STRUCTURE OF SYMPLECTIC LIE GROUPS
AND MOMENTUM MAP

ALBERTO MEDINA

Abstract. We describe the structure of the Lie groups endowed with a left-invariant symplectic form, called symplectic Lie groups, in terms of semi-direct products of Lie groups, symplectic reduction and principal bundles with affine fiber. This description is particularly nice if the group is Hamiltonian, that is, if the left canonical action of the group on itself is Hamiltonian. The principal tool used for our description is a canonical affine structure associated with the symplectic form. We also characterize the Hamiltonian symplectic Lie groups among the connected symplectic Lie groups. We specialize our principal results to the cases of simply connected Hamiltonian symplectic nilpotent Lie groups or Frobenius symplectic Lie groups. Finally we pursue the study of the classical affine Lie group as a symplectic Lie group.

1. Introduction. A closed nondegenerate smooth 2-form Ω on a manifold M is called a symplectic form and the pair (M, Ω) is called a symplectic manifold. A smooth action Ψ of a Lie group G on (M, Ω) is symplectic if Ω is Ψ-invariant. This means that $i(\overline{x})\Omega$ is a closed 1-form on M for any x in the Lie algebra $\mathcal{G} := T_\varepsilon(G) = L(G)$ of G, where $\overline{x}$ denotes the fundamental vector field associated with x and ε is the unit of G. When $i(\overline{x})\Omega$ is exact on M for any $x \in \mathcal{G}$, the action Ψ is Hamiltonian i.e. there exists a smooth map $J : M \to \mathcal{G}^*$, $p \mapsto J(p)$ such that $i(\overline{x})\Omega$ is the derivative of the smooth map $J_x : M \to \mathbb{R}$ associated with each $x \in \mathcal{G}$ by $p \mapsto J_x(p) := J(p)(x)$. Such a map is called a momentum map for Ψ.

DEFINITION 1.1. A Lie group G endowed with a left-invariant symplectic form ω^+ is called a symplectic Lie group. The pair (G, ω^+) is Hamiltonian if the left canonical action L_G of G on G (i.e. by means of left translations) is Hamiltonian. The pair $(\mathcal{G}, \omega)$ with $\omega = \omega_\varepsilon^+$, where ε is the identity of G, is called a symplectic Lie algebra.

DEFINITION 1.2. A pair (G, ∇^+), where ∇^+ is a left-invariant affine structure on G, will be called an affine Lie group.

A left-invariant affine structure on G corresponds to a bilinear product $(x, y) \mapsto xy$ on $\mathcal{G}$ such that

(1) $$(xy)z - x(yz) = (yx)z - y(xz)$$

and (product compatible with the Lie bracket of $\mathcal{G}$ in the sense that)

$$xy - yx = [x, y]$$

for any $x, y, z \in \mathcal{G}$.

A vector space A endowed with a bilinear product that satisfies (1) is called a left-symmetric algebra. In this case $[x, y] := xy - yx$ is a Lie bracket on A and the derived ideal $[A, A]$ is different from A ([8]).

REMARK 1.3. 1. If G admits a left-invariant symplectic form its Lie algebra is different from its derived ideal $[\mathcal{G}, \mathcal{G}]$. In particular a semisimple Lie group is not a symplectic Lie group.

2. A symplectic Lie group (G, ω^+) is endowed with a left-invariant affine structure given by the left-invariant torsion-free and flat connection ∇^+ defined by the formula:

$$(2) \qquad \omega^+(\nabla_{x^+}^+ y^+, z^+) = -\omega^+(y^+, [x^+, z^+])$$

where a^+ is the left-invariant vector field on G associated with $a \in \mathcal{G} = T_\varepsilon(G)$.

The aim of this paper is to describe the structure of symplectic Lie groups in terms of semi-direct products of Lie groups or symplectic reduction and principal fiber bundles with affine fibers improving results given in [6]. This description can be bettered if the canonical left action L_G of G on G is Hamiltonian.

The affine structure associated with ω^+ plays a very important role in our study. Obviously a better understanding of the structure of symplectic Lie groups is important in Mathematics and Physics.

2. Structure of symplectic Lie groups. In this section we will describe the general structure of the symplectic Lie groups by means of Theorems 2.3 and 2.4 which complete and improve results given in [6].

Let (G, ω^+) be a connected symplectic Lie group. Consider a Lie subgroup H of G with Lie algebra $L(H)$ and denote by L_H, respectively R_H, the left (resp. right) canonical action of H on G given by the product of G. Obviously L_H is symplectic.

Study of L_H and R_H. Let $O_L(\sigma) = H\sigma$ be the orbit of $\sigma \in G$ and let F be the subbundle of TG, tangent to the orbits of L_H. Denote by $\mathcal{F}$ the foliation associated with F. It is well known that the subbundle $F^\perp$, orthogonal to F, is integrable. Let $\mathcal{F}^\perp$ be the corresponding foliation. The fiber of F above $\sigma \in G$ is given by $F_\sigma = \{x_\sigma^-; x \in L(H)\}$ with x^- the right-invariant vector field on G associated with $x \in L(H)$. The subbundle $F^\perp$ is defined by the intersection of the kernels of the closed 1-forms $i(x^-)\omega^+$ where $x \in L(H)$. On the other hand the orbits of R_H are the leaves of the integrable distribution $D_\sigma^+ := \{x_\sigma^+; x \in L(H)\}$. For $\tau \in H$ we have:

$$O_L(\tau) = H = O_R(\tau).$$

From the formula $\omega_\sigma^+(x_\sigma^+, y_\sigma^+) = \omega(x, y)$ for $x, y \in \mathcal{G}$, it follows that if $L(H)$ is symplectic (resp. totally isotropic or coisotropic or Lagrangian) then R_H determines a left-invariant symplectic (resp. totally isotropic or coisotropic or Lagrangian) foliation on G. Moreover if H is normal in G we obtain:

LEMMA 2.1. *Let H be a connected normal Lie subgroup of a connected symplectic Lie group (G, ω^+). Then $\mathcal{F}^\perp$ is left invariant and its leaves are affine submanifolds of (G, ∇^+). Moreover if L_H is Hamiltonian, the leaves of $\mathcal{F}^\perp$ are closed.*

PROOF. For $a, b \in L(H)^\perp$ and $x \in L(H)$, we have

$$\omega(ab, x) = -\omega(b, [a, x]) = 0.$$

Consequently $L(H)^\perp$ is a subalgebra of the left-symmetric algebra $L(G)$ given by (2).

Suppose that L_H is Hamiltonian and let $J : G \to L(H)^*$ be a momentum map for L_H. As the orbits of L_H have the same dimension, the rank of J is constant on G and, for every $\sigma \in G$, the connected component of $J^{-1}(J(\sigma))$ containing σ is the leaf of $\mathcal{F}^\perp$ through σ. Hence the leaves of $\mathcal{F}^\perp$ are closed. $\square$

LEMMA 2.2. *Every symplectic Lie group (G, ω^+) contains a non-trivial connected normal and closed subgroup H isomorphic to a vector group or a torus.*

PROOF. Since G is an affine group, the Lie algebra $\mathcal{G}$ is different from its derived ideal $[\mathcal{G}, \mathcal{G}]$ (see [8]). Consequently G contains a subgroup verifying the conditions required in the Lemma (see for example [8]). $\square$

THEOREM 2.3. *Let (G, ω^+) be a connected symplectic Lie group and H a non-trivial connected normal and closed subgroup of G. Denote by $H^\perp$ the connected Lie subgroup of G orthogonal to H and put $K = H \cap H^\perp$. We have*

1. If K is discrete, H and $H^\perp$ are symplectic Lie subgroups of (G, ω^+) and G contains an open connected symplectic Lie subgroup isomorphic to group $(H \times H^\perp)/Ker f$ where $H \times H^\perp$ is a semi-direct product, $H^\perp$ acts on H by symplectomorphisms and f is the homomorphism of Lie groups from $H \times H^\perp$ into G given by $f(\sigma, \tau) = \sigma\tau$.

2. If K is not discrete then it is a closed Abelian and normal subgroup of $H^\perp$, the quotient group $R = H^\perp/K$ is a reduced symplectic Lie group of (G, ω^+) and the canonical sequence

(3) $$\{\varepsilon\} \to K \to H^\perp \to R \to \{\varepsilon\}$$

is an exact sequence of affine Lie groups where the affine structures are deduced from those of (G, ∇^+), that of K being trivial.

PROOF. The Lie algebra I of H is a Lie ideal of $\mathcal{G}$. The identity

$$\omega(ab, x) = -\omega(b, [a, x]) = 0$$

for $a, b \in I^\perp$ and $x \in I$, shows that $I^\perp$ is a left-symmetric subalgebra of $\mathcal{G}$. Hence $H^\perp$ is an affine Lie subgroup of (G, ∇^+).

Suppose K is discrete. In this case $\mathcal{G} = I \times I^\perp$(semi-direct product). Obviously, the connected subgroup K is closed in $H^\perp$. Moreover, for $a \in I \cap I^\perp$, $b \in I^\perp$ and $i \in I$, the identity

$$\omega([a, b], i) + \omega([b, i], a) + \omega([i, a], b) = 0$$

becomes $\omega([a, b], i) = 0$ and thus $[a, b] \in I^\perp$. Furthermore as

$$\omega([a, b], c) = 0 \quad \text{for } c \in I^\perp$$

we have $[a, b] \in I$. Hence K is normal in $H^\perp$.

On the other hand, if K is not discrete it is clear that R is connected and $\dim(R) < \dim(H^\perp)$. Also, R is a reduced symplectic manifold of (G, ω^+) because $I \cap I^\perp$ is the nilradical of the restriction of ω to $I^\perp$.

From the identity for $b \in I \cap I^\perp$, $x \in I^\perp$ and $z \in I$ (or $z \in I^\perp$),

$$\omega(bx, z) = -\omega(x, [b, z]) = 0$$

it follows that $I \cap I^\perp$ is a 2-sided ideal of $I^\perp$. All this implies that (3) is an exact sequence of affine Lie groups.

Finally the formula

$$\omega(bx, a) = -\omega(x, [b, a]) = 0$$

where $b, x \in I \cap I^\perp$ and $a \in L(G)$ shows that $bx = 0$. This means that the affine structure of K deduced from ∇^+ is trivial.   □

THEOREM 2.4. *Let (G, ω^+) be a connected symplectic Lie group and H a non-trivial connected normal Lie subgroup of G. If L_H is Hamiltonian on G then $H^\perp$ is an affine closed subgroup of (G, ∇^+), the homogeneous manifold $G/H^\perp$ is endowed with a commutative parallelism (and consequently with an affine structure) and the canonical maps in*

(4) $$H^\perp \to G \to G/H^\perp$$

are affine maps.

PROOF. Since L_H is Hamiltonian, the leaves of $\mathcal{F}^\perp$ are closed. In particular $H^\perp$ is closed in G and (4) is a principal bundle whose fiber is the affine Lie group $H^\perp$. Let $\{x_1, \ldots, x_r\}$ be a basis of $I = L(H)$. We know that the subbundle $F^\perp$ is given by the intersection of the kernels of the left-invariant forms $\eta_j^+ := i(x_j^+)\omega^+$ but also by the intersection of the closed 1-forms $\eta_j := i(x_j^-)\omega^+$ with $1 \le j \le r$. The 1-forms η_j are exact and basic relatively to the projection p. So, the projected 1-forms η_j define a local parallelism on $G/H^\perp$. But this parallelism is global and commutative because the η_j are exact. Hence the injection i of $H^\perp$ in G and the projection p in (4) are affine maps.   □

3. Momentum maps in symplectic Lie groups. Let (G, ω^+) be a connected symplectic Lie group. The scalar 2-cocycle ω determines a representation of the Lie algebra $\mathcal{G}$ by endomorphisms of the affine space $\mathcal{G}^*$ given by $x \to (i(x)\omega, ad^*(x))$, where $x \to ad^*(x)$ is the coadjoint representation of $\mathcal{G}$. This affine representation provides a homomorphism of Lie groups of $\tilde{G}$, the universal covering group of G, into the classical affine group $Aff(\mathcal{G}^*) \simeq \mathcal{G}^* \times GL(\mathcal{G}^*)$ (semi-direct product) given for $\sigma = \exp(x)$ with $x \in \mathcal{G}$ by the formula

(5) $$\rho(\sigma) = (\tilde{Q}(\sigma), Ad^*(\sigma))$$

where

$$(6) \qquad \tilde{Q}(\sigma) = \Sigma_{k=1}^{\infty}(1/k!)(ad^*(x))^{k-1}.i(x)\omega.$$

Notice that $\tilde{G}$ is not necessarily a exponential Lie group.

As ω is non-degenerate, the orbit O of $0 \in \mathcal{G}^*$ by the ρ action, is open. Let ∇^0 be the connection induced in O by the usual connection of the affine space $\mathcal{G}^*$. The pullback of ∇^0 by the orbital map from $\tilde{G}$ in O is a left-invariant affine structure. If Π is the covering map from $\tilde{G}$ to G, there exists a unique left-invariant affine structure on G such that Π is affine. This latter affine structure on G coincides with that given by formula (2). Moreover, there exists a representation h of the fundamental group $\Pi_1(G)$ of G by affine transformations of $\mathcal{G}^*$ such that $h(\gamma) \circ \tilde{Q} = \tilde{Q} \circ \gamma$ for every γ in $\Pi_1(G) := \Pi^{-1}(\varepsilon)$.

The map $\tilde{Q}$ is a developing map and h is the holonomy representation of the affine structure ∇^+. The geodesical completeness of ∇^+ is equivalent to the fact that $\tilde{Q}$ is a diffeomorphism.

Suppose that $J : G \to \mathcal{G}^*$ is a momentum map for L_G. This is case if $H^1(G, \mathbb{R}) = 0$ or $H^1(\mathcal{G}^*) = H^2(\mathcal{G}^*) = 0$, for instance. This means that, for every $x \in \mathcal{G}$, the derivative of the map $J_x : G \to \mathbb{R}$ with $J_x(\sigma) := J(\sigma)(x)$, is given by $i(x^-)\omega^+$. The map $\psi_{(\sigma,x)} : G \to \mathbb{R}$ given by

$$(7) \qquad \psi_{(\sigma,x)}(\tau) := J_x(L_\sigma(\tau)) - J_{Ad(\sigma^{-1})(x)}(\tau)$$

with $\sigma \in G$ and $x \in \mathcal{G}$ is constant, because G is connected. Taking $\tau = \varepsilon$ we obtain a mapping $Q : G \to \mathcal{G}^*$, that is the 1-cocycle of G relative to the coadjoint representation, associated with J. For $\sigma \in G$ putting

$$J'(\sigma) := J(\sigma) - J(\varepsilon)$$

defines another momentum map with associated 1-cocycle $Q' = J'$.

We put:

DEFINITION 3.1.   Let (G, ω^+) be a Hamiltonian symplectic Lie group. If J is a momentum map for L_G with J as 1-cocycle we will say that J is a momentum-cocycle.

Hence we have

LEMMA 3.2.   *Let (G, ω^+) be a symplectic Lie group. If L_G is Hamiltonian we have:*

1. *There exists a momentum-cocycle J for L_G such that $J(\varepsilon) = 0 \in \mathcal{G}^*$.*

2. *If J is a momentum-cocycle then the image of J is diffeomorphic to the manifold G/D where $D := \{\sigma \in G; J(\sigma) = J(\varepsilon)\}$ is a discrete subgroup of G. Consequently every momentum map for L_G is a local diffeomorphism.*

PROOF.   Let $J = Q$ be a momentum-cocycle with $J(\varepsilon) = 0 \in \mathcal{G}^*$. The map given by $\rho(\sigma) := (Q(\sigma), Ad^*(\sigma))$ with $\sigma \in G$, is a representation of G by affine transformations of $\mathcal{G}^*$ such that the orbit O of $0 \in \mathcal{G}^*$ is open. Hence, $Im J = O$ can be identified with G/D where D is the ρ-stabilizer in $0 \in \mathcal{G}^*$. $\qquad\square$

REMARK 3.3.   If L_G is Hamiltonian there exists a momentum-cocycle for L_G which is the developing map of the affine structure given by formula (2).

A direct calculation shows the following fact

LEMMA 3.4.   *If $(\tilde{G}, \omega^+)$ is a simply connected symplectic Lie group, the map given by formula* (6) *is a momentum-cocycle for $L_{\tilde{G}}$.*

In what follows we want to find necessary and sufficient conditions to have Hamiltonian symplectic Lie groups.

Let $J : G \to \mathcal{G}^*$ be a momentum map for L_G and $\Pi : \tilde{G} \to G$ the universal covering map. A simple calculation shows that $\tilde{J} := J \circ \Pi$ is a momentum map for $L_{\tilde{G}}$. Moreover if we suppose that $J(\varepsilon) = 0$, it follows that

$$(8) \qquad\qquad \Pi^{-1}(\varepsilon) =: \Pi_1(G) \subset \tilde{J}^{-1}(\{0\}) \,.$$

We have also

LEMMA 3.5.   *Let $\tilde{J}$ be a momentum-cocycle for $L_{\tilde{G}}$ with $\tilde{J}(\tilde{\varepsilon}) = 0 \in \mathcal{G}^*$. Then $\tilde{J}$ factors, by means of Π, into a momentum map J for L_G if and only if condition* (8) *is verified.*

PROOF.   Put $J(\sigma) := \tilde{J}(\tilde{\sigma})$ where $\tilde{\sigma} \in \Pi^{-1}(\sigma)$. We must show that this expression defines a map. In other words, we must verify that we have $\tilde{J}(\tilde{\sigma}) = \tilde{J}(\tilde{\tau})$ for every $\tilde{\tau} \in \Pi^{-1}(\sigma)$. It is clear that it suffices to consider the case where $\sigma = \varepsilon$. As $\tilde{\sigma}, \tilde{\tau} \in \Pi^{-1}(\varepsilon) = \Pi_1(G)$ it follows that $\tilde{\sigma}^{-1}\tilde{\tau} \in \Pi^{-1}(\varepsilon)$ and hence $\tilde{\sigma}^{-1}\tilde{\tau} \in \tilde{J}^{-1}(\{0\})$. Since $\tilde{J}$ is a 1-cocycle, $\tilde{J}(\tilde{\sigma}^{-1}\tilde{\tau}) = 0$ can be written as

$$\tilde{J}(\tilde{\sigma}^{-1}) + Ad^*_{\tilde{G}}(\tilde{\sigma}^{-1})(\tilde{J}(\tilde{\tau})) = 0 \,.$$

Applying $Ad^*_{\tilde{G}}(\tilde{\sigma})$ to this equality we get

$$Ad^*_{\tilde{G}}(\tilde{\sigma})(\tilde{J}(\tilde{\sigma}^{-1})) + \tilde{J}(\tilde{\tau}) = 0 \,.$$

But the first term on the left hand side is equal to $-\tilde{J}(\tilde{\sigma})$, consequently we have $\tilde{J}(\tilde{\sigma}) = \tilde{J}(\tilde{\tau})$ and the map J is well defined. Finally a simple calculation shows that J is a momentum-map for L_G.   □

These preliminary facts imply the following important result

THEOREM 3.6.   *Let (G, ω^+) be a connected symplectic Lie group. Then, L_G is Hamiltonian if and only if the holonomy representation of the affine Lie group (G, ∇^+), is trivial.*

PROOF.   Suppose L_G Hamiltonian. There exists a momentum map J for L_G such that $J(\varepsilon) = 0$. If we put $\tilde{J} := J \circ \Pi$ we get a momentum-cocycle for $L_{\tilde{G}}$ such that $\tilde{J}(\tilde{\sigma}) = 0$ and (8) is verified. So, $h(\gamma) = id_{\mathcal{G}^*}$ for every $\gamma \in \Pi_1(G)$. This means that for each $\tilde{\sigma} \in \tilde{G}$,

$$\tilde{J}(\gamma\tilde{\sigma}) = \tilde{J}(\tilde{\sigma}) \,.$$

In fact we have

$$\tilde{J}(\gamma\tilde{\sigma}) = \tilde{J}(\gamma) + Ad^*_{\tilde{G}}(\gamma)(\tilde{J}(\tilde{\sigma})) = \tilde{J}(\gamma) + \tilde{J}(\tilde{\sigma}) = \tilde{J}(\tilde{\sigma}) \,.$$

Reciprocally, if $h(\gamma) = id_{\mathcal{G}^*}$ for every $\gamma \in \Pi_1(G)$, it follows that we have for $\tilde{\tau} = \gamma\tilde{\sigma}$,

$$\tilde{J}(\tilde{\tau}) = \tilde{J}(\gamma\tilde{\sigma}) = h(\gamma)\tilde{J}(\tilde{\sigma}) = \tilde{J}(\tilde{\sigma}).$$

Hence, we get a momentum-cocycle for L_G by setting $J(\sigma) := \tilde{J}(\tilde{\sigma})$ where $\tilde{\sigma} \in \Pi^{-1}(\sigma)$ and $\tilde{J}$ is a momentum-cocycle for $L_{\tilde{G}}$. □

REMARK 3.7. Let (T^{2n}, ω^+) be a symplectic torus. That this group is not Hamiltonian follows from the compactness of T^{2n} or from the fact that the holonomy of (T^{2n}, ∇^+) is not trivial.

For the sake of completeness let us recall at this point that Theorem 3 in [10] can be presented as follows

THEOREM 3.8. *Let (G, ω^+) be a connected symplectic Lie group, $\Pi : \tilde{G} \to G$ the universal covering of G and $\tilde{J}$ a momentum-cocycle for $L_{\tilde{G}}$. Then, $\tilde{J}$ is a global diffeomorphism i.e. ∇^+ is geodesically complete, if and only if G is unimodular. Moreover, if this condition is verified then G is solvable.*

EXAMPLE 3.9. Consider the manifold $G = \mathbb{R}^*_+ \times \mathbb{R}$ endowed with the product $(a, b)(c, d) := (ac, ad + b)$. The corresponding Lie algebra $\mathcal{G} = \text{span}\{e_1, e_2\}$ with $[e_1, e_2] = e_2$ endowed with the scalar 2-cocycle $\omega = e_1^* \wedge e_2^*$ is a symplectic Lie algebra, where $\{e_1^*, e_2^*\}$ denotes the dual basis. As G is simply connected it is a Hamiltonian Lie group. We will determine a moment-cocycle for L_G using the formula (6).

For $x = x_1 e_1 + x_2 e_2 \in \mathcal{G}$ and $t \in \mathbb{R}$ a direct calculation gives:

$$\exp(tx) = (1, tx_2) \quad \text{if } x_1 = 0; \exp(tx) = \left(e^{x_1}, \frac{x_2}{x_1}(e^{x_1 t} - 1)\right) \quad \text{if } x_1 \neq 0$$

$$ad_x^*(e_1) = 0; ad_x^*(e_2) = x_2 e_1^* - x_1 e_2^*.$$

This implies

$$Q(\exp x) = -x_2 e_1^*, \quad \text{if } x_1 = 0$$

$$Q(\exp x) = \frac{x_2}{x_1}(e^{-x_1} - 1)e_1^* + (1 - e^{-x_1})e_2^*, \quad \text{if } x_1 \neq 0.$$

Hence for $(a, b) \in G$ we get then

$$Q(1, b) = -be_1^*$$

$$Q(a, b) = \frac{b}{a - 1}\left(\frac{1}{a}\right)e_1^* + \left(1 - \frac{1}{a}\right)e_2^*, \quad \text{otherwise.}$$

Recall that Q is also the developing map of the canonical affine structure. Taking $H = \{(1, b); b \in \mathbb{R}\}$, as in Theorem 2.3, we get $K = R = \{0\}$. Moreover H is an affine Lie subgroup of G and the canonical sequence, $H \to G \to \mathbb{R}e_1$ is an exact sequence of affine Lie groups. Also $(\mathcal{G}, \omega, j)$, with $j(e_1) = e_2$ and $j(e_2) = -e_1$, is a positive Kahlerian Lie algebra non Ricci flat (see [10]).

EXAMPLE 3.10. Let $\mathcal{G}_1 := \mathrm{Span}\{e_1, e_2\}$ the real Abelian Lie algebra endowed with the scalar 2-cocycle $\omega_1(e_1, e_2) = -1$ and the complex structure $j_1(e_1) = e_2$, $j_1(e_2) = -e_1$. As $g_1(x, x) = \omega_1(j_1(x), x)$ for $x \in \mathcal{G}_1$ is an Euclidian structure, the triple $(\mathcal{G}_1, \omega_1, j_1)$ is a Kahlerian Lie algebra ([5]). Denote by $(\mathcal{G}_2, \omega_2, j_2)$ another copy of the same algebra with $\mathcal{G}_2 = \mathrm{Span}\{e_3, e_4\}$ and by η the representation of $\mathcal{G}_2$ in $so(\mathcal{G}_1)$ given by $\eta(e_3)(e_1) = e_2$, $\eta(e_3)(e_2) = -e_1$, $\eta(e_4)(e_1) = -e_2$, $\eta(e_4)(e_2) = e_1$. The Lie algebra $\mathcal{G}$, semi-direct product of $\mathcal{G}_1$ by $\mathcal{G}_2$ via η, becomes a positive Kahlerian Lie algebra taking $\omega = \omega_1 + \omega_2$ and $j = j_1 + j_2$. The Riemann metric is $g = g_1 + g_2$; this metric is flat. Hence a priori, on the corresponding simply connected Lie group $G = G_1 \times G_2 = \mathbb{R}^4$ there are two left invariant affine structures and consequently two developing maps and (G, ω^+) is Hamiltonian. It is clear that the orthogonal symplectic of $H := G_1$ is G_2 and a direct calculation shows that the product in G for $\sigma = (a, b, c, d)$ and $\tau = (a', b', c', d')$ is given by,

$$\sigma\tau = (a + a'\cos(c - d) - b'\sin(c - d), b + a'\sin(c - d) + b'\cos(c - d), c + c', d + d') .$$

As G is not a exponential Lie group we can using the closed 1-formes on G given by $\omega^+(x^-,)$ for find the momentum map.

4. Structure of simply connected nilpotent symplectic Lie groups.

Let (G, ω^+) be a simply connected nilpotent symplectic Lie group. Consider the exponential Lie subgroup $H := \{\exp tz; t \in \mathbb{R}\}$ where z is a non-zero central element of $\mathcal{G}$. Since the cohomology group $H^1(G, \mathbb{R})$ is trivial, the action L_G is Hamiltonian. Obviously $J_H := i^* \circ J$, where i the inclusion map of H in G, is a momentum map for L_H and $J_H(\exp x)$ is the linear map on $\mathbb{R}z$ given by $z \to \omega(x, z)$. Hence $J_H^{-1}(\{0\}) =: H^\perp$ is a closed and connected subgroup of G which is the (connected) subgroup of G orthogonal to H relatively to ω^+. Applying Theorem 2.3 and Theorem 2.4 we obtain the exact canonical sequence of affine groups:

$$(9) \qquad\qquad \{\varepsilon\} \to H \to H^\perp \to R := H^\perp/H \to \{\varepsilon\}$$

and the canonical principal fiber bundle

$$(10) \qquad\qquad \{\varepsilon\} \to H^\perp \to G \to G/H^\perp$$

where the projection can be identified with the momentum map J_H.

But as z is central we can deduce that the Lie algebra of $H^\perp$ is an ideal of the Lie algebra $\mathcal{G}$. As a consequence $G/H^\perp$ is a (connected) one-dimensional affine subgroup of (G, ∇^+).

In conclusion we have proved the following result:

THEOREM 4.1. *Let (G, ω^+) be a simply connected nilpotent symplectic Lie group. If H is an exponential 1-dimensional central Lie subgroup of G then H and $H^\perp$ are affine subgroups of (G, ∇^+) and G is a semi-direct product of the normal subgroup $H^\perp$ by the group $G/H^\perp$ where $H^\perp$ is a central extension of the reduced symplectic Lie group $R = H^\perp/H$.*

REMARK 4.2. At the level of Lie algebras, Theorem 4.1 corresponds to a particular case of the notion of symplectic double extension of symplectic Lie algebras introduced by Medina and Revoy in [13] (see also [5], [6]).

For Lie groups we put

DEFINITION 4.3.   A symplectic Lie group (G, ω^+) is called a symplectic double extension of a symplectic Lie group (R, η^+) if (G, ω^+) contains a totally isotropic Lie subgroup H such that $H^\perp/H$ is a symplectic Lie group isomorphic to (R, η^+).

In these terms we can say that every simply connected nilpotent symplectic Lie group G of dimension $2(n+1)$ is a symplectic double extension of a $2n$-dimensional simply connected nilpotent symplectic Lie group. Moreover G is obtained by a sequence of symplectic double extensions starting from the simply connected Abelian Lie group.

5.   The classic affine group as Frobenius Lie group.

DEFINITION 5.1.   A symplectic Lie group (G, ω^+) is called a Frobenius Lie group if $\omega^+ = d\nu^+$ where ν^+ is a left-invariant 1-form on G.

Every Frobenius Lie group $(G, d\nu^+)$ is a Hamiltonian Lie group. In fact the map defined by

(11)
$$J(\sigma)(x) := -\nu_\sigma^+(x_\sigma^-)$$

for $\sigma \in G$ and $x \in \mathcal{G}$ is a Ad_G^*-equivariant momentum map and $J'(\sigma) := J(\sigma) - J(\varepsilon)$ is a momentum-cocycle for L_G.

Consequently if H is a non trivial connected normal and closed subgroup of $(G, d\nu^+)$ such that $K := H^\perp \cap H$ is non-discrete, then G is a symplectic double extension of the symplectic Lie group $R := H^\perp/K$. The case where $K = H$ is particularly nice and simple. To illustrate this situation we will study the group of the affine transformations of the real (or complex) affine space of dimension n.

Consider the classical affine group $G = Aff(\mathbb{R}^n) \equiv \mathbb{R}^n \times GL(\mathbb{R}^n)$ (semi-direct product by means of the trivial representation) and let $\mathcal{G} \equiv \mathbb{R}^n \times gl(\mathbb{R}^n)$ be its Lie algebra. To any $\alpha \in L(G)^*$ we can associate a unique pair (g, M) with $g \in (\mathbb{R}^n)^*$ and $M \in gl(\mathbb{R}^n)$ given by:

$$\alpha(x, u) = g(x) + \mathrm{tr}(M \circ u)$$

for $x \in \mathbb{R}^n$ and $u \in gl(\mathbb{R}^n)$.

The associated 2-coboundary $\delta\alpha$ can be written

$$\delta\alpha((x, u); (y, v)) = -g(u(x)) - v(y) - \mathrm{tr}(M \circ [u, v]).$$

Suppose that the coadjoint orbit of $\alpha \equiv (g, M)$ is open (see Theorem 2.5 of [2]) and take $\omega^+ = \delta\alpha^+$ and $H = \mathbb{R}^n$. The momentum map $J : G \to \mathcal{G}^*$, $J(\sigma) := Ad_\sigma^*(\alpha)$ can be written

$$J(\sigma)(x) = \alpha_\sigma^+(x_\sigma^-)$$

where x^- denotes the right-invariant vector field on G associated with $x \in \mathcal{G}$.

Remember that $J_H(\sigma)$ is given by the restriction of $J(\sigma)$ to the Lie algebra of H.

Dardié and Medina have described the architecture of the classical affine group in the following terms (result not yet published).

THEOREM 5.2.   *If the classical affine Lie group $G = Aff(\mathbb{R}^n)$ is endowed with the left-invariant symplectic form $\omega^+ = \delta\alpha^+$ with $\alpha \equiv (g, M)$ and $H = \mathbb{R}^n$ then we have:*

1.   *$J_H^{-1}(g)$ is a closed subgroup of G containing H.*
2.   *The exact canonical sequence of Lie groups*

$$\{\varepsilon\} \to H \to J_H^{-1}(g) \to J_H^{-1}(g)/H \to \{\varepsilon\}$$

splits and is a sequence of affine Lie groups.

3.   *The symplectic reduced Lie group $J_H^{-1}(g)/H$ is isomorphic to $Aff(\mathbb{R}^{n-1})$.*
4.   *In the canonical fiber bundle*

$$J_H^{-1}(g) \to Aff(\mathbb{R}^n) \to (\mathbb{R}^n)^* - \{0\}$$

the fiber is an affine Lie subgroup of $Aff(\mathbb{R}^n)$ and the projection is affine relatively to the usual affine structure of $(\mathbb{R}^n)^* - \{0\}$.

The proof of this Theorem is rather technical. Basically it requires a good understanding of the ideas and tools developed in the above sections and the following technical Lemma.

LEMMA 5.3.   *For G and H as in Theorem 5.2 the momentum map J_H is a surjective submersion on $(\mathbb{R}^n)^* - \{0\}$ and*

(12) $$J_H(x, T) = {}^tT^{-1}g$$

for $(x, T) \in G$.

PROOF.   The Lemma is a consequence of the following formulas describing the coadjoint representation of G:

$$Ad^*_{(0,U)}(h, N) = ({}^tU^{-1}(h), U \circ N \circ U^{-1})$$
$$Ad^*_{(x,I)}(h, N) = (h, N + h \times x)$$

where U is taken in the connected component in $GL(n, \mathbb{R})$ of the unit I and $(h \times x)(y) := h(y)x$ for $x, y \in \mathbb{R}^n$.

Let us now sketch the proof of the Theorem. Formula (12) implies that $J_H^{-1}(g) = \{(x, T) \in G, {}^tT^{-1}g = g\}$ is a closed subgroup of G. The factor group $J_H^{-1}(g)/H$ can be identified with subgroup $\{(0, T) \in G, {}^tT^{-1}g = g\}$ of G and consequently it can be interpreted as the group of affine transformations of $Ker(g)$. On the other hand we have $L(J_H^{-1}(g)) = L(H)^\perp$. Hence assertions 2 and 4 are consequences of Theorems 2.3 and 2.4. Finally Lemma 5.3 implies assertion 3 and Theorem 5.2 is proved.   □

Denote now by α_1 and α_2 the elements of $\mathcal{G}^*$ given by $\alpha_1(x, T) = g(x)$ and $\alpha_2(x, T) = \text{tr}(M \circ T)$. We have $\delta\alpha = \delta\alpha_1 + \delta\alpha_2$ and $\mathcal{G}$ is the orthogonal direct sum of the symplectic subalgebras $\mathcal{G}_1 := \text{Rad}(\delta\alpha_2)$ and $\mathcal{G}_2 := \text{Rad}(\delta\alpha_1)$(see [2]). Moreover $\mathcal{G}_2$ is isomorphic to the Lie algebra $aff(\text{Ker}(g))$ and $\mathcal{G}_1$ is isomorphic to the natural semidirect product of $\mathbb{R}^n$ by the commutant of M in $gl(\mathbb{R}^n)$ denoted in the following by $C(M)$.

Consequently the infinitesimal version of Theorem 5.2 is given by the following assertion

PROPOSITION 5.4. *In the notations of Theorem 5.2 if $I = L(H)$, the canonical sequence*

$$0 \to I \to I^\perp \to I^\perp/I \to 0$$

is a split exact sequence of Lie algebras with $I \cong \mathbb{R}^n$, $I^\perp = I \times \mathcal{G}_2 \cong \mathbb{R}^n \times aff(\mathbb{R}^{n-1})$ and $I^\perp/I \cong \mathcal{G}_2 \cong aff(\mathbb{R}^{n-1})$. It is also an exact sequence of left-symmetric algebras.

Moreover the vector space $aff(\mathbb{R}^n)$ is a direct sum of the Lie subalgebras $I^\perp$ and $C(M)$.

Let us be more precise about the morphisms of Lie algebras in Proposition 5.4. The isomorphism between $\mathcal{G}_2$ and $aff(\mathbb{R}^{n-1})$ is given by the choice of a supplementary of $\mathrm{Ker}(g)$ into $\mathbb{R}^n$. In fact if $x \in \mathbb{R}^n$ verifies $g \circ M^i x = 0$ for $0 \le i \le n-2$ and $g \circ M^{n-1} x = 1$, then the map $(0, u) \to (u(M^{n-1}x), u')$, where u' is the restriction of u to $\mathrm{Ker}(g)$, defines such an isomorphism. Moreover the symplectic reduced form corresponds to the 2-coboundary associated with $(g_1, M_1) \in aff((\mathbb{R}^{n-1})^*)$ where g_1 and M_1 are given by

$$\mathrm{tr}(M \circ u) = g_1(u(M^{n-1}(x))) + \mathrm{tr}(M_1 \circ u').$$

Suppose M is nilpotent (this is always possible, see [2]), then g_1 can be identified with ${}^t M(g) \in (\mathbb{R}^n)^*$ and M_1 is also nilpotent.

Hence in the case where $\alpha \equiv (g, N)$ with N nilpotent, an iteration of the (infinitesimal) procedure of symplectic reduction yields a decomposition of the Lie algebra $aff(\mathbb{R}^n)$ as a sum of Abelian subalgebras,

$$aff(\mathbb{R}^n) = K_n \oplus K_{n-1} \oplus \cdots \oplus K_1 \oplus C(N) \oplus C(N_1) \oplus \cdots \oplus C(N_{n-1})$$

where K_i and $C(N_{i-1})$ are the subalgebras corresponding respectively to the (totally) isotropic ideal and the commutant of N_{i-1} found at the moment of the i-reduction.

Let us explain in few words this decomposition.

The elements of $C(N_i)$ are the polynomials in N_i with N_i the endomorphism defined by $N_i^i = 0$ and $N_i^j x = N^j x$ for $j < i$ and $0 \le i \le n-1$.

Let $B = \{x, Nx, \ldots, N^{n-1}x\}$ be the dual basis of

$$B^* = \{N^{n-1}g, {}^t N^{n-2}g, \ldots, g\}.$$

Using B we can identify the Lie algebra $aff(\mathbb{R}^n)$ with the Lie algebra of $(n+1) \times (n+1)$-matrices with the last row equal to zero.

In this context the coefficients of the $n \times n$ linear part $(a_{i,j})$ of any element of K_i are null except for $a_{1,i}, a_{2,i}, \ldots, a_{i-1,i}$ of the i-column.

Consequently, relatively to the basis B dual of B^*, the subspace $L = K_n \oplus K_{n-1} \oplus \cdots \oplus K_1$ can be identified with the Lie subalgebra of $gl(\mathbb{R}^{n+1})$ consisting of strict upper triangular matrices.

Finally, in the basis B given above, $L' = C(N) \oplus C(N_1) \oplus \cdots \oplus C(N_{n-1})$ could be identified with the subalgebra of $(n+1) \times (n+1)$ lower triangular matrices having the last row equal to zero.

As at each symplectic reduction K_i and $C(N_i)$ are totally isotropic subspaces, it follows that L and L' are Lagrangian subalgebras of $aff(\mathbb{R}^n)$ relatively to $\delta\alpha$ with $\alpha \equiv (g, N)$ and N nilpotent.

Denote by Λ and Λ' the connected Lie subgroups of $Aff(\mathbb{R}^n)$ corresponding to subalgebras L and L'. The canonical left actions L_Λ and $L_{\Lambda'}$ on $Aff(\mathbb{R}^n)$ are Hamiltonian and consequently Λ and Λ' are closed (Theorem 3.1 of [2]).

However we have (see [4]):

PROPOSITION 5.5.   (i) $Aff(\mathbb{R}^n)$ *has exactly two open orbits for the coadjoint representation. Moreover, two elements* (g, M) *and* (h, N) *having an open orbit are in the same orbit if and only if the bases* $\{{}^tM^{n-1}g, {}^tM^{n-2}g, \ldots, g\}$ *and* $\{{}^tN^{n-1}g, {}^tN^{n-2}g, \ldots, g\}$ *give the same orientation.*

(ii) *If* Ω *and* Ω' *are two left invariant symplectic forms on* $Aff(\mathbb{R}^n)$, *there exists an automorphism* φ *of the Lie algebra* $aff(\mathbb{R}^n)$ *such that*

$$\Omega_\varepsilon(., .) = \Omega'_\varepsilon(\varphi., \varphi.).$$

Proposition 5.4 and the above analysis imply

THEOREM 5.6.   *The symplectic Lie group* $(Aff(\mathbb{R}^n), d\alpha^+)$ *is endowed with a pair of left invariant transversal Lagrangian foliations whose leaves are closed and given by* Λ *and* Λ' *as above.*

REFERENCES

[1]   C. BENSON AND GORDON, Kahler and symplectic structures on nilmanifolds. Topology 27 (1988), no. 4, 513–518.

[2]   M. BORDEMANN, A. MEDINA AND A. OUADEFEL, Le groupe affine comme variété symplectique. Tohoku Math. J. (2) 45 (1993), no. 3, 423–436.

[3]   B. CHU, Symplectic homogeneous spaces. Trans. Am. Math. Soc. 197 (1974), 145–159.

[4]   J. M. DARDIÉ, Groupes de Lie symplectiques ou kählériens et double extension. Ph.D. Thesis, Université de Montpellier II, 1993.

[5]   J. M. DARDIÉ AND A. MEDINA, Algèbres de Lie kählériennes et double extension. J. Algebra, 185 (1996), no. 3, 774–795.

[6]   J. M. DARDIÉ AND A. MEDINA, Double extension symplectique d'un groupe de Lie symplectique. Adv. Math. 117 (1996), no. 2, 208–227.

[7]   D. FRIED, W. GOLDMAN AND M. W. HIRSCH, Affine manifolds with nilpotent holonomy. Comm. Math. Helv. 56 (1981), no. 4, 487–523.

[8]   J. HELMSTETTER, Radical d'une algèbre symétrique à gauche. Ann. Inst. Fourier (Grenoble) 29 (1979), no. 4, 17–35.

[9]   J. L. KOSZUL, Variétés localement plates et convexité. Osaka J. Math. 2 (1965), 285–290.

[10]   A. LICHNEROWICZ AND A. MEDINA, On Lie groups with left invariant symplectic or Kählerian structures. Lett. Math. Phys. 16 (1988), no. 3, 225–235.

[11]   J. MARSDEN AND A. WEINSTEIN, Reduction of symplectic manifolds with symmetry. Rep. Mathematical Phys. 5 (1974), no. 1, 121–130.

[12] A. MEDINA, Structures de Poisson affines. Symplectic geometry and mathematical physics (Aix-en-Provence, 1990), 288–302. Progr. Math., 99, Birkhauser Boston, Boston, MA, 1991.

[13] A. MEDINA AND PH. REVOY, Groupes de Lie à structure symplectique invariante, Symplectic geometry, groupoids, and integrable systems, (Berkeley, CA, 1989), 247–266. Math. Sci. Res. Inst. Publ., 20, Springer, New York, 1991.

[14] A. MEDINA AND PH. REVOY, Lattices in symplectic Lie Groups. J. Lie Theory 17 (2007), no. 1, 27–39.

[15] M. RAIS, La représentation coadjointe du groupe affine. Ann. Inst. Fourier (Grenoble) 28 (1978), no. 1, 207–237.

[16] J. M. SOURIAU, Structures des systèmes dynamiques. Collection Dunod Université, Dunod Paris, 1970.

INSTITUT DES MATHÉMATIQUES
UMR 5149 DU CNRS
UNIVERSITÉ DE MONTPELLIER 2
34095 MONTPELLIER CEDEX 5
FRANCE

E-mail address: alberto.medina@univ.montp2.fr