TORIC MODIFICATIONS OF CYCLIC ORBIFOLDS AND AN EXTENDED ZAGIER RECIPROCITY FOR DEDEKIND SUMS

Dedicated to the memory of Professor Masayoshi Nagata

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Abstract. We study a toric modification of Fujiki-Oka type for cyclic quotient singularities. Especially the behavior of rational Chow rings, orbifold signatures and so on are explicitly calculated. As a result, we extend Zagier's reciprocity for higher-dimensional Dedekind sums. Namely, we define Dedekind sums with weight by using Atiyah-Singer's equivariant signature with non-isolated fixed point locus, and then prove our reciprocity among them.

Introduction. This paper concerns with singularity theory and number theory via the methods of toric geometry. By extending the notion of Hilbert modular cusps, Tsuchihashi [Tsu] defined higher-dimensional cusp singularities via toric geometry. Ogata [Og1] [Og2] studied zeta functions, eta functions and signature defects for these cusp singularities. Ishida [I2] also studied the special value of the zeta functions of these singularities. Our studies are on the same line as the above works in the spirit, but our objects are not cusp singularities but cyclic quotient singularities. We start to explain it from the viewpoint of singularity.

0.1. Let X be an n-dimensional analytic space which has as isolated cyclic quotient singularity at the origin $\mathbf{o} \in X$. If $n = 2$, the resolution process of the singularity $(X, \mathbf{o})$ is classically well-known via Hirzebruch-Jung method. If $n \geq 3$, Fujiki [Fuj] first constructed a certain resolution process for $(X, \mathbf{o})$, and other types of processes are known in these days.

Oka [Ok] constructed a certain resolution process for toric singularities. Note that cyclic quotient singularities belong to a class of toric singularities which come from semi-unimodular cones (§1.2). If one applies Oka's subdivision lemma ([Ok, p. 410]) to a semi-umimodular cone, then the resulting toric modification $\mu : Y \to X$ is nothing but the birational holomorphic map which is used by Fujiki as the key lemma [Fuj, Lemma 3] of his construction. We call μ *Fujiki-Oka modification*.

The aim of this paper is to study geometric and number-theoretic properties of Fujiki-Oka modifications. We show the following:

  (i) We previously define a natural multi-dimensional Euclidian algorithm (Definition 1.1.1). Then the space Y has cyclic quotient singularities whose types essentially

match with this algorithm (Lemma 1.2.5). The singularities of Y are no more isolated in general.

(ii) The morphism μ has a relative projective orbifold compactification $\overline{\mu} : \overline{Y} \to \overline{X}$ (Proposition 1.3.1), and the relative Chow ring for $\overline{\mu}$ is described (Remark 3.2.2).

Now we assume that n is even, and study the behavior via $\overline{\mu}$ of invariants related to signatures. For even dimensional complex orbifold M in general, two types of signatures are defined. The one is the (usual) signature $\mathrm{Sign}\,M$ as a rationally homology manifold, and the other is the orbifold signature $\mathrm{Sign}^{\mathrm{orb}}\,M$ which is defined by the integral of orbifold L-class. The difference $\mathrm{Sdef}\,M := \mathrm{Sign}\,M - \mathrm{Sign}^{\mathrm{orb}}\,M$ is called *S-defect*[1], which is essentially described by Lefschetz fixed point formula for the signature operator ([AS], [Ka]). We calculate the relative contributions of $\overline{\mu}$ to these three invariants.

(iii) The difference of signatures $\mathrm{Sign}\,\overline{Y} - \mathrm{Sign}\,\overline{X}$ is calculated by using Leung-Reiner's theorem [LR] (Corollary 2.1.2).

(iv) The difference of orbifold signatures $\mathrm{Sign}\,\overline{Y}^{\mathrm{orb}} - \mathrm{Sign}^{\mathrm{orb}}\,\overline{X}$ is calculated by using the method of Pommersheim [P] (Proposition 3.2.1).

(v) The difference of S-defects $\mathrm{Sdef}\,\overline{Y} - \mathrm{Sdef}\,\overline{X}$ is calculated according to Kawasaki's formula [Ka] (Proposition 5.1.2).

By substituting the results of (iii)–(v) to the equality

$$(1) \qquad (\mathrm{Sign}\,\overline{Y} - \mathrm{Sign}\,\overline{X}) - (\mathrm{Sign}^{\mathrm{orb}}\,\overline{Y} - \mathrm{Sign}^{\mathrm{orb}}\,\overline{X}) = \mathrm{Sdef}\,\overline{Y} - \mathrm{Sdef}\,\overline{X},$$

we have our reciprocity formula for generalized Dedekind sums (Theorem 4.2.2). We explain a number-theoretic meaning of it in the next subsection.

0.2. Historically, in order to express the explicit $SL_2(\mathbf{Z})$ actions to his eta function, Dedekind introduced the notion of Dedekind sums

$$s(a; d) = \sum_{k=1}^{d-1} \left(\left(\frac{k}{d}\right)\right) \left(\left(\frac{ka}{d}\right)\right)$$

where d, a are mutually prime natural numbers and $((x)) = x - [x] - 1/2$ $(x \notin \mathbf{Z})$, $((x)) = 0$ $(x \in \mathbf{Z})$. He also proved the 2-dimensional inhomogeneous reciprocity

$$(2) \qquad s(a; d) + s(d; a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{d} + \frac{a}{d} + \frac{1}{da}\right).$$

Rademacher extended (2) to the 2-dimensional homogeneous reciprocity for the homogeneous Dedekind sum $s(a_1, a_2; d) = \sum_{k=1}^{d-1} ((ka_1/d)) ((ka_2/d))$. He expressed it as

$$(3) \qquad s(a_1, a_2; d) = \frac{1}{4d} \sum_{k=1}^{d-1} \cot\frac{\pi a_1 k}{d} \cot\frac{\pi a_2 k}{d}$$

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[1] We may call it *signature defect*. Although this word is sometimes used in more delicate sense assuming a framing of the tangent bundle, we avoid it here.

by using Fourier expansion (cf. [RG]). As an extended form of (3), Hirzebruch-Zagier [HZ] and Zagier [Z] introduced the notion of higher-dimensional Dedekind sums

$$(4) \qquad s(a_1, \ldots, a_n; d) = \frac{(-1)^{n/2}}{d} \sum_{k=1}^{d-1} \prod_{i=1}^{n} \cot\left(\frac{\pi k a_i}{d}\right)$$

for an even natural number n. Zagier [Z] also proved the n-dimensional homogeneous reciprocity

$$(5) \qquad \sum_{i=0}^{n} s(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n; a_i) = 1 - \frac{L_n(a_0, \ldots, a_n)}{\prod_{i=0}^{n} a_i},$$

where $a_0, a_1, \ldots, a_n$ are pairwise prime natural numbers and L_n is Hirzebruch's L-polynomial of degree n.

Beck [B] extended the notion of Dedekind sums which relates to higher derivatives of the cotangent function, and the reciprocity was studied by [B] and [BR]. See for instance [GS], [BH1], [BH2] for other number theoretic related topics.

From the topological viewpoint, the invariant (4) is directly related to the G-signature theorem with an isolated fixed point, i.e., the Lefschetz fixed point formula ([AB, p. 473]) for the signature operator. For the reciprocity related to the Lefschetz fixed point formula of other Dirac operators, see [FFU], [Fuk] etc.

From the algebro-geometric viewpoint, Pommersheim [P] gave an interesting proof for the 2-dimesional homogenenous reciprocity by using toric geometry. Reid [R] defined another type of higher-dimensional Dedekind sum which relates to orbifold Riemann-Roch problem, and this direction was developed by [BRZ]. See for instance [U], [A1] for other related topics.

Now our reciprocity is as follows. We drop the symmetry condition in (5) to the effect that $a_0, a_1, \ldots, a_n$ are pairwise prime, and prove a reciprocity similar to (5). Geometrically, this situation naturally leads us to using the G-signature theorem (cf. [AS], [HZ]) for non-isolated fixed point locus. Our formulation is the following.

We first define the notion of Dedekind sums with weight as follows. By substituting the variables for the data of the normal bundle of the fixed point locus of equivariant L-class, we first define the Atiyah-Singer cotangent function. This is a meromorphic function whose constant term coincides with the usual cotangent product and whose other coefficients are written in terms of higher derivatives of usual cotangent functions and the Bernoulli numbers as in (35). Then the Dedekind sum with weight is defined by substituting the Atiyah-Singer cotangent function for the cotangent function in (4).

We define the notion of n-dimensional proper fraction $\mathbf{a}/d = (a_1, \ldots, a_n)/d$ $(0 \le a_i \le d - 1)$ and higher-dimensional analogs of the classical Euclidean algorithm in the set of these proper fractions, as mentioned in §0.1. We call them i-th remainder maps and i-th round down maps for $1 \le i \le n$.

Then our reciprocity (37) is stated as a relation among the remainder maps of the proper fraction $\mathbf{a}/d$ with respect to the Dedekind sums with weight under the assumption that

$\gcd(d, a_i) = 1$ $(1 \le i \le n-1)$ and $a_n = 1$. If we further assume that $a_1, \ldots, a_{n-1}$ are pairwise prime, then our reciprocity coincides with inhomogeneous Zagier reciprocity (Corollary 4.2.3). In this sense, our reciprocity is a generalization of Zagier reciprocity.

Lastly we note that our trials are on the first stage and many related problems are open, and some of them are commented in Remark 5.5.1.

1. Fujiki-Oka modification. We express a cyclic orbipoint (a cyclic quotient singularity) as the toric variety corresponding to a semi-unimodular simplicial cone, and define a toric modification via Oka's subdivision of this cone. The algorithm for this subdivision essentially coincides with the Euclidean algorithm for a higher-dimensional proper fraction defined in §1.1.

1.1. Let $\mathbf{N}$ and $\mathbf{Z}$ be the set of natural numbers and integers, respectively. We call

$$(6) \qquad \frac{\mathbf{a}}{d} = \frac{(a_1, \ldots, a_n)}{d}$$

an *n-dimensional proper fraction* $(n \ge 1)$ if $d \in \mathbf{N}$ and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ satisfy $0 \le a_i \le d-1$ for $1 \le i \le n$. Here we call $\mathbf{a}$ and d the *numerator* and the *denominator* of $\mathbf{a}/d$, respectively, and call a_i the *i-th component* of the numerator. The prototype of this notion comes from Reid [R].

Let $\mathbf{Q}_n^{\mathrm{prop}}$ be the set of n-dimensional proper fractions, and let $\overline{\mathbf{Q}}_n^{\mathrm{prop}}$ be the set of union of $\mathbf{Q}_n^{\mathrm{prop}}$ and the symbol ∞. Similarly, we set $\overline{\mathbf{Z}}^n = \mathbf{Z}^n \cup \{\infty\}$. If the n-th component of the numerator satisfies $a_n = 1$, then $\mathbf{a}/d$ is said to be *inhomogeneous*. If $\mathbf{a}/d \in \mathbf{Q}_n^{\mathrm{prop}}$ satisfies

$$\gcd(d, a_i) = 1 \quad (1 \le i \le n),$$

then it is said to be *irreducible*. Moreover an irreducible proper fraction $\mathbf{a}/d$ satisfying

$$\gcd(a_i, a_j) = 1 \quad (1 \le i < j \le n)$$

is said to be *absolutely irreducible*. The multiple of $\mathbf{a}/d$ by $k \in \mathbf{Z}$ is defined by

$$k \cdot \frac{\mathbf{a}}{d} = \frac{(\overline{ka_1}, \ldots, \overline{ka_n})}{d} \in \mathbf{Q}_n^{\mathrm{prop}}$$

where $\overline{ka_i} \equiv ka_i \pmod{d}$ with $0 \le \overline{ka_i} \le d-1$ (cf. [R, p. 372]). We sometimes write the numerator of $\mathbf{a}/d$ by $\mathbf{a} = \mathrm{Num}(\mathbf{a}/d)$. On the other hand, the *i-th shifted numerator* of $\mathbf{a}/d$ is defined by

$$(7) \qquad N_i\left(\frac{\mathbf{a}}{d}\right) = (a_1, \ldots, a_{i-1}, -d, a_{i+1}, \ldots, a_n) \in \mathbf{Z}^n.$$

Now we define an analog for $\mathbf{Q}_n^{\mathrm{prop}}$ of the classical Euclidean algorithm as follows:

DEFINITION 1.1.1. (i) For $1 \le i \le n$, the *i-th round down map* $Z_i : \overline{\mathbf{Q}}_n^{\mathrm{prop}} \to \overline{\mathbf{Z}}^n$ is defined by

$$Z_i\left(\frac{(a_1, \ldots, a_n)}{d}\right) = \begin{cases} \left(\left\lfloor \dfrac{a_1}{a_i} \right\rfloor, \ldots, \left\lfloor \dfrac{a_{i-1}}{a_i} \right\rfloor, \left\lfloor \dfrac{-d}{a_i} \right\rfloor, \left\lfloor \dfrac{a_{i+1}}{a_i} \right\rfloor, \ldots, \left\lfloor \dfrac{a_n}{a_i} \right\rfloor\right) & \text{if } a_i \ne 0 \\ \infty & \text{if } a_i = 0 \end{cases}$$

and $Z_i(\infty) = \infty$. Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(ii) The *i-th remainder map* $R_i : \overline{\mathbf{Q}}_n^{\mathrm{prop}} \to \overline{\mathbf{Q}}_n^{\mathrm{prop}}$ is defined by

$$
R_i\left(\frac{(a_1, \ldots, a_n)}{d}\right) = \begin{cases} \dfrac{\left(\overline{a_1}, \ldots, \overline{a_{i-1}}, \overline{(-d)}, \overline{a_{i+1}}, \ldots, \overline{a_n}\right)}{a_i} & \text{if } a_i \neq 0 \\ \infty & \text{if } a_i = 0 \end{cases}
$$

and $R_i(\infty) = \infty$.

For instance, if $\alpha = (2, 6, 3, 1)/7$, then $Z_3(\alpha) = (0, 2, -3, 0)$ and $R_3(\alpha) = (2, 0, 2, 1)/3$.

In the usual addition on $\mathbf{Z}^n$, we have

(8) $$\mathrm{Num}(R_i(\mathbf{a}/d)) + a_i Z_i(\mathbf{a}/d) = N_i(\mathbf{a}/d), \quad \text{if } a_i \neq 0.$$

Applications of the iteration of this algorithm are discussed in [A2].

1.2. We fix an inhomogeneous proper fraction $\mathbf{a}/d = (a_1, \ldots, a_{n-1}, 1)/d \in \mathbf{Q}_n^{\mathrm{prop}}$ and a primitive d-th root of unity ξ. Let $(z_1, \ldots, z_n)$ be a complex coordinate of $\mathbf{C}^n$ and let $\varphi_{\mathbf{a}/d} : \mathbf{C}^n \to \mathbf{C}^n$ be the holomorphic automorphism defined by

(9) $$\varphi_{\mathbf{a}/d} : (z_1, \ldots, z_{n-1}, z_n) \longmapsto (\xi^{a_1} z_1, \ldots, \xi^{a_{n-1}} z_{n-1}, \xi z_n).$$

The germ at the origin of the quotient space $\mathbf{C}^n / \langle \phi_{\mathbf{a}/d} \rangle$ with respect to the cyclic group $\langle \phi_{\mathbf{a}/d} \rangle$ generated by $\phi_{\mathbf{a}/d}$ is called a *cyclic orbipoint* of type $\mathbf{a}/d$. If $\mathbf{a}/d$ is an irreducible fraction, then this singularity is isolated. In general, it need not be isolated.

Now we interpret this quotient space in terms of toric geometry. We follow Oda [Od] for the basic terminology of toric geometry. We fix a free abelian group $N \simeq \mathbf{Z}^n$ of rank n. We put $M = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ and denote by

$$\langle \ \rangle : M \times N \longrightarrow \mathbf{Z}$$

the canonical pairing. We put $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$, and also denote by $\langle \ \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}$ the natural pairing. Let $P_i = {}^t(P_{i1}, \ldots, P_{in}) \in N$ be primitive elements for $1 \leq i \leq n$, and let

(10) $$\sigma = \mathbf{R}_{\geq 0} P_1 + \cdots + \mathbf{R}_{\geq 0} P_n$$

be the n-dimensional simplicial cone generated by $P_1, \ldots, P_n$. Let $\triangle_\sigma$ be the fan consisting of all the faces of σ, and let U_σ be the affine toric variety corresponding to $\triangle_\sigma$. Namely U_σ is defined to be the space of semi-group homomorphisms $\mathrm{Hom}_{\mathrm{semigr}}\left(\mathbf{C}[\check{\sigma} \cap M], \mathbf{C}\right)$ sending 0 to 0, where $\check{\sigma} = \{x \in M_{\mathbf{R}}; \ \langle x, y \rangle \geq 0, \ y \in \sigma\}$ is the dual cone of σ.

For an m-dimensional face $\tau = \mathbf{R}_{\geq 0} P_{i_1} + \cdots + \mathbf{R}_{\geq 0} P_{i_m}$ of σ (for $m \leq n$), the greatest common divisor of the absolute values of the $m \times m$ minors of the matrix $\{p_{i_k j}\}_{1 \leq k \leq m, 1 \leq j \leq n}$ is called the *multiplicity* of τ, and is denoted by $\mathrm{mult}\ \tau$. Note that $\mathrm{mult}\ \tau = 1$ if and only if $P_{i_1}, \ldots, P_{i_m}$ forms a part of a $\mathbf{Z}$-basis of N. If $\mathrm{mult}\ \sigma = 1$, then σ is said to be *nonsingular*

(or *unimodular*). In this case, the variety U_σ is non-singular. As the simplest simplicial cone but singular, we define the following:

DEFINITION 1.2.1.   *A simplicial cone σ as in* (10) *is said to be* semi-unimodular over P_n *if the multiplicity of the face* $\mathbf{R}_{\geq 0}P_1 + \cdots + \mathbf{R}_{\geq 0}P_{n-1}$ *is one, i.e., the set of primitive vectors* $\{P_1, \ldots, P_{n-1}\}$ *is a part of* $\mathbf{Z}$-*basis of N. In this case, P_n is called the* exceptional element.

The above notion intrinsically appear in Oka [Ok], where an inductive method for a general toric resolution was studied. The following proposition is due to Oka.

POPOSITION 1.2.2 ([Ok, p. 410]).   *Suppose $\sigma = \mathbf{R}_{\geq 0}P_1 + \cdots + \mathbf{R}_{\geq 0}P_n$ is semi-unimodular over P_n with* mult $\sigma = d > 1$. *Let $Q \in N$ be a primitive element such that*
(a)   *The ray $\mathbf{R}_{\geq 0}Q$ is contained in σ, i.e., Q is written as $\sum_{i=1}^{n} r_i P_i$ for $r_i \in \mathbf{R}_{\geq 0}$.*
(b)   *The cone $\mathbf{R}_{\geq 0}P_1 + \cdots + \mathbf{R}_{\geq 0}P_{n-1} + \mathbf{R}_{\geq 0}Q$ is non-singular.*
Then the following (i) *and* (ii) *hold:*
(i)   *There exist non-negative integers $a_1, \ldots, a_{n-1}$ such that Q is written as*

$$(11) \qquad Q = \frac{\sum_{i=1}^{n-1} a_i P_i + P_n}{d} .$$

(ii)   *The integers $a_1, \ldots, a_{n-1}$ in* (11) *are unique modulo d. Especially there exists a unique element $Q \in N$ which satisfies* (11) *under the assumption $0 \leq a_i \leq d - 1$.*

By Proposition 1.2.2, (ii), the inhomogeneous proper fraction $\mathbf{a}/d = (a_1, \ldots, a_{n-1}, 1)/d$ is uniquely determined for a given semi-unimodular cone σ over P_n, which we call *the proper fraction* of σ. The primitive element $Q \in N$ in (11) determined by the proper fraction of σ is called *Oka center* of σ.

PROPOSITION 1.2.3.   *Let $\sigma = \mathbf{R}_{\geq 0}P_1 + \cdots + \mathbf{R}_{\geq 0}P_n$ be semi-unimodular over P_n with the proper fraction $\mathbf{a}/d$. Then the germ at the origin of the toric variety U_σ coincides with the cyclic orbipoint of type $\mathbf{a}/d$.*

PROOF.   Let Q be Oka center of σ. Since the simplex $\mathbf{R}_{\geq 0}P_1 + \cdots + \mathbf{R}_{\geq 0}P_{n-1} + \mathbf{R}_{\geq 0}Q$ is non-singular, $\{P_1, \ldots, P_{n-1}, Q\}$ forms a $\mathbf{Z}$-basis of N. We put $P_n = \sum_{i=1}^{n-1} x_i P_i + x'Q$ for $x_i, x' \in \mathbf{Z}$. Then

$$d = |\det\left(P_1, \ldots, P_{n-1}, \sum_{i=1}^{n-1} x_i P_i + x'Q \right)| = x'|\det(P_1, \ldots, P_{n-1}, Q)| = x' .$$

Therefore the expression $Q = (- \sum_{i=1}^{n-1} x_i P_i + P_n)/d$ implies that $x_i = -a_i$. Then we have

$$(12) \qquad P_n = - \sum_{i=1}^{n-1} a_i P_i + dQ .$$

Let N' be the $\mathbf{Z}$-submodule of N of index d generated by $P_1, \ldots, P_{n-1}, P_n$. Let $\{m_1, \ldots, m_n\}$ be the dual $\mathbf{Z}$-basis of M with respect to $\{P_1, \ldots, P_{n-1}, Q\}$, and let $\{m'_1, \ldots, m'_n\}$ be the dual $\mathbf{Z}$-basis of $M' = \mathrm{Hom}_{\mathbf{Z}}(N', \mathbf{Z})$ with respect to $\{P_1, \ldots, P_{n-1}, P_n\}$. By easy calculation using

(12), we have

(13)
$$m_i = m_i' - a_i m_n' \quad (1 \le i \le n-1), \quad m_n = d m_n'.$$

We identify $N_{\mathbf{R}}' = N' \otimes_{\mathbf{Z}} \mathbf{R}$ with $N_{\mathbf{R}}$, and let $(N', \Delta_\sigma) \to (N, \Delta_\sigma)$ be the map of fans of the identical Δ_σ with respect to the mutually dinstinct lattices N' and N. Let

(14)
$$U_\sigma' = \mathrm{T}_{N'}\mathrm{emb}\,(\Delta_\sigma) \longrightarrow U_\sigma = \mathrm{T}_N\mathrm{emb}\,(\Delta_\sigma)$$

be the natural equivariant morphism. Then $U_\sigma' \simeq \mathbf{C}^n$ and (14) is the quotient map by the kernel of the map $T_{N'} \to T_N$, which is the cyclic group of order d. By the same argument as in the proof of [Od, Prop.1.24], it follows from (13) that (14) is nothing but the quotient map by the action (9). $\square$

REMARK 1.2.4. The notions of the proper fraction and Oka center for a semi-unimodular cone depend on the exceptional element. Namely the cone σ as above may possibly be semi-unimodular over another P_k for $k \ne n$. Then it may have another proper fraction and another Oka center if one changes the exceptional element. In the viewpoint of Proposition 1.2.3, the choice of the exceptional element of a fixed semi-unimodular cone relates to the change of the primitive d-th root of unity of the action (9). In fact, if ξ^{a_k} in (9) is another primitive d-th root of unity for some k, then σ is also semi-unimodular over P_k.

LEMMA 1.2.5. Let $\sigma = \mathbf{R}_{\ge 0} P_1 + \cdots + \mathbf{R}_{\ge 0} P_n$ be semi-unimodular over P_n whose proper fraction is $\mathbf{a}/d = (a_1, \ldots, a_{n-1}, 1)/d$ with Oka center Q. If $a_i \ne 0$, then the simplicial cone

$$\sigma_i = \mathbf{R}_{\ge 0} P_1 + \cdots + \mathbf{R}_{\ge 0} P_{i-1} + \mathbf{R}_{\ge 0} Q + \mathbf{R}_{\ge 0} P_{i+1} + \cdots + \mathbf{R}_{\ge 0} P_n$$

is semi-unimodular over P_n with multiplicity a_i. Oka center Q_i of σ_i coincides with

$$Q_i = \frac{\sum_{j=1}^{i-1} \overline{a_j} P_j + \overline{(-d)} Q + \sum_{j=i+1}^{n-1} \overline{a_j} P_j + P_n}{a_i},$$

where $\overline{a_j} \equiv a_j \pmod{a_j}$ and $0 \le \overline{a_j} \le a_i - 1$ etc. In other words, the proper fraction of σ_i coincides with the image $R_i(\mathbf{a}/d)$ of the i-th remainder map.

PROOF. The multiplicity of σ_i coincides with

$$\left| \det\left(P_1, \ldots, P_{i-1}, \frac{\sum_{i=1}^{n-1} a_i P_i + P_n}{d}, P_{i+1}, \ldots, P_n \right) \right| = \frac{a_i}{d} \, |\det(P_i, \ldots, P_n)| = a_i \,.$$

Since Q_i is Oka center, there exist x_j $(j \ne i)$ and x' with $0 \le x_j \le a_i - 1$, $0 \le x' \le a_i - 1$ such that

(15)
$$a_i Q_i = \sum_{j \ne i, n} x_j P_j + x' Q + P_n \,.$$

It follows from (12) and (15) that

$$a_i Q_i = \sum_{j \ne i, n} (x_j - a_j) P_j - a_i P_i + (x' + d) Q \,.$$

Since Q_i is contained in the lattice $\mathbf{Z}P_1 + \cdots + \mathbf{Z}P_{n-1} + \mathbf{Z}Q$, we have

$$x_j \equiv a_j, \quad x' \equiv -d \pmod{a_i}.$$

Therefore the assertion follows. □

1.3. From now on, we assume that the proper fraction $\mathbf{a}/d$ of this semi-unimodular cone σ is irreducible. We put $X := U_\sigma = \mathrm{T}_N\mathrm{emb}\,(\Delta_\sigma)$. Then X has an isolated orbipoint of type $\mathbf{a}/d$ at the origin. Moreover Oka center Q of σ is in the interior $\mathrm{supp}(\sigma)^{(0)}$ of σ, i.e., $\mathbf{R}_{\geq 0}Q$ is not contained in any proper face of σ.

Now let $\widetilde{\sigma}$ be the subdivision of σ at the center Q. Then $\widetilde{\sigma}$ consists of n simplicial cones. Let $\Delta_{\widetilde{\sigma}}$ be the fan consisting of all the faces of $\widetilde{\sigma}$. We consider the natural birational equivariant morphism

$$\mu : Y = \mathrm{T}_N\mathrm{emb}\,(\Delta_{\widetilde{\sigma}}) \longrightarrow X.$$

We call μ *Fujiki-Oka modification* of X. Note that this modification essentially appears in Fujiki [Fuj, Lemma 3] without using the terminology of toric geometry. Then μ has the following good compactification $\overline{\mu} : \overline{Y} \to \overline{X}$, the existence of whose proof is postponed to §3.3

PROPOSITION 1.3.1. *Let $\overline{\Delta_\sigma}$ be a simplicial completion of Δ_σ, and let $\overline{\Delta_{\widetilde{\sigma}}}$ be the simplical completion of $\Delta_{\widetilde{\sigma}}$ which is obtained by the subdivision of σ to $\widetilde{\sigma}$ in $\overline{\Delta_\sigma}$.*

We can choose such a pair $\overline{\Delta_\sigma}$, $\overline{\Delta_{\widetilde{\sigma}}}$ so that $\overline{\Delta_\sigma}$ and $\overline{\Delta_{\widetilde{\sigma}}}$ are both projective. Namely $\overline{Y} = \mathrm{T}_N\mathrm{emb}\,(\overline{\Delta_{\widetilde{\sigma}}})$ and $\overline{X} = \mathrm{T}_N\mathrm{emb}\,(\overline{\Delta_\sigma})$ are both projective.

2. Signature as rational homology manifold. For the compactified Fujiki-Oka $\overline{\mu} : \overline{Y} \to \overline{X}$ as in §1.3, we describe the difference of the signatures $\mathrm{Sign}\,\overline{Y} - \mathrm{Sign}\,\overline{X}$.

2.1. From now on to the last section (§§2–5), the natural number n is assumed to be even otherwise stated.

Let Δ be an n-dimensional projective simplicial fan, and put $W = \mathrm{T}_N\mathrm{emb}\,(\Delta)$. Since W is simplicial, W is automatically a complex orbifold. Especially W is a rational homology manifold (cf. [St, p. 527]). Let $\mathrm{Sign}\,W$ be the signature of the intersection form on $H^n(W, \mathbf{Q})$ as a rational homology manifold.

For $0 \leq j \leq n$, let $\Delta(j)$ be the set of j-dimensional cones of Δ and put $|\Delta(j)|$ the cardinality of $\Delta(j)$.

The following theorem is proved by Leung-Reiner [LR, Theorem 1.1] by using the result of McMullen [Mc, Theorem 8.6]. (See also [MS, §3] for its generalization.) Here we give another proof, whose method is analogous to [Od, §3.3].

THEOREM 2.1.1. ([LR], [Mc]). *For a simplicial projective toric variety $W = \mathrm{T}_N\mathrm{emb}\,(\Delta)$, we have*

$$\mathrm{Sign}\,W = \sum_{j=0}^{n} (-2)^j |\Delta(n-j)|.$$

PROOF. Let $\mathrm{Sing}(W)$ be the singular locus of W as an analytic space, and let $i : W - \mathrm{Sing}(W) \hookrightarrow W$ be the inclusion map. Let $\Omega^{\bullet}_{W-\mathrm{Sing}(W)}$ be the de Rham complex on $W - \mathrm{Sing}(W)$, and put $\widetilde{\Omega}^{\bullet}_W = i_* \Omega^{\bullet}_{W-\mathrm{Sing}(W)}$. By Danilov [D, Thm.12.5], the spectral sequence

$$E_1^{p,q} = H^q(W, \widetilde{\Omega}^p_W) \Longrightarrow H^{p+q}(W, \mathbf{C})$$

degenerates at E_1-term. Especially, by putting

$$h^{p,q}(W) = \dim H^q(W, \widetilde{\Omega}^p_W),$$

the k-th Betti number is written as $b_k(W) = \sum_{p+q=k} h^{p,q}(W)$.

By the projectivity of W, there exists an ample line bundle on W, and we denote by $L \in H^2(W, \mathbf{C})$ its cohomology class. By Beilinson-Bernstein-Deligne [BBD] and Goresky-MacPherson [GM] the hard Lefschetz theorem holds for W, i.e., the map $\omega \mapsto L^q \wedge \omega$ induces an isomorphism from $H^{n-q}(W, \mathbf{C})$ to $H^{n+q}(W, \mathbf{C})$ for any $q \in \mathbf{N}$ (see also [CLS, p. 620 and the comment in p. 794]). We denote the k-th primitive cohomology by

$$P^k(W, \mathbf{C}) = \mathrm{Ker}\left(L^{n-k+1} : H^k(W, \mathbf{C}) \to H^{2n-k+1}(W, \mathbf{C})\right).$$

Then we have the Lefschetz decomposition

$$H^q(W, \mathbf{C}) = \oplus_{r \geq 0} L^r P^{q-2r}(W, \mathbf{C})$$

for $q \geq 0$. By Steenbrink [St, p. 531], the Hermitian inner product on $H^{n-q}(W, \mathbf{C})$ defined by

$$Q(x, y) = (-1)^{(n-q)(n-q-1)/2} C x \wedge L^q \bar{y} [W]$$

is positive definite, where C is the Weil operator and $\bar{y}$ is the complex conjugation of y with respect to $H^{n-q}(W, \mathbf{R})$. Therefore by the same argument as in [GH, pp. 125–126], we have the "Hirzebruch-type" formula

$$(16) \qquad \mathrm{Sign}\, W = \sum_{p+q \equiv 0 \bmod 2} (-1)^q h^{p,q}(W).$$

On the other hand, we define the Poincaré series of W as

$$P(t) = \sum_{p=0}^{n} h^{p,p}(W)\, t^p \in \mathbf{Z}[t].$$

By using the Ishida complex [I1] for a simplicial toric variety, Oda [Od, Thm.3.11,(2)] shows that

$$(17) \qquad P(t) = \sum_{j=0}^{n} |\Delta(j)|(t-1)^{n-j}.$$

Since $h^{p,q}(W) = 0$ for $p \neq q$ (cf. [D, Cor.12.7], [Od, Thm.3.11, (1)]), it follows from (16) that $\mathrm{Sign}\, W$ coincides with $P(-1)$. Therefore the assersion follows from (17) by putting $t = -1$. $\qquad \square$

COROLLARY 2.1.2. *The compactified Fujiki-Oka modification $\overline{\mu} : \overline{Y} \longrightarrow \overline{X}$ satisfies*

$$\text{Sign}\, \overline{Y} - \text{Sign}\, \overline{X} = -1\,.$$

PROOF. Since $\overline{X}$ and $\overline{Y}$ have the same fans outsides σ and the subdivision $\widetilde{\sigma}$ of σ is barycentric, it follows from Theorem 2.1.1 that

$$\text{Sign}\, \overline{Y} - \text{Sign}\, \overline{X} = \sum_{j=0}^{n-1} (-2)^j \left(|\Delta_{\widetilde{\sigma}}(n-j)| - |\Delta_{\sigma}(n-j)| \right)$$

$$= \sum_{j=0}^{n-1} (-2)^j {}_n C_{j+1} = \frac{-(1-2)^n - 1}{2} = -1\,.$$

□

3. Orbifold signature. In this section, we describe the difference of the orbifold signatures $\text{Sign}^{\text{orb}}\, \overline{Y} - \text{Sign}^{\text{orb}}\, \overline{X}$ for the modification $\overline{\mu} : \overline{Y} \to \overline{X}$. For this purpose, we calculate the difference between rational Chow rings $A^*(\overline{Y})_{\mathbf{Q}}$ and $A^*(\overline{X})_{\mathbf{Q}}$. By using this, we also prove the claim of the projectivity in Proposition 1.3.1.

3.1. For an n-dimensional simplicial toric variety $W = T_N \text{emb}\,(\Delta)$, let $A^*(W)_{\mathbf{Q}}$ be the rational Chow ring of W. For each $\rho \in \Delta$, let $V(\rho)$ be the T_N-invariant subvariety corresponding to ρ, i.e., $V(\rho)$ is the closure $\overline{\text{orb}(\rho)}$ of the T_N-orbit corresponding to ρ. By essentially Jurkewicz and Danilov [D, p.131], $A^*(W)_{\mathbf{Q}}$ is generated by the T_N-invariant divisors $\{V(\rho)\,;\ \rho \in \Delta(1)\}$ with the following relations:

(i) For any $m \in W$, we have

(18)
$$\sum_{\rho \in \Delta(1)} \langle m, \rho \rangle V(\rho) = 0\,.$$

(ii) If $\rho_1, \ldots, \rho_k \in \Delta(1)$ are mutually distinct, then

$$V(\rho_1) \cdot V(\rho_2) \cdots V(\rho_k)$$

(19)
$$= \begin{cases} \dfrac{1}{\text{mult}(\rho_1 + \cdots + \rho_k)} V(\rho_1 + \cdots + \rho_k) & \text{if } \rho_1 + \cdots + \rho_k \in \Delta \\[2mm] 0 & \text{otherwise,} \end{cases}$$

where $\rho_1 + \cdots + \rho_k$ is the face of Δ generated by $\rho_1, \ldots, \rho_k$.

The rational total Chern class of W is given by

$$c_{\mathbf{Q}}(W) = \prod_{\rho \in \Delta(1)} (1 + V(\rho)) \in A^*(W)_{\mathbf{Q}} = \sum_{\rho \in \Delta} \frac{1}{\text{mult}(\rho)} V(\rho)\,,$$

and the orbifold signature of W is the rational number defined by

(20)
$$\text{Sign}^{\text{orb}}(W) = \kappa_n \left(\prod_{\rho \in \Delta(1)} V(\rho) t \cdot \coth(V(\rho)t) \right) \in A^n(W)_{\mathbf{Q}} \simeq \mathbf{Q}\,.$$

As will be discussed in §5.1, the definition (20) coincides with the V-signature of Satake [Sa].

3.2. We go back to the compactified Fujiki-Oka modification $\overline{\mu} : \overline{Y} \longrightarrow \overline{X}$. The following argument is analogous to that of Pommersheim [P, §3].

PROPOSITION 3.2.1.

$$\text{Sign}^{\text{orb}}(\overline{Y}) - \text{Sign}^{\text{orb}}(\overline{X}) = -\frac{L_n(d, a_1, \ldots, a_{n-1}, 1)}{d \prod_{i=1}^{n-1} a_i}.$$

PROOF. As in the proof of Proposition 1.2.3, let $\{m_1, \ldots, m_n\}$ be the **Z**-basis dual to the **Z**-basis $\{P_1, \ldots, P_{n-1}, Q\}$ of the lattice N in this order. Let $\{P_1, \ldots, P_n, P_{n+1}, \ldots, P_s\}$ be the primitive elements of the rays of $\overline{\Delta_\sigma}$ so that $\{P_{n+1}, \ldots, P_s\}$ are lying on $\mathbf{R}^n \setminus \text{supp}(\sigma)$. Then $\{Q, P_1, \ldots, P_n, P_{n+1}, \ldots, P_s\}$ are the primitive elements of rays of $\overline{\Delta_{\widetilde\sigma}}$, and the rational total Chern classes are

$$(21) \qquad c_{\mathbf{Q}}(\overline{X}) = \prod_{i=1}^{s}(1 + V(P_i)_{\overline{X}}), \quad c_{\mathbf{Q}}(\overline{Y}) = (1 + V(Q)) \prod_{i=1}^{s}(1 + V(P_i)_{\overline{Y}}),$$

where $V(P_i)_{\overline{X}}$ (resp. $V(P_i)_{\overline{Y}}$) are the rational equivalence class of $V(\mathbf{R}_{\geq 0} P_i)$ in $A^*(\overline{X})_{\mathbf{Q}}$ (resp. $A^*(\overline{Y})_{\mathbf{Q}}$), etc. Now it follows from (12) and (18) that

$$0 = \langle m_n, Q \rangle V(Q) + \sum_{i=1}^{s} \langle m_n, P_i \rangle V(P_i)_{\overline{Y}} = V(Q) + d\, V(P_n)_{\overline{Y}} + \varepsilon_n\,,$$

$$0 = \langle m_j, Q \rangle V(Q) + \sum_{i=1}^{s} \langle m_j, P_i \rangle V(P_i)_{\overline{Y}} = V(P_j) - a_i\, V(P_i)_{\overline{Y}} + \varepsilon_j \quad (1 \leq j \leq n-1),$$

where $\varepsilon_j = \sum_{i=n+1}^{s} \langle m_j, P_i \rangle V(P_i)_{\overline{Y}}$ for $1 \leq j \leq n$. Therefore on $A^*(\overline{Y})_{\mathbf{Q}}$, we have

$$(22) \qquad V(P_n)_{\overline{Y}} = -\frac{1}{d} \cdot V(Q) - \frac{\varepsilon_n}{d}, \quad V(P_j)_{\overline{Y}} = -\frac{a_j}{d} \cdot V(Q) - \frac{a_j \varepsilon_n}{d} + \varepsilon_j \quad (1 \leq j \leq n-1).$$

On $A^*(\overline{X})_{\mathbf{Q}}$, we similarly have

$$(23) \qquad V(P_n)_{\overline{X}} = -\frac{\varepsilon_n}{d}, \quad V(P_j)_{\overline{Y}} = -\frac{a_j \varepsilon_n}{d} + \varepsilon_j \quad (1 \leq j \leq n-1).$$

Here $\varepsilon_j = \sum_{i=n+1}^{s} \langle m_j, P_i \rangle V(P_i)_{\overline{X}}$ in (23) more precisely, although we use the same symbol ε_j as in (22) by identifying them through the isomorphism $\overline{Y} \setminus \text{supp}(\widetilde\sigma) \longrightarrow \overline{X} \setminus \text{supp}(\sigma)$.

For $n + 1 \leq i \leq s$, since P_i and Q are not in a same simplicial cone in $\overline{\Delta_{\widetilde\sigma}}$, we have $V(Q)V(P_i) = 0$. Especially

$$(24) \qquad V(Q) \cdot \varepsilon_j = 0 \quad (1 \leq j \leq n).$$

By an easy calculation using (21)–(24), we have

$$\mathrm{Sign}^{\mathrm{orb}}(\overline{Y}) - \mathrm{Sign}^{\mathrm{orb}}(\overline{X})$$

$$= \kappa_n \Bigg\{ V(Q)t \cdot \coth(V(Q)t) \cdot \prod_{j=1}^{n-1} \left(\left(-\frac{a_j}{d} V(Q)t \right) \coth \left(-\frac{a_j}{d} V(Q)t \right) \right)$$

$$\times \left(-\frac{1}{d} V(Q)t \right) \coth \left(-\frac{1}{d} V(Q)t \right) \Bigg\}$$

$$= V(Q)^n \cdot L_n \left(1, -\frac{a_1}{d}, \dots, -\frac{a_{n-1}}{d}, -\frac{1}{d} \right)$$

(25) $$= \frac{V(Q)^n}{d^n} \cdot L_n(d, a_1, \dots, a_{n-1}, 1) .$$

Since the simplicial cone $\mathbf{R}_{\geq 0} P_1 + \cdots + \mathbf{R}_{\geq 0} P_{n-1} + \mathbf{R}_{\geq 0} Q$ is nonsingular, it follows from (19), (22), (24) that

$$1 = V(Q) \, V(P_1) \, \cdots \, V(P_{n-1}) = V(Q)^n \cdot \frac{(-1)^{n-1} \, a_1 \cdots a_{n-1}}{d^{n-1}} .$$

Namely

(26) $$V(Q)^n = -\frac{d^{n-1}}{a_1 \cdots a_{n-1}} .$$

From (25), (26), the assertion follows.

REMARK 3.2.2. From the argument in the proof of Proposition 3.2.1, we have the following: Let $\mathbf{Q}[\mathbf{q}, \mathbf{p}_1, \dots, \mathbf{p}_n]$ be the polynomial ring in $n + 1$ variables $\mathbf{q}, \mathbf{p}_1, \dots, \mathbf{p}_n$ with rational coefficients, and let $f(\mathbf{q}, \mathbf{p}_1, \dots, \mathbf{p}_n) \in \mathbf{Q}[\mathbf{q}, \mathbf{p}_1, \dots, \mathbf{p}_n]$ be the homogeneous term of degree n of the power series of the function $\mathbf{q} \coth \mathbf{q} \cdot (\prod_{i=1}^{n} \mathbf{p}_i \coth \mathbf{p}_i)$. Then the invariant $\mathrm{Sign}^{\mathrm{orb}}(\overline{Y}) - \mathrm{Sign}^{\mathrm{orb}}(\overline{X})$ is nothing but the rational number obtained from $f(\mathbf{q}, \mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{p}_n)$ by substituting the relation

(27) $$\mathbf{p}_i = -\frac{a_i}{d}\mathbf{q} \quad (1 \leq i \leq n-1), \quad \mathbf{p}_n = -\frac{1}{d}\mathbf{q}, \quad \mathbf{q}^n = -\frac{d^{n-1}}{a_1 \cdots a_{n-1}} .$$

3.3. We prove Proposition 1.3.1. The existence of simplicial completions of X, Y is obvious. Since the discussion in §3.2 is independent of the projectivity of $\overline{X}$, $\overline{Y}$, we can use the results in §3.2.

Let ℓ_σ be a $|\sigma|$-linear support function of σ, and let $[\ell_\sigma]$ be the element of the dual lattice M which corresponds to ℓ_σ. Let $\mathcal{P}$ be an integral polytope containing $[\ell_\sigma]$ as one of the vertices. By the Galois correspondence [Od, §2.4], there exists a projective fan $\overline{\Delta}_\sigma$ which corresponds to $\mathcal{P}$, i.e., $\overline{X} = \mathrm{T}_N \mathrm{emb}(\overline{\Delta}_\sigma)$ is projective.

Now let $\overline{\Delta}_{\widetilde{\sigma}}$ be the fan which is obtained from the Oka subdivision of the simplicial cone σ in $\overline{\Delta}_\sigma$. It suffices to prove that $\overline{\Delta}_{\widetilde{\sigma}}$ is also a projective fan.

For a $\mathbf{Q}$-ample divisor D_1 and an effective divisor D_2 on a complex orbifold in general, the divisor $r_1 D_1 + r_2 D_2$ $(r_1, r_2 \in \mathbf{Q}_{\geq 0})$ is also $\mathbf{Q}$-ample. Therefore there exists a set of

positive rational numbers $\{\alpha_i\}_{1 \le i \le s}$ such that for any set of rational numbers $\{x_i\}_{1 \le i \le s}$ which satisfy

$$(28) \qquad\qquad x_i \ge \alpha_i \quad (1 \le i \le s),$$

the divisor $\sum_{i=1}^{s} x_i V(P_i)_{\overline{X}}$ is **Q**-ample on $\overline{X}$.

Next we deal with $\overline{Y}$. We set $P_0 = Q$ and $V(P_i) = V(P_i)_{\overline{Y}}$ for simplicity. We claim that, if we choose some rational numbers $\{x_i\}_{1 \le i \le s}$ which satisfiy (28), then there exists a rational number $x_0 \in \mathbf{Q}_{\ge 0}$ such that the divisor

$$(29) \qquad\qquad D = \sum_{i=0}^{s} x_i V(P_i)$$

is **Q**-ample on $\overline{Y}$.

Let C be any T_N-invariant curve on $\overline{Y}$. In order to prove the ampleness of D, it suffices to prove $DC > 0$ by the toric Nakai criterion [Od, p. 95]. The curve C is written as $V(P_{j_1}) \cdots V(P_{j_{n-1}})$ for some $0 \le j_1 < \cdots < j_{n-1} \le s$. Our discussion is divided into the following cases.

(i) Assume $j_{n-1} \ge n+1$. Then C is contained in the closure of $\overline{Y} \setminus \mathrm{supp}\,\widetilde{\sigma}$, and therefore $C \to \overline{\mu}(C)$ is an isomorphism. Since we may assume that $\overline{\mu}_*(D) = \sum_{i=1}^{s} x_i V(P_i)_{\overline{X}}$ is ample on $\overline{X}$ by (28), we have $DC = \overline{\mu}_*(D)\overline{\mu}_*(C) > 0$.

(ii) Assume $j_{n-1} \le n$. Our discussion is divided into the following sub-cases.

(ii-a) Suppose $j_1 = 0$. Then C is written as $V(P_0) \cdots V(P_{i-1}) V(P_{i+1}) \cdots V(P_{j-1}) V(P_{j+1}) \cdots V(P_n)$ for some $1 \le i < j \le n$. Since $V(P_0)V(P_k) = 0$ in $A^*(\overline{Y})_\mathbf{Q}$ for $k \ge n+1$, we have $DC = \sum_{k=0}^{n} V(P_k)\, C$. Moreover in this case, the equations (22), (24), (26) say that we may numerically assume

$$(30) \qquad V(P_k) \equiv -\frac{a_k}{d} \cdot V(P_0) \quad (1 \le k \le n), \qquad V(P_0)^n \equiv -\frac{d^{n-1}}{a_1 \cdots a_n}.$$

By an easy calculation using (30), we have

$$DC = \frac{1}{a_i a_j} \left(\sum_{k=1}^{n} a_k x_k - dx_0 \right).$$

Hence the condition $DC > 0$ is equivalent to

$$(31) \qquad\qquad \sum_{k=1}^{n} a_k x_k - dx_0 > 0.$$

(ii-b) Suppose $j_1 \ge 1$. Then C is written as $V(P_1) \cdots V(P_{i-1}) V(P_{i+1}) \cdots V(P_n)$ for some $1 \le i \le n$. By an easy calculation using (22), (24), (26), we have

$$DC = \frac{1}{a_i d} \left(- \sum_{k=1}^{n} a_k x_k + dx_0 \right) + \sum_{k=1}^{n} x_k D_1 \cdots D_n$$

$$+ \sum_{k=n+1}^{s} x_k V(P_k) V(P_1) \cdots V(P_{i-1}) V(P_{i+1}) \cdots V(P_n),$$

where $D_1, \ldots, D_n$ are some $\mathbf{Q}$-divisors whose supports are contained in the support of $\sum_{k=n+1}^{s} V(P_k)$. Therefore the condition $DC > 0$ is equivalent to

(32)
$$\sum_{k=n+1}^{s} x_k V(P_k) V(P_1) \cdots V(P_{i-1}) V(P_{i+1}) \cdots V(P_n)$$
$$> \frac{1}{a_i d} \left(\sum_{k=1}^{n} a_k x_k - dx_0 \right) - \sum_{k=1}^{n} x_k D_1 \cdots D_n .$$

The linear inequalties (28), (31), (32) with respect to $x_0, x_1, \ldots, x_s$ have simultaneous rational solutions. Indeed, as the first step, we choose $x_0, x_1, \ldots, x_n$ so that (28) (31) are satisfied. For any i $(1 \le i \le n)$, there exists some k $(n + 1 \le k \le s)$ such that $\mathbf{R}_{\ge 0} P_k + \mathbf{R}_{\ge 0} P_1 + \cdots + \mathbf{R}_{\ge 0} P_{i-1} + \mathbf{R}_{\ge 0} P_{i+1} + \cdots + \mathbf{R}_{\ge 0} P_n$ is a simplicial cone in $\overline{\Delta}_{\widetilde{\sigma}}$. Therefore, as the second step, we can choose $x_{n+1}, \ldots, x_s$ sufficiently large so that (32) is also satisfied. □

4. Reciprocity of Dedekind sums with weight. By using Atiyah-Singer cotangent function, we define the notion of Dedekin sums with weight and state our reciprocity.

4.1. We set $\mathrm{I} = \{1, 2, \ldots, n\}$. For $\mathbf{a}/d \in \mathbf{Q}_n^{\mathrm{prop}}$ as in (6), we denote by I the disjoint union $\mathrm{I}_0(\mathbf{a}/d) \coprod \mathrm{I}_{\mathrm{ref}}(\mathbf{a}/d) \coprod \mathrm{I}_*(\mathbf{a}/d)$, where

(33)
$$\mathrm{I}_0(\mathbf{a}/d) = \{i \in \mathrm{I} \,;\, a_i = 0\}, \quad \mathrm{I}_{\mathrm{ref}}(\mathbf{a}/d) = \left\{ i \in \mathrm{I} \,;\, \frac{a_i}{d} = \frac{1}{2} \right\},$$
$$\mathrm{I}_*(\mathbf{a}/d) = \mathrm{I} \backslash \left(\mathrm{I}_0(\mathbf{a}/d) \coprod \mathrm{I}_{\mathrm{ref}}(\mathbf{a}/d) \right) .$$

For a fixed $\mathbf{a}/d$, we sometimes simply write $\mathrm{I}_0 = \mathrm{I}_0(\mathbf{a}/d)$ and so on if there is no confusion. We denote by $|\mathrm{I}_0|$ the cardinality of I_0 and so on.

For a Laurent series $f(x_1, \ldots, x_n) = \sum_{-\infty < i_1, \ldots, i_n < \infty} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$ in the variables $x_1, \ldots, x_n$ with rational coefficients, we denote by

(34)
$$\kappa_m \left(f(x_1, \ldots, x_n) \right) = \sum_{i_1 + \cdots + i_n = m} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} \quad (m \in \mathbf{Z})$$

its homogeneous part of degree m. For simplicity, we write $\mathbf{x} = (x_1, \ldots, x_n)$.

DEFINITION 4.1.1. For $\mathbf{a}/d \in \mathbf{Q}_n^{\mathrm{prop}}$, Atiyah-Singer cotangent function $\cot(\mathbf{a}/d)(\mathbf{x})$ is the meromorphic function defined by

$$\frac{(-1)^n \cdot 2^{|\mathrm{I}_0| - |\mathrm{I}_{\mathrm{ref}}|}}{d(|\mathrm{I}_0| + 1) \left(\prod_{k \in \mathrm{I}_0} x_k \right)} \cdot \kappa_{|\mathrm{I}_0|} \left\{ \prod_{k \in \mathrm{I}_*} \coth \left(\frac{x_k}{2} + \frac{\sqrt{-1} \pi a_k}{d} \right) \prod_{k \in \mathrm{I}_0} \frac{x_k}{2} \coth \frac{x_k}{2} \prod_{k \in \mathrm{I}_{\mathrm{ref}}} 2 \tanh \frac{x_k}{2} \right\} .$$

This function appears as an interpretation in our situation of Atiyah-Singer's equivariant L-class [AS, p. 582]. Taylor expansion says

$$(-1)^n 2^{|\mathrm{I}_{\mathrm{ref}}| - |\mathrm{I}_0|} d(|\mathrm{I}_0| + 1) \left(\prod_{k \in \mathrm{I}_0} x_k \right) \cot(\mathbf{a}/d)(\mathbf{x})$$

is its homogeneous part of degree $|I_0|$ of

$$
\prod_{k \in I_*} \sum_{j=0}^{\infty} \frac{(-\sqrt{-1})^{j+1}}{j! \, 2^j} \cot^{(j)} \left(\frac{\pi a_k}{d} \right) x_k^j
$$

(35)

$$
\cdot \prod_{k \in I_0} \sum_{j=0}^{\infty} \frac{B_{2j}}{(2j)!} x_k^{2j} \cdot \prod_{k \in I_{\text{ref}}} \sum_{j=1}^{\infty} \frac{4(2^{2j} - 1) B_{2j}}{(2j)!} x_k^{2j-1},
$$

where B_{2j} is 2j-th Bernoulli number so that $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30, \ldots$, and $\cot^{(j)} x = (d^j/dx^j)(\cot x)$ (see e.g. [B] for the notation). Therefore Atiyah-Singer cotangent function is essentially written in terms of the derivatives of the usual cotangent function and Bernoulli numbers.

EXAMPLE 4.1.2. If $I_0(\mathbf{a}/d) = \emptyset$, then $\cot(\mathbf{a}/d)(\mathbf{x})$ is the constant function

$$
\cot \left(\frac{\mathbf{a}}{d} \right)(\mathbf{x}) = \frac{(-1)^{n/2}}{d} \prod_{k=1}^{n} \cot \left(\frac{\pi a_k}{d} \right).
$$

EXAMPLE 4.1.3. If $\mathbf{a}/d = (a_1, a_2, 0, 0)/d \in \mathbf{Q}_4^{\text{prop}}$ with $d \geq 3$, $\gcd(d, a_1) = \gcd(d, a_2) = 1$, then

$$
\cot \left(\frac{\mathbf{a}}{d} \right)(\mathbf{x}) = \frac{1}{18 d x_3 x_4} \left(3 x_1^2 \cot^{(2)} \frac{\pi a_1}{d} \cot \frac{\pi a_2}{d} + 6 x_1 x_2 \cot^{(1)} \frac{\pi a_1}{d} \cot^{(1)} \frac{\pi a_2}{d} \right.
$$

$$
\left. + 3 x_2^2 \cot \frac{\pi a_1}{d} \cot^{(2)} \frac{\pi a_2}{d} - 2(x_3^2 + x_4^2) \cot \frac{\pi a_1}{d} \cot \frac{\pi a_2}{d} \right).
$$

4.2. We define the following.

DEFINITION 4.2.1. For $\mathbf{a}/d \in \mathbf{Q}_n^{\text{prop}}$ and an element $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbf{Z}^n$, we define the Dedekind sum of $\mathbf{a}/d$ with weight $\mathbf{w}$ by

$$
\mathbf{s} \left(\frac{\mathbf{a}}{d}, \mathbf{w} \right) = \sum_{k=1}^{d-1} \cot \left(k \cdot \frac{\mathbf{a}}{d} \right)(\mathbf{w}).
$$

If $\mathbf{a}/d$ is irreducible, then $k \cdot \mathbf{a}/d$ is also irreducible for $1 \leq k \leq d - 1$. In this case, Example 4.1.2 says that

(36)

$$
\mathbf{s} \left(\frac{\mathbf{a}}{d}, \mathbf{w} \right) = \frac{(-1)^{n/2}}{d} \cdot \sum_{k=1}^{d-1} \prod_{i=1}^{n} \cot \frac{\pi a_i k}{d}
$$

independently of the weight $\mathbf{w}$, and we simply write it as $\mathbf{s}(\mathbf{a}/d)$. The definition (36) for an irreducible fraction $\mathbf{a}/d$ is essentially nothing but the Dedekind sum defined by Hirzebruch-Zagier [HZ]. In this sense, the notion of Dedekind sums with weight is a generalization of the one in [HZ]. Now let

$$
L_n(x_1, \ldots, x_r) = \kappa_n \left(\prod_{i=1}^{r} x_i \coth x_i \right)
$$

be the Hirzebruch L-polynomial in the variables $x_1, \ldots, x_r$. The following theorem is our main result, whose proof will be accomplished in §5.

THEOREM 4.2.2 (Reciprocity law).   *Let* $\mathbf{a}/d = (a_1, \ldots, a_{n-1}, 1)/d \in \mathbf{Q}_n^{\mathrm{prop}}$ *be an irreducible inhomogeneous element. Then we have*

$$(37) \qquad \mathbf{s}\left(\frac{\mathbf{a}}{d}\right) - \sum_{i=1}^{n-1} \mathbf{s}\left(R_i\left(\frac{\mathbf{a}}{d}\right), N_i\left(\frac{\mathbf{a}}{d}\right)\right) = 1 - \frac{L_n(d, a_1, \ldots, a_{n-1}, 1)}{d \prod_{i=1}^{n-1} a_i}.$$

If $\mathbf{a}/d$ is absolutely irreducible, then $R_i(\mathbf{a}/d)$ is also (absolutely) irreducible for any i. Therefore Theorem 4.2.2 implies the following:

COROLLARY 4.2.3.   *If* $\mathbf{a}/d = (a_1, \ldots, a_{n-1}, 1)/d \in \mathbf{Q}_n^{\mathrm{prop}}$ *is an absolutely irreducible inhomogeneous element, then*

$$(38) \qquad \mathbf{s}\left(\frac{\mathbf{a}}{d}\right) - \sum_{i=1}^{n-1} \mathbf{s}\left(R_i\left(\frac{\mathbf{a}}{d}\right)\right) = 1 - \frac{L_n(d, a_1, \ldots, a_{n-1}, 1)}{d \prod_{i=1}^{n-1} a_i}.$$

We easily check that Corollary 4.2.3 coincides with the inhomogeneous Zagier reciprocity. Namely if we put $a_n = 1$ in the formula (47) in [Z, p. 158], then it coincides with (38). In this sense, the formula (37) is a generalization of the inhomogeneous Zagier reciprocity.

EXAMPLE 4.2.4.   If $\mathbf{a}/d = (a_1, a_2, a_3, a_4)/d \in \mathbf{Q}_4^{\mathrm{prop}}$, where $a_4 = 1$, $\gcd(d, a_i) = 1$ $(1 \leq i \leq 3)$ and $d \geq 3$ is odd, then

$$\mathbf{s}\left(\frac{\mathbf{a}}{d}\right) - \sum_{i=1}^{3} \mathbf{s}\left(R_i\left(\frac{\mathbf{a}}{d}\right), N_i\left(\frac{\mathbf{a}}{d}\right)\right) = 1 - \frac{1}{45d \prod_{i=1}^{4} a_i}\left(\sum_{1 \leq i < j \leq 4} 15a_i^2 a_j^2 - \sum_{i=1}^{4} a_i^4\right).$$

Here by putting $c_{1i} = \gcd(a_1, a_i)$ $(i = 2, 3)$, $c_{123} = \gcd(a_1, a_2, a_3)$ and $d' = -d$, we have

$$\mathbf{s}\left(R_1\left(\frac{\mathbf{a}}{d}\right), N_1\left(\frac{\mathbf{a}}{d}\right)\right) = \frac{1}{a_1} \sum_{1 \leq k \leq a_1 - 1}' \cot\frac{\pi d' k}{a_1} \cot\frac{\pi a_2 k}{a_1} \cot\frac{\pi a_3 k}{a_1} \cot\frac{\pi k}{a_1}$$

$$+ \sum_{i=2}^{3} \frac{1}{2a_1 a_i} \sum_{1 \leq k \leq c_{1i} - 1}' \left(d' \cot^{(1)}\frac{\pi d' k}{c_{1i}} \cot\frac{\pi a_{i*} k}{c_{1i}} \cot\frac{\pi k}{c_{1i}}\right.$$

$$+ a_{i*} \cot\frac{\pi d' k}{c_{1i}} \cot^{(1)}\frac{\pi a_{i*} k}{c_{1i}} \cot\frac{\pi k}{c_{1i}}$$

$$+ \cot\frac{\pi d' k}{c_{1i}} \cot\frac{\pi a_{i*} k}{c_{1i}} \cot^{(1)}\frac{\pi k}{c_{1i}}\right) + \frac{1}{18a_1 a_2 a_3} \sum_{k=1}^{c_{123}-1} \left\{ 3d'^2 \cot^{(2)}\frac{\pi d' k}{c_{123}} \cot\frac{\pi k}{c_{123}}\right.$$

$$+ 6d' \cot^{(1)}\frac{\pi d' k}{c_{123}} \cot^{(1)}\frac{\pi k}{c_{123}} + 3 \cot\frac{\pi d' k}{c_{123}} \cot^{(2)}\frac{\pi k}{c_{123}} - 2(a_2^2 + a_3^2) \cot\frac{\pi d' k}{c_{123}} \cot\frac{\pi k}{c_{123}}\right\}.$$

In the first (dashed) summation $\sum'$, k moves from 1 to $a_1 - 1$ such that $(\alpha k)/a_1 \notin \mathbf{Z}$ for $\alpha = a_2, a_3$. In the second (dashed) summation $\sum'$, the suffix i^* is determined by $2^* = 3$ and $3^* = 2$, and k moves from 1 to $c_{1i} - 1$ such that $(a_{i*}k)/c_{1i} \notin \mathbf{Z}$.

The terms $\mathbf{s}(R_i(\mathbf{a}/d), N_i(\mathbf{a}/d))$ for $i = 2, 3$ are similar.

5. S-defect. In this section, we calculate the difference $\mathrm{Sdef}(\overline{Y}) - \mathrm{Sdef}(\overline{X})$ of the singature defects for the modification $\overline{\mu} : \overline{Y} \to \overline{X}$, and prove Theorem 4.2.2.

5.1. Let Z be an n-dimensional complex abelian orbifold. Namely there exists an orbifold chart

$$(39) \qquad Z = \bigcup U_i , \quad \{\widetilde{U}_i \xrightarrow{\pi_i} U_i, \ G_i = \mathrm{Gal}(\widetilde{U_{\sigma_i}}/U_i)\}$$

so that any G_i is an abelian group. First, we briefly recall the V-signature theorem of Kawasaki [Ka] restricted to Z for later use.

Let $L(\widetilde{U}_i)$ be the Hirzebruch L-form on $\widetilde{U}_i$. Let $\{\widetilde{\rho}_i, \rho_i\}$ be the partition of unity subordinate to the chart (39). Namely, $\widetilde{\rho}_i$ is a C^∞ function whose support is contained in $\widetilde{U}_i$ so that the system of function ρ_i on U_i with $\pi_i^* \rho_i = \widetilde{\rho}_i$ satisfies $\sum \rho_i = 1$. Satake's V-signature ([Sa], [Ka]) of Z is defiend by

$$(40) \qquad \mathrm{Sign}^V(Z) = \sum \frac{1}{|G_i|} \int_{\widetilde{U}_i} \widetilde{\rho}_i L(\widetilde{U}_i) .$$

The right-hand side of (40) is sometimes written as $\int_Z^{\mathrm{orb}} L(Z)$.

On the other hand, for any $g \in G_i$, let $L(g, \widetilde{U}_i)$ be the equivariant L-form on $\widetilde{U}_i$ with respect to g (cf. [AS]). The S-defect of Z is defined by

$$(41) \qquad \mathrm{Sdef}(Z) = \sum_i \frac{1}{|G_i|} \sum_{g \in G_i, \ g \neq id} \int_{\widetilde{U}_i} \widetilde{\rho}_i L(g, \widetilde{U}_i) .$$

Then the V-signature theorem says that

$$(42) \qquad \mathrm{Sign}(Z) - \mathrm{Sign}^V(Z) = \mathrm{Sdef}(Z) .$$

Kawasaki [Ka, §1] describes $\mathrm{Sdef}(Z)$ in the following global formulation. Here we also use the terminology of Chen-Ruan [CR, §3] at the same time.

Let ΣZ be the singular locus of the orbifold Z, i.e., ΣZ consists of the points $x \in Z$ whose isotropy groups G_x are non-trivial. Since G_x is abelian, the centralizer of each element of G_x and the conjugacy class of G_x are both isomorphic to G_x.

Let $\widetilde{\Sigma} Z$ be the *twisted sector* of Z [CR, p. 7]. Namely, $\widetilde{\Sigma} Z$ consists of the pairs $\{(x, g) ; \ x \in \Sigma Z, \ g \in G_x, \ g \neq id\}$ which has the orbifold structure as follows: Let $\{\widetilde{U}_x \to U_x, \ G_x = \mathrm{Gal}(\widetilde{U}_x/U_x)\}$ be a local uniformazing system of Z containing x, and let $\widetilde{U}_x{}^g$ be the fixed point locus of g on $\widetilde{U}_x$. Since G_x coincides with the centralizer of g in G_x, G_x itself acts on $\widetilde{U}_x{}^g$ possibly non-effectively in general. Then the systems $\{\widetilde{U}_x{}^g \to \widetilde{U}_x{}^g/G_x\}$ are patched globally and defines the orbifold chart on $\widetilde{\Sigma} Z$. Note that this orbifold structure need not be reduced.

Let $\widetilde{\Sigma}Z_k$ ($k = 1, \ldots, \alpha$) be the connected components of $\widetilde{\Sigma}Z$. For $(x, g) \in \widetilde{\Sigma}Z_k$, let K_x be the kernel of the natural homomorphism from G_x to the diffeomorphism group $\mathrm{Diffeo}(\widetilde{U_x}^g)$ of $\widetilde{U_x}^g$. Then the cardinality $|K_x|$ is independent of the choice of (x, g). We set $m_k = |K_x|$ and call it the *multiplicity* of the connected component Z_k [Ka, p. 77]. If we change each local group from G_x to G_x/K_x, then the system $\{\widetilde{U_x}^g \to \widetilde{U_x}^g/(G_x/K_x)\}$ defines a reduced orbifold structure on Z_k, which is written as Z_k^{red}.

Through the immersion $\widetilde{\Sigma}Z_k^{\mathrm{red}} \longrightarrow Z$ defined by $(x, g) \mapsto x$, the orbifold normal bundle on $\widetilde{\Sigma}Z_k^{\mathrm{red}}$ is well-defined. Then by patching the natural local equivariant L-forms, we can define the orbifold characteristic L^{Σ} on $\widetilde{\Sigma}Z_k^{\mathrm{red}}$. We set

$$\langle \mathrm{L}^{\Sigma}, \widetilde{\Sigma}Z_k \rangle = \frac{1}{m_k} \int_{\widetilde{\Sigma}Z_k^{\mathrm{red}}}^{\mathrm{orb}} \mathrm{L}^{\Sigma} .$$

By [Ka, p. 78], we have

$$(43) \qquad\qquad \mathrm{Sdef}(Z) = \sum_{k=1}^{\alpha} \langle \mathrm{L}^{\Sigma}, \widetilde{\Sigma}Z_k \rangle .$$

Next let W be a simplicial toric variety. Then W automatically has the structure of a complex orbifold. The following lemma seems to be well-known:

LEMMA 5.1.1.   *The V-signature* $\mathrm{Sign}^V(W)$ *coincides with the orbifold signature* $\mathrm{Sign}^{\mathrm{orb}}(W)$ *defined in §3.1.*

PROOF.   In Satake's sense, since the total V-Chern form of W is generated by the first Chern form of V-line bundles of the torus invariant divisors, the V-signature is defined as the orbifold integral of the (Hirzebruch's L type) wedge product of these forms.

On the other hand, the Chow ring $A^*(W)$ is also generated by the torus invariant divisors and the intersection product of these divisors essentially comes from the homological intersection in the local uniformization systems.

In the local uniformization systems, since torus invariant divisors cross transversally with each other, the wedge product of the Chern forms and the intersection products of these divisors have the classical direct correspondence with natural compatibility (cf. [GH, Chap.0]). Therefore the assertion easily follows. We omit the details.

We go back to the situation in §1.3.

PROPOSITION 5.1.2.   *For the compactified Fujiki-Oka modification* $\overline{\mu} : \overline{Y} \to \overline{X}$, *we have*

$$\mathrm{Sdef}(\overline{X}) - \mathrm{Sdef}(\overline{Y}) = \mathbf{s}\left(\frac{\mathbf{a}}{d}\right) - \sum_{i=1}^{n-1} \mathbf{s}\left(R_i\left(\frac{\mathbf{a}}{d}\right), N_i\left(\frac{\mathbf{a}}{d}\right)\right).$$

5.2.   As a preparation for the proof of Proposition 5.1.2, we describe a connected component of the twisted sector of Y, and also give an explicit description of the characteristic L^{Σ} on it.

The fan $\widetilde{\sigma}$ in §1.3 consists of the union of the faces of a nonsingular simplex $\sigma_n = \mathbf{R}_{\geq 0}P_1 + \cdots + \mathbf{R}_{\geq 0}P_{n-1} + \mathbf{R}_{\geq 0}Q$ and semi-unimodular simplexes $\sigma_i = \mathbf{R}_{\geq 0}P_1 + \cdots +$

$\mathbf{R}_{\geq 0} P_{i-1} + \mathbf{R}_{\geq 0} Q + \mathbf{R}_{\geq 0} P_{i+1} + \cdots + \mathbf{R}_{\geq 0} P_n$ over P_n $(1 \leq i \leq n-1)$. The minimal orbifold chart of $Y = T_N \mathrm{emb}\,(\Delta_{\widetilde{\sigma}})$ is given by

$$(44) \qquad Y = \bigcup_{1 \leq i \leq n} U_{\sigma_i}\,, \quad \{\widetilde{U_{\sigma_i}} \xrightarrow{\pi_i} U_{\sigma_i},\ G_i = \mathrm{Gal}(\widetilde{U_{\sigma_i}}/U_{\sigma_i})\}\,.$$

Here π_n is the identity map of $\widetilde{U_{\sigma_n}} = U_{\sigma_n}$ and the local uniformization map π_i $(1 \leq i \leq n-1)$ is the cyclic covering of order $a_i = \mathrm{mult}(\sigma_i)$ described as follows. Let N_i be the sub-lattice of the original lattice N generated by the primitive vectors $P_1, \ldots, P_{i-1}, Q, P_{i+1}, \ldots, P_n$. Then the natural equivariant morphism arising from $(N_i, \Delta_\sigma) \to (N, \Delta_\sigma)$

$$\widetilde{U_{\sigma_i}} = T_{N_i} \mathrm{emb}\,(\Delta_\sigma) \simeq \mathbf{C}^n \longrightarrow U_{\sigma_i} = T_N \mathrm{emb}\,(\Delta_\sigma)$$

is nothing but the map π_i. The local Galois group G_i is isomorphic to N/N_i, which is the cyclic group of order a_i.

We denote the torus invariant divisors on $\widetilde{U_{\sigma_i}}$ by

$$D_{1,i} = V(P_1)|_{\widetilde{U_{\sigma_i}}}, \ldots, D_{i,i} = V(Q)|_{\widetilde{U_{\sigma_i}}}, \ldots, D_{n,i} = V(P_n)|_{\widetilde{U_{\sigma_i}}}\,.$$

Let $(z_1, \ldots, z_n)$ be the coordinate of $\widetilde{U_{\sigma_i}} \simeq \mathbf{C}^n$ so that $D_{k,i}$ is defined by $z_k = 0$ $(1 \leq k \leq n)$. We fix the generator g_i of $G_i \simeq \mathbf{Z}/a_i\mathbf{Z}$. By Lemma 1.2.5, the proper fraction of σ_i coincides with

$$R_i(\mathbf{a}/d) = \frac{(\overline{a_1}, \ldots, \overline{a_{i-1}}, \overline{(-d)}, \overline{a_{i+1}}, \ldots, \overline{a_{n-1}}, 1)}{a_i}\,.$$

We set

$$(45) \qquad a_{i,i} = -d\,, \quad a_{n,i} = 1\,, \quad a_{k,i} = a_k \quad (1 \leq k \leq n-1, k \neq i)\,.$$

By the same argument as in the proof of Proposition 1.2.3, the element $g_i^j \in G_i$ $(0 \leq j \leq a_i - 1)$ acts on $\widetilde{U_{\sigma_i}}$ as

$$(z_1, \ldots, z_n) \longmapsto \left(\mathbf{e}\left(\frac{j a_{1,i}}{a_i}\right) z_1, \ldots, \mathbf{e}\left(\frac{j a_{n,i}}{a_i}\right) z_n\right)$$

where $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}\,x)$. In the notation of (33), we define the subsets of I by

$$\mathrm{I}_{j,i}^0 = \mathrm{I}_0\,(j R_i(\mathbf{a}/d))\,, \quad \mathrm{I}_{j,i}^{\mathrm{ref}} = \mathrm{I}_{\mathrm{ref}}\,(j R_i(\mathbf{a}/d))\,, \quad \mathrm{I}_{j,i}^* = \mathrm{I}_*\,(j R_i(\mathbf{a}/d))\,.$$

Then the fixed point locus $V(j, i) = (\widetilde{U_{\sigma_i}})^{g_i^j}$ of the action g_i^j on $\widetilde{U_{\sigma_i}}$ coincides with the $|\mathrm{I}_{j,i}|$-dimensional torus invariant nonsingular subvariety

$$(46) \qquad V(j, i) = \bigcap_{k \in \mathrm{I} \setminus \mathrm{I}_{j,i}^0} D_{k,i}\,.$$

The tangent bundle and the normal bundle in $\widetilde{U_{\sigma_i}}$ of $V(j, i)$, respectively, have the natural direct sum decompositions

$$(47) \qquad T_{V(j,i)} \simeq \bigoplus_{k \in \mathrm{I}_{j,i}^0} [D_{k,i}]\,, \quad N_{V(j,i)/\widetilde{U_{\sigma_i}}} \simeq \bigoplus_{k \in \mathrm{I} \setminus \mathrm{I}_{j,i}^0} [D_{k,i}]$$

where $[D_{k,i}]$ is the line bundle which corresponds to the divisor $D_{k,i}$. The element g_i^j acts on the direct factor $[D_{k,i}]$ ($k \in I\backslash I_{j,i}^0$) of the normal bundle with the eigenvalue $\mathbf{e}\left(ja_{k,i}/a_i\right)$. More precisely, if $k \in I_{j,i}^*$, then g_i^j acts $[D_{k,i}]$ as the rotation of angle $2\pi j a_{k,i}/a_i$. If $k \in I_{j,i}^{\mathrm{ref}}$, then g_i^j acts on $[D_{k,i}]$ as the reflection.

By [AS, pp. 581–582], the equivariant L-form $L(g_i^j, \widetilde{U_{\sigma_i}})$ is the differential form on $V(j, i)$ given by

$$
\begin{aligned}
L(g_i^j, \widetilde{U_{\sigma_i}}) = &\prod_{k \in I_{j,i}^*} \coth \left(\frac{c_1([D_{k,i}]|_{V(j,i)})}{2} + \frac{\sqrt{-1}\pi j a_{k,i}}{a_i} \right) \\
&\times \prod_{k \in I_{j,i}^0} c_1([D_{k,i}]|_{V(j,i)}) \coth \frac{c_1([D_{k,i}]|_{V(j,i)})}{2} \\
&\times \prod_{k \in I_{j,i}^{\mathrm{ref}}} \tanh \frac{c_1([D_{k,i}]|_{V(j,i)})}{2} ,
\end{aligned}
\tag{48}
$$

where $c_1([D_{k,i}]|_{V(j,i)})$ is the first Chern form of the restricted line bundle $[D_{k,i}]|_{V(j,i)}$ and the product is the wedge product of the Chern forms.

As a preparation for the global formulation, we rewrite (48) in the following way. Since $\gcd(d, a_i) = 1$ and $a_{n,i} = 1$, the suffices i and n are not contained in $I_{j,i}^0$. We put $r = |I_{j,i}^0|$, and set

$$
I_{j,i}^0 = \{i_1, \ldots, i_r\} \subset I\backslash\{i, n\}, \quad I\backslash I_{j,i}^0 = \{i_{r+1}, \ldots, i_{n-r-2}\} \cup \{i, n\} .
$$

Then $ja_{i_k} \equiv 0 \pmod{a_i}$ for $1 \leq k \leq r$, and $ja_k \not\equiv 0 \pmod{a_i}$ for $k \in I\backslash I_{j,i}^0$. Moreover we have

$$
c := \gcd(a_i, a_{i_1}, \ldots, a_{i_r}) > 1 .
\tag{49}
$$

We put $a_i' = a_i/c$. There exists j' ($1 \leq j' \leq c - 1$) such that $j = a_i' j'$. Then (48) is rewritten as

$$
\begin{aligned}
L(g_i^j, \widetilde{U_{\sigma_i}}) = &\prod_{k \in I_{j,i}^*} \coth \left(\frac{c_1([D_{k,i}]|_{V(j,i)})}{2} + \frac{\sqrt{-1}\pi j' a_{k,i}}{c} \right) \\
&\times \prod_{k \in I_{j,i}^0} c_1([D_{k,i}]|_{V(j,i)}) \coth \frac{c_1([D_{k,i}]|_{V(j,i)})}{2} \\
&\times \prod_{k \in I_{j,i}^{\mathrm{ref}}} \tanh \frac{c_1([D_{k,i}]|_{V(j,i)})}{2} .
\end{aligned}
\tag{50}
$$

Note that (50) is symmetric with respect to the $r + 1$ suffices $i, i_1, \ldots, i_r$.

Now the group G_i acts on $V(j, i)$ possibly non-effectively. The kernel $K_{j,i}$ of the natural homomorphism $G_i \to \mathrm{Diffeo}(V(j, i))$ coincides with the subgroup $\langle g_i^j \rangle$ of G_i generated by

g_i^j. Therefore the chart

(51) $\left\{ V(j,i) \longrightarrow \pi_i(V(j,i)), \;\; \mathrm{Gal}(V(j,i)/\pi_i(V(j,i)) \simeq G_i/K_{j,i} \right\}$

induces on $\pi_i(V(j,i))$ a reduced orbifold structure (cf. [CR, p. 3]) with $|G_i/K_{j,i}| = a_i/c$. By definition, the multiplicity $m_{j,i}$ of $V(j,i)$ coincides with

(52) $m_{j,i} = |K_{j,i}| = c \,,$

which is also symmetric with respect to the suffices $i, i_1, \ldots, i_r$ by (49).

5.3.   We globalize the observation in §5.2. We denote by $\overline{V(j,i)}$ the closure of $\pi_i(V_{j,i})$ in $\overline{Y}$, which coincides with the T_N-invariant subvariety

(53) $\overline{V(j,i)} = V\left(\sum_{\ell=r+1}^{n-r-2} \mathbf{R}_{\geq 0} P_{i_\ell} + \mathbf{R}_{\geq 0} Q + \mathbf{R}_{\geq 0} P_n \right).$

As the restriction of $r+1$ local uniformazing systems of (44), we have the covering $\overline{V(j,i)} = \sum_{\ell=0}^{r} \left(U_{i_\ell} \cap \overline{V(j,i)} \right)$ where we put $U_{i_0} = U_i$ and $U_{i_\ell} = U_{\sigma_{i_\ell}}$. This also induces the orbifold chart on $\overline{V(j,i)}$. More precisely, the system on $U_{i_0} \cap \overline{V(j,i)} = V(j,i)$ is given by (51) and the other systems $U_{i_\ell} \cap \overline{V(j,i)}$ are essentially of the same type as this.

Then $\overline{V(j,i)}$ is one of the connected components of the twisted sector of $\overline{Y}$. The multiplicity $m_{j,i}$ of $\overline{V(j,i)}$ is already given by (52). The equivariant L-form on $\widetilde{U_{i_\ell}}$ ($1 \leq \ell \leq r$) which is patched together with (48) into the global form on $\overline{V(j,i)}$ is essentially of the same type as (48). The well-definedness of this globalization is clear from the symmetry property of (50) with respect to the $r+1$ suffices. We denote by $L^{\Sigma}(g_i^j)$ this orbifold differential form on $\overline{V(j,i)}$. This is nothing but the restriction of L^{Σ} to the component $\overline{V(j,i)}$.

LEMMA 5.3.1.   *We have an equality*

$\langle \mathrm{L}^{\Sigma}(\overline{g_i^j}), \overline{V(j,i)} \rangle$

$\qquad = \dfrac{1}{a_i} \cdot \kappa_n \Bigg\{ \prod_{k \in \mathrm{I}_{j,i}^* \setminus \{i\}} \coth\left(\dfrac{[V(P_k)]}{2} + \dfrac{\sqrt{-1}\pi j a_k}{a_i} \right) \coth\left(\dfrac{[V(Q)]}{2} - \dfrac{\sqrt{-1}\pi j d}{a_i} \right)$

$\qquad\qquad \times \prod_{k \in \mathrm{I}_{j,i}^0} \left([V(P_k)] \cdot \coth \dfrac{[V(P_k)]}{2} \right) \prod_{k \in \mathrm{I}_{j,i}^{\mathrm{ref}}} \tanh \dfrac{[V(P_k)]}{2}$

$\qquad\qquad \times \left(\prod_{k \in \mathrm{I}_{j,i}^* \setminus \{i\}} [V(P_k)] \right) [V(Q)] \Bigg\} [\overline{Y}]$

where the product is the intersection product in the Chow ring $A_{\mathbf{Q}}^*(\overline{Y})$ and κ_n is the degree n part in the Taylor expansion evaluated at the fundamental class of $\overline{Y}$, which is the element in $H^n(\overline{Y}, \mathbf{Q}) \simeq \mathbf{Q}$.

PROOF.   We denote by $(T_{\overline{V(j,i)}})_{\mathbf{Q}}$, $(N_{\overline{V(j,i)}/Y})_{\mathbf{Q}}$ the orbifold tangent bundle and the orbifold normal bundle in Y, respectively. The local decomposition (47) induces the global

decomposition

$$(54) \quad (T_{\overline{V(j,i)}})_{\mathbf{Q}} \simeq \bigoplus_{k \in \mathrm{I}^0_{j,i}} [V(P_k)], \quad (N_{\overline{V(j,i)}/\overline{Y}})_{\mathbf{Q}} \simeq \left(\bigoplus_{k \in (\mathrm{I} \setminus \mathrm{I}^0_{j,i}) \setminus \{i\}} [V(P_k)] \right) \bigoplus [V(Q)],$$

where $[V(P_k)]$ is the $\mathbf{Q}$-line bundle (orbifold line bundle) associated to the $\mathbf{Q}$-divisor $V(P_k)$, etc. For each local system of $\overline{V(j,i)}$, say $V(j,i) = \overline{V(j,i)} \cap U_i$, we claim that

$$(55) \qquad \frac{1}{m_{j,i}} \int_{V(j,i)} L^{\Sigma}(\overline{g_i^j})|_{V(j,i)} = \frac{1}{a_i} \int_{V(j,i)} L(g_i^j, \widetilde{U}_i).$$

Indeed, the integral $\int_{V(j,i)} L(g_i^j, \widetilde{U}_i)$ is calculated on the non-reduced orbifold structure with the restriction to $V_{j,i}$ of the Galois group $G_i = \mathrm{Gal}(\widetilde{U}_i/U_i)$, it is a_i/c times the reduced integral which is nothing but $\int_{V(j,i)} L^{\Sigma}(\overline{g_i^j})|_{V(j,i)}$ (cf. [CR, p. 5]). Since we have $m_{j,i} = c$ by (52), we have the claim (55).

By the same argument as in Lemma 5.1.1, the integral of the orbifold characteristic class is interpreted to be that on the rational Chow ring. Therefore the assertion follows from (45), (46), (50), (54) and (55).

Now we write down the explicit value $\langle \mathrm{L}^{\Sigma}(\overline{g_i^j}), \overline{V(j,i)} \rangle$ by using the terminology defined in §1.

LEMMA 5.3.2.   *We have an equality*

$$\langle \mathrm{L}^{\Sigma}(\overline{g_i^j}), \overline{V(j,i)} \rangle = (|\mathrm{I}^0_{j,i}| + 1) \cot \left(j \cdot R_i \left(\frac{\mathbf{a}}{d} \right) \right) \left(N_i \left(\frac{\mathbf{a}}{d} \right) \right).$$

PROOF.   Since $[V(Q)]$ is contained in the product of the right-hand term of Lemma 5.3.1, by the same reason as in the proof of Proposition 3.2.1 and Remark 3.2.2, we may numerically assume that

$$(56) \qquad \begin{aligned} [V(P_i)] &\equiv -\frac{a_i}{d}[V(Q)] \quad (1 \leq i \leq n-1), \\ [V(P_n)] &\equiv -\frac{1}{d}[V(Q)], \quad [V(Q)]^n = -\frac{d^{n-1}}{a_1 \cdots a_{n-1}}. \end{aligned}$$

Therefore, it follows from Lemma 5.3.1 and (56) that

$$(57) \qquad \begin{aligned} &\langle \mathrm{L}^{\Sigma}(\overline{g_i^j}), \overline{V(j,i)} \rangle \\ &= \frac{1}{a_i \left(\prod_{k \in \mathrm{I}^0_{ji}} a_k \right)} \cdot \kappa_{|\mathrm{I}^0_{j,i}|} \left\{ \prod_{k \in \mathrm{I}^*_{j,i} \setminus \{i\}} \coth \left(\frac{a_k x}{2} + \frac{\sqrt{-1}\pi j a_k}{a_i} \right) \right. \\ &\qquad \times \coth \left(\frac{-dx}{2} - \frac{\sqrt{-1}\pi j d}{a_i} \right) \prod_{k \in \mathrm{I}^0_{j,i}} \left(a_k x \cdot \coth \frac{a_k x}{2} \right) \prod_{k \in \mathrm{I}^{\mathrm{ref}}_{j,i}} \tanh \frac{a_k x}{2} \right\} \end{aligned}$$

where $\kappa_{|I^0_{j,i}|}$ is the coefficient of the degree $|I^0_{j,i}|$ term of Tayler expansion with respect to the variable x. The right-hand side of (57) coincides with

$$(|I^0_{j,i}| + 1) \cot\left(j \cdot R_i\left(\frac{\mathbf{a}}{d}\right)\right)\left(N_i\left(\frac{\mathbf{a}}{d}\right)\right)$$

by an easy calculation.

5.4 We prove Proposition 5.1.2. Since the restriction $\overline{Y}\backslash Y \longrightarrow \overline{X}\backslash X$ of $\overline{\mu}$ is an isomorphism, the contribution to the S-defects of both loci are cancelled. Since X has an isolated cyclic orbipoint of type $\mathbf{a}/d$ such that the action (9) is small (i.e., any power of the map (9) is reflection free), we have

$$(58) \qquad\qquad\qquad \mathrm{Sdef}(X) = \mathbf{s}\left(\frac{\mathbf{a}}{d}\right).$$

Now the set of the connected components of the twisted sector on Y is written as

$$(59) \qquad\qquad\qquad \{\overline{V(j_1, i_1)}, \dots, \overline{V(j_\alpha, i_\alpha)}\}$$

for some $1 \le i_k \le n-1$ and some $1 \le j \le a_{i_k}$ ($1 \le k \le \alpha-1$). Since $\mathrm{L}^\Sigma = \bigoplus_{k=1}^\alpha \mathrm{L}^\Sigma(g^{\overline{j_k}}_{i_k})$, it follows from (43), (58) that

$$\mathrm{Sdef}(\overline{X}) - \mathrm{Sdef}(\overline{Y}) = \mathbf{s}\left(\frac{\mathbf{a}}{d}\right) - \sum_{k=1}^\alpha \langle \mathrm{L}^\Sigma(g^{\overline{j_k}}_{i_k}), \overline{V(j_k, i_k)} \rangle.$$

Hence by Lemma 5.3.2, it suffices to prove

$$(60) \qquad \sum_{k=1}^\alpha (|I^0_{j_k,i_k}| + 1) \cot\left(j_k \cdot R_{i_k}\left(\frac{\mathbf{a}}{d}\right)\right)\left(N_{i_k}\left(\frac{\mathbf{a}}{d}\right)\right) = \sum_{i=1}^{n-1} \mathbf{s}\left(R_i\left(\frac{\mathbf{a}}{d}\right), N_i\left(\frac{\mathbf{a}}{d}\right)\right).$$

For any (j, i) with $1 \le i \le n - 1$, $1 \le j \le a_i - 1$, the variety $\overline{V(j, i)}$ is one of the elements in the set (59). On the other hand, any element $\overline{V(j_k, i_k)}$ in the set (59) is counted $(|I^0_{j_k,i_k}| + 1)$ times if (j, i) runs through all the pairs with $1 \le i \le n - 1$, $1 \le j \le a_i - 1$. From this, the assertion (60) follows by an easy calculation. Thus we complete the proof of Proposition 5.1.2.

5.5 The proof of Theorem 4.2.2. is straightforward from Corollary 2.1.2, Proposition 3.2.1, Lemma 5.1.1, Proposition 5.1.2, the formulas (42) and (1).   □

REMARK 5.5.1.   With respect to related open questions, one may ask the following:
(1) What is the homogeneous version of (37) ? Namely can one drop the assumption that the last component of the numerator is 1 ?
(2) What is the reciprocity without the assumption of irreducibility of $\mathbf{a}/d$? This question is related to the computation problem of Hirzebruch-Zagier's Dedekind sum by iterated applications of reciprocities.
(3) Does the formula (37) relate to Beck reciprocity [B] ?
(4) What is the analogy for (37) of the relation between Dedekind sum and Dedekind eta function ?

(5)   Can one apply the method of Fujiki-Oka modification to detect other types of reci-
procity for Fourier-Dedekind sums (cf. [BR]) ?

(6)   Do other types of quotient singularities or toric singularities have their "own re-
ciprocities" ?

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