(\phi_l-\phi_k)^2(x,y)N(x,dy)\mu_H(dx)\,.$$

By the diagonal method, we may select a subsequence $l_k \to \infty$ such that for each n when $k \ge n$,

$$\int_{E_n}\tilde{\hat{h}}(x)\int_{E_\Delta}(\phi_{l_{k+1}}-\phi_{l_k})^2(x,y)N(x,dy)\mu_H(dx) \le \frac{1}{2^{3k}}\,.$$

Then

$$P_\nu\left(\sup_{0\le s\le T}|M^{l_{k+1}}_{s\wedge\tau_{E_n}} - M^{l_k}_{s\wedge\tau_{E_n}}| > \frac{1}{2^k}\right) \le \frac{C_\nu(1+T)}{2^k}\,.$$

Define $\Lambda_0^n = \{\omega \in \Omega \mid M^{l_k}_{s\wedge\tau_{E_n}}$ converges uniformly in s on each finite interval$\}$. Then, $\Lambda_0^{n_1} \supset \Lambda_0^{n_2}$ for $n_1 \le n_2$. By the Borel-Cantelli lemma, we get

$$P_\nu((\Lambda_0^n)^c) = 0 \quad \text{for } \nu \in \hat{S}^*_{00} \text{ with } \nu(E) < \infty\,.$$

Therefore $P_x((\Lambda_0^n)^c) = 0$ for $\mathcal{E}$-q.e. $x \in E$ (cf. [16, Theorem A.3]). Let Γ_k be the defining set of the MAF M^{l_k}, denote $\Gamma = \cap_k\Gamma_k$ and $\Lambda^n = \Lambda_0^n \cap \Gamma$. Then we have $P_x((\Lambda^n)^c) = 0$ for $\mathcal{E}$-q.e. $x \in E$. For each $\omega \in \Lambda^n$, $M^{l_k}_{t\wedge\tau_{E_n}}$ converges uniformly in t on each finite interval and for each k,

$$M^{l_k}_{(t+s)\wedge\tau_{E_n}} = M^{l_k}_{t\wedge\tau_{E_n}} + M^{l_k}_{s\wedge\tau_{E_n}} \circ \theta_{t\wedge\tau_{E_n}}\,, \quad \text{if } 0 \le t\,,\ \ s < \infty\,.$$

Thus, L^n, the limit of $\{M^{l_k}_{s\wedge\tau_{E_n}}\}_{k=1}^\infty$, is a P_x-square integrable purely discontinuous martingale for $\mathcal{E}$-q.e. $x \in E$ and satisfies:

$$L^n_{(t+s)\wedge\tau_{E_n}} = L^n_{t\wedge\tau_{E_n}} + L^n_{s\wedge\tau_{E_n}} \circ \theta_{t\wedge\tau_{E_n}}\,, \quad \text{if } 0 \le t\,,\ \ s < \infty\,.$$

By the above construction, we find that $L^{n_1}_{t\wedge\tau_{E_{n_1}}} = L^{n_2}_{t\wedge\tau_{E_{n_1}}}$ for $n_1 \le n_2$. We define $M^{[u],d}_t = L^n_t, t \le \tau_{E_n}$, and $M^{[u],d}_t = L^n_t, t \ge \zeta$, if for some n, $\tau_{E_n} = \zeta < \infty$; $M^{[u],d}_t = 0, t \ge \zeta$, otherwise. Then $M^{[u],d} \in \mathcal{M}^{I(\zeta)}_{loc}$, which gives all the jumps of $\tilde{u}(X_t) - \tilde{u}(X_0)$ on $I(\zeta)$. Since $\{M^l_t\}$ is an MAF for each l, we find that $\{M^{[u],d}_t\}$ is a local MAF by the uniform convergence on $I(\zeta)$.

We define $N^{[u]}_{t\wedge\tau_{E_n}} := \tilde{u}(X_{t\wedge\tau_{E_n}}) - \tilde{u}(X_0) - M^{[u],c}_{t\wedge\tau_{E_n}} - M^{[u],d}_{t\wedge\tau_{E_n}}$ for each $n \in \mathbb{N}$. Then $N^{[u]}$ is a local AF of $\mathbf{M}$ and $t \mapsto N^{[u]}_{t\wedge\tau_{E_n}}$ is continuous. Now we show that $\{N^{[u]}_{t\wedge\tau_{E_n}}\}$ has zero quadratic variation and hence $N^{[u]} \in \mathcal{L}_c$. By Fukushima's decomposition for part processes, we have that for $k \ge n$,

$$\begin{aligned}\widetilde{u_kf_k}(X_{t\wedge\tau_{E_n}}) - \widetilde{u_kf_k}(X_0) &= \widetilde{u_kf_k}(X^{V_k}_{t\wedge\tau_{E_n}}) - \widetilde{u_kf_k}(X^{V_k}_0)\\ &= M^{k,[u_kf_k]}_{t\wedge\tau_{E_n}} + N^{k,[u_kf_k]}_{t\wedge\tau_{E_n}}\end{aligned}$$

$$= M^{k,[u_k f_k],c}_{t\wedge\tau_{E_n}} + M^{k,[u_k f_k],d}_{t\wedge\tau_{E_n}} + N^{k,[u_k f_k]}_{t\wedge\tau_{E_n}}$$

and

$$\tilde{u}(X_{t\wedge\tau_{E_n}}) - \tilde{u}(X_0) = M^{[u],c}_{t\wedge\tau_{E_n}} + M^{[u],d}_{t\wedge\tau_{E_n}} + N^{[u]}_{t\wedge\tau_{E_n}}.$$

Then

$$\begin{aligned} N^{[u]}_{t\wedge\tau_{E_n}} &= N^{k,[u_k f_k]}_{t\wedge\tau_{E_n}} + M^{k,[u_k f_k],d}_{t\wedge\tau_{E_n}} - M^{[u],d}_{t\wedge\tau_{E_n}} + \tilde{u}(X_{t\wedge\tau_{E_n}}) - \widetilde{u_k f_k}(X_{t\wedge\tau_{E_n}}) \\ &= N^{k,[u_k f_k]}_{t\wedge\tau_{E_n}} + M^{k,[u_k f_k],d}_{t\wedge\tau_{E_n}} - M^{[u],d}_{t\wedge\tau_{E_n}} + [\tilde{u}(X_{\tau_{E_n}}) - \widetilde{u_k f_k}(X_{\tau_{E_n}})]1_{\{\tau_{E_n}\le t\}}. \end{aligned}$$

We define $A_t = [\tilde{u}(X_{\tau_{E_n}}) - \widetilde{u_k f_k}(X_{\tau_{E_n}})]1_{\{\tau_{E_n}\le t\}}$ and $\varrho(x,y) = (\tilde{u}(y)-\tilde{u}(x))1_{\{y\in(E_n^c\cup\{\Delta\})\}}$. Let $\nu \in S^*_{00}$ satisfying $\nu(E) < \infty$. Then, by [16, Lemma A.9], we get

$$\begin{aligned} E_\nu[(\tilde{u}(X_{\tau_{E_n}}) - \tilde{u}(X_{\tau_{E_n}-}))^2 1_{\{\tau_{E_n}\le t\}}] &= E_\nu\left[\sum_{0<s\le t\wedge\tau_{E_n}} \varrho^2(X_{s-}, X_s)\right] \\ &= E_\nu\left[\int_0^{t\wedge\tau_{E_n}} \int_{E_\Delta} \varrho^2(X_s, y) N(X_s, dy)\, dH_s\right] \\ &\le C_\nu(1+t)\int_{E_n} \tilde{\hat{h}} \int_{E_\Delta} \varrho^2(x,y) N(x,dy)\mu_H(dx) \\ &\le C_\nu(1+t)\int_{E_n} \tilde{\hat{h}} \int_{E_\Delta} (\tilde{u}(x)-\tilde{u}(y))^2 N(x,dy)\mu_H(dx) \\ &< \infty, \end{aligned}$$

where C_ν is a positive constant. Thus $E_\nu[A_t^2] < \infty$. Note that τ_{E_n} is an $\{\mathcal{F}_{t\wedge\tau_{E_n}}\}$-stopping time and $\{A_t\}$ is an adapted quasi-left continuous bounded variation processes. We denote by $\{A_t^p\}$ the dual predictable projection of $\{A_t\}$. Then $\{A_t^p\}$ is an adapted continuous bounded variation process (cf. [8, Theorem A.3.5]). Moreover, $(A - A^p)$ is an $\{\mathcal{F}_{t\wedge\tau_{E_n}}\}$-purely discontinuous P_ν-square-integrable martingale. Since both $\{M^{k,[u_k f_k],d}_{t\wedge\tau_{E_n}}\}$ and $\{M^{[u],d}_{t\wedge\tau_{E_n}}\}$ are $\{\mathcal{F}_{t\wedge\tau_{E_n}}\}$-purely discontinuous martingales and

$$N^{[u]}_{t\wedge\tau_{E_n}} = N^{k,[u_k f_k]}_{t\wedge\tau_{E_n}} + (M^{k,[u_k f_k],d}_{t\wedge\tau_{E_n}} - M^{[u],d}_{t\wedge\tau_{E_n}} + A_t - A_t^p) + A_t^p,$$

we find that $\{M^{k,[u_k f_k],d}_{t\wedge\tau_{E_n}} - M^{[u],d}_{t\wedge\tau_{E_n}} + A_t - A_t^p\}$ is a purely discontinuous martingale with zero jump, which must be equal to zero. Hence

$$N^{[u]}_{t\wedge\tau_{E_n}} = N^{k,[u_k f_k]}_{t\wedge\tau_{E_n}} + A_t^p. \tag{10}$$

Since $m(E_n) < \infty$ and the quadratic variations of $N^{k,[u_k f_k]}_{t\wedge\tau_{E_n}}$ and A_t^p vanish in $P_{\bar{h}_n\cdot m}$-measure and $P_{\phi\cdot m}$-measure, respectively, we conclude that the quadratic variation of $\{N^{[u]}_{t\wedge\tau_{E_n}}\}$ vanishes in P_m-measure, i.e., $\{N^{[u]}_{t\wedge\tau_{E_n}}\}$ has zero quadratic variation.

Finally, we prove the uniqueness of decomposition (4). Suppose that $M^1 \in \mathcal{M}^{I(\zeta)}_{loc}$ and $N^1 \in \mathcal{L}_c$ such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^1 + N_t^1, \quad t \ge 0, \quad P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in E.$$

By Proposition 2.4 below, we can choose an $\{E_n\} \in \Theta$ such that $I(\zeta) = \bigcup_n [\![0, \tau_{E_n}]\!]$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Then, for each $n \in \mathbb{N}$, $\{(M^{[u]} - M^1)^{\tau_{E_n}}\}$ is a locally square integrable martingale and a zero quadratic variation process. This implies that $P_m(\langle (M^{[u]} - M^1)^{\tau_{E_n}} \rangle_t \neq 0, \exists\, t \in [0, \infty)) = 0$. Consequently by the analog of [8, Lemma 5.1.10] in the setting of semi-Dirichlet forms, $P_x(\langle (M^{[u]} - M^1)^{\tau_{E_n}} \rangle_t \neq 0, \exists\, t \in [0, \infty)) = 0$ for $\mathcal{E}$-q.e. $x \in E$. Therefore $M_t^{[u]} = M_t^1$, $0 \le t \le \tau_{E_n}$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Since n is arbitrary, we obtain the uniqueness of decomposition (4) up to the equivalence of local AFs.

(b) Let $u \in D(\mathcal{E})_{loc}$ and suppose that the decomposition (4) holds. We shall show that u satisfies Condition (S). First, $M^{[u],d} \in \mathcal{M}_{loc}^{d,I(\zeta)}$ implies that there exist a sequence of increasing stopping times $\{T_n\}$ such that $\bigcup_n [\![0, T_n]\!] = I(\zeta)$ and a sequence of L^2-martingales $\{M^n\}$ such that $(M^{[u],d} 1_{I(\zeta)})^{T_n} = (M^n 1_{I(\zeta)})^{T_n}$. Hence $(M^{[u],d})^{T_n}$ is an L^2-martingale and its square bracket equals $\sum_{0<s\le t\wedge T_n} (u(X_s) - u(X_{s-}))^2$ and is an integrable increasing process. We use $[M^{[u],d}](t, \omega)$ to denote $(\sum_{0<s\le t} (u(X_s(w)) - u(X_{s-}(w)))^2) 1_{I(\zeta)}(t, w)$. Then, $[M^{[u],d}] \in (\mathcal{A}_{loc,0})^{I(\zeta)}$ (cf. [11, §8.3]) and is a local AF. Therefore $\langle M^{[u],d} \rangle_t = (\int_0^t \int_{E_\Delta} (\tilde{u}(X_s) - \tilde{u}(y))^2 N(X_s, dy) dH_s) 1_{I(\zeta)}$ is a PCAF on $I(\zeta)$ and can be extended to a PCAF by [2, Remark 2.2]. By Lemma 1.1, its Revuz measure $\mu_u'(dx) = \int_{E_\Delta} (\tilde{u}(x) - \tilde{u}(y))^2 N(x, dy) \mu_H(dx)$ is a smooth measure. Thus $\mu_u(dx) = \int_E (\tilde{u}(x) - \tilde{u}(y))^2 J(dy, dx)$, which is controlled by $\mu_u'(dx)$, is also a smooth measure. This implies that u satisfies Condition (S). □

1.3. Local energies of $M^{[u]}$ and $N^{[u]}$. Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption 1.3. Suppose $u \in D(\mathcal{E})_{loc}$ and u satisfies Condition (S). Then, by Theorem 1.4, u admits the Fukushima type decomposition (4). In this subsection, we study the local energies of $M^{[u]}$ and $N^{[u]}$. Our result shows that $N^{[u]}$ is locally of zero energy in the weak* sense (cf. Theorem 1.15 (ii)) below. We are grateful to the referee whose comments stimulated us to study this subject.

We continue with the above setting for Subsections 1.1 and 1.2. Let $B = (B_t)_{t \ge 0}$ be a local AF of X and V a quasi-open set of E. Define

$$\bar{h}^{V,*} := e^{-2} \hat{T}_1^V (\hat{G}_2^V \phi),$$

and

$$e^{V,*}(B) := \lim_{t \downarrow 0} \frac{1}{2t} E_{\bar{h}^{V,*} \cdot m}(B_{t \wedge \tau_V}^2)$$

whenever the limit exists in $[0, \infty]$. In this paper $e^{V,*}(B)$ is called the *local energy of B on V in the weak* sense*.

One can check that $\bar{h}^{V,*} \in D(\mathcal{E})_V$ and $\bar{h}^{V,*} \le \bar{h}^V$ (cf. (3)).

THEOREM 1.15. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption* 1.3. *Suppose $u \in D(\mathcal{E})_{loc}$ and u satisfies Condition* (*S*). *Let $M^{[u]}$ and $N^{[u]}$ be the martingale and the zero quadratic variation parts of the Fukushima type decomposition* (4), *respectively. Then, there exists an $\{E_n\} \in \Theta$ such that for $n \in \mathbb{N}$,*

(i) $\{M^{[u]}_{t\wedge\tau_{E_n}}\}$ *is a* P_x*-square-integrable martingale for* $\mathcal{E}$*-q.e.* $x\in E$ *and* $e^{E_n,*}(M^{[u]})<\infty$.

(ii) $E_x[(N^{[u]}_{t\wedge\tau_{E_n}})^2]<\infty$ *for* $t\geq 0$, $\mathcal{E}$*-q.e.* $x\in E$, *and* $e^{E_n,*}(N^{[u]})=0$.

PROOF. Let $\{V_n\}$, $\{E_n\}$, $\{f_n\}$ and $\{u_n\}$, etc. be defined the same as in Subsection 1.2. We first prove Assertion (i). From the above proof of Theorem 1.4, we find that $\{M^{[u]}_{t\wedge\tau_{E_n}}\}$ is a P_x-square-integrable martingale for $\mathcal{E}$-q.e. $x\in E$ and $n\in\mathbb{N}$. Denote by $\mu^c_{\langle u_nf_n\rangle}$ the Revuz measure of $\langle M^{[u_nf_n],c}\rangle$ w.r.t. X^{V_n}; and denote by $\mu^{n,c}_{\langle u\rangle}$ and $\mu^{n,d}_{\langle u\rangle}$ the Revuz measures of $\langle M^{[u],c}_{\cdot\wedge\tau_{E_n}}\rangle$ and $\langle M^{[u],d}_{\cdot\wedge\tau_{E_n}}\rangle$ w.r.t. X^{E_n}, respectively. By the proof of Theorem 1.4 and [5, Proposition 4.1.10] (note that the assertion of the latter holds true also in the setting of semi-Dirichlet forms), we get

$$\mu^{n,c}_{\langle u\rangle}(dx)=1_{E_n}(x)\cdot\mu^c_{\langle u_nf_n\rangle}(dx)\,, \tag{11}$$

and

$$\mu^{n,d}_{\langle u\rangle}(dx)=1_{E_n}(x)\cdot\left(2\int_E(\tilde u(x)-\tilde u(y))^2J(dy,dx)+\tilde u^2(x)K(dx)\right). \tag{12}$$

Then, we obtain by Lemma 1.1, (11), (12) and Lemma 3.2 below that

$$\begin{aligned}
e^{E_n,*}(M^{[u]})&=\lim_{t\downarrow 0}\frac{1}{2t}\Big\{E_{\bar h^{E_n,*}\cdot m}\big[\langle M^{[u],c}_{\cdot\wedge\tau_{E_n}}\rangle\big]+E_{\bar h^{E_n,*}\cdot m}\big[\langle M^{[u],d}_{\cdot\wedge\tau_{E_n}}\rangle\big]\Big\}\\
&=\frac12\int_{E_n}\widetilde{\bar h^{E_n,*}}(x)\mu^c_{\langle u_nf_n\rangle}(dx)+\int_{E_n}\widetilde{\bar h^{E_n,*}}(x)\int_E(\tilde u(x)-\tilde u(y))^2J(dy,dx)\\
&\quad+\frac12\int_{E_n}\widetilde{\bar h^{E_n,*}}(x)\tilde u^2(x)K(dx)\\
&\leq\mathcal{E}(u_nf_n,u_nf_n\hat G_2^{E_n}\phi)-\frac12\mathcal{E}((u_nf_n)^2,\hat G_2^{E_n}\phi)+\|\tilde{\bar h}|_{E_n}\|_\infty\mu_u(E_n)\\
&\quad+\frac12\|\tilde{\bar h}|_{E_n}\|_\infty\|\widetilde{u_nf_n}\|^2_\infty K(E_n)\\
&<\infty,
\end{aligned}$$

which verifies Assertion (i).

We now prove Assertion (ii). By (10), we get

$$N^{[u]}_{t\wedge\tau_{E_n}}=N^{n,[u_nf_n]}_{t\wedge\tau_{E_n}}+A^p_t\,, \tag{13}$$

where $A_t=[\tilde u(X_{\tau_{E_n}})-\widetilde{u_nf_n}(X_{\tau_{E_n}})]1_{\{\tau_{E_n}\leq t\}}$. From the proof of Theorem 1.4, we find that $E_\nu[A_t^2]<\infty$ for $t\geq 0$ and $\nu\in S^*_{00}$ satisfying $\nu(E)<\infty$. Then, we obtain by [16, Theorem A.3] that $E_x[(N^{[u]}_{t\wedge\tau_{E_n}})^2]<\infty$ for $t\geq 0$ and $\mathcal{E}$-q.e. $x\in E$.

Note that $(\hat G_2^{E_n}\phi)\cdot m\in\hat S^*_{00}$ (cf. (9)) and $\int_E\hat G_2^{E_n}\phi dm<\infty$. Hence $E_{(\hat G_2^{E_n}\phi)\cdot m}[(A^p)^2_t]<\infty$ for $t\geq 0$, which implies that the quadratic variation of A^p vanishes in $L^1(P_{(\hat G_2^{E_n}\phi)\cdot m})$. By (8) and the boundedness of u_nf_n, we find that the quadratic variation of $N^{n,[u_nf_n]}_{\cdot\wedge\tau_{E_n}}$ vanishes in $L^1(P_{\bar h_n\cdot m})$ and hence in $L^1(P_{(\hat G_2^{E_n}\phi)\cdot m})$. Thus, we obtain by (13) that the quadratic variation of $N^{[u]}_{\cdot\wedge\tau_{E_n}}$ vanishes in $L^1(P_{(\hat{G}_2^{E_n}\phi)\cdot m})$, i.e.,

$$E_{(\hat{G}_2^{E_n}\phi)\cdot m}\left\{\sum_{k=0}^{[T/\varepsilon_l]}(N^{[u]}_{\{(k+1)\varepsilon_l\}\wedge\tau_{E_n}}-N^{[u]}_{\{k\varepsilon_l\}\wedge\tau_{E_n}})^2\right\}\to 0 \quad \text{as } l\to\infty\,, \tag{14}$$

for any $T>0$ and any sequence $\{\varepsilon_l\}_{l\in\mathbb{N}}$ converging to 0.

Note that $A_t^p = A^p_{t\wedge\tau_{E_n}}$ for $t\geq 0$ and $n\in\mathbb{N}$. By (14), we get

$$\begin{aligned}
0 &= \lim_{s\downarrow 0}\sum_{k=0}^{[1/s]}E_{(\hat{G}_2^{E_n}\phi)\cdot m}\big[(N^{[u]}_{\{(k+1)s\}\wedge\tau_{E_n}}-N^{[u]}_{\{ks\}\wedge\tau_{E_n}})^2\big]\\
&= \lim_{s\downarrow 0}\sum_{k=0}^{[1/s]}\int_{E_n}\hat{T}^{E_n}_{ks}\hat{G}_2^{E_n}\phi(x)E_x\big[(N^{[u]}_{s\wedge\tau_{E_n}})^2\big]m(dx)\\
&\geq \lim_{s\downarrow 0}\sum_{k=0}^{[1/s]}\int_{E_n}\bar{h}^{E_n,*}(x)E_x\big[(N^{[u]}_{s\wedge\tau_{E_n}})^2\big]m(dx)\\
&= 2e^{E_n,*}(N^{[u]})\,.
\end{aligned}$$

The proof is complete. □

REMARK 1.16. Let $g\in D(\mathcal{E})_{E_n}$ be a γ-co-excessive function ($\gamma\geq 0$) of X^{E_n}. By Lemma 1.1, (11) and (12) (cf. the above proof of Theorem 1.15), we obtain that

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}E_{g\cdot m}[(\tilde{u}(X_{t\wedge\tau_{E_n}})-\tilde{u}(X_0))^2]\\
&\qquad=\int_{E_n}\tilde{g}(x)\mu^c_{\langle u\rangle}(dx)+2\int_{E_n}\tilde{g}(x)\int_E(\tilde{u}(x)-\tilde{u}(y))^2J(dy,dx)\\
&\qquad\quad+\int_{E_n}\tilde{g}(x)\tilde{u}^2(x)K(dx)\,,
\end{aligned}\tag{15}$$

where $\mu^c_{\langle u\rangle}$ is the Revuz measure of $\langle M^{[u],c}\rangle$. (15) seems to be a new localization formula, which can be compared with the localization formula for symmetric Markov processes obtained by Chen and Fukushima recently (see [4, Theorem 1.1]).

2. Remarks on stochastic sets of interval type. For the convenience of the reader, we recall first some concepts and results concerning sets of interval type given in [11, §8.3]. Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual condition. A subset $B\subset\Omega\times[0,\infty)$ is said to be a set of interval type if there exists a nonnegative random variable T such that for each $\omega\in\Omega$, the section B_ω is either $[0,T(\omega)[$ or $[0,T(\omega)]$ and $B_\omega\neq\emptyset$. B is called an optional (resp. predictable) set of interval type, if it is an optional (resp. predictable) set and is of interval type.

Let B be an optional set of interval type. A stochastic process Y defined on B is called a special semi-martingale on B, denoted by $(\mathcal{S}_p)^B$, if there exist a sequence of increasing stopping times $\{T_n\}$ with $T_n\uparrow T$ (T is the debut of B^c), and a sequence of special semi-martingales

$\{Y^n\}$ such that, $\bigcup_n [\![0, T_n]\!] \supset B$ and for each n and $t > 0$, $(Y 1_B)_{t\wedge T_n} = (Y^n 1_B)_{t\wedge T_n}$. In the same manner one can define local martingale on B (denoted by $(\mathcal{M}_{loc})^B$), adapted process with locally integrable variation on B (denoted by $(\mathcal{A}_{loc})^B$), and others (cf. [11, Definition 8.19]).

The assertion below, which is referred as the Doob-Meyer decomposition on sets of interval type, was stated in [11, Theorem 8.26].

Assertion. Let B be an optional set of interval type and $Y \in (\mathcal{S}_p)^B$. Then Y can be uniquely decomposed as: $Y = M + A$, where $M \in (\mathcal{M}_{loc})^B$ and $A \in (\mathcal{A}_{loc,0})^B$ is a predictable process (i.e., A is the restriction of a predictable process on B.).

Although the above assertion has been employed by several papers (including our previous paper [16]), during the course of our research we observed the following remark.

REMARK 2.1. In the above assertion if B is not a predictable set of interval type, then the uniqueness of the decomposition $Y = M + A$ may fail to be true.

PROOF. We take just the counterexample stated in [11, Remark 8.24] to illustrate our remark. Let $T > 0$ be a totally inaccessible time with $P(T < \infty) > 0$, e.g., the first jump time of a Poisson process. We consider the stochastic interval $B = [\![0, T[\![$. Then B is an optional set of interval type but not a predictable set. Let $A_t := 1_{[\![T,\infty[\![}(t)$ and $\tilde{A}_t$ be its dual predictable projection. Let $\{Y_t, 0 \le t < T\}$ be the restriction of $\tilde{A}$ on B. Then we have decomposition $Y = M + 0$ where $M \in (\mathcal{M}_{loc})^B$ is the restriction of $\tilde{A} - A$ on B. But we have also another decomposition $Y = 0 + Y$ where $Y \in (\mathcal{A}_{loc,0})^B$ is the restriction of $\tilde{A}$ on B. Therefore the decomposition stated in the above assertion is not unique. □

The above remark reveals that the Doob-Meyer decomposition may fail to be unique on an optional set of interval type. In the same manner, we observe that the Fukushima type decomposition may fail to be unique on an optional set of interval type. Note that with the notation of Theorem 1.4, $[\![0, \zeta[\![$ is an optional set of interval type but is not necessarily a predictable set.

REMARK 2.2. In Theorem 1.4 if we use $\mathcal{M}_{loc}^{[\![0,\zeta[\![}$ instead of $\mathcal{M}_{loc}^{I(\zeta)}$, then the uniqueness of the decomposition may fail to be true.

PROOF. We provide below a counterexample to illustrate the remark. Suppose that we have a decomposition

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \ge 0, \quad P_x\text{-a.s.} \quad \text{for } \mathcal{E}\text{-q.e. } x \in E,$$

with $M^{[u]} \in \mathcal{M}_{loc}^{[\![0,\zeta[\![}$ and $N^{[u]} \in \mathcal{L}_c$, and suppose that $\zeta_i = \zeta$ with $P_x(\zeta < \infty) > 0$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. We write $A_t := 1_{\{\zeta \le t\}}$ (i.e., $A_t = I_\Delta(X_t)$) and denote by $\tilde{A}_t$ the dual predictable projection of A_t. Then it is clear that $\tilde{A} \in \mathcal{L}_c$. But we have also $\tilde{A} \in (\mathcal{M}_{loc})^{[\![0,\zeta[\![}$, because $\{\tilde{A} 1_{[\![0,\zeta[\![}\}^\zeta = \{(\tilde{A} - A) 1_{[\![0,\zeta[\![}\}^\zeta$. Therefore, we have another decomposition:

$$\tilde{u}(X_t) - \tilde{u}(X_0) = (M_t^{[u]} - \tilde{A}_t) + (N_t^{[u]} + \tilde{A}_t), \quad t \ge 0, \quad P_x\text{-a.s.} \quad \text{for } \mathcal{E}\text{-q.e. } x \in E,$$

which violates the uniqueness. □

With the above discussion, we see that the existence of a suitable predictable set of interval type is important for the uniqueness of the Fukushima type decomposition. Fortunately in Theorem 1.4 we find such a suitable set $I(\zeta) = [\![0, \zeta[\![\cup[\![\zeta_i]\!]$. In Proposition 2.4 below we shall provide a proof for the existence and uniqueness of such ζ_i. We shall need the following characterizations for a set of interval type to be predictable. For their proofs we refer to [11].

LEMMA 2.3 ([11, Theorems 8.18]). *The following statements are equivalent:*

(i) *B is a predictable set of interval type.*

(ii) $1_B = 1_F 1_{[\![0,T[\![} + 1_{F^c} 1_{[\![0,T]\!]}$, *where T is a stopping time, $F \in \mathcal{F}_{T-}$ and $T_F > 0$ is a predictable time.*

(iii) $B = \bigcup_n [\![0, T_n]\!]$, *where $\{T_n\}$ is an increasing sequence of stopping times.*

Below we consider a quasi-regular semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E; m)$. Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with lifetime ζ be the associated m-tight special standard process.

PROPOSITION 2.4. (i) *There exists an $\{\mathcal{F}_t\}$-stopping time ζ_i (may be identically equal to ∞) which is the totally inaccessible part of ζ w.r.t. P_x for $\mathcal{E}$-q.e. $x \in E$. Such a ζ_i is unique in the sense of P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$.*

(ii) *Denote by $I(\zeta) := [\![0, \zeta[\![\cup[\![\zeta_i]\!]$. Then $I(\zeta)$ is a predictable set of interval type, and there exists a sequence $\{V_n\} \in \Theta$ such that for any $\{U_n\} \in \Theta$, $I(\zeta) = \bigcup_n [\![0, \tau_{V_n\cap U_n}]\!]$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$.*

PROOF. The uniqueness of ζ_i follows from [11, Theorem 4.20]. Below we show the existence of ζ_i and the assertion (ii). By the local compactification method (cf. [12, Theorem 3.5], see also [18, Theorem VI.1.6]) in the setting of semi-Dirichlet forms, we may assume without loss of generality that $(X_t)_{t\geq 0}$ is a Hunt process and E is a locally compact separable metric space.

We take a fixed sequence $\{V_n\} \in \Theta$ such that each V_n is a relatively compact open set and $E = \bigcup_n V_n$. Denote by $B := \bigcup_n [\![0, \tau_{V_n}]\!]$ and $T := \lim_{n\to\infty} \tau_{V_n}$. Set $F = \{\omega \mid T(\omega) < \infty, (\omega, T(\omega)) \in B^c\}$. By Lemma 2.3, for each P_x, it holds that B is a predictable set of interval type, T is an $\{\mathcal{F}_t\}$-stopping time, $F \in \mathcal{F}_{T-}$, $T_F := T I_F + (+\infty) I_{F^c}$ is a predictable time, and $1_B = 1_F 1_{[\![0,T[\![} + 1_{F^c} 1_{[\![0,T]\!]} = 1_{[\![0,T[\![} + 1_{[\![T_{F^c}]\!]}$. Let ζ be the lifetime of $(X_t)_{t\geq 0}$, we define

$$\zeta_i = \zeta_{F^c} := \zeta I_{F^c} + (+\infty) I_F .$$

Note that for $\mathcal{E}$-q.e. $x \in E$, we have $\tau_{V_n} \uparrow \zeta = T$ P_x-a.s., therefore $I(\zeta) = [\![0, \zeta[\![\cup[\![\zeta_i]\!] = [\![0, T[\![\cup[\![T_{F^c}]\!] = B$ is a predictable set of interval type. Moreover, by the quasi-left continuity of Hunt process and the assumption that V_n has compact closure, we find that for any n and $x \in E$, $P_x\{S = \tau_{V_n} = \zeta < \infty\} = 0$ for any predictable time S. Hence $\zeta_i = T_{F^c}$ is the totally inaccessible part of ζ w.r.t. P_x for $\mathcal{E}$-q.e. $x \in E$. Finally, for arbitrary $\{U_n\} \in \Theta$, we have $\tau_{V_n\cap U_n} \uparrow \zeta = T$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Therefore $I(\zeta) = \bigcup_n [\![0, \tau_{V_n\cap U_n}]\!]$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$, which completes the proof. □

REMARK 2.5. We thank Professor Z.Q. Chen who brought to our attention that $I(\zeta)$ had been used in [3]. In a private communication Chen told us that they were aware of the distinction between $I(\zeta)$ and $[\![0, \zeta[\![$, and the authors of the paper [3] had made such a point in Section 3 of [3]. Somehow after several iterations, that point got lost in the paper [2].

3. Transformation formula for MAFs. In this section, we give a transformation formula for MAFs. We adopt the setting of Section 1. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption 1.3. From the proof of Theorem 1.4, we can see that $M^{[u],c}$ is well defined whenever $u \in D(\mathcal{E})_{loc}$. Below is the main result of this section.

THEOREM 3.1. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption* 1.3. *Let $m \in \mathbb{N}$, $\Phi \in C^1(\mathbb{R}^m)$, and $u = (u_1, u_2, \ldots, u_m)$ with $u_i \in D(\mathcal{E})_{loc}$, $1 \le i \le m$. Then $\Phi(u) \in D(\mathcal{E})_{loc}$ and*

$$M^{[\Phi(u)],c} = \sum_{i=1}^{m} \Phi_{x_i}(u) \cdot M^{[u_i],c} \quad on \ \ I(\zeta)\,, \qquad P_x\text{-}a.s. \ for\ \mathcal{E}\text{-}q.e. \ x \in E\,.$$

The proof of the theorem will be accomplished at the end of this section by employing Theorem 3.3 below.

We fix a $\{V_n\} \in \Theta$ satisfying Assumption 1.3 and such that $\tilde{\hat{h}}$ is bounded on each V_n. Let X^{V_n}, $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$, $\bar{h}_n$, etc. be the same as in Section 1. For $u \in D(\mathcal{E})_{V_n,b}$, we denote by $\mu^{(n)}_{\langle u\rangle}$ the Revuz measure of $\langle M^{n,[u]}\rangle$. For $u, v \in D(\mathcal{E})_{V_n,b}$, we define

$$\mu^{(n)}_{\langle u,v\rangle} := \frac{1}{2}(\mu^{(n)}_{\langle u+v\rangle} - \mu^{(n)}_{\langle u\rangle} - \mu^{(n)}_{\langle v\rangle})\,. \tag{16}$$

Similar to [16, Lemma 3.1], we can prove the following lemma.

LEMMA 3.2. *Let $u, v, f \in D(\mathcal{E})_{V_n,b}$. Then*

$$\int_{V_n} \tilde{f} d\mu^{(n)}_{\langle u,v\rangle} = \mathcal{E}(u, vf) + \mathcal{E}(v, uf) - \mathcal{E}(uv, f)\,.$$

For $u \in D(\mathcal{E})_{V_n,b}$, we denote by $M^{n,[u],c}$ and $M^{n,[u],d}$ the continuous and purely discontinuous parts of $M^{n,[u]}$, respectively; and denote by $\mu^{n,c}_{\langle u\rangle}$ and $\mu^{n,d}_{\langle u\rangle}$ the Revuz measures of $\langle M^{n,[u],c}\rangle$ and $\langle M^{n,[u],d}\rangle$, respectively. Then $M^{n,[u]} = M^{n,[u],c} + M^{n,[u],d}$ and

$$\mu^{(n)}_{\langle u\rangle} = \mu^{n,c}_{\langle u\rangle} + \mu^{n,d}_{\langle u\rangle}\,. \tag{17}$$

Let $(N^{(n)}(x, dy), H^{(n)})$ be a Lévy system of X^{V_n} and $\nu^{(n)}$ the Revuz measure of $H^{(n)}$. Similar to [8, (5.3.7)–(5.3.10)], we can show that

$$\begin{aligned}\langle M^{n,[u],d}\rangle_t &= \Bigg(\sum_{0<s\le t} (\triangle M^{n,[u],d}_s)^2\Bigg)^p \\ &= \int_0^t \int_{V_n\cup\{\Delta\}} (\tilde{u}(x) - \tilde{u}(y))^2 N^{(n)}(X^{V_n}_s, \Delta) dH^{(n)}_s \end{aligned} \tag{18}$$

and

$$\mu^{n,d}_{\langle u\rangle}(dx)=\int_{V_n\cup\{\Delta\}}(\tilde u(x)-\tilde u(y))^2N^{(n)}(x,dy)\nu^{(n)}(dx)\,, \tag{19}$$

where $\triangle M^{n,[u],d}_s=M^{n,[u],d}_s-M^{n,[u],d}_{s-}$. For $u,v\in D(\mathcal{E})_{V_n,b}$, we define

$$\mu^{n,c}_{\langle u,v\rangle}:=\frac12(\mu^{n,c}_{\langle u+v\rangle}-\mu^{n,c}_{\langle u\rangle}-\mu^{n,c}_{\langle v\rangle})\,,\quad \mu^{n,d}_{\langle u,v\rangle}:=\frac12(\mu^{n,d}_{\langle u+v\rangle}-\mu^{n,d}_{\langle u\rangle}-\mu^{n,d}_{\langle v\rangle})\,. \tag{20}$$

THEOREM 3.3. *Let $u,v,w\in D(\mathcal{E})_{V_n,b}$. Then*

$$d\mu^{n,c}_{\langle uv,w\rangle}=\tilde u d\mu^{n,c}_{\langle v,w\rangle}+\tilde v d\mu^{n,c}_{\langle u,w\rangle}\,. \tag{21}$$

PROOF. The argument for the proof of this theorem is similar to that of [16, Theorem 3.2]. We shall only emphasize the differences caused by the jump part.

By quasi-homeomorphism (cf. [12, Theorem 3.8]) and the polarization identity, (21) holds for $u,v,w\in D(\mathcal{E})_{V_n,b}$ is equivalent to

$$\int_{V_n}\tilde f d\mu^{n,c}_{\langle u^2,w\rangle}=2\int_{V_n}\tilde f\tilde u d\mu^{n,c}_{\langle u,w\rangle}\,,\quad \forall f,u,w\in D(\mathcal{E})_{V_n,b}\,. \tag{22}$$

For $u,w\in D(\mathcal{E})_{V_n,b}$, we define

$$\eta^{(n)}_{u,w}(dx)=\int_{V_n\cup\{\Delta\}}(\tilde u(x)-\tilde u(y))^2(\tilde w(x)-\tilde w(y))N^{(n)}(x,dy)\nu^{(n)}(dx)\,.$$

Then, by (16)–(20), we find that (22) is equivalent to

$$\int_{V_n}\tilde f d\mu^{(n)}_{\langle u^2,w\rangle}=2\int_{V_n}\tilde f\tilde u d\mu^{(n)}_{\langle u,w\rangle}+\int_{V_n}\tilde f d\eta^{(n)}_{u,w},\ \ \forall f,u,w\in D(\mathcal{E})_{V_n,b}\,. \tag{23}$$

For $k\in\mathbb{N}$, we define $v_k:=kR^{V_n}_{k+1}u$. Then $v_k\to u$ in $D(\mathcal{E})_{V_n}$ as $k\to\infty$. By Assumption 1.3 and [18, Corollary I.4.15], we can show that $\sup_{k\ge1}\mathcal{E}(v_kw,v_kw)<\infty$. Then, by [18, Lemma I.2.12], there exists a subsequence $\{(v_{k_l})\}_{l\in\mathbb{N}}$ of $\{v_k\}_{k\in\mathbb{N}}$ such that $u_kw\to uw$ in $D(\mathcal{E})_{V_n}$ as $k\to\infty$, where $u_k:=\frac1k\sum_{l=1}^k v_{k_l}$. Note that $u_k\to u$ in $D(\mathcal{E})_{V_n}$ as $k\to\infty$ and $\|u_k\|_\infty\le\|u\|_\infty$ for $k\in\mathbb{N}$. Moreover, $\|L^{V_n}u_k\|_\infty<\infty$ for $k\in\mathbb{N}$, where L^{V_n} is the generator of X^{V_n}. For $k,l\in\mathbb{N}$, we define $f_k:=f\wedge(k\bar h_n)$ and $f_{k,l}:=l\hat G^{V_n}_{l+1}f_k$.

Similar to [16, Theorem 3.2], to prove (23), we may assume without loss of generality that $f\ge0$, $u=u_k$ and $f=f_{k,l}$.

For $0<\delta<1$, we have

$$\begin{aligned}
&\lim_{t\downarrow0}\frac1t E_{f_{k,l}\cdot m}[\langle M^{n,[u_k]}\rangle_t^2]\\
&=\lim_{t\downarrow0}\frac2t E_{f_{k,l}\cdot m}\left[\int_0^t\langle M^{n,[u_k]}\rangle_{(t-s)}\circ\theta_s d\langle M^{n,[u_k]}\rangle_s\right]\\
&=\lim_{t\downarrow0}\frac2t E_{f_{k,l}\cdot m}\left[\int_0^t E_{X^{V_n}_s}\big[\langle M^{n,[u_k]}\rangle_{(t-s)}\big]d\langle M^{n,[u_k]}\rangle_s\right]\\
&\le 2\Big\langle E_\cdot\big[\langle M^{n,[u_k]}\rangle_\delta\big]\cdot\mu^{(n)}_{\langle u_k\rangle},\widetilde{f_{k,l}}\Big\rangle.
\end{aligned}$$

Note that by our choice of u_k, there exists a constant $D_k > 0$ such that $E_x(\langle M^{n,[u_k]}\rangle_\delta) = E_x[(M_\delta^{n,[u_k]})^2] = E_x[(\widetilde{u_k}(X_\delta^{V_n}) - \widetilde{u_k}(X_0^{V_n}) - \int_0^\delta L^{V_n}u_k(X_s^{V_n})ds)^2] \le D_k$ for $\mathcal{E}$-q.e. $x \in V_n$. Letting $\delta \to 0$, we obtain by the dominated convergence theorem that

$$\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[\langle M^{n,[u_k]}\rangle_t^2] = 0\,. \tag{24}$$

We have

$$\begin{aligned}
\int_{V_n}\widetilde{f_{k,l}}d\mu^{(n)}_{\langle u_k^2,w\rangle} &= \lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\big[\langle M^{n,[u_k^2]}, M^{n,[w]}\rangle_t\big]\\
&= \lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\Big[\big(\widetilde{u_k}^2(X_t^{V_n}) - \widetilde{u_k}^2(X_0^{V_n})\big)\big(\tilde{w}(X_t^{V_n}) - \tilde{w}(X_0^{V_n})\big)\Big]\\
&= \lim_{t\downarrow 0}\frac{2}{t}E_{(f_{k,l}u_k)\cdot m}\Big[\big(\widetilde{u_k}(X_t^{V_n}) - \widetilde{u_k}(X_0^{V_n})\big)\big(\tilde{w}(X_t^{V_n}) - \tilde{w}(X_0^{V_n})\big)\Big]\\
&\quad + \lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\Big[\big(\widetilde{u_k}(X_t^{V_n}) - \widetilde{u_k}(X_0^{V_n})\big)^2\big(\tilde{w}(X_t^{V_n}) - \tilde{w}(X_0^{V_n})\big)\Big]\\
&:= \lim_{t\downarrow 0}[I(t) + II(t)]\,.
\end{aligned}$$

Similar to [16, Theorem 3.2], we can show that

$$\lim_{t\downarrow 0} I(t) = 2\int_{V_n}\widetilde{f_{k,l}}\widetilde{u_k}d\mu^{(n)}_{\langle u_k,w\rangle}\,.$$

Note that

$$\begin{aligned}
\lim_{t\downarrow 0} II(t) &= \lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\big[(M_t^{n,[u_k],c})^2M_t^{n,[w]}\big]\\
&\quad +2\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\big[(M_t^{n,[u_k],c})(M_t^{n,[u_k],d})M_t^{n,[w],c}\big]\\
&\quad +2\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\big[(M_t^{n,[u_k],c})(M_t^{n,[u_k],d})M_t^{n,[w],d}\big]\\
&\quad +\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\big[(M_t^{n,[u_k],d})^2M_t^{n,[w],c}\big]\\
&\quad +\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\big[(M_t^{n,[u_k],d})^2M_t^{n,[w],d}\big]\\
&:= \lim_{t\downarrow 0}\{III_1(t) + 2III_2(t) + 2III_3(t) + III_4(t) + IV(t)\}\,.
\end{aligned}$$

Similar to [16, Theorem 3.2], we can show that

$$\lim_{t\downarrow 0} III_1(t) = 0\,. \tag{25}$$

By Itô's formula and the orthogonality of the continuous and purely discontinuous martingales, we get

$$\lim_{t\downarrow 0}|III_2(t)| \le \left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\left[\langle M^{n,[u_k],c}, M^{n,[w],c}\rangle_t^2\right]\right\}^{\frac{1}{2}}$$
$$\cdot\left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[M_t^{n,[u_k],d}]^2\right\}^{\frac{1}{2}}.$$

Similar to (25), we can show that $\lim_{t\downarrow 0} III_2(t) = 0$.

By Itô's formula and the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\lim_{t\downarrow 0}|III_4(t)| &= \lim_{t\downarrow 0}\left|\frac{1}{t}E_{f_{k,l}\cdot m}\left\{\sum_{0<s\le t} M_s^{n,[w],c}(\triangle M_s^{n,[u_k],d})^2\right\}\right| \\
&= \lim_{t\downarrow 0}\left|\frac{1}{t}E_{f_{k,l}\cdot m}\left\{\int_0^t M_s^{n,[w],c} d\langle M^{n,[u_k],d}\rangle_s\right\}\right| \\
&\le \lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\{M_t^{n,[w],c,*}\langle M^{n,[u_k],d}\rangle_t\} \\
&\le \left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}(M_t^{n,[w],c,*})^2\right\}^{\frac{1}{2}}\left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\left(\langle M^{n,[u_k],d}\rangle_t\right)^2\right\}^{\frac{1}{2}} \\
&\le C\left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}(M_t^{n,[w],c})^2\right\}^{\frac{1}{2}}\left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\left(\langle M^{n,[u_k],d}\rangle_t\right)^2\right\}^{\frac{1}{2}} \\
&= C\left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\langle M^{n,[w],c}\rangle_t\right\}^{\frac{1}{2}}\left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\left(\langle M^{n,[u_k],d}\rangle_t\right)^2\right\}^{\frac{1}{2}},
\end{aligned}$$

where $M_t^{n,[w],c,*}$ denotes the maximum of $\{M_s^{n,[w],c}, 0\le s\le t\}$ and C is a positive constant. Hence $\lim_{t\downarrow 0} III_4(t) = 0$. Similarly, we can show that $\lim_{t\downarrow 0} III_3(t) = 0$.

Finally, we estimate $IV(t)$. By Itô's formula and the dual predictable projection, we get

$$\begin{aligned}
IV(t) &= \frac{1}{t}E_{f_{k,l}\cdot m}\left\{(M_t^{n,[u_k],d})^2 M_t^{n,[w],d}\right\} \\
&= \frac{1}{t}E_{f_{k,l}\cdot m}\left\{\sum_{0<s\le t}(M_s^{n,[u_k],d})^2 M_s^{n,[w],d} - (M_{s-}^{n,[u_k],d})^2 M_{s-}^{n,[w],d}\right. \\
&\quad \left. -2M_{s-}^{n,[u_k],d}M_{s-}^{n,[w],d}(M_s^{n,[u_k],d} - M_{s-}^{n,[u_k],d}) - (M_{s-}^{n,[u_k],d})^2(M_s^{n,[w],d} - M_{s-}^{n,[w],d})\right\} \\
&= \frac{1}{t}E_{f_{k,l}\cdot m}\left\{\sum_{0<s\le t}(\triangle M_s^{n,[u_k],d})^2 \triangle M_s^{n,[w],d}\right. \\
&\quad \left. + \sum_{0<s\le t} M_{s-}^{n,[w],d}(\triangle M_s^{n,[u_k],d})^2 + \sum_{0<s\le t} M_{s-}^{n,[u_k],d} \triangle M_s^{n,[u_k],d} \triangle M_s^{n,[w],d}\right\} \\
&= \frac{1}{t}E_{f_{k,l}\cdot m}\left\{\int_0^t\int_{V_n\cup\{\Delta\}}\left(u_k(X_s^{V_n}) - u_k(y)\right)^2\left(w(X_s^{V_n}) - w(y)\right)N^{(n)}(X_s^{V_n}, dy)dH_s^{(n)}\right.
\end{aligned}$$

$$+\sum_{0<s\le t} M_{s-}^{n,[w],d}(\triangle M_s^{n,[u_k],d})^2+\sum_{0<s\le t} M_{s-}^{n,[u_k],d}\triangle M_s^{n,[u_k],d}\triangle M_s^{n,[w],d}\Bigg\}$$

$$:= IV_1(t)+IV_2(t)+IV_3(t)\,.$$

We have

$$\lim_{t\downarrow 0} IV_1(t)=\int_{V_n} f_{k,l}d\eta^{(n)}_{u_k,w},$$

and by Lemma 1.12 and (24),

$$\begin{aligned}\lim_{t\downarrow 0}|IV_2(t)| &= \lim_{t\downarrow 0}\left|\frac{1}{t}E_{f_{k,l}\cdot m}\left\{\int_0^t M_{s-}^{n,[w],d}d\langle M^{n,[u_k],d}\rangle_s\right\}\right|\\ &\le \lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\big\{(M_t^{n,[w],d,*})\langle M^{n,[u_k],d}\rangle_t\big\}\\ &\le C\left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\langle M^{n,[w],d}\rangle_t\right\}^{\frac12}\left\{\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}\langle M^{n,[u_k],d}\rangle_t^2\right\}^{\frac12}\\ &=0\,,\end{aligned}$$

where $M_t^{n,[w],d,*}$ denotes the maximum of $\{M_s^{n,[w],d}, 0\le s\le t\}$. Similarly, we get $\lim_{t\downarrow 0} IV_3(t)=0$. Therefore, the proof is complete. □

PROOF OF THEOREM 3.1. By virtue of Theorem 3.3, following the argument of the proof of [16, Theorem 3.10], we can prove Theorem 3.1. We omit the details here. □

4. Examples. In this section, we consider some concrete examples. Note that our Theorems 1.4 and 3.1 are generalization of the corresponding results of [16], which were only given for local semi-Dirichlet forms without jump.

EXAMPLE 4.1 (see [9] and cf. also [22]). Let (E,d) be a locally compact separable metric space, m a positive Radon Measure on E with full topological support, and $k(x,y)$ a nonnegative Borel measurable function on $\{(x,y)\in E\times E\,|\,x\neq y\}$. Set $k_s(x,y)=\frac12(k(x,y)+k(y,x))$ and $k_a(x,y)=\frac12(k(x,y)-k(y,x))$. Denote by $C_0^{lip}(E)$ the family of all uniformly Lipschitz continuous functions on E with compact support. Suppose that the following conditions hold:

(A.I) $x\mapsto\int_{y\neq x}(1\wedge d(x,y)^2)k_s(x,y)m(dy)\in L^1_{loc}(E;m)$.

(A.II) $\sup_{x\in E}\int_{\{y:k_s(x,y)\neq 0\}}\frac{k_a^2(x,y)}{k_s(x,y)}m(dy)<\infty$.

Define for $u,v\in C_0^{lip}(E)$,

$$\eta(u,v)=\iint_{x\neq y}(u(x)-u(y))(v(x)-v(y))k_s(x,y)m(dx)m(dy)$$

and

$$\mathcal{E}(u,v)=\frac12\eta(u,v)+\iint_{x\neq y}(u(x)-u(y))v(y)k_a(x,y)m(dx)m(dy)\,.$$

Then, there exists $\alpha > 0$ such that $(\mathcal{E}_\alpha, C_0^{lip}(E))$ is closable on $L^2(E; m)$ and its closure $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a regular semi-Dirichlet form on $L^2(E; m)$. Moreover, there exists $C > 1$ such that for any $u \in D(\mathcal{E}_\alpha)$,

$$\frac{1}{C}\eta_\alpha(u, u) \le \mathcal{E}_\alpha(u, u) \le C\eta_\alpha(u, u)\,.$$

Therefore, our Theorems 1.4 and 3.1 hold for any $u \in D(\mathcal{E})_{loc}$ which satisfies Condition (S), in particular, for any $u \in D(\mathcal{E})$ by noting that $|k_a(x, y)| \le k_s(x, y)$.

EXAMPLE 4.2 (see [27]). Let G be an open set of $\mathbb{R}^d$. Suppose that the following conditions hold:

(B.I) There exist $0 < \lambda \le \Lambda$ such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for } x \in G\,,\ \ \xi \in \mathbb{R}^d\,.$$

(B.II) $b_i \in L^d(G; dx)$, $i = 1, \dots, d$.

(B.III) $c \in L_+^{d/2}(G; dx)$.

(B.IV) $x \mapsto \int_{y\neq x}(1 \wedge |x - y|^2)k_s(x, y)dy \in L^1_{loc}(G; dx)$.

(B.V) $\sup_{x\in G}\int_{\{|x-y|\ge 1, y\in G\}}|k_a(x, y)|dy < \infty$, $\sup_{x\in G}\int_{\{|x-y|<1, y\in G\}}|k_a(x, y)|^\gamma dy < \infty$ for some $0 < \gamma \le 1$, and $|k_a(x, y)|^{2-\gamma} \le C_1 k_s(x, y)$, $x, y \in G$ with $0 < |x - y| < 1$ for some constant $C_1 > 0$.

Define for $u, v \in C_0^1(G)$,

$$\begin{aligned}\eta(u, v) &= \frac{1}{2}\sum_{i=1}^{d}\int_G \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)dx \\ &\quad + \frac{1}{2}\iint_{x\neq y}(u(x) - u(y))(v(x) - v(y))k_s(x, y)dxdy\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}(u, v) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_G a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)dx + \sum_{i=1}^{d}\int_G b_i(x)u(x)\frac{\partial v}{\partial x_i}(x)dx \\ &\quad + \int_G u(x)v(x)c(x)dx \\ &\quad + \frac{1}{2}\iint_{x\neq y}(u(x) - u(y))(v(x) - v(y))k_s(x, y)dxdy \\ &\quad + \iint_{x\neq y}(u(x) - u(y))v(x)k_a(x, y)dxdy.\end{aligned}$$

Then, when λ is sufficiently large, there exists $\alpha > 0$ such that $(\mathcal{E}_\alpha, C_0^1(G))$ is closable on $L^2(G; dx)$ and its closure $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a regular semi-Dirichlet form on $L^2(G; dx)$.

Moreover, there exists $C' > 1$ such that for any $u \in D(\mathcal{E}_\alpha)$,

$$\frac{1}{C'}\eta_\alpha(u,u) \leq \mathcal{E}_\alpha(u,u) \leq C'\eta_\alpha(u,u)\,.$$

Therefore, our Theorems 1.4 and 3.1 hold for any $u \in D(\mathcal{E})_{loc}$ which satisfies Condition (S), in particular, for any $u \in D(\mathcal{E})$ by noting that $|k_a(x,y)| \leq k_s(x,y)$.

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REFERENCES

[1] C. Z. CHEN, L. MA AND W. SUN, Stochastic calculus for Markov processes associated with non-symmetric Dirichlet forms, Sci. China Math. 55 (2012), 2195–2203.
[2] Z. Q. CHEN, P. J. FITZSIMMONS, K. KUWAE AND T. S. ZHANG, Stochastic calculus for symmetric Markov processes, Ann. Probab. 36 (2008), 931–970.
[3] Z. Q. CHEN, P.J. FITZSIMMONS, M. TAKEDA, J. YING AND T. S. ZHANG, Absolute continuity of symmetric Markov processes, Ann. Probab. 32 (2004), 2067–2098.
[4] Z. Q. CHEN AND M. FUKUSHIMA, A localization formula in Dirichlet form theory, Proc. Amer. Math. Soc. 140 (2012), 1815–1822.
[5] Z. Q. CHEN AND M. FUKUSHIMA, Symmetric Markov processes, time change, and boundary theory, Princeton University Press, 2012.
[6] Z. Q. CHEN, Z. M. MA AND M. RÖCKNER, Quasi-homeomorphisms of Dirichlet forms, Nagoya Math. J. 136 (1994), 1–15.
[7] M. FUKUSHIMA, A decomposition of additive functionals of finite energy, Nagoya Math. J. 74 (1979), 137–168.
[8] M. FUKUSHIMA, Y. OSHIMA AND M. TAKEDA, Dirichlet forms and symmetric Markov processes, second revised and extended edition, Walter de Gruyter, 2011.
[9] M. FUKUSHIMA AND T. UEMURA, Jump-type Hunt processes generated by lower bounded semi-Dirichlet forms, Ann. Probab. 40 (2012), 858–889.
[10] X.F. HAN, Z.M. MA AND W. SUN, $h\hat{h}$-transforms of positivity preserving semigroups and associated Markov processes, Acta Math. Sin. (Engl. Ser.) 27 (2011), 369–376.
[11] S. W. HE, J. G. WANG AND J. A. YAN, Semimartingale theory and stochastic calculus, Science Press, Beijing, 1992.
[12] Z.C. HU, Z.M. MA AND W. SUN, Extensions of Lévy-Khintchine formula and Beurling-Deny formula in semi-Dirichlet forms setting, J. Funct. Anal. 239 (2006), 179–213.
[13] O. KALLENBERG, Foundations of modern probability, second edition, Springer, 2002.
[14] K. KUWAE, Maximum principles for subharmonic functions via local semi-Dirichlet forms, Can. J. Math. 60 (2008), 822–874.
[15] K. KUWAE, Stochastic calculus over symmetric Markov processes without time reversal, Ann. Probab. 38 (2010), 1532–1569.
[16] L. MA, Z.M. MA AND W. SUN, Fukushima's decomposition for diffusions associated with semi-Dirichlet forms, Stoch. Dyn. 12 (2012), 1250003.
[17] Z. M. MA, L. OVERBECK AND M. RÖCKNER, Markov processes associated with semi-Dirichlet forms, Osaka J. Math. 32 (1995), 97–119.
[18] Z. M. MA AND M. RÖCKNER, Introduction to the theory of (non-symmetric) Dirichlet forms, Springer-Verlag, 1992.

[19] Z.M. MA AND M. RÖCKNER, Markov processes associated with positivity preserving coercive forms, Can. J. Math. 47 (1995), 817–840.

[20] Y. OSHIMA, Lecture on Dirichlet Spaces, Univ. Erlangen-Nürnberg, 1988.

[21] Y. OSHIMA, Semi-Dirichlet forms and Markov processes, Walter de Gruyter, 2013.

[22] R. L. SCHILLING AND J. WANG, Lower bounded semi-Dirichlet forms associated with Lévy type operators, Festschrift Masatoshi Fukushima, 507–526, Interdiscip. Math. Sci. 17, World Sci. Publ., Hackensack, NJ, 2015.

[23] W. STANNAT, The theory of generalized Dirichlet forms and its applications in analysis and stochastics, Memoirs of the AMS, Vol. 142, No. 678, 1999.

[24] W. SUN, The convergence theory of Dirichlet forms and some topics concerning generalized Dirichlet forms, Ph.D. thesis, Chinese Academy of Sciences, 1998.

[25] M. TAKEDA AND T. S. ZHANG, Asymptotic properties of additive functionals of Brownian motion, Ann. Probab. 25 (1997), 940–952.

[26] G. TRUTNAU, Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions, Osaka J. Math. 37 (2000), 315–343.

[27] T. UEMURA, On multidimensional diffusion processes with jumps, Osaka J. Math. 51 (2014), 969–992.

[28] L. F. WANG, Fukushima's decomposition of semi-Dirichlet forms and some related research, Ph.D. thesis, Chinese Academy of Sciences, 2013.

ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCES
NO. 55, ZHONG-GUAN-CUN EAST ROAD,
BEIJING, 100190
CHINA

E-mail address: mazm@amt.ac.cn

DEPARTMENT OF MATHEMATICS AND STATISTICS
CONCORDIA UNIVERSITY
MONTREAL, H3G 1M8
CANADA

E-mail address: wei.sun@concordia.ca

COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE
HEBEI NORMAL UNIVERSITY
SHIJIAZHUANG, 050024
CHINA

E-mail address: flywit1986@163.com