ALEXANDROV'S ISODIAMETRIC CONJECTURE AND THE CUT LOCUS OF A SURFACE

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Abstract. We prove that Alexandrov's conjecture relating the area and diameter of a convex surface holds for the surface of a general ellipsoid. This is a direct consequence of a more general result which estimates the deviation from the optimal conjectured bound in terms of the length of the cut locus of a point on the surface. We also prove that the natural extension of the conjecture to general dimension holds among closed convex spherically symmetric Riemannian manifolds. Our results are based on a new symmetrization procedure which we believe to be interesting in its own right.

1. Introduction. Let Σ be a closed oriented surface. (By a *surface* we always mean a connected smooth 2-dimensional Riemannian manifold; *closed* means compact and without boundary.) Unless otherwise stated, we assume that Σ is *convex*, *i.e.* the Gauss curvature K is non-negative and not identically equal to zero. Let A and D denote the surface *area* and the (intrinsic) *diameter* of Σ, respectively.

This paper is concerned with the following conjecture raised by A. D. Alexandrov in 1955 [2]:

CONJECTURE 1. *For any closed oriented convex surface Σ,*

$$(1) \qquad \frac{A}{D^2} \leq \frac{\pi}{2} \approx 1.5708 \,.$$

It can easily be seen that the value on the right hand side of this inequality is attained for the respective quotient by the doubly-covered disk, *i.e.* a degenerate surface formed by gluing two flat disks along their boundaries.

Due to the inequality relating the area and the diameter of a planar convex domain, namely, $A/D^2 \leq \pi/4$, with equality attained only for the disk, it follows that the conjecture holds among degenerate surfaces made of two copies of the same convex planar domain glued at the boundary. However, for general convex surfaces the conjecture remains open with the strongest result so far being, to the best of our knowledge, that by Calabi and Cao [3] yielding

$$\frac{A}{D^2} \leq \frac{8}{\pi} \approx 2.5465 \,,$$

while other (weaker) bounds may be found in [18, 19]. We note that the proof of the above inequality by Calabi and Cao is done indirectly by means of eigenvalue estimates.

Even within restricted families of surfaces, such as the boundary of parallelepipeds, the problem does not seem to be easy, mainly due to the difficulties arising in the computation of the intrinsic diameter of a given surface. The classes for which the conjecture has been proved are tetrahedra (Makai [13] and Zalgaller [22]), rectangular parallelepipeds (Nikonorov and Nikonorova [15]) and surfaces of revolution (Makuha [14] and Abreu and the first author [1]). Except for the last family of surfaces, which contains the double disk, the optimizers in the first and second cases are not degenerate, being the regular tetrahedron and the parallelepiped with edge lengths 1, 1 and $\sqrt{2}$, respectively, with the corresponding inequalities satisfied for surfaces within each family being

$$\frac{A}{D^2} \leq \frac{3\sqrt{3}}{4} \approx 1.299 \qquad \text{and} \qquad \frac{A}{D^2} \leq \frac{1+2\sqrt{2}}{3} \approx 1.276\,.$$

We point out that the result for surfaces of revolution given in [14, 1] is actually stronger, in that the convexity restriction is replaced by the condition that the surface is diffeomorphic to the sphere and isometric to a closed surface in $\mathbb{R}^3$. In fact, it is suggested in [1] that it might be possible to replace convexity by this weaker condition also in the case of general surfaces.

We note that by the Kuiper-Nash embedding theorem, the example in Section 3.1 in [1] may be C^1 isometrically embedded in $\mathbb{R}^3$, and we thus have that there exists a surface in $\mathbb{R}^3$, homeomorphic to the sphere, for which the quotient A/D^2 may be made to be arbitrarily large. An interesting question is thus whether the conjecture may hold for more general surfaces as in the case of surfaces of revolution, or whether having non-negative curvature is an essential condition.

Here we shall go one step further and, having proven the result for spherically symmetric d-manifolds in $\mathbb{R}^{d+1}$ – see Theorem 3 in Appendix B –, we conjecture that, under some assumptions, a similar result will hold in any dimension.

In this paper we relate the quotient between A and D^2 to the cut-locus geometry of the surface Σ, bounding it by the conjectured optimal value plus a deviation term proportional to the 1-dimensional Hausdorff measure of the cut-locus. This allows us to prove the conjecture in the case of closed convex surfaces having one point for which the cut locus reduces to a single point – see Corollary 1 below. Our proof is based on a symmetrization procedure which transforms a given surface into a surface of revolution, while ensuring that the quotient A/D^2 does not decrease. However, the resulting surface is not necessarily closed, and the extra term depending on the measure of the cut locus appears as a way of estimating the length of the boundary circle.

In order to state our results, we need some standard notions related to the geometry of a surface. Given a point $p \in \Sigma$, let us consider a family of geodesics γ_θ emanating from p with initial direction $\theta \in S^1$. The *distance to the cut point of p along γ_θ*, denoted here by $d_p(\theta)$, is defined as the supremum over all distances t for which γ_θ is the minimizing geodesic, *i.e.* dist$(p, \gamma_\theta(t)) = t$. We clearly have the global bounds $\rho \leq d_p(\theta) \leq D$, where ρ denotes the *injectivity radius* of Σ. The *cut locus* of p, *i.e.* the set of all cut points of p, is denoted

by $\mathcal{C}_p$. A subset of $\mathcal{C}_p$ is formed by *conjugate cut points*, *i.e.*, roughly, those points $q \in \mathcal{C}_p$ which can "almost" be joined with p via a 1-parameter family of geodesics $\theta \mapsto \gamma_\theta$ (the precise definition is given by means of the vanishing of a *Jacobi field* along γ_θ).

It is well known that $\mathcal{C}_p$ is a set of Riemannian measure zero. Moreover, $\mathcal{C}_p$ has finite 1-dimensional Hausdorff measure [8, 10] that we denote by $|\mathcal{C}_p|$ and call the *total length* of $\mathcal{C}_p$. One of the results of this paper is the following bound.

THEOREM 1. *Let Σ be a closed oriented convex surface. Suppose that there is a point $p \in \Sigma$ such that the set of conjugate points in the cut locus $\mathcal{C}_p$ is countable. Then*

$$(2) \qquad \frac{A}{D^2} \le \frac{\pi}{2} + \frac{|\mathcal{C}_p|}{\rho} .$$

As a consequence of this theorem, we get an estimate on the deviation from the conjectured optimal value in terms of the cut locus of a point in Σ with the smallest measure.

COROLLARY 1. *Conjecture 1 holds true for any closed oriented convex surface Σ for which there is a point p whose cut locus $\mathcal{C}_p$ consists of a single point.*

As a large class of surfaces to which the result applies, let us mention rotationally symmetric surfaces; indeed, the cut locus of a pole reduces to its antipodal point.

More significantly, the same situation happens for umbilical points on any ellipsoid [11] and we thus have

COROLLARY 2. *Conjecture 1 holds true for ellipsoids.*

This paper is organized as follows. In the forthcoming Section 2 we summarize some basic facts about the geometry of (not necessarily convex) surfaces which will be needed throughout the paper. In particular, we recall the notion of geodesic polar coordinates based on an arbitrary point of the surface and present a formula for the total length of the cut locus of the point. As we were unable to find a direct reference for this formula, we provide a proof in Appendix A. The symmetrization procedure is performed in Section 3, where Theorem 1 is proved as a consequence of a stronger and more general result (Theorem 2). The latter does not require the hypothesis that the conjugate cut points are countable.

In Appendix B, we prove a higher-dimensional analogue of Alexandrov's conjecture for spherically symmetric manifolds in any dimension $d \ge 2$ (Theorem 3). In this proof we only require that the spherically symmetric manifold is diffeomorphic to the sphere S^d and embedded in $\mathbb{R}^{d+1}$. This situation includes convex spherically symmetric manifolds diffeomorphic to the sphere S^d due to the embeddability characterization of [17, Thm. 2.1] combined with a classical result [9] (in fact, even less restrictive, integral-type conditions on the positivity of the curvature can be required [5]). Let us remark that surfaces considered in Theorems 1 and 2 are necessarily diffeomorphic to S^2 and embeddable in $\mathbb{R}^3$.

2. Preliminaries. Let p be any point on Σ. A useful parameterization of Σ (regardless of the sign restriction of its curvature K) is given by the *geodesic polar coordinates* [4,

Sec. III.1] based on p, *i.e.* by the diffeomorphism

$$\Phi_p : U_p \to \Sigma \setminus (C_p \cup \{p\}) : \left\{(t, \theta) \mapsto \exp_p\left(t \cos\theta\, e_1 + t \sin\theta\, e_2\right)\right\}.$$

Here $\exp_p$ denotes the exponential map on $T_p\Sigma$, $\{e_1, e_2\}$ is an orthonormal basis of $T_p\Sigma$ and

$$U_p := \left\{(t, \theta) \,\big|\, 0 < t < d_p(\theta),\ \theta \in S^1\right\}.$$

In these coordinates, the Riemannian metric of Σ becomes

$$(3) \qquad\qquad d\ell^2 = dt^2 + F_p(t, \theta)^2\, d\theta^2, \qquad (t, \theta) \in U_p.$$

The function F_p^2 admits a smooth non-negative square root F_p that plays the role of Jacobian of Φ_p and satisfies the Jacobi equation

$$(4) \qquad \begin{cases} \partial_1^2 F_p(t, \theta) + K_p(t, \theta)\, F_p(t, \theta) = 0 & \forall (t, \theta) \in U_p, \\[4pt] F_p(0, \theta) = 0 & \forall \theta \in S^1, \\[4pt] \partial_1 F_p(0, \theta) = 1 & \forall \theta \in S^1. \end{cases}$$

Here K_p is the Gauss curvature expressed in the geodesic polar coordinates based on p. Note that the family of geodesics γ_θ mentioned in the introduction can be related to the exponential map via $\gamma_\theta(t) = \Phi_p(t, \theta)$.

Since the cut locus $\mathcal{C}_p$ is a set of measure zero with respect to the Riemannian measure of Σ (induced by the Euclidean Lebesgue measure on the tangent plane), the geodesic polar coordinates represent a useful chart for calculation of integrals. In particular, for the surface area we have

$$(5) \qquad\qquad A := \int_\Sigma d\Sigma = \int_{U_p} F_p(t, \theta)\, dt\, d\theta,$$

irrespectively of the choice of the point p.

The diameter of Σ,

$$D := \max_{p, q \in \Sigma} \operatorname{dist}(p, q),$$

where $\operatorname{dist}(\cdot, \cdot)$ stands for the geodesic distance on Σ, is a less accessible quantity. Using the geodesic polar coordinates, let us introduce

$$(6) \qquad\qquad D_p := \max_{\theta \in S^1} d_p(\theta),$$

i.e., the maximal distance of p to its cut locus measured among all the geodesics γ_θ emanating from p. Then we have

$$(7) \qquad\qquad D = \max_{p \in \Sigma} D_p.$$

Similarly, the injectivity radius of Σ can be defined through the formulae

$$(8) \qquad\qquad \rho := \min_{p \in \Sigma} \rho_p, \qquad \text{where} \qquad \rho_p := \min_{\theta \in S^1} d_p(\theta).$$

Note that $\rho > 0$, because the geodesic $\gamma_\theta(t)$ is always minimizing for sufficiently small t and Σ is compact.

By the Gauss-Bonnet theorem for closed surfaces, one has the following identity for the *total Gauss curvature*

$$(9) \qquad \int_{\Sigma} K \, d\Sigma = \int_{U_p} K_p(t, \theta) \, F_p(t, \theta) \, dt \, d\theta = 2\pi \, \chi_{\Sigma} \,,$$

where χ_{Σ} denotes the Euler characteristic of Σ. For orientable surfaces $\chi_{\Sigma} = 2(1 - g_{\Sigma})$, where g_{Σ} is the genus of Σ. In our case, when K is in addition non-negative and non-trivial, we necessarily have $g_{\Sigma} = 0$ (Σ is diffeomorphic to the sphere S^2) and the total Gauss curvature thus equals 4π.

Since K is supposed to be non-negative, it follows from (4) that

$$(10) \qquad F_p(t, \theta) \leq t$$

for every $(t, \theta) \in U_p$. In particular, F_p is bounded on U_p. (For a general closed surface, regardless of the sign restriction of K, the particular bound (10) cannot be ensured, but a similar bound with the right hand side being replaced by Ct with a suitable constant C does hold, and F_p is bounded on U_p in any case.) Using in addition that F_p is uniformly continuous on U_p, we know that the function F_p admits unique boundary values on ∂U_p by the continuous extension. Hence, $F_p(d_p(\theta), \theta)$ is well defined for every $\theta \in S^1$. Let us also mention that $\theta \mapsto d_p(\theta)$ is a Lipschitz continuous function [12]. As a consequence, we have the following formula for the total length of the cut locus

$$(11) \qquad |\mathcal{C}_p| = \frac{1}{2} \int_{S^1} \sqrt{F_p\big(d_p(\theta), \theta\big)^2 + d_p'(\theta)^2} \, d\theta \,.$$

We were unable to find a direct reference for this formula, and thus present a proof in Appendix A – *cf* Proposition 1. For this proof we need the hypothesis of Theorem 1 about the structure of conjugate points in the cut locus.

Finally, we introduce the quantity

$$(12) \qquad M_p := \int_{S^1} \frac{F_p\big(d_p(\theta), \theta\big)}{d_p(\theta)} \, d\theta = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{S^1} \int_{d_p(\theta)-\varepsilon}^{d_p(\theta)} \frac{F_p(t, \theta)}{t} \, dt \, d\theta \,.$$

Note that the integrand is well defined because $d_p(\theta) > \rho > 0$. From (10) we deduce a crude bound

$$(13) \qquad M_p \leq 2\pi \,.$$

Estimating d_p by the injectivity radius of Σ, we have

$$(14) \qquad M_p \leq \frac{1}{\rho} \int_{S^1} F_p\big(d_p(\theta), \theta\big) \, d\theta = \frac{1}{\rho} \lim_{\varepsilon \to 0} \frac{|\Omega_{\varepsilon}|}{\varepsilon} \,,$$

where

$$|\Omega_{\varepsilon}| := \int_{S^1} \int_{d_p(\theta)-\varepsilon}^{d_p(\theta)} F_p(t, \theta) \, dt \, d\theta$$

denotes the 2-dimensional Riemannian measure of an ε-tubular neighbourhood of $\mathcal{C}_p$ (note that Ω_{ε} differs from the ε-tubular neighbourhood of $\mathcal{C}_p$ defined by parallel curves!). We

are more interested in an estimate of the quantity M_p by means of the total length of $\mathcal{C}_p$; combining the inequality (14) with (11), we obtain

$$(15) \qquad\qquad M_p \leq \frac{2\,|\mathcal{C}_p|}{\rho}\,.$$

3. Proof of Theorem 1. Our proof of Theorem 1 is based on a symmetrization procedure. For every $s \in [0, D_p)$, we introduce

$$(16) \qquad\qquad f_p(s) := \frac{1}{2\pi} \int_{S^1} \frac{D_p}{d_p(\theta)}\, F_p\!\left(\frac{d_p(\theta)}{D_p}\, s, \theta\right) d\theta\,.$$

Employing (4), it is easy to check that $f_p \in C^\infty((0, D_p))$ satisfies

$$(17) \qquad\qquad \begin{cases} f_p''(s) + k_p(s) f_p(s) = 0 & \forall s \in (0, D_p)\,, \\ \qquad\qquad f_p(0) = 0\,, \\ \qquad\qquad f_p'(0) = 1\,, \end{cases}$$

with

$$k_p(s) := \frac{\displaystyle\int_{S^1} \frac{d_p(\theta)}{D_p}\, K_p\!\left(\frac{d_p(\theta)}{D_p}s, \theta\right) F_p\!\left(\frac{d_p(\theta)}{D_p}s, \theta\right) d\theta}{\displaystyle\int_{S^1} \frac{D_p}{d_p(\theta)}\, F_p\!\left(\frac{d_p(\theta)}{D_p}s, \theta\right) d\theta}\,.$$

Hence, through the formula for the Riemannian metric on the cross-product manifold $(0, D_p) \times S^1$

$$d\ell^2 = ds^2 + f_p(s)^2\, d\theta^2\,, \qquad (s, \theta) \in (0, D_p) \times S^1\,,$$

f_p defines a rotationally symmetric surface Σ_p of the Gauss curvature k_p. The curvature k_p is non-negative if it is the case for K. The surface area of Σ_p reads

$$(18) \qquad\qquad A_p := 2\pi \int_0^{D_p} f_p(s)\, ds = \int_{U_p} \frac{D_p^2}{d_p(\theta)^2}\, F_p(t, \theta)\, dt\, d\theta\,.$$

At the same time, it is easily seen that the symmetrization (16) preserves the total Gauss curvature

$$(19) \qquad\qquad 2\pi \int_0^{D_p} k_p(s)\, f_p(s)\, ds = \int_{U_p} K_p(t, \theta)\, F_p(t, \theta)\, dt\, d\theta = 4\pi\,,$$

the last identity following from (9). On the other hand, it is absolutely necessary to stress that (contrary to Σ) Σ_p may not be complete.

Regardless of whether Σ_p is closed or not, it can be represented as a surface of revolution embedded in $\mathbb{R}^3$. The embedding is explicitly provided by the mapping

$$\phi_p : (0, D_p) \times S^1 \to \mathbb{R}^3 : \left\{(s, \vartheta) \mapsto \big(r_p(s) \cos\theta, r_p(s) \sin\theta, z_p(s)\big)\right\}\,,$$

where

$$r_p(s) := f_p(s)\,, \qquad z_p(s) := \int_0^s \sqrt{1 - f_p'(\tau)^2}\, d\tau\,.$$

By virtue of (17), we have

$$(20) \qquad f_p'(s) = 1 - \int_0^s k_p(\tau)\, f_p(\tau)\, d\tau\,.$$

Since k_p is non-negative, it follows with help of (19) that $-1 \le f_p' \le 1$ (with the boundary values attained at D_p and 0, respectively). Consequently, z_p is well defined, non-decreasing, and ϕ_p is thus indeed an embedding. As mentioned above, however, ϕ_p does not necessarily represent a closed surface in general. The closedness depends on the value of $r_p(D_p) = f_p(D_p)$, which might be different from zero.

As a matter of fact, the boundary of ϕ_p is formed by a circle Γ_p of extrinsic radius $f_p(D_p)$ (by extrinsic we mean that Γ_p is regarded as a curve in $\mathbb{R}^3$). Let us exclude for a moment the "singular" situation $f_p(D_p) = 0$ (in which case the surface Σ_p is closed). By (19) and the Gauss-Bonnet theorem for surfaces with boundary applied to Σ_p, it then follows that

$$(21) \qquad \oint_{\Gamma_p} \kappa_p\, dl = -2\pi\,,$$

where κ_p denotes the geodesic curvature of Γ_p (as a curve on $\Sigma_p \subset \mathbb{R}^3$). Consequently, $\kappa_p = \pm f_p(D_p)^{-1}$ (with the sign depending on the parameterization of Γ_p) and the normal curvature of Γ_p is necessarily zero. At the same time, the length of the boundary circle equals

$$(22) \qquad L_p := |\Gamma_p| = 2\pi\, f_p(D_p) = D_p\, M_p\,,$$

where M_p was introduced in (12). This formula remains trivially valid for $f_p(D_p) = 0$. We have the crude bound

$$(23) \qquad L_p \le 2\pi\, D_p\,,$$

which follows from (13).

By definition (7), $D_p \le D$. At the same time, comparing (5) with (18) and using definition (6), we have $A_p \ge A$. Consequently, we arrive at an intermediate bound

$$(24) \qquad \frac{A}{D^2} \le \frac{A_p}{D_p^2}$$

valid for any $p \in \Sigma$. The problem of Conjecture 1 is thus reduced to rotationally symmetric surfaces (albeit possibly not complete).

To prove Theorem 1, we divide A_p into two parts

$$A_p^{(1)}(\tau) := 2\pi \int_0^\tau f_p(s)\, ds\,, \qquad A_p^{(2)}(\tau) := 2\pi \int_\tau^{D_p} f_p(s)\, ds\,,$$

with $\tau \in (0, D_p]$, and use the following comparison arguments, respectively.

LEMMA 1. $A_p^{(1)}(\tau) \le \pi \tau^2$.

PROOF. Since k_p is non-negative, it follows from (17) (or more directly from (20)) that $f_p(s) \le s$ for all $s \in [0, D_p]$. Using this estimate in the integral defining $A_p^{(1)}(\tau)$, we get the desired inequality. □

LEMMA 2. $A_p^{(2)}(\tau) \leq \pi(D_p - \tau)^2 + L_p(D_p - \tau)$.

PROOF.   Here the idea is to use the *geodesic parallel (Fermi) coordinates* [4, Sec. III.6] based on Γ_p rather than the geodesic polar coordinates. In these coordinates, the Riemannian metric of Σ_p becomes

$$d\ell^2 = dr^2 + h_p(r)^2 \, du^2 \,, \qquad (r, u) \in (0, D_p) \times \Gamma_p \,,$$

where the Jacobian $h_p \in C^\infty((0, D_p))$ satisfies

$$(25) \qquad \begin{cases} h_p''(r) + k_p(D_p - r)h_p(r) = 0 & \forall r \in (0, D_p) \,, \\ \qquad\qquad\quad h_p(0) = 1 \,, \\ \qquad\qquad\quad h_p'(0) = -\kappa_p \,. \end{cases}$$

Here the coordinate r is measured from Γ_p, rather than from p, and the minus sign in front of κ_p is required due to the consistency with the convexity of Σ_p, *cf* (21). Consequently,

$$(26) \qquad\qquad A_p^{(2)}(\tau) = \int_{\Gamma_p} \int_0^{D_p - \tau} h_p(r) \, dr \, du \,.$$

Since k_p is non-negative, it follows from (25) that $h_p(r) \leq 1 - \kappa_p \, r$ for all $r \in [0, D_p]$. Using this estimate in (26) and recalling (21), we get the desired inequality.   □

Putting these lemmata together, we get

$$A_p \leq \pi\tau^2 + \pi(D_p - \tau)^2 + L_p(D_p - \tau)$$

for any $\tau \in (0, D_p]$. The minimum of the right hand side as a function of τ is achieved for

$$\tau_{\min} := \frac{D_p}{2} + \frac{L_p}{4\pi} \,,$$

which is an admissible point from $(0, D_p]$ due to (23). Using this value and expressing L_p in terms of M_p, cf. (22), we get the bound

$$(27) \qquad\qquad \frac{A_p}{D_p^2} \leq \frac{\pi}{2} + \frac{1}{2} M_p \left(1 - \frac{M_p}{4\pi} \right) \,.$$

Plugging this estimate into (24), we obtain the main result of this paper.

THEOREM 2.   *Let Σ be a closed oriented convex surface. For any point $p \in \Sigma$, we have*

$$(28) \qquad\qquad \frac{A}{D^2} \leq \frac{\pi}{2} + \frac{1}{2} M_p \left(1 - \frac{M_p}{4\pi} \right) \,.$$

Theorem 1 follows as a weaker version of this theorem by neglecting the non-positive term in the brackets and applying the crude bound (15).

Appendix A.   The total length of the cut locus.   This appendix is devoted to a proof of formula (11) for a general (*i.e.* not necessarily convex) surface.

PROPOSITION 1.   *Let Σ be closed surface*, *and let p be a point in Σ. Assume that the set of conjugate points in the cut locus $\mathcal{C}_p$ is countable. Then the total length of the cut locus is given by formula* (11).

First we recall some definitions and facts from [8], to where we refer for more information. Fix $p \in \Sigma$. A cut point $q \in \mathcal{C}_p$ is said to be a *conjugate cut point* if it is conjugate to p along at least one minimizing geodesic γ_θ joining p to q (*i.e.* $F_p(t, \theta) = 0$ for $t \neq 0$, $\theta \in S^1$, corresponding to q), and is said to be a *non-conjugate cut point* otherwise. The *order* of a non-conjugate cut point $q \in \mathcal{C}_p$ is the number of minimizing geodesics joining p to q. The order is always finite and at least two. A cut point is said to be a *cleave point* if it is a non-conjugate cut of order two; it is a *non-cleave point* otherwise.

The cut locus $\mathcal{C}_p$ is a closed, compact, connected and non-empty subset of Σ. The set of cleave points is a relatively open subset of $\mathcal{C}_p$ forming a smooth 1-dimensional submanifold of Σ. The set of non-cleave points in $\mathcal{C}_p$ is a closed subset whose 1-dimensional Hausdorff measure is zero (in fact, the Hausdorff dimension is 0). It is also known that at a cleave point q, the two minimizing geodesics joining p to q make the same angle with the tangent plane at q to the submanifold of cleave points, but from opposite sides.

Moreover, by a theorem of Myers [8, Thm. 2.3], the topological structure of $\mathcal{C}_p$ is that of a local tree. To recall the notion, two other topological definitions are needed. An *arc* is a topological space homeomorphic to the unit interval [0, 1]. A *tree* is is a topological space with the property that every pair of points q_1, q_2 is contained in a unique arc with endpoints q_1, q_2. A *local tree* is a topological space in which every point is contained in arbitrarily small closed neighbourhoods which are themselves trees. For simply connected surfaces, $\mathcal{C}_p$ is a tree. (Convex oriented surfaces considered in this paper are simply connected due to the Gauss-Bonnet theorem (9) and the text below it; for general convex surfaces the tree property is also mentioned in [23].)

Each point of a tree is either an *endpoint*, *ordinary point* or *branch point* depending upon the number (one, two, or more respectively) of connected components possessed by a deleted neighbourhood of the point. Every endpoint is a conjugate cut point; every cleave point is ordinary; and every non-conjugate, non-cleave point is a branch point. There are at most countably many branch points in the cut locus $\mathcal{C}_p$, but the set of endpoints may be uncountable. For real-analytic manifolds the endpoints are at most finite [16], but there exist examples of smooth surfaces with an infinite number of endpoints (non-triangulable cut loci) [6]. We refer to [20] for an interesting numerical study of the structure of cut loci of rotationally symmetric surfaces.

PROOF OF PROPOSITION 1.   The map $w : S^1 \to \mathcal{C}_p$ defined by $w(\theta) := \Phi_p(d_p(\theta), \theta)$ is a continuous proper map (*i.e.* the inverse image of compact sets is compact). Let

$$E := \{\theta \in S^1 \mid w(\theta) \text{ is a non-cleave point of } \mathcal{C}_p\},$$

$$E_0 := \{\theta \in E \mid w(\theta) \text{ is a conjugate cut point of } \mathcal{C}_p\},$$

$$E_1 := \{\theta \in E \mid w(\theta) \text{ is a non-conjugate cut point of } \mathcal{C}_p\}.$$

We have $\mathcal{C}_p = w(S^1)$ and we know that the 1-dimensional Hausdorff measure of the image $w(E)$ is zero and that $V := \mathcal{C}_p \setminus w(E)$ is the union of at most countably many smooth connected 1-dimensional manifolds V_n of cleave points. Consequently, $|\mathcal{C}_p| = |V|$.

• Cleave points. Since the Hausdorff 1-measure of a smooth curve is its arclength, the standard calculus formula, employing the fact that d_p is Lipschitz continuous [12] (in fact, the absolute continuity established in [8, 10] would be sufficient), gives

$$|V_n| = \int_{I_n^{(1)}} \sqrt{F_p\big(d_p(\theta), \theta\big)^2 + d_p'(\theta)^2} \, d\theta \,.$$

Here $I_n^{(1)} \subset S^1$ is one interval from the disjoint union $w^{-1}(V_n) = I_n^{(1)} \cup I_n^{(2)}$; $w^{-1}(V_n)$ has exactly *two* connected components because V_n is composed of cleave points. Taking this multiplicity into account, we conclude with

$$|V| = \frac{1}{2} \int_{S^1 \setminus E} \sqrt{F_p\big(d_p(\theta), \theta\big)^2 + d_p'(\theta)^2} \, d\theta \,.$$

• Non-conjugate non-cleave points. As recalled above, every non-conjugate non-cleave point is a branch point and there are at most countably many branch points $\{q_n\}_{n=1}^N \subset \mathcal{C}_p$, with $N \in \{1, \ldots, \infty\}$. Moreover, since the order m_n of each q_n is finite, it follows that the preimage $w^{-1}(q_n)$ is composed of m_n distinct angles $\theta_n^{(1)} < \theta_n^{(2)} < \cdots < \theta_n^{(m_n)}$. Consequently, using in addition that the cut locus $\mathcal{C}_p$ is compact, the Lebesgue 1-measure of $E_1 = \cup_{n=1}^\infty w^{-1}(q_n)$ is equal to zero and we have

$$\int_{E_1} \sqrt{F_p\big(d_p(\theta), \theta\big)^2 + d_p'(\theta)^2} \, d\theta = 0 \,.$$

• Conjugate cut points. For a general closed (even convex) surface the set of conjugate cut points can be uncountable, although it must be a totally disconnected set with Hausdorff dimension being zero [8, Rem. 2.4]. Under our assumption, however, there are at most countably many conjugate cut points $\{q_n\}_{n=1}^N \subset \mathcal{C}_p$, with $N \in \{1, \ldots, \infty\}$. The preimage E_0 does not necessarily have the Lebesgue 1-measure equal to zero, however, by definition of conjugate cut points, we have

$$\forall \theta \in w^{-1}(q_n), \qquad F_p\big(d_p(\theta), \theta\big) = 0 \quad \text{and} \quad d_p(\theta) = c_n \,,$$

where c_n is a constant. Consequently,

$$\int_{E_0} \sqrt{F_p\big(d_p(\theta), \theta\big)^2 + d_p'(\theta)^2} \, d\theta = \sum_{n=1}^N \int_{w^{-1}(q_n)} \sqrt{F_p\big(d_p(\theta), \theta\big)^2 + d_p'(\theta)^2} \, d\theta = 0 \,.$$

Summing up, the total length of the cut locus $\mathcal{C}_p$ is determined by $|V|$, which coincides with the formula (11) because the integration over E does not contribute. □

Appendix B.   Alexandrov's conjecture in higher dimensions.   Let Σ be the sphere S^d with a *spherically symmetric* Riemannian metric, that is, such that there exists a point $p \in \Sigma$ for which the metric in the geodesic spherical coordinates about p reads

$$(29) \qquad d\ell^2 = ds^2 + f_p(s)^2 d\theta^2 \,, \qquad (s, \theta) \in U_p := (0, D_p) \times S^{d-1} \,.$$

Here $d\theta^2$ is the standard Euclidean metric on the unit sphere S^{d-1}, D_p is a positive number and $f_p \in C^\infty((0, D_p))$ is a positive function satisfying the Jacobi equation (17), where $k_p(s)$ should be interpreted as the *radial curvature* of Σ at q for any q such that $\mathrm{dist}(p, q) = s$. By the radial curvature we mean the restriction of the sectional curvature functions to all the planes containing the unit vector field W such that, for any point $q \in \Sigma \setminus (\{p\} \cup \mathcal{C}_p)$, $W(q)$ is the unit vector tangent to the unique geodesic joining p to q (this notion is standardly defined for manifolds with a pole [7], but it clearly extends as given here to an arbitrary manifold when parameterized in the geodesic spherical coordinates).

Because of the spherical symmetry (29), the geodesics starting from p have the same behaviour in all directions, implying that if one geodesic is minimising in one direction up to time t, the same must happen to all other geodesics emanating from p. Thus the exponential map depends only on the radial variable and we conclude that while it remains a diffeomorphism, the boundary of its image for each such (positive) t must be a $(d-1)$-sphere. At a point for which the exponential map stops being a diffeomorphism, the same must happen at all other points at the same distance from p and thus the cut locus is either a $(d-1)$-sphere or a point. Since the closure of $\Sigma \setminus \mathcal{C}_p$ is Σ, which is a closed manifold, we conclude that the cut locus of p reduces to the single point $p^* := \exp_p(\{D_p\} \times S^{d-1}) \in \Sigma$, *i.e.*

$$(30) \qquad\qquad\qquad\qquad \mathcal{C}_p = \{p^*\} \,.$$

We note that the particular two-dimensional case of (30) is covered in [21, Lem. 2.1].

Since Σ is diffeomorphic to S^d, it follows from (29) and (30) that we may respectively identify p and p^* with the South and North Poles of S^d. We have $\mathcal{C}_{p^*} = \{p\}$ and formulae analogous to (29) and (17) hold for p^*.

We say that Σ is *convex* if all the sectional curvatures at all the points on the manifold Σ are non-negative. In this case, it follows from (17) that

$$(31) \qquad\qquad f_p(s) \leq s \qquad \text{and} \qquad f_{p^*}(s) \leq s$$

for every $s \in (0, D_p)$.

We note that convexity is not necessary to obtain the above estimates. In fact, both inequalities in (31) follow from $|f_p'(s)| \leq 1$ for every $s \in [0, D_p]$, which is equivalent to the embeddability of Σ to $\mathbb{R}^{d+1}$, which in turn can be ensured by assuming mere positivity of integrals of the radial curvature over polar caps of S^d (which automatically holds under the convexity assumption); *cf* [17, Thm. 2.1] and [5]. Let us therefore assume only that Σ is a spherically symmetric manifold (in the sense of the definition given above) that is isometrically embedded in $\mathbb{R}^{d+1}$. The manifold Σ can be obtained by rotating a curve around the x^1 axis in $\mathbb{R}^{d+1}$ by the action of $SO(d)$.

Applying the comparison estimates (31) to the Riemannian volume of Σ, that we denote again by A, we arrive at

$$A = |S^{d-1}| \int_0^{D_p} f_p^{d-1}(s) \, ds = |S^{d-1}| \int_0^{D_p} f_{p^*}^{d-1}(s) \, ds$$

$$= |S^{d-1}| \int_0^{D_p/2} f_p^{d-1}(s) \, ds + |S^{d-1}| \int_0^{D_p/2} f_{p^*}^{d-1}(s) \, ds$$

(32) $$\leq \frac{|S^{d-1}|}{2^{d-1}d} D_p^d = \frac{|B^d|}{2^{d-1}} D_p^d \, .$$

Here $|S^{d-1}|$ denotes the $(d-1)$-dimensional volume of the $(d-1)$-dimensional sphere S^{d-1} and $|B^d|$ denotes the d-dimensional volume of the d-dimensional unit ball B^d.

Since Σ is spherically symmetric, its diameter D equals the distance between the poles p and p^*. To see this, it is enough to consider that for any two points a and b in Σ, we have

$$\begin{aligned} \text{dist}(a, b) \quad &\leq \quad \min\{\text{dist}(a, p) + \text{dist}(p, b), \text{dist}(a, p^*) + \text{dist}(p^*, b)\} \\ &\leq \quad \tfrac{1}{2}\big[\text{dist}(a, p) + \text{dist}(p, b) + \text{dist}(a, p^*) + \text{dist}(p^*, b)\big] \\ &= \quad \text{dist}(p, p^*) \, , \end{aligned}$$

where the equality uses the symmetry assumption through the minimizing property of geodesics joining p with p^*, namely,

$$\text{dist}(a, p) + \text{dist}(a, p^*) = \text{dist}(p, p^*) = \text{dist}(p, b) + \text{dist}(p^*, b) \, .$$

We have thus proven the following result.

THEOREM 3. *Let Σ be a closed spherically symmetric Riemannian manifold of dimension d diffeomorphic to S^d that is isometrically embedded in $\mathbb{R}^{d+1}$. Then*

(33) $$\frac{A}{D^d} \leq \frac{|B^d|}{2^{d-1}} \, .$$

Again, it can be verified that the value on the right hand side of this inequality is attained for the respective quotient by the degenerate convex surface formed by gluing two balls of diameter D along their boundaries.

It is thus natural to propose the following extension of Conjecture 1 to any dimension.

CONJECTURE 2. *Inequality (33) holds for any closed oriented convex Riemannian manifold of dimension d.*

REMARK 1. Conjecture 2 makes sense also in $d = 1$, where its validity is trivial.

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