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MULTIPLE AND NODAL SOLUTIONS FOR NONLINEAR EQUATIONS WITH A NONHOMOGENEOUS DIFFERENTIAL OPERATOR AND CONCAVE-CONVEX TERMS

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Abstract. In this paper we consider a nonlinear parametric Dirichlet problem driven by a nonhomogeneous differential operator (special cases are the p-Laplacian and the (p, q)-differential operator) and with a reaction which has the combined effects of concave ($(p-1)$-sublinear) and convex ($(p-1)$-superlinear) terms. We do not employ the usual in such cases AR-condition. Using variational methods based on critical point theory, together with truncation and comparison techniques and Morse theory (critical groups), we show that for all small $\lambda > 0$ (λ is a parameter), the problem has at least five nontrivial smooth solutions (two positive, two negative and the fifth nodal). We also prove two auxiliary results of independent interest. The first is a strong comparison principle and the second relates Sobolev and Hölder local minimizers for C^1 functionals.

1. Introduction. Let $\Omega \subseteq \boldsymbol{R}^n$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Dirichlet problem:

$$\left\{\begin{array}{l} -\mathrm{div}\ (a(Du(z)) = \lambda|u(z)|^{q-2}u(z) + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \quad \lambda > 0. \end{array}\right\} \tag{1}$$

Here the map $a : \boldsymbol{R}^n \to \boldsymbol{R}^n$ involved in the differential operator of (1), is strictly monotone and satisfies certain regularity conditions (see hypotheses H_0). The p-Laplacian ($p > 1$) defined by $\triangle_p u = \mathrm{div}(\|Du\|^{p-2}Du)$ for all $u \in W_0^{1,p}(\Omega)$ and the (p, τ)-differential operator ($2 \le \tau < p$) defined by $\triangle_p u + \mu\triangle_\tau u$ for all $u \in W_0^{1,p}(\Omega)$ with $\mu \ge 0$, are special cases of the differential operator in problem (1). We stress that the differential operator in (1) need not be homogeneous and this is the source of many technical difficulties. In (1), $q \in (1, p)$ and so the first term in the right-hand side of (1) is "concave" (i.e., $(p-1)$-sublinear). On the other hand, for $f(z, x)$ we assume that it is a Caratheodory function (i.e., for all $x \in \boldsymbol{R}$ $x \to f(z, x)$ is measurable and for a.e. $z \in \Omega$ $x \to f(z, x)$ is continuous), which exhibits $(p-1)$-superlinear growth near $\pm\infty$ in the x-variable. So, in the reaction of problem (1) we have the combined effects of "concave" and "convex" nonlinearities and a special case of the right-hand side of (1), is the following function which we encounter in the literature

$$g(x, \lambda) = \lambda|x|^{q-2}x + |x|^{r-2}x \quad \text{with } 1 < q < p < r < p^*,$$

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where

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases} \quad \text{(the critical Sobolev exponent)}.$$

This particular reaction can be found in the semilinear works (i.e., equations driven by the Laplacian) of Ambrosetti-Brezis-Cerami [2], Bartsch-Willem [5] and Tang [31], who focus on the existence and multiplicity of positive solutions. Their work was extended to equations driven by the p-Laplacian by Garcia Azorero-Manfredi-Peral Alonso [15], Guo-Zhang [20] and Kyritsi-Papageorgiou [23].

In this paper, using a combination of variational methods based on critical point theory, with suitable truncation and comparison techniques and with Morse Theory (critical groups), we produce five nontrivial smooth solutions and provide precise sign information for all of them (two positive, two negative and the fifth is nodal (sign changing)). We mention that all previous results concerning the existence of nodal solutions, deal with equations driven by the Laplacian or p-Laplacian and the reaction satisfies the well known Ambrosetti-Rabinowitz condition (AR-condition for short), see Bartsch-Liu-Weth [4], Dancer-Du [10] and Filippakis-Kristaly-Papageorgiou [14]. The fact that the differential operator in problem (1) is not homogeneous, does not allow the use of the techniques employed in the aforementioned papers. It seems that our result here is the first one on the existence of nodal solutions for nonlinear equations driven by a nonhomogeneous differential operator.

2. Mathematical background. In this section, we briefly review the main mathematical tools which we will use in the sequel. We also prove two auxiliary results, which are of independent interest. So, let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$, we denote the duality brackets for the pair (X, X^*). Let $\varphi \in C^1(X)$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following holds "every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \boldsymbol{R}$ is bounded and $(1 + \|x_n\|)\varphi'(x_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence."

This compactness type condition, is in general weaker that the usual Palais-Smale condition (PS-condition for short). However, the C-condition suffices to prove a deformation theorem and from it derive the minimax theory for certain critical values of $\varphi \in C^1(X)$ (see, for example, Papageorgiou-Kyritsi [29]). In particular, we have the following result, known in the literature as the "mountain pass theorem."

THEOREM 2.1. *If $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$, $\rho > 0$, $\|x_1 - x_0\| > \rho$*

$$\max\{\varphi(\mathrm{x}_0), \varphi(\mathrm{x}_1)\} < \inf[\varphi(\mathrm{x}); \|\mathrm{x} - \mathrm{x}_0\| = \rho] = \eta_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], X); \gamma(0) = x_0, \gamma(1) = x_1\}$, then $c \geq \eta_\rho$ and c is a critical value of φ (i.e., there exists $x \in X$ such that $\varphi'(x) = 0$ and $\varphi(x) = c$).

Given $\varphi \in C^1(X)$ and $c \in \boldsymbol{R}$, we introduce the following sets:

$$K_\varphi = \{x \in X; \varphi'(x) = 0\},\ K_\varphi^c = \{x \in K_\varphi; \varphi(x) = c\}$$
$$\text{and } \varphi^c = \{x \in X; \varphi(x) \le c\}\,.$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \ge 0$, by $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. The critical groups at an isolated point $x \in K_\varphi^c$ are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, \varphi^c \cap U\backslash\{x\}) \quad \text{for all } \ k \ge 0\,,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The excision property of singular homology, implies that this definition is independent of the particular choice of the neighborhood U.

In the analysis of problem (1), in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the ordered Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}); u|_{\partial\Omega} = 0\}$. The order cone of $C_0^1(\overline{\Omega})$ is $C_+ = \{u \in C_0^1(\overline{\Omega}); u(z) \ge 0 \ \text{ for all } \ z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+; u(z) > 0 \quad \text{for all } \ z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \quad \text{for all } \ z \in \partial\Omega\right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

Next we prove two auxiliary results, which are of independent interest. The first is a comparison principle, while the second relates local $C_0^1(\overline{\Omega})$ and $W_0^{1,p}(\Omega)$ minimizers for a large class of C^1-functionals. To this end we introduce the following hypotheses:

<u>H</u>: $G : \overline{\Omega} \times \boldsymbol{R}^n \to \boldsymbol{R}$ is a C^1-function such that for all $z \in \overline{\Omega}$, $G(z, 0) = 0$, $\nabla_y G(z, y) = a(z, y)$, $a(z, 0) = 0$ and

(i) $a \in C^1(\overline{\Omega} \times (\boldsymbol{R}^n\backslash\{0\}), \boldsymbol{R}^n)$ and for every $K \subseteq \boldsymbol{R}^n\backslash\{0\}$ compact, there exists $a \in (0, 1)$ such that $a \in C^a(\overline{\Omega} \times K, \boldsymbol{R}^n)$;

(ii) for every $z \in \overline{\Omega}$ and every $y \in \boldsymbol{R}^n\backslash\{0\}$, we have

$$c_0(\eta + \|y\|)^{p-2}\|\xi\|^2 \le (\nabla_y a(z, y)\xi, \xi)_{\boldsymbol{R}^n} \ \text{ for all } \ \xi \in \boldsymbol{R}^n$$

and some $c_0 > 0$, $\eta \ge 0$;

(iii) for every $z \in \overline{\Omega}$ and every $y \in \boldsymbol{R}^n\backslash\{0\}$, we have

$$\|\nabla_y a(z, y)\| \le c_1(\eta + \|y\|)^{p-2} \ \text{ for some } \ c_1 > 0$$

and with $\eta \ge 0$ as in (ii);

(iv) for every $\rho > 0$, there exists $c_2 = c_2(\rho) > 0$ such that

$$|a(z, y) - a(z', y)| \le c_2(1 + \|y\|)^{p-1}\|z - z'\|$$

for all $z, z' \in \Omega$, all $\|y\| \le \rho$.

From these hypotheses and using the integral form of the mean value theorem we can have:

LEMMA 2.2. *If hypotheses* H *hold, then for all* $z \in \overline{\Omega}$, $a(z,\cdot)$ *is strictly monotone and there exist* $c_3, c_4 > 0$ *such that*

$$\|a(z,y)\| \le c_3(1+\|y\|)^{p-1} \quad \textit{and} \quad (a(z,y),y) \ge c_4\|y\|^p \quad \textit{for all} \quad (z,y) \in \overline{\Omega} \times \boldsymbol{R}^n\,.$$

An immediate consequence of this lemma is the following growth estimate for the potential function $G(z,\cdot)$:

COROLLARY 2.3. *If hypotheses* H *hold, then for all* $z \in \overline{\Omega}$, $G(z,\cdot)$ *is strictly convex and there exist* $c_5, c_6 > 0$ *such that*

$$c_5\|y\|^p \le G(z,y) \le c_6(1+\|y\|^p) \quad \textit{for all} \quad (z,y) \in \overline{\Omega} \times \boldsymbol{R}^n\,.$$

As we already mentioned, the first auxiliary result is a strong comparison principle. It extends Proposition 2.2 of Guedda-Veron [19] and Proposition 2.6 of Arcoya-Ruiz [3], who deal with the p-Laplacian (i.e., $G(y) = \frac{1}{p}\|y\|^p$ for all $y \in \boldsymbol{R}^n$.) Other comparison results, can be found in the works of Cuesta-Takac [9] and Lucia-Prashanth [26]. So, we consider the following two nonlinear Dirichlet problems:

$$-\mathrm{div}\, a(z, Du(z)) + \gamma|u(z)|^{p-2}u(z) = h_1(z) \quad \text{in} \ \ \Omega, \ \ u|_{\partial\Omega} = 0\,. \tag{2}$$

$$-\mathrm{div}\, a(z, Dv(z)) + \gamma|v(z)|^{p-2}v(z) = h_2(z) \quad \text{in} \ \ \Omega, \ \ v|_{\partial\Omega} = 0\,, \tag{3}$$

where $\gamma \ge 0$ and $h_1, h_2 \in L^\infty(\Omega)$. We assume that $h_1 \prec h_2$ meaning that for every compact $K \subseteq \Omega$, we can find $\varepsilon = \varepsilon(K) > 0$ such that $h_1(z) + \varepsilon \le h_2(z)$ for a.e. $z \in K$. Note that if $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$, then $h_1 \prec h_2$.

PROPOSITION 2.4. *If* $u, v \in W_0^{1,p}(\Omega)$ *are nontrivial solutions of* (2) *and* (3) *respectively*, $u, v \ge 0$ *and* $h_1 \prec h_2$, *then* $v - u \in \mathrm{int}C_+$.

PROOF. From Theorem 7.1 (p. 286) of Ladyzhenskaya-Uraltseva [24], we know that $u, v \in L^\infty(\Omega)$. So, applying Theorem 1 of Lieberman [25], we have $u, v \in C_+\backslash\{0\}$. Moreover, invoking Theorem 2.4.1 (p. 30) of Pucci-Serrin [30], we have that

$$u(z) \le v(z) \quad \text{for all} \ \ z \in \overline{\Omega}\,.$$

For small $\delta > 0$ let $\Omega_\delta \subseteq \Omega$ be the δ-neighborhood of $\partial\Omega$ in Ω defined by

$$\Omega_\delta = \{z \in \Omega; d(z) < \delta\}\,,$$

where $d(z) = d(z, \partial\Omega)$ (the distance of $z \in \Omega$ from $\partial\Omega$). From the proof of Lemma 14.16 (p. 355) of Gilbarg-Trundinger [18], we know that $d \in C^2(\overline{\Omega_\delta})$. Hence, $\partial\Omega_\delta$ is a C^2-manifold. From Cuesta-Takac [9] (see (2.2) in the proof of Proposition 2.4), we know that $w = v - u$ satisfies in the sense of distributions the following nonlinear elliptic inequality

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\left(a_{ij}(z)\frac{\partial w}{\partial z_j}\right) + \theta(z)w = h_2 - h_1 \ge 0 \quad \text{in} \ \ \Omega_\delta\,, \tag{4}$$

where the coefficients a_{ij} belong in $C(\overline{\Omega_\delta})$, the differential operator in (4) is uniformly elliptic in Ω_δ and $\theta \in C(\overline{\Omega_\delta})$, $\theta \geq 0$. Invoking Theorem 4 of Vazquez [32], we infer that

$$\frac{\partial w}{\partial n}(z) < 0 \quad \text{for all } z \in \partial\Omega \subseteq \partial\Omega_\delta\,. \tag{5}$$

Let $C = \{z \in \Omega; u(z) = v(z)\}$ (the coincidence set). From (5) it follows that C is a compact set in Ω. Let Ω' be an open set such that $C \subseteq \Omega' \subseteq \overline{\Omega'} \subseteq \Omega$. For small $\varepsilon \in (0, 1)$, we have

$$u(z) + \varepsilon \leq v(z) \quad \text{for all } z \in \partial\Omega' \text{ and } h_1(z) + \varepsilon \leq h_2(z) \text{ for all } z \in \Omega'\,. \tag{6}$$

We choose $\delta \in (0, \varepsilon)$ such that

$$\gamma\big||x|^{p-2}x - |y|^{p-2}y\big| \leq \varepsilon \text{ for all } x, y \in \left[\min_{\overline{\Omega'}} u, \max_{\overline{\Omega'}} u + 1\right] \text{ with } |x - y| \leq 2\delta\,. \tag{7}$$

Then we have

$$\begin{aligned} -\text{div}\, a(z, D(u + \delta)) + \gamma(u + \delta)^{p-1} &= -\text{div}\, a(z, Du) + \gamma(u + \delta)^{p-1} \\ &= \gamma[(u + \delta)^{p-1} - u^{p-1}] + h_1 \quad \text{(see (2))} \\ &\leq \varepsilon + h_1 \quad \text{(see (7))} \\ &\leq h_2 \quad \text{(see (6))} \\ &= -\text{div}\, a(z, Dv) + \gamma v^{p-1} \quad \text{a.e. in } \Omega' \\ &\text{(see (2))}\,. \end{aligned} \tag{8}$$

Using once more Theorem 2.4.1 (p. 30) of Pucci-Serrin [30], we infer that $u(z)+\delta \leq v(z)$ for all $z \in \Omega'$. Since $C \subseteq \Omega'$, it follows that $C = \emptyset$. Therefore

$$w(z) = v(z) - u(z) > 0 \quad \text{for all } z \in \Omega\,. \tag{9}$$

From (5) and (9), we conclude that $w = v - u \in \text{int}\, C_+$. □

REMARK 2.5. A careful reading of the above proof reveals that we may replace the term $\gamma|x|^{p-2}x$ by a Caratheodory function $\beta(z, x)$ which is locally Lipschitz in $x \in \boldsymbol{R}$, $\frac{\partial}{\partial x}\beta(z, x) \geq 0$ for a.e. $z \in \Omega \times \boldsymbol{R}$ and there exists $r > 0$ such that

$$\frac{\partial \beta}{\partial x}(z, x) = \begin{cases} \eta|x|^{p-2} & \text{if } 1 < p \leq 2 \\ \eta & \text{if } 2 < p \end{cases} \qquad \text{for a.e. } (z, x) \in \Omega \times (0, r)$$

with $\eta \geq 0$ as in hypotheses H(ii)(iii) (see Cuesta-Takac [9]).

The next auxiliary result compares local $C_0^1(\overline{\Omega})$ and $W_0^{1,p}(\Omega)$-minimizers for a large class of C^1-functionals. Our result generalizes those of Brezis-Nirenberg [6], where $G(y) = \frac{1}{2}\|y\|^2$ for all $y \in \boldsymbol{R}^n$ and of Garcia Azorero-Manfredi-Peral Alonso [15] and Guo-Zhang [20], where $G(y) = \frac{1}{p}\|y\|^p$ (in [15] $1 < p < \infty$ and in [20] $p \geq 2$). Moreover, our proof is simpler.

Let $f_0 : \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ be a Caratheodory function with subcritical growth in $x \in \boldsymbol{R}$, i.e.,

$$|f_0(z,x)| \le a(z) + c|x|^{r-1} \quad \text{for a.e. } z \in \Omega,\ \text{all } x \in \boldsymbol{R}\,,$$
$$\text{with} \quad a \in L^\infty(\Omega)_+, c > 0, 1 < r < p^*\,.$$

We let $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional

$$\psi_0 : W_0^{1,p}(\Omega) \to \boldsymbol{R}$$

defined by

$$\psi_0(u) = \int_\Omega G(z, Du(z))dz - \int_\Omega F_0(z,u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

PROPOSITION 2.6. *If hypotheses* H *hold and* $u_0 \in W_0^{1,p}(\Omega)$ *is a local* $C_0^1(\overline{\Omega})$*-minimizer of* ψ_0*, i.e., there exists* $\rho_0 > 0$ *such that*

$$\psi_0(u_0) \le \psi_0(u_0 + h) \quad \textit{for all } h \in C_0^1(\overline{\Omega}) \textit{ with } \|h\|_{C_0^1(\overline{\Omega})} \le \rho_0\,,$$

then $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$ *and it is also a local* $W_0^{1,p}(\Omega)$*-minimizer of* ψ_0*, i.e., there exists* $\rho_1 > 0$ *such that*

$$\psi_0(u_0) \le \psi_0(u_0 + h) \quad \textit{for all } h \in W_0^{1,p}(\Omega) \textit{ with } \|h\| \le \rho_1\,.$$

PROOF. Let $h \in C_0^1(\overline{\Omega})$ and let $t > 0$ be small. By hypothesis we have

$$\psi_0(u_0) \le \psi_0(u_0 + th) \Rightarrow 0 \le \langle \psi_0'(u_0), h\rangle\,. \tag{10}$$

Since $h \in C_0^1(\overline{\Omega})$ is arbitrary and $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$, from (10) it follows that

$$\psi_0'(u_0) = 0, \Rightarrow V(u_0) = N_{f_0}(u_0)\,, \tag{11}$$

where $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $(1/p + 1/p' = 1)$ is the map defined by

$$\langle V(u), y\rangle = \int_\Omega (a(z,Du), Dy)_{\boldsymbol{R}^n} dz \quad \text{for all } u, y \in W_0^{1,p}(\Omega)$$

and $N_{f_0}(u)(\cdot) = f_0(\cdot, u(\cdot))$ for all $u \in W_0^{1,p}(\Omega)$. From (11) it follows that

$$-\text{div}\, a(z, Du_0(z)) = f_0(z, u_0(z)) \quad \text{a.e. in } \Omega,\ u_0|_{\partial\Omega} = 0\,. \tag{12}$$

As before, invoking Theorem 7.1 (p. 286) of Ladyzhenskaya-Uraltseva [24], we have that $u_0 \in L^\infty(\Omega)$. So, we can apply Theorem 1 of Lieberman [25] and conclude that $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

Next we show that u_0 is also a $W_0^{1,p}(\Omega)$-minimizer of ψ_0. We argue by contradiction. So, we suppose that u_0 is not a local $W_0^{1,p}(\Omega)$-minimizer of ψ_0. Let $\varepsilon > 0$ and set $\overline{B}_\varepsilon^r = \{u \in W_0^{1,p}(\Omega); \|u\|_r \le \varepsilon\}$. We consider the following minimization problem

$$\inf\,[\psi_0(u_0 + h); h \in \overline{B}_\varepsilon^r] = m_0^\varepsilon > -\infty\,. \tag{13}$$

Since we have assumed that u_0 is not a local $W_0^{1,p}(\Omega)$-minimizer of ψ_0, we have

$$m_0^\varepsilon < \psi_0(u_0)\,. \tag{14}$$

Let $\{h_n\}_{n\geq 1} \subseteq \overline{B}_\varepsilon^r$ be a minimizing sequence for problem (13). From Corollary 2.3 and the growth condition on $f_0(z,\cdot)$ we see that $\{h_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$h_n \xrightarrow{w} h_\varepsilon \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad h_n \to h_\varepsilon \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{15}$$

Exploiting the compact embedding of $W_0^{1,p}(\Omega)$ into $L^r(\Omega)$ (recall that $r < p^*$), we can easily check that ψ_0 is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. So, from (15) it follows that

$$\begin{aligned} &\psi_0(u_0+h_\varepsilon) \leq \liminf_{n\to\infty} \psi_0(u_0+h_n) \quad \text{and} \quad h_\varepsilon \in \overline{B}_\varepsilon^r \\ \Rightarrow\ &\psi_0(u_0+h_\varepsilon) = m_0^\varepsilon \quad \text{and so} \quad h_\varepsilon \neq 0 \text{ (see (14))}. \end{aligned}$$

Hence, in problem (13) the infimum is attained at some $h_\varepsilon \in \overline{B}_\varepsilon^r \setminus\{0\}$. By virtue of the Lagrange multiplier rule (see, for example, Ioffe-Tichomirov [21] (p. 74)), we can find $\lambda_\varepsilon \leq 0$ such that

$$\begin{aligned} &\psi'(u_0+h_\varepsilon) = \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon \\ \Rightarrow\ &V(u_0+h_\varepsilon) = N_{f_0}(u_0+h_\varepsilon) + \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon. \end{aligned}$$

Hence

$$\left\{ \begin{array}{l} -\text{div}\, a(z, D(u_0+h_\varepsilon)(z)) = f_0(z,(u_0+h_\varepsilon)(z)) + \lambda_\varepsilon |h_\varepsilon(z)|^{r-2} h_\varepsilon(z), \\ \text{a.e. in } \Omega,\ h_\varepsilon|_{\partial\Omega} = 0. \end{array} \right\} \tag{16}$$

From (12) and (16) we have

$$\left\{ \begin{array}{l} -\text{div}\, (a(z, D(u_0+h_\varepsilon)(z)) - a(z, Du_0(z))) \\ = f_0(z,(u_0+h_\varepsilon)(z)) - f_0(z,u_0(z)) + \lambda_\varepsilon |h_\varepsilon(z)|^{p-2} h_\varepsilon(z) \text{ a.e. in } \Omega. \end{array} \right\} \tag{17}$$

<u>Case 1:</u> Suppose that $\lambda_\varepsilon \in [-1,0]$ for all $\varepsilon \in (0,1]$. Set $v_\varepsilon(z) = (u_0+h_\varepsilon)(z)$ and $\xi_\varepsilon(z,y) = a(z,y) - a(z,Du_0(z))$. Then (17) becomes

$$\left\{ \begin{array}{l} -\text{div}\, \xi_\varepsilon(z, Dv_\varepsilon(z)) = f_0(z,v_\varepsilon(z)) - f_0(z,u_0(z)) \\ +\lambda_\varepsilon |(v_\varepsilon-u_0)(z)|^{p-2}(v_\varepsilon-u_0)(z) \text{ a.e. in } \Omega. \end{array} \right\} \tag{18}$$

By virtue of Theorem 7.1 (p. 286) of Ladyzhenskaya-Uraltseva [24], we can find $M_1 > 0$ such that

$$\|v_\varepsilon\|_\infty \leq M_1 \quad \text{for all } \varepsilon \in (0,1]. \tag{19}$$

Using Lemma 2, we can easily check that $\xi_\varepsilon(z,y)$ verifies hypotheses H. This fact and (19), permit the use of Theorem 1 of Lieberman [25]. So, we can find $\gamma \in (0,1)$ and $M_2 > 0$ such that

$$v_\varepsilon \in C_0^{1,\gamma}(\overline{\Omega}) \text{ and } \|v_\varepsilon\|_{C_0^{1,\gamma}(\overline{\Omega})} \leq M_2 \quad \text{for all } \varepsilon \in (0,1]. \tag{20}$$

Case 2: Suppose that $\lambda_{\varepsilon_n} < -1$ for all $n \geq 1$, with $\varepsilon_n \downarrow 0$. In this case, we set $\widehat{\xi}_{\varepsilon_n}(z, y) = \frac{1}{|\lambda_{\varepsilon_n}|}[a(z, y) - a(z, Du_0(z))]$. Then (17) becomes

$$\left\{ \begin{array}{l} -\mathrm{div}\, \widehat{\xi}_{\varepsilon_n}(z, Dv_{\varepsilon_n}(z)) \\ = \frac{1}{|\lambda_{\varepsilon_n}|}[f_0(z, v_{\varepsilon_n}(z)) - f_0(z, u_0(z))] - |(v_{\varepsilon_n} - u_0)(z)|^{p-2}(v_{\varepsilon_n} - u_0)(z) \\ \text{a.e. in } \Omega \end{array} \right\} \tag{21}$$

where $v_{\varepsilon_n} = u_0 + h_{\varepsilon_n}$. For all $w \in W_0^{1,p}(\Omega)$, we have

$$\langle V(u_0), w \rangle = \int_\Omega f_0(z, u_0(z))w(z)dz \quad \text{(see (11))} \tag{22}$$

$$\langle V(v_{\varepsilon_n}), w \rangle = \int_\Omega f_0(z, v_{\varepsilon_n}(z))w(z)dz + \lambda_{\varepsilon_n} \int_\Omega |(v_{\varepsilon_n} - u_0)|^{r-2}(v_{\varepsilon_n} - u_0)(z)dz \tag{23}$$

(see (16)).

For $\tau \geq 1$ we consider the function $|v_{\varepsilon_n} - u_0|^\tau (v_{\varepsilon_n} - u_0)$. We have

$$\begin{aligned} &D(|v_{\varepsilon_n} - u_0|^\tau (v_{\varepsilon_n} - u_0)) \\ &\quad = |v_{\varepsilon_n} - u_0|^\tau D(v_{\varepsilon_n} - u_0) + \tau(v_{\varepsilon_n} - u_0)\frac{v_{\varepsilon_n} - u_0}{|v_{\varepsilon_n} - u_0|}|v_{\varepsilon_n} - u_0|^{\tau-1}D(v_{\varepsilon_n} - u_0) \\ &\quad = (\tau + 1)|v_{\varepsilon_n} - u_0|^\tau D(v_{\varepsilon_n} - u_0) \\ &\qquad \Rightarrow |v_{\varepsilon_n} - u_0|^\tau (v_{\varepsilon_n} - u_0) \in W_0^{1,p}(\Omega) \quad \text{(recall that } v_{\varepsilon_n}, u_0 \in C_0^1(\overline{\Omega})\text{)}. \end{aligned}$$

We use $|v_{\varepsilon_n} - u_0|^\tau (v_{\varepsilon_n} - u_0)$ as the test function w in both (22) and (23). Then we subtract (22) from (23). We obtain

$$\begin{aligned} 0 &\leq (\tau + 1)\int_\Omega (a(z, Dv_{\varepsilon_n}) - a(z, Du_0), Dv_{\varepsilon_n} - Du_0)_{\mathbf{R}^n}|v_{\varepsilon_n} - u_0|^\tau dz \\ &= \int_\Omega (f_0(z, v_{\varepsilon_n}) - f_0(z, u_0))(v_{\varepsilon_n} - u_0)|v_{\varepsilon_n} - u_0|^\tau dz \\ &\quad + \lambda_{\varepsilon_n}\int_\Omega |v_{\varepsilon_n} - u_0|^{\tau+r}dz \quad \text{for all } n \geq 1. \end{aligned} \tag{24}$$

Recall that $\|v_{\varepsilon_n}\|_\infty \leq M_1$ for all $n \geq 1$ (see (19)) and that $u_0 \in C_0^1(\overline{\Omega})$. Therefore

$$\begin{aligned} &\int_\Omega (f_0(z, v_{\varepsilon_n}) - f_0(z, u_0))(v_{\varepsilon_n} - u_0)|v_{\varepsilon_n} - u_0|^\tau dz \\ &\quad \leq M_3\int_\Omega |v_{\varepsilon_n} - u_0|^{\tau+1}dz \quad \text{for some } M_3 > 0, \text{ all } n \geq 1, \\ &\quad \leq M_3|\Omega|_N^{\frac{r-1}{\tau+r}}\|v_{\varepsilon_n} - u_0\|_{\tau+r}^{\tau+1} \\ &\quad \left(\text{using Hölder's inequality with exponents } \frac{\tau+r}{\tau+1} \text{ and } \frac{\tau+r}{r-1}\right), \end{aligned} \tag{25}$$

where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbf{R}^n$. Using (25) in (24), we obtain

$$-\lambda_{\varepsilon_n}\|v_{\varepsilon_n} - u_0\|_{\tau+r}^{\tau+r} \leq M_3|\Omega|_N^{\frac{r-1}{\tau+r}}\|v_{\varepsilon_n} - u_0\|_{\tau+r}^{\tau+1}$$

$$\Rightarrow -\lambda_{\varepsilon_n}\|v_{\varepsilon_n}-u_0\|_{\tau+r}^{r-1} \le M_3|\Omega|_N^{\frac{r-1}{\tau+r}} \quad \text{for all } \tau \ge 1\,, \quad \text{all } n \ge 1\,.$$

Let $\tau \to +\infty$. We obtain

$$-\lambda_{\varepsilon_n}\|v_{\varepsilon_n}-u_0\|_\infty^{r-1} \le M_3 \text{ for all } n \ge 1$$

$$\Rightarrow \|v_{\varepsilon_n}-u_0\|_\infty^{r-1} \le \frac{M_3}{|\lambda_{\varepsilon_n}|} \quad \text{for all } n \ge 1\,. \tag{26}$$

We return to (21) and denote the right-hand side function by $g_{\varepsilon_n}(z,x)$. Then for a.e. $z \in \Omega$ and all $x \in [-M_4, M_4,]$ where $M_4 = \|u_0\|_\infty + M_1$, we have

$$|g_{\varepsilon_n}(z,x)| \le \frac{1}{|\lambda_{\varepsilon_n}|}[M_5 + M_3] \quad \text{for some } M_5 > 0, \text{ all } n \ge 1\,.$$

Therefore, we can apply Theorem 1 of Lieberman [25] and obtain $\beta_0 \in (0,1)$ and $M_6 > 0$ such that

$$h_{\varepsilon_n} \in C_0^{1,\beta_0}(\overline{\Omega}) \quad \text{and} \quad \|h_{\varepsilon_n}\|_{C_0^{1,\beta_0}(\overline{\Omega})} \le M_6 \text{ for all } n \ge 1\,. \tag{27}$$

Recall that for every $\beta' \in (0,1)$, the space $C_0^{1,\beta'}(\overline{\Omega})$ is embedded compactly in $C_0^1(\overline{\Omega})$. Then, from (20) and (27) it follows that for a suitable subsequence, we have

$$u_0 + h_{\varepsilon_n} \to u_0 \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty\,.$$

Since by hypothesis u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of ψ_0, we can find $n_0 \ge 1$ such that

$$\psi_0(u_0) \le \psi_0(u_0 + h_{\varepsilon_n}) \quad \text{for all } n \ge n_0\,. \tag{28}$$

On the other hand, since h_{ε_n} $n \ge 1$ solves problem (13) and because of (14), we have

$$\psi_0(u_0 + h_{\varepsilon_n}) < \psi_0(u_0) \quad \text{for all } n \ge 1\,. \tag{29}$$

Comparing (28) and (29) we reach a contradiction. This concludes the proof. □

As in the above proof, let $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear map defined by

$$\langle V(u), y\rangle = \int_\Omega (a(z,Du), Dy)_{\boldsymbol{R}^n}dz \quad \text{for all } u, y \in W_0^{1,p}(\Omega)\,. \tag{30}$$

From Gasinski-Papageorgiou [16] (p. 591), we have:

PROPOSITION 2.7. *If $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is the nonlinear map defined by* (30), *then V is bounded, continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, i.e., if $u_n \overset{w}{\to} u$ in $W_0^{1,p}(\Omega)$ and* $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$, *then $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

In what follows, for every $r \in \boldsymbol{R}$, we set $r^\pm = \max\{\pm r, 0\}$. Recall that, if $u \in W_0^{1,p}(\Omega)$, then $|u|, u^+, u^- \in W_0^{1,p}(\Omega)$ and we have $|u| = u^+ + u^-$, $u = u^+ - u^-$. As we already mentioned in the proof of Proposition 2.6, by $|\cdot|_N$ we denote the Lebesgue measure on $\boldsymbol{R}^n$. Note that by $\|\cdot\|$ we will denote both the norm of the Sobolev space $W_0^{1,p}(\Omega)$ ($\|u\| = \|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$, by Poincare's inequality) and the norm of $\boldsymbol{R}^n$. It will always be clear from the context which one is used. Finally, let $\widehat{\lambda}_1 > 0$ be the principal eigenvalue of $(-\triangle_p, W_0^{1,p}(\Omega))$. Recall that $\widehat{\lambda}_1 \|u\|_p^p \leq \|Du\|_p^p$ for all $u \in W_0^{1,p}(\Omega)$.

3. Solutions of constant sign. In this section, for $\lambda > 0$ sufficiently small we show that problem (1) has at least four nontrivial smooth solutions of constant sign (two positive and two negative). To this end we introduce the following conditions on the map $a(y)$ $y \in \boldsymbol{R}^n$ and on the nonlinearity $f(z, x)$ $(z, x) \in \Omega \times \boldsymbol{R}$.

$\underline{\mathrm{H}_0}$: $G \in C^1(\boldsymbol{R}^n, \boldsymbol{R})$ such that $G(0) = 0$, $\nabla G(y) = a(y) = a_0(\|y\|)y$ with $a_0(t) > 0$ for all $t > 0$, $a(0) = 0$, a satisfies hypotheses H(i), (ii), (iii) (without the z dependence) and

(iv) $pG(y) - (a(y), y)_{\boldsymbol{R}^n} \geq \beta$ for some $\beta \in \boldsymbol{R}$ and all $y \in \boldsymbol{R}^n$.

REMARK 3.1. Hypotheses H_0 are a restricted version of hypotheses H. So, in particular, the estimates in Lemma 2.2 and Corollary 2.3 remain valid and $G(\cdot)$ is strictly convex. Since $G(0) = 0$ and $\nabla G(y) = a(y)$, we have

$$(a(y), y)_{\boldsymbol{R}^n} \geq G(y) \quad \text{for all } \ y \in \boldsymbol{R}^n \,. \tag{31}$$

<u>Examples:</u> The following maps satisfy hypotheses H_0:

$a_1(y) = \|y\|^{p-2}y$ with $1 < p < \infty$ (corresponds to the p-Laplacian),

$a_2(y) = \|y\|^{p-2}y + \mu\|y\|^{\tau-2}y$ with $2 \leq \tau < p < +\infty$

(corresponds to the (p, τ)-equations)

$a_3(y) = (1 + \|y\|^2)^{\frac{p-2}{2}} y$ with $p \geq 2$

(corresponds to the generalized mean curvature operator),

$a_4(y) = \|y\|^{p-2}y + ln(1 + \|y\|^{p-2})y$ with $p \geq 2$.

Now the hypotheses for $f(z, x)$ are the following:

$\underline{\mathrm{H}_1}$: $f : \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is Carathodory function such that $f(z, 0) = 0$ for a.e. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z) + c|x|^{r-1}$ for a.e. $z \in \Omega$, all $x \in \boldsymbol{R}$ with $a \in L^\infty(\Omega)_+$, $c > 0$ and $p < r < p^*$;

(ii) If $F(z, x) = \int_0^x f(z, s)ds$, then $\displaystyle\lim_{x\to\pm\infty} \frac{F(z, x)}{|x|^p} = +\infty$ uniformly for a.e. $z \in \Omega$;

(iii) there exist $\tau \in ((r - p)\max\{1, \frac{N}{p}\}, p^*)$, $\tau > q$ and $\beta_0 > 0$ such that

$$\beta_0 \leq \liminf_{x\to\pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^\tau} \quad \text{uniformly for a.e. } z \in \Omega\,;$$

(iv)

$$\lim_{x\to 0} \frac{f(z, x)}{|x|^{p-2}x} = 0 \ \text{ uniformly for a.e. } \ z \in \Omega;$$

(v) $f(z, x)x > 0$ for a.e. $z \in \Omega$, all $x \neq 0$ and for every $\rho > 0$, there exists $\gamma_\rho > 0$ such that for a.e. $z \in \Omega$, $x \to f(z, x) + \gamma_\rho|x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

REMARK 3.2. Hypothesis H_1(iii) implies that for a.e. $z \in \Omega$, the primitive $x \to F(z,x)$ is p-superlinear. Clearly H_1(iii) is satisfied if

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for a.e. } z \in \Omega\,,$$

i.e., for a.e. $z \in \Omega$, the nonlinearity $x \to f(z,x)$ is $(p-1)$-superlinear. We emphasize that we do not employ the usual in such cases Ambrosetti-Rabinowitz condition (AR-condition for short). We recall that the AR-condition says that there exist $\mu > p$ and $M > 0$ such that

$$(32)\quad 0 < \mu F(z,x) \le f(z,x)x \quad \text{for a.e. } z \in \Omega,\ \text{all } |x| \ge M \text{ and } \operatorname*{ess\,inf}_{\Omega} F(\cdot, \pm M) > 0\,.$$

Integrating (32), we obtain the following weaker condition

$$(33)\qquad c_7|x|^{\mu} \le F(z,x) \quad \text{for a.e. } z \in \Omega,\ \text{ all } |x| \ge M \text{ and some } c_7 > 0\,.$$

Evidently (33) implies the much weaker condition

$$(34)\qquad \lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} = +\infty \quad \text{uniformly for a.e. } z \in \Omega \text{ (see } H_1\text{(ii))}\,.$$

The AR-condition although quite natural and very helpful in verifying the PS-condition for the energy functional of the problem, is rather restrictive and excludes from consideration $(p-1)$-superlinear functions with "slow" growth near $\pm\infty$ (see (33)). For this reason there have been efforts to replace (32). A survey of the relevant literature, can be found in Miyagaki-Souto [27]. Here, instead of the AR-condition, we use H_1(iii), which covers new situations. Similar conditions were first used by Costa-Magalhaes [8] and Fei [13]. The second part of hypothesis H_1(v) is more general than assuming that for a.e. $z \in \Omega$, $x \to f(z,x)$ is nondecreasing.

Examples: The following functions satisfy hypotheses H_1 (for the sake of simplicity we drop the z dependence):

$$f_1(z) = |x|^{r-2}x \quad \text{where } p < r < p^*$$
$$f_2(z) = |x|^{p-2}x \log(1+|x|^p)\,.$$

Note that f_1 satisfies the AR-condition, but f_2 does not. For $\lambda > 0$, let $\varphi_\lambda^{\pm} : W_0^{1,p}(\Omega) \to \boldsymbol{R}$ be the C^1-functionals defined by

$$\varphi_\lambda^{\pm}(u) = \int_\Omega G(Du(z))dz - \frac{\lambda}{q}\|u^{\pm}\|_q^q - \int_\Omega F(z, \pm u^{\pm}(z))dz \ \text{ for all } u \in W_0^{1,p}(\Omega)\,.$$

PROPOSITION 3.3. *If hypotheses H_0 and H_1 hold and $\lambda > 0$ then $\varphi_\lambda^{\pm}$ satisfy the C-condition.*

PROOF. We do the proof for φ_λ^+, the proof for φ_λ^- being similar.

Let $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$(35)\qquad |\varphi_\lambda^+(u_n)| \le M_1 \quad \text{for some } M_1 > 0\,,\ \text{ all } n \ge 1\,,$$

$$(36)\qquad \text{and } (1+\|u_n\|)(\varphi_\lambda^+)'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty\,.$$

From (36) we have

$$|\langle(\varphi_\lambda^+)'(u_n), h\rangle| \le \frac{\varepsilon_n}{1+\|u_n\|}\|h\| \quad \text{for all } \ h \in W_0^{1,p}(\Omega) \text{ with } \varepsilon_n \downarrow 0$$

$$\Rightarrow |\langle V(u_n), h\rangle - \lambda \int_\Omega (u_n^+)^{q-1} h dz - \int_\Omega f(z, u_n^+) h dz|$$

$$\le \frac{\varepsilon_n}{1+\|u_n\|}\|h\| \quad \text{for all } \ n \ge 1 \,. \tag{37}$$

In (37) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ and have

$$\int_\Omega (a(-Du_n^-), -Du_n^-)_{\mathbb{R}^n} dz \le \varepsilon_n \quad \text{for all } \ n \ge 1$$

$$\Rightarrow c_4\|Du_n^-\|_p^p \le \varepsilon_n \quad \text{for all } \ n \ge 1 \text{ (see Lemma 2.2)}$$

$$\Rightarrow u_n^- \to 0 \quad \text{in } \ W_0^{1,p}(\Omega) \text{ as } n \to \infty \,. \tag{38}$$

From (35) we have

$$\int_\Omega pG(Du_n)dz - \frac{\lambda p}{q}\|u_n^+\|_q^q - \int_\Omega pF(z, u_n^+)dz \le pM_1 \quad \text{for all } \ n \ge 1 \,. \tag{39}$$

Also, if in (37) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$, then

$$-\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^n} dz + \lambda\|u_n^+\|_q^q + \int_\Omega f(z, u_n^+)u_n^+ dz \le \varepsilon_n \quad \text{for all } \ n \ge 1 \,. \tag{40}$$

Adding (39) and (40) and since $G \ge 0$ (see Corollary 2.3), we obtain

$$\int_\Omega (pG(Du_n^+) - (a(Du_n^+), Du_n^+)_{\mathbb{R}^n})dz + \int_\Omega (f(z, u_n^+)u_n^+ - pF(z, u_n^+))dz$$

$$\le M_2 + \lambda\left(\frac{p}{q} - 1\right)\|u_n^+\|_q^q \quad \text{for some } \ M_2 > 0, \quad \text{all } \ n \ge 1$$

$$\Rightarrow \int_\Omega (f(z, u_n^+)u_n^+ - pF(z, u_n^+))dz \le M_3 + \lambda\left(\frac{p}{q} - 1\right)\|u_n^+\|_q^q \tag{41}$$

for some $M_3 > 0$, all $n \ge 1$ (see H_0(iv)).

By virtue of hypotheses H_1(i), (iii), we can find $\beta_1 \in (0, \beta_0)$ and $c_8 > 0$ such that

$$\beta_1 x^\tau - c_8 \le f(z, x)x - pF(z, x) \quad \text{for a.e. } \ z \in \Omega, \ \text{ all } x \ge 0 \,. \tag{42}$$

If we use (42) in (41), we obtain

$$\beta_1\|u_n^+\|_\tau^\tau \le M_4 + \lambda\left(\frac{p}{q} - 1\right)\|u_n^+\|_q^q \quad \text{for some } M_4 > 0, \text{ all } n \ge 1$$

$$\Rightarrow \|u_n^+\|_\tau^\tau \le M_5 + c_9\|u_n^+\|_\tau^q$$

for some $M_4 > 0$, $c_9 > 0$ and all $n \ge 1$

$$\Rightarrow \{u_n^+\}_{n\ge 1} \subseteq L^\tau(\Omega) \ \text{ is bounded (since } q < \tau) \,. \tag{43}$$

It is clear that in hypothesis H_1(iii), we may assume that $\tau \leq r < p^*$. First suppose that $N \neq p$ and let $t \in [0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}.$$

By virtue of the interpolation inequality (see for example, Gasinski-Papageorgiou [17] (p. 905)), we have

$$\begin{aligned} &\|u_n^+\|_r \leq \|u_n^+\|_\tau^{1-t} \|u_n^+\|_{p^*}^t \\ &\Rightarrow \|u_n^+\|_r^r \leq M_{10}\|u_n^+\|^{tr} \quad \text{for some } M_{10} > 0, \text{ all } n \geq 1 \text{ (see (43))}. \end{aligned} \tag{44}$$

Hypothesis H_1(i) implies that

$$|f(z, x)x| \leq \widehat{a}(z) + \widehat{c}|x|^r \quad \text{for a.e. } z \in \Omega,\ \text{all } x \in \mathbb{R},\ \text{with } \widehat{a} \in L^\infty(\Omega)_+, \widehat{c} > 0. \tag{45}$$

In (37) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ and have

$$\int_\Omega (a(Du^+), Du^+)_{\mathbb{R}^n} dz - \lambda\|u_n^+\|_q^q - \int_\Omega f(z, u_n^+)u_n^+ dz \leq \varepsilon_n \text{ for all } n \geq 1$$

$$\begin{aligned} &\Rightarrow\ c_4\|Du_n^+\|_p^p \leq c_{10} + \lambda\|u_n^+\|_q^q + c_{11}\|u_n^+\|_r^r \\ &\qquad \text{for some } c_{10}, c_{11} > 0, \text{ all } n \geq 1 \text{ (see Lemma 2.2 and (45))} \end{aligned}$$

$$\Rightarrow\ \|u_n^+\|^p \leq c_{12}(1 + \lambda\|u_n^+\|^q + \|u_n^+\|^{tr}) \quad \text{for some } c_{12} > 0, \text{ all } n \geq 1 \text{ (see (44))}. \tag{46}$$

From the hypothesis on τ (see H_1(iii)) and an easy calculation, we infer that $tr < p$. So, from (46) it follows that

$$\begin{aligned} &\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \ \text{ is bounded} \\ &\Rightarrow \{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \ \text{ is bounded (see (38))}. \end{aligned} \tag{47}$$

If $N = p$, then by definition $p^* = +\infty$ and the Sobolev embedding theorem implies that $W_0^{1,p}(\Omega)$ is embedded compactly in $L^\eta(\Omega)$ for all $\eta \in [1, +\infty)$. Let $\tau \leq r < \eta$ and let $t \in [0, 1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\tau} \Rightarrow tr = \frac{\eta(r-\tau)}{\eta - \tau}.$$

Note that $\frac{\eta(r-\tau)}{\eta-\tau} \to r - \tau$ as $\eta \to +\infty = p^*$ and by hypothesis H_1(iii) $r - \tau < p$ (recall $N = p$). So, for $\eta > r$ large enough, we will have $tr < p$. Hence, if in the previous argument we replace p^* by such large $\eta \in (r, +\infty)$, then as above we reach (47). Because of (47) and by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \ \text{ in } \ W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^r(\Omega) \ \text{ as } \ n \to \infty. \tag{48}$$

In (37) we choose $h = u_n - u$, pass to the limit as $n \to \infty$ and use (48). Then

$$\begin{aligned} &\limsup_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0 \\ &\Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) \ \text{ (see Proposition 2.7)} \\ &\Rightarrow \varphi_\lambda^+ \ \text{ satisfies the C-condition}. \end{aligned}$$

A similar argument works for φ_λ^-. □

PROPOSITION 3.4. *If hypotheses* H_0 *and* H_1 *hold, then there exists* $\lambda^* > 0$ *such that to every* $\lambda \in (0, \lambda^*)$ *there corresponds* $\rho_\lambda^\pm > 0$ *for which*

$$\inf[\varphi_\lambda^\pm(u); \|u\| = \rho_\lambda^\pm] = \eta_\lambda^\pm > 0\,.$$

PROOF. We do the proof for φ_λ^+, the proof for φ_λ^- being similar. By virtue of hypotheses H_1(i), (iv), given $\varepsilon > 0$, we can find $c_{13} = c_{13}(\varepsilon) > 0$ such that

$$\begin{aligned} & f(z,x) \le \varepsilon x^{p-1} + c_{13}x^{r-1} \quad \text{for a.e. } z \in \Omega,\ \text{all } x \ge 0 \\ & \Rightarrow F(z,x) \le \frac{\varepsilon}{p}x^p + \frac{c_{13}}{r}x^r \quad \text{for a.e. } z \in \Omega,\ \text{all } x \ge 0\,. \end{aligned} \tag{49}$$

Then for $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \varphi_\lambda^+(u) &= \int_\Omega G(Du)dz - \frac{\lambda}{q}\|u^+\|_q^q - \int_\Omega F(z,u^+)dz \\ &\ge c_5\|Du\|_p^p - \frac{\lambda}{q}\|u\|_q^q - \frac{\varepsilon}{p}\|u\|_p^p - \frac{c_{13}}{r}\|u\|_r^r \ \text{(see Corollary 2.3 and (44))} \\ &\ge \left(c_5 - \frac{\varepsilon}{\widehat{\lambda}_1 p}\right)\|u\|^p - c_{14}(\lambda\|u\|^q + \|u\|^r) \quad \text{for some } c_{14} > 0\,. \end{aligned}$$

Choosing $\varepsilon \in (0, \widehat{\lambda}_1 pc_5)$, we infer that

$$\begin{aligned} & \varphi_\lambda^+(u) \ge c_{15}\|u\|^p - c_{14}(\lambda\|u\|^q + \|u\|^r) \quad \text{for some } c_{15} > 0, \text{all } u \in W_0^{1,p}(\Omega) \\ & \Rightarrow \varphi_\lambda^+(u) \ge (c_{15} - c_{14}(\lambda\|u\|^{q-p} + \|u\|^{r-p}))\|u\|^p \quad \text{for all } u \in W_0^{1,p}(\Omega)\,. \end{aligned} \tag{50}$$

Consider the function

$$\xi(t) = \lambda t^{q-p} + t^{r-p} \quad \text{for all } t > 0\,. \tag{51}$$

Evidently $\xi(\cdot)$ is continuous on $(0, +\infty)$ and since $q < p < r$, we see that

$$\xi(t) \to +\infty \ \text{ as } \ t \to 0^+ \quad \text{and as } \ t \to +\infty\,.$$

So, we can find $t_0 \in (0, +\infty)$ such that

$$\begin{aligned} & \xi(t_0) = \min[\xi(\mathrm{t}); \mathrm{t} > 0] > 0 \\ & \Rightarrow \xi'(t_0) = (q-p)\lambda t_0^{q-p-1} + (r-p)t_0^{r-p-1} = 0 \\ & \Rightarrow \lambda(p-q) = (r-p)t_0^{r-q} \\ & \Rightarrow t_0 = t_0(\lambda) = \left[\frac{\lambda(p-q)}{r-p}\right]^{\frac{1}{r-q}}\,. \end{aligned}$$

Evidently $t_0(\lambda) \to 0^+$ as $\lambda \to 0^+$ and so from (51) it is clear that we can find $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ we have $\xi(t_0) < c_{15}/c_{14}$. Hence from (49) we infer that

$$\inf[\varphi_\lambda^+(u); \|u\| = \rho_\lambda^+] = \eta_\lambda^+ > 0\,.$$

A similar argument works for the functional φ_λ^-. □

PROPOSITION 3.5. *If hypotheses* H_0 *and* H_1 *hold and* $u \in C_+\backslash\{0\}$ *with* $\|u\|_p = 1$, *then* $\varphi_\lambda^\pm(tu) \to -\infty$ *as* $t \to \pm\infty$.

PROOF. Again we do the proof for φ_λ^+, the proof for φ_λ^- being similar. By virtue of hypotheses H_1(i), (ii), given $\mu > 0$, we can find $c_{16} = c_{16}(\mu) > 0$ such that

$$F(z,x) \geq \mu x^p - c_{16} \quad \text{for a.e. } z \in \Omega\,, \quad \text{all } x \geq 0\,. \tag{52}$$

Then for $u \in C_+\backslash\{0\}$ with $\|u\|_p = 1$ and $t > 0$, we have

$$\begin{aligned}\varphi_\lambda^+(tu) &= \int_\Omega G(D(tu))dz - \frac{\lambda t^q}{q}\|u\|_q^q - \int_\Omega F(z,tu)dz\\ &\leq c_{17}(1 + t^p\|u\|^p) - \mu t^p \quad \text{(see Corollary 2.3 and (52))}\\ &= t^p(c_{17}\|u\|^p - \mu) + c_{17}\,. \end{aligned}\tag{53}$$

Choose $\mu > c_{17}\|u\|^p$. Then, from (53) it is clear that $\varphi_\lambda^+(tu) \to -\infty$ as $t \to +\infty$.
A similar argument works for the functional φ_λ^-. □

Now we are ready to produce the constant sign smooth solutions of problem (1), $\lambda \in (0, \lambda^*)$.

PROPOSITION 3.6. *If hypotheses* H_0 *and* H_1 *hold and* $\lambda \in (0, \lambda^*)$ *(*$\lambda^* > 0$ *as in Proposition* 3.4), *then problem* (1) *has at least four nontrivial smooth solutions of constant sign*

$$\begin{aligned}&u_0, \widehat{u} \in \operatorname{int} C_+\,, \quad u_0 \leq \widehat{u}\,, \quad u_0 \neq \widehat{u}\\ &\textit{and}\ \ v_0, \widehat{v} \in -\operatorname{int} C_+\,, \quad \widehat{v} \leq v_0\,, \quad \widehat{v} \neq v_0\,.\end{aligned}$$

PROOF. First we establish the existence of the two positive smooth solutions. Propositions 3.3, 3.4 and 3.5, permit the application of Theorem 2.1 (the mountain pass theorem). So, we obtain $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda^+(0) = 0 < \eta_\lambda^+ \leq \varphi_\lambda^+(u_0) \tag{54}$$

$$\text{and} \ \ (\varphi_\lambda^+)'(u_0) = 0\,. \tag{55}$$

From (54) we see that $u_0 \neq 0$. From (55) we have

$$V(u_0) = \lambda(u_0^+)^{q-1} + N_f(u_0^+) \quad \text{where} \ \ N_f(u)(\cdot) = f(\cdot, u(\cdot)) \ \ \text{for all} \ \ u \in W_0^{1,p}(\Omega)\,. \tag{56}$$

Acting on (56) with $-u_0^- \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned}&c_4\|Du_0^-\|_p^p \leq 0 \ \ \text{(see Lemma 2.2)}\\ &\Rightarrow u_0 \geq 0,\ u_0 \neq 0\,.\end{aligned}$$

So (56) becomes

$$\begin{aligned}&V(u_0) = \lambda u_0^{q-1} + N_f(u_0)\\ &\Rightarrow -\operatorname{div} a(Du_0(z)) = \lambda u_0(z)^{q-1} + f(z, u_0(z)) \ \ \text{a.e. in} \ \ \Omega,\ u_0|_{\partial\Omega} = 0\,.\end{aligned}$$

Hence, u_0 is a nontrivial solution of (1) and the nonlinear regularity theory (see [24] and [25]) implies that $u_0 \in C_+\backslash\{0\}$. We have

$$-\mathrm{div}\, a(Du_0(z)) = \lambda u_0(z)^{p-1} + f(z, u_0(z)) \geq 0 \ \text{ a.e. in } \Omega$$
$$\Rightarrow u_0 \in \mathrm{int}\, C_+ \ \text{(see Montenegro [28], Theorem 6).}$$

Next consider the following truncation of the reaction

$$h_\lambda^+(z,x) = \begin{cases} \lambda u_0(z)^{q-1} + f(z, u_0(z)) & \text{if } x \leq u_0(z) \\ \lambda x^{q-1} + f(z,x) & \text{if } u_0(z) < x\,. \end{cases} \tag{57}$$

This is a Caratheodory function. Let $H_\lambda^+(z,x) = \int_\Omega h_\lambda^+(z,s)ds$ and consider the C^1-functional $\psi_\lambda^+ : W_0^{1,p}(\Omega) \to \boldsymbol{R}$ defined by

$$\psi_\lambda^+(u) = \int_\Omega G(Du(z))dz - \int_\Omega H_\lambda^+(z,u(z))dz \quad \text{for all } \ u \in W_0^{1,p}(\Omega)\,.$$

CLAIM 3.7. *u_0 is a local minimizer of the functional ψ_λ^+.*

PROOF. Let $\tilde{u} \in K_{\psi_\lambda^+}$. Then

$$V(\tilde{u}) = N_{h_\lambda^+}(u)(\cdot) \text{ where } N_{h_\lambda^+}(\tilde{u}) = h_\lambda^+(\cdot, u(\cdot)) \quad \text{for all } \ u \in W_0^{1,p}(\Omega)\,. \tag{58}$$

On (58) we act with $(u_0 - \tilde{u})^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} \langle V(\tilde{u}), (u_0-\tilde{u})^+\rangle &= \int_\Omega h_\lambda^+(z,\tilde{u})(u_0-\tilde{u})^+dz \\ &= \int_\Omega (\lambda u_0^{q-1} + f(z,u_0))(u_0-\tilde{u})^+dz \ \ \text{(see (57))} \\ &= \langle V(u_0), (u_0-\tilde{u})^+\rangle \end{aligned}$$

$$\Rightarrow \int_{\{u_0>\tilde{u}\}} (a(Du_0) - a(D\tilde{u}), Du_0 - D\tilde{u})_{\boldsymbol{R}^n} dz = 0$$
$$\Rightarrow u_0 \leq \tilde{u} \ \ \text{(since } a(\cdot) \text{ is strictly monotone, see Lemma 2.2)}\,.$$

Let $\theta \in (\lambda, \lambda^*)$ and let $\overline{u}_0 \in \mathrm{int}\, C_+$ be a solution of problem (1) with the parameter being θ obtained as above via the use of the mountain pass theorem (see Theorem 2.1). We have

$$-\mathrm{div}\, a(D\overline{u}_0(z)) = \theta\overline{u}_0(z) + f(z,\overline{u}_0(z)) > \lambda\overline{u}_0(z) + f(z,\overline{u}_0(z)) \ \text{ a.e. in } \Omega\,. \tag{59}$$

Therefore $\overline{u}_0 \in \mathrm{int}\, C_+$ is an upper solution for (1) with the parameter being λ. Then by truncating at $\overline{u}_0(z)$ and using the direct method, we can produce a solution of (1) (with the parameter being λ), which is less than or equal to $\overline{u}_0$. So, we may assume $u_0 \leq \overline{u}_0$.

Let $[u_0) = \{u \in W_0^{1,p}(\Omega);\, u_0(z) \leq u(z) \ \text{ a.e. in } \ \Omega\}$ and $[0, \overline{u}_0] = \{u \in W_0^{1,p}(\Omega);\, 0 \leq u(z) \leq \overline{u}_0(z) \ \text{ a.e. in } \ \Omega\}$. We have just seen that $K_{\psi_\lambda^+} \subseteq [u_0)$. Also we may assume that $K_{\psi_{\lambda+}} \cap [0, \overline{u}_0] = \{u_0\}$ (otherwise we already have a second smooth positive solution as it is evident from (57)). We consider the following truncation of $h_\lambda^+(z,\cdot)$:

$$\widehat{h}_\lambda^+(z,x)=\begin{cases} h_\lambda^+(z,u_0(z)) & \text{if } x<u_0(z)\\ h_\lambda^+(z,x) & \text{if } u_0(z)\le x\le \overline{u}_0(z)\\ h_\lambda^+(z,\overline{u}_0(z)) & \text{if } \overline{u}_0(z)<x\,.\end{cases} \tag{60}$$

This is a Caratheodory function. We set $\widehat{H}_\lambda^+(z,x)=\int_0^x \widehat{h}_\lambda^+(z,s)ds$ and consider the C^1-functional $\widehat{\psi}_\lambda^+ : W_0^{1,p}(\Omega)\to \boldsymbol{R}$ defined by

$$\widehat{\psi}_\lambda^+(u)=\int_\Omega G(Du(z))dz-\int_\Omega \widehat{H}_\lambda^+(z,u(z))dz \quad \text{for all } u\in W_0^{1,p}(\Omega)\,.$$

It is clear from (60) and Corollary 2.3 that $\widehat{\psi}_\lambda^+$ is coercive. Also exploiting the compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, we can easily check that $\widehat{\psi}_\lambda^+$ is sequentially weakly lower semicontinuous. So, from the Weierstrass theorem, we can find $\widehat{u}_0\in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\widehat{\psi}_\lambda^+(\widehat{u}_0)=\inf[\widehat{\psi}_\lambda^+(u); u\in W_0^{1,p}(\Omega)]\\
&\Rightarrow (\widehat{\psi}_\lambda^+)'(\widehat{u}_0)=0\\
&\Rightarrow V(\widehat{u}_0)=N_{\widehat{h}_\lambda^+}(\widehat{u}_0),\quad \text{where } N_{\widehat{h}_\lambda^+}(u)(\cdot)=\widehat{h}_\lambda^+(\cdot,u(\cdot)) \text{ for all } u\in W_0^{1,p}(\Omega)\,.
\end{aligned} \tag{61}$$

On (61) we act with $(u_0-\widehat{u}_0)^+\in W_0^{1,p}(\Omega)$. Using (60) and (57), as before we obtain $u_0\le \widehat{u}_0$. Also, on (61) we act with $(\widehat{u}_0-\overline{u}_0)^+\in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
\langle V(\widehat{u}_0),(\widehat{u}_0-\overline{u}_0)^+\rangle &=\int_\Omega \widehat{h}_\lambda^+(z,\widehat{u}_0)(\widehat{u}_0-\overline{u}_0)^+dz\\
&=\int_\Omega (\lambda\overline{u}_0^{q-1}+f(z,\overline{u}_0))(\widehat{u}_0-\overline{u}_0)^+dz \ \text{ (see (60) and (57))}\\
&<\int_\Omega (\theta\overline{u}_0^{q-1}+f(z,\overline{u}_0))(\widehat{u}_0-\overline{u}_0)^+dz \ \text{ since } \lambda<\theta\\
&=\langle V(\overline{u}_0),(\widehat{u}_0-\overline{u}_0)^+\rangle \ \text{ (see (59))},
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \int_{\{\widehat{u}_0>\overline{u}_0\}}(a(D\widehat{u}_0)-a(D\overline{u}_0),D\widehat{u}_0-D\overline{u}_0)_{\boldsymbol{R}^n}dz<0\\
&\Rightarrow \widehat{u}_0\le\overline{u}_0 \text{ (by virtue of the strict monotonicity of } a(\cdot)\text{, see Lemma 2.2)}\,.
\end{aligned}$$

So, $\widehat{u}_0\in[u_0,\overline{u}_0]$ and (61) becomes

$$\begin{aligned}
&V(\widehat{u}_0)=N_{h_\lambda^+}(\widehat{u}_0) \ \text{ (see (60))}\\
&\Rightarrow \widehat{u}_0\in K_{\psi_\lambda^+}\cap[0,\widehat{u}_0]\\
&\Rightarrow \widehat{u}_0=u_0\,.
\end{aligned}$$

Let $\rho=\|\overline{u}_0\|_\infty$ and let $\gamma_\rho>0$ be as postulated by hypothesis H_1(v). We have

$$-\operatorname{div} a(Du_0(z))+\gamma_\rho u_0(z)^{p-1}=\lambda u_0(z)^{q-1}+f(z,u_0(z))+\gamma_\rho u_0(z)^{p-1}$$

$$
\begin{aligned}
&< \theta u_0(z)^{q-1} + f(z, u_0(z)) + \gamma_\rho u_0(z)^{p-1} \quad (\text{since } \lambda < \theta)\\
&\le \theta \overline{u}_0(z)^{q-1} + f(z, \overline{u}_0(z)) + \gamma_\rho \overline{u}_0(z)^{p-1} \quad (\text{sence } u_0 \le \overline{u}_0, \text{ see } \mathrm{H}_1(\mathrm{v}))\\
&= -\mathrm{div}\, a(D\overline{u}_0(z)) + \gamma_\rho \overline{u}_0(z)^{p-1} \quad \text{a.e. in } \Omega\,.
\end{aligned}
$$

Invoking Proposition 2.4, we infer that $\overline{u}_0 - u_0 \in \operatorname{int} C_+$. Since $\psi_\lambda^+|_{[0,\overline{u}_0]} = \widehat{\psi}_\lambda^+|_{[0,\overline{u}_0]}$ and $u_0 \in \operatorname{int} C_+$, $\overline{u}_0 - u_0 \in \operatorname{int} C_+$, we infer that u_0 is a local $C_0^1(\overline{\Omega})$ minimizer of ψ_λ^+. Invoking Proposition 2.6, we have that u_0 is a local $W_0^{1,p}(\Omega)$ minimizer of ψ_λ^+. This proves the Claim. □

We may assume that u_0 is isolated in $K_{\psi_\lambda^+}$ (otherwise and since $K_{\psi_\lambda^+} \subseteq [u_0)$, we have a whole sequence of positive smooth solutions of problem (1), see (57)). Then reasoning as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 29) we can find $\rho \in (0,1)$ small such that

$$\psi_\lambda^+(u_0) < \inf[\psi_\lambda^+(u);\ \|u - u_0\| = \rho] = \tilde{\eta}_\lambda^+\,. \tag{62}$$

Let $u \in C_+\backslash\{0\}$, $\|u\|_p = 1$. As in the proof of Proposition 3.4, using hypothesis H_1(ii) and (57) we can show that

$$\psi_\lambda^+(tu) = -\infty \text{ as } t \to +\infty\,. \tag{63}$$

Finally, a slight modification of the proof of Proposition 3.3, reveals that ψ_λ^+ satisfies the C-condition. This fact together with (62) and (63), permit the use of Theorem 2.1 (the mountain pass theorem). So, we obtain $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda^+(u_0) < \tilde{\eta}_\lambda^+ \le \widehat{\psi}_\lambda^+(\widehat{u}) \quad (\text{see } (62)) \tag{64}$$

$$\text{and} \quad (\psi_\lambda^+)'(\widehat{u}) = 0\,. \tag{65}$$

From (64) we have that $u_0 \ne \widehat{u}$. From (65) and the nonlinear regularity theory we have $\widehat{u} \in K_{\psi_\lambda^+} \subseteq [u_0) \cap \operatorname{int} C_+$. Hence $\widehat{u} \in \operatorname{int} C_+$ solves problem (1) and $u_0 \le \widehat{u}$, $u_0 \ne \widehat{u}$.

Similarly, working with φ_λ^-, we obtain two nontrivial negative solutions of (1), $\widehat{v}, v_0 \in -\operatorname{int} C+$, with $\widehat{v} \le v_0$, $\widehat{v} \ne v_0$. □

4. Nodal solutions. In this section we look for nodal solutions (i.e., sign changing) solutions. We start by considering the following auxiliary problem

$$-\mathrm{div}\, a(Du(z)) = \widehat{f}_0(z, u(z)) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0\,. \tag{66}$$

We strengthen hypotheses H_0 as follows:

$\underline{\mathrm{H}_0'}$: Hypotheses H_0 hold and

(v) if $G_0(t) = \int_0^t a_0(s)s\,ds$, then there exists $\tau \in (q,p)$ such that $t \to G_0(t^{\frac{1}{\tau}})$ is convex on $(0,+\infty)$ (note that $G(y) = G_0(\|y\|)$ for all $y \in \boldsymbol{R}^n$).

The hypotheses on $\widehat{f}_0(z,x)$ are the following:

$\underline{\mathrm{H}_2}$: $\widehat{f}_0 : \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is Caratheodory function such that $\widehat{f}_0(z,0) = 0$ for a.e. in Ω and

(i) $|\widehat{f}_0(z,x)| \le a(z) + c|x|^{r-1}$ for a.e. in Ω, for all $x \in \boldsymbol{R}$ with $a \in L^\infty(\Omega)_+$, $c > 0$ and $1 < r < p^*$;

(ii) with $q \in (1, p)$ as in $\mathrm{H}_0'(v)$, for a.e. $z \in \Omega$ $x \to \widehat{f_0}(z,x)/x^{q-1}$ is strictly decreasing on $(0, +\infty)$, $x \to \widehat{f_0}(z,x)/|x|^{q-2}x$ is strictly increasing on $(-\infty, 0)$ and $\widehat{f_0}(z,x)x \geq -\widehat{c}|x|^\theta$ for all $x \in \boldsymbol{R}$, with $\widehat{c} > 0, \theta \geq p$.

The next result establishes the uniqueness of solutions of constant sign (when they exist) and extends an analogous result of Diaz-Saa [11], where the differential operator is p-Laplacian.

PROPOSITION 4.1. *If hypotheses* H_0' *and* H_2 *hold, then problem* (65) *has at most one nontrivial positive solution and at most one nontrivial negative solution.*

PROOF. We show the uniqueness of the nontrivial positive solution (when it exists) the proof for the uniqueness of the nontrivial negative solution being similar.

Let $\xi : L^1(\Omega) \to \overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{+\infty\}$ be the integral functional defined by

$$\xi(u) = \begin{cases} \displaystyle\int_\Omega G(Du^{\frac{1}{\tau}})dz & \text{if } u \geq 0,\ u^{\frac{1}{\tau}} \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise}. \end{cases}$$

Then ξ is convex (see H_0'(v) and Diaz-Saa [11]) and lower semicontinuous (by Fatou's lemma).

Suppose u is a nontrivial positive solution of (66). As before, the nonlinear regularity theory (see [24], [25]) and the nonlinear maximum principle (see [32] and hypothesis H_2(ii)), imply $u \in \operatorname{int} C_+$. Note that $u^\tau \geq 0$ and $(u^\tau)^{\frac{1}{\tau}} = u \in W_0^{1,p}(\Omega)$. So u^τ is in the effective domain of the $\overline{\boldsymbol{R}}$-valued functional ξ. Let $h \in C_0^1(\overline{\Omega})$ and $\lambda > 0$ small. Then $u^\tau + \lambda h \in C_+$ and so the Gateuax derivative of ξ at u^τ in the direction h exists. Moreover, using the chain rule we have

$$\xi'(u^\tau)(h) = \int_\Omega \frac{-\operatorname{div} a(Du)}{u^{\tau-1}} h dz. \tag{67}$$

Let y be another nontrivial positive solution of (66). Again we have $y \in \operatorname{int} C_+$. By virtue of the convexity of $\xi(\cdot)$ and using (67), we have

$$\begin{aligned} &\int_\Omega \left(\frac{-\text{div } a(Du)}{u^\tau - 1} + \frac{\text{div } a(Dy)}{y^\tau - 1} \right)(u^\tau - y^\tau)dz \geq 0 \\ &\Rightarrow 0 \geq \int_\Omega \left(\frac{\widehat{f_0}(z,u)}{u^{\tau-1}} - \frac{\widehat{f_0}(z,y)}{y^{\tau-1}} \right)(u^\tau - y^\tau)dz \geq 0 \ \ \text{(see (66) and } \mathrm{H}_2\text{(ii))} \\ &\Rightarrow u = y \ \ \text{(see } \mathrm{H}_2\text{(ii))}. \end{aligned}$$

Similarly we can obtain the uniqueness of the nontrivial negative solution. □

Now, let $\widehat{f_0}(z,x) = |x|^{q-2}x$ and consider the following auxiliary problem:

$$-\text{div } a(Du(z)) = \lambda |u(z)|^{q-2}u(z) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0, \quad \lambda > 0. \tag{68}$$

PROPOSITION 4.2. *If hypotheses* H_0' *hold,* $1 < q < p$ *and* $\lambda > 0$, *then problem* (68) *has a unique nontrivial positive solution* $\tilde{u}_+^\lambda \in \operatorname{int} C_+$ *and a unique nontrivial negative solution* $-\tilde{u}_+^\lambda \in -\operatorname{int} C_+$.

PROOF. We do the proof for $\tilde{u}_+^\lambda$, and from this follows the uniqueness of a negative solution. Let $\theta_\lambda^+ : W_0^{1,p}(\Omega) \to \boldsymbol{R}$ be the C^1-functional defined by

$$\theta_\lambda^+(u) = \int_\Omega G(Du(z))dz - \frac{\lambda}{q}\|u^+\|_q^q \quad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

Corollary 2.3 and the fact that $q < p$, imply that θ_λ^+ is coercive. Also θ_λ^+ is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_+^\lambda \in W_0^{1,p}(\Omega)$ such that

$$\theta_\lambda^+(\tilde{u}_+^\lambda) = \inf[\theta_\lambda^+(u); u \in W_0^{1,p}(\Omega)] = \tilde{m}_\lambda^+\,. \tag{69}$$

If $u \in \operatorname{int} C_+$ and $t \in (0,1)$ is small, then since $q < p$ we have

$$\begin{aligned} &\theta_\lambda^+(tu) < 0 \\ \Rightarrow\ &\theta_\lambda^+(\tilde{u}_+^\lambda) = \tilde{m}_\lambda^+ < 0 = \theta_\lambda^+(0) \\ \Rightarrow\ &\tilde{u}_+^\lambda \neq 0\,. \end{aligned}$$

From (69) we have

$$\begin{aligned} &(\theta_\lambda^+)'(\tilde{u}_+^\lambda) = 0 \\ \Rightarrow\ &V(\tilde{u}_+^\lambda) = \lambda((\tilde{u}_+^\lambda)^+)^{q-1}\,. \end{aligned} \tag{70}$$

On (70) we act with $-(\tilde{u}_+^\lambda)^- \in W_0^{1,p}(\Omega)$ and via Lemma 2.2 we obtain $\tilde{u}_+^\lambda \geq 0$, $\tilde{u}_+^\lambda \neq 0$. Then (70) becomes

$$\begin{aligned} &V(\tilde{u}_+^\lambda) = \lambda(\tilde{u}_+^\lambda)^{q-1} \\ \Rightarrow\ &\tilde{u}_+^\lambda \in \operatorname{int} C_+ \text{ (see [25], [28]) solves (68)}\,. \end{aligned}$$

The uniqueness of $\tilde{u}_+^\lambda$ follows from Proposition 4.1 (recall that $\tau > q$).

Similarly we have $-\tilde{u}_+^\lambda \in -\operatorname{int} C_+$. □

Using this proposition, we can establish the existence of extremal constant sign smooth solutions for (1) ($\lambda \in (0,\lambda^*)$), i.e., we show that there exist a smallest nontrivial positive solution and a biggest nontrivial negative solution for (1) ($\lambda \in (0,\lambda^*)$, where $\lambda^* > 0$ is as in Proposition 3.4).

PROPOSITION 4.3. *If hypotheses* H_0' *and* H_1 *hold and* $\lambda \in (0,\lambda^*)$ ($\lambda^* > 0$ *as in Proposition* 3.4), *then problem* (1) *has a smallest nontrivial positive solution* $u_*^\lambda \in \operatorname{int} C_+$ *and a biggest nontrivial negative solution* $v_*^\lambda \in -\operatorname{int} C_+$.

PROOF. We do the proof for u_*^λ, the proof for v_*^λ being similar.

First we show that if u is a nontrivial positive solution of (1) ($\lambda \in (0,\lambda^*)$), then $u \geq \tilde{u}_+^\lambda$. To this end, first note that $u \in \operatorname{int} C_+$ (as before) and

$$-\operatorname{div} a(Du(z)) = \lambda u(z)^{q-1} + f(z,u(z)) \geq \lambda u(z)^{q-1} \ \text{ a.e. in } \Omega \ \text{ (see } \mathrm{H}_1(\mathrm{v}))\,. \tag{71}$$

We consider the following function

$$\gamma_\lambda^+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda x^{q-1} & \text{if } 0 \le x \le u(z) \\ \lambda u(z)^{q-1} & \text{if } u(z) < x\,. \end{cases} \tag{72}$$

This is a Caratheodory function. Let $\Gamma_\lambda^+(z,x) = \int_0^x \gamma_\lambda^+(z,s)ds$ and consider the C^1-functional $\sigma_\lambda^+ : W_0^{1,p}(\Omega) \to \boldsymbol{R}$ defined by

$$\sigma_\lambda^+(u) = \int_\Omega G(Du(z))dz - \int_\Omega \Gamma_\lambda^+(z,u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

It is clear from Lemma 2.2 and (72), that σ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\lambda^+(\tilde{u}) = \inf[\sigma_\lambda^+(u); u \in W_0^{1,p}(\Omega)] = \widehat{m}_\lambda^+\,. \tag{73}$$

As before, since $q < p$, we have

$$\begin{aligned} &\sigma_\lambda^+(\tilde{u}) = \widehat{m}_\lambda^+ < 0 = \sigma_\lambda^+(0) \\ &\Rightarrow \tilde{u} \neq 0\,. \end{aligned}$$

From (73) we have

$$\begin{aligned} &(\sigma_\lambda^+)'(\tilde{u}) = 0 \\ &\Rightarrow V(\tilde{u}) = N_{\gamma_\lambda^+}(\tilde{u}) \end{aligned} \tag{74}$$

$$\text{where } N_{\gamma_\lambda^+}(u)(\cdot) = \gamma_\lambda^+(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

On (74) we act with $-\tilde{u}^- \in W_0^{1,p}(\Omega)$ and obtain $\tilde{u} \ge 0$ and in fact $\tilde{u} \in \operatorname{int} C_+$ by nonlinear regularity theory and the nonlinear maximum principle (see [24] and [25]). Also acting on (74) with $(\tilde{u}-u)^+ \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned} \langle V(\tilde{u}), (\tilde{u}-u)^+\rangle &= \int_\Omega \gamma_\lambda^+(z,\tilde{u})(\tilde{u}-u)^+dz \\ &= \int_\Omega \lambda u^{q-1}(\tilde{u}-u)^+dz \ \ \text{(see (72))} \\ &\le \langle V(u), (\tilde{u}-u)^+\rangle \ \ \text{(see (71))} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \int_{\{\tilde{u}>u\}} (a(D\tilde{u}) - a(Du), D\tilde{u} - Du)_{\boldsymbol{R}^n} dz \le 0 \\ &\Rightarrow \tilde{u} \le u \ \text{(recall that } a(\cdot) \text{ is strictly monotone, see Lemma 2.2)}\,. \end{aligned}$$

Then $\tilde{u} \in [0,u] \cap \operatorname{int} C_+$ and so (74) becomes $V(\tilde{u}) = \lambda \tilde{u}^{q-1}$, hence $\tilde{u} = \tilde{u}_+^\lambda$ by virtue of Proposition 4.2. Therefore $\tilde{u}_+^\lambda \le u$.

We introduce the set

$$S_+(\lambda) = \{\lambda > 0;\ \text{problem (1) has a nontrivial positive solution}\}\,.$$

From Proposition 3.6, we know that $S_+(\lambda) \neq \emptyset$. Also, $S_+(\lambda) \subseteq [\tilde{u}_+^\lambda) \cap \operatorname{int} C_+$ where $[\tilde{u}_+^\lambda) = \{u \in W_0^{1,p}(\Omega);\ \tilde{u}_+^\lambda(z) \leq u(z) \text{ a.e. in } \Omega\}$. Moreover, as in Filippakis-Kristaly-Papageorgiou [14] we can check that $S_+(\lambda)$ is downward directed (i.e., if $u_1, u_2 \in S_+(\lambda)$, we can find $u \in S_+(\lambda)$ such that $u \leq \min\{u_1, u_2\}$). Let $C \subseteq S_+(\lambda)$ be a chain (i.e., a totally ordered subset of $S_+(\lambda)$). From Dunford-Schwartz [12] (p. 336), we know that we can find $\{u_n\}_{n\geq 1} \subseteq C$ such that

$$\inf_{n\geq 1} u_n = \inf C\,.$$

Evidently, we may assume that $u_n \leq u_0$ for all $n \geq 1$, with $u_0 \in \operatorname{int} C_+$ from Proposition 3.6. We have

$$\begin{aligned} & V(u_n) = \lambda u_n^{q-1} + N_f(u_n) \text{ for all } n \geq 1 && (75)\\ \Rightarrow\ & \{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded}. \end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} \overline{u} \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to \overline{u} \text{ in } L^p(\Omega) \text{ as } n \to \infty\,. \tag{76}$$

On (75) we act with $u_n - \overline{u} \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (76). We obtain

$$\begin{aligned} & \lim \langle V(u_n), u_n - \overline{v}\rangle = 0 \\ \Rightarrow\ & u_n \to \overline{u} \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see Proposition 2.7)}. && (77) \end{aligned}$$

From the first part of the proof we have $\tilde{u}_+^\lambda \leq \overline{u}$. Also, passing to the limit as $n \to \infty$ in (75) and using (77), we obtain

$$\begin{aligned} & V(\overline{u}) = \lambda \overline{u}^{q-1} + N_f(\overline{u}) \\ \Rightarrow\ & \overline{u} \in S_+(\lambda) \cap \operatorname{int} C_+ \quad \text{and} \quad \overline{u} = \inf C\,. \end{aligned}$$

Since C is an arbitrary chain in $S_+(\lambda)$, from the Kuratowski-Zorn lemma, we infer that $S_+(\lambda)$ admits a minimal element $u_*^\lambda \geq \tilde{u}_+^\lambda$. Since $S_+(\lambda)$ is downward directed, it follows that u_*^λ is the smallest nontrivial positive solution of problem (1). Similarly, we produce $v_*^\lambda \in -\operatorname{int} C_+$, $v_*^\lambda \leq -\tilde{u}_+^\lambda$ the biggest nontrivial negative solution of (1). □

Having these two extremal constant sign smooth solutions, we can now implement the strategy of Dancer-Du [10] (see also [14]). Namely, we focus on the order interval $[v_*^\lambda, u_*^\lambda] = \{u \in W_0^{1,p}(\Omega);\ v_*^\lambda(z) \leq u(z) \leq u_*^\lambda(z) \text{ a.e. in } \Omega\}$. Using suitable truncation and variational techniques coupled with Morse Theory (critical groups), we show that problem (1) has a solution $y_0 \in [v_*^\lambda, u_*^\lambda] \cap C_0^1(\overline{\Omega})$, $y_0 \neq 0$. The extremality of v_*^λ, u_*^λ, implies that y must be nodal.

PROPOSITION 4.4. *If hypotheses* H_0' *and* H_1 *hold and* $\lambda \in (0, \lambda^*)$, *then problem* (1) *has a nodal solution* $y_0 \in C_0^1(\overline{\Omega})$.

PROOF. Let $u_*^\lambda \in \operatorname{int} C_+$ and $v_*^\lambda \in -\operatorname{int} C_+$ be the two extremal constant sign solutions of (1) produced in Proposition 4.3. We introduce the following truncation of the reaction:

$$\widehat{\beta}_\lambda(z,x) = \begin{cases} \lambda|v_*^\lambda(z)|^{q-2}v_*^\lambda(z) + f(z,v_*^\lambda(z)) & \text{if } x < v_*^\lambda(z) \\ \lambda|x|^{q-2}x + f(z,x) & \text{if } v_*^\lambda(z) \le x \le u_*^\lambda(z) \\ \lambda u_*^\lambda(z)^{q-1} + f(z,u_*^\lambda(z)) & \text{if } u_*^\lambda(z) < x\,. \end{cases} \tag{78}$$

This is a Caratheodory function. We set $\widehat{\mathrm{B}}_\lambda(z,x) = \int_0^x \widehat{\beta}_\lambda(z,s)ds$ and consider the C^1-functional $j_\lambda : W_0^{1,p}(\Omega) \to \boldsymbol{R}$ defined by

$$j_\lambda(u) = \int_\Omega G(Du(z))dz - \int_\Omega \widehat{\mathrm{B}}_\lambda(z,u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

Also set $\widehat{\beta}_\lambda^\pm(z,x) = \widehat{\beta}_\lambda(z,\pm x^\pm)$, $\widehat{\mathrm{B}}_\lambda^\pm(z,x) = \int_0^x \widehat{\beta}_\lambda^\pm(z,s)ds$ and consider the C^1-functionals $j_\lambda^\pm : W_0^{1,p}(\Omega) \to \boldsymbol{R}$ defined by

$$j_\lambda^\pm(u) = \int_\Omega G(Du(z))dz - \int_\Omega \widehat{\mathrm{B}}_\lambda^\pm(z,u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

Reasoning as in the proof of Proposition 4.3, we can show that

$$K_{j_\lambda} \subseteq [v_*^\lambda, u_*^\lambda]\,, \quad K_{j_\lambda^+} \subseteq [0, u_*^\lambda]\,, \quad K_{j_\lambda^-} \subseteq [v_*^\lambda, 0] \ \text{(see (78))}\,.$$

Taking into account the extremality of the solutions u_*^λ and v_*^λ (see Proposition 4.3), we have

$$K_{j_\lambda} \subseteq [v_*^\lambda, u_*^\lambda]\,, \quad K_{j_\lambda^+} = \{0, u_*^\lambda\}\,, \quad K_{j_\lambda^-} = \{v_*^\lambda, 0\}\,. \tag{79}$$

CLAIM 4.5. *$u_*^\lambda \in \operatorname{int} C_+$ and $v_*^\lambda \in -\operatorname{int} C_+$ are local minimizers of j_λ.*

PROOF. Evidently j_λ^+ is coercive and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0^\lambda \in W_0^{1,p}(\Omega)$ such that

$$j_\lambda^+(\tilde{u}_0^\lambda) = \inf[j_\lambda^+(u)\,;\ u \in W_0^{1,p}(\Omega)] = \overline{m}_\lambda^+\,. \tag{80}$$

As before, the presence of the "concave" term λx^{q-1}, $x \in [0, u_*^\lambda(z)]$, implies that

$$\begin{aligned} &j_\lambda^+(\tilde{u}_0^\lambda) = \overline{m}_+^\lambda < 0 = j_\lambda^+(0) \ \text{(see (80))}\,, \\ &\tilde{u}_0^\lambda \neq 0 \ \text{ and so } \ \tilde{u}_0^\lambda = u_*^\lambda \ \text{(see (80))}\,. \end{aligned}$$

Note that $j_\lambda|_{W_+} = j_\lambda^+|_{W_+}$ $W_+ = \{u \in W_0^{1,p}(\Omega);\, u(z) \ge 0 \ \text{a.e. in } \Omega\}$ and recall that $u_*^\lambda \in \operatorname{int} C_+$ (see Proposition 4.3). So, it follows that u_*^λ is a local $C_0^1(\overline{\Omega})$-minimizer of j_λ. Invoking Proposition 2.6 we have that u_*^λ is a local $W_0^{1,p}(\Omega)$-minimizer of j_λ. Similarly for $v_*^\lambda \in -\operatorname{int} C_+$ using this time the functional j_λ^-. This proves the claim. □

We may assume that $j_\lambda(v_*^\lambda) \le j_\lambda(u_*^\lambda)$ (the analysis is similar if the opposite inequality holds). Also because of the claim and reasoning as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0,1)$ small such that

$$j_\lambda(v_*^\lambda) \le j_\lambda(u_*^\lambda) < \inf[j_\lambda(u);\, \|u - u_*^\lambda\| = \rho] = \eta_\lambda,\ \|v_*^\lambda - u_*^\lambda\| > \rho\,. \tag{81}$$

From Lemma 2.2 and (78), it is clear that j_λ is coercive, hence it satisfies the PS-condition. This fact together with (81) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in K_{j_\lambda} \backslash \{v_*^\lambda, u_*^\lambda\} \subseteq [v_*^\lambda, u_*^\lambda] \backslash \{v_*^\lambda, u_*^\lambda\}$ (see (79) and (81)). Then $y_0 \in C_0^1(\overline{\Omega})$ and solves (1). Also, since y_0 is a critical point of j_λ of mountain pass type, we have

$$C_1(j_\lambda, y_0) \neq 0 \ \text{(see Chang [7] (p. 89))}. \tag{82}$$

On the other hand, from Jiu-Su [22] (Proposition 2.4), we know that

$$C_k(j_\lambda, 0) = 0 \quad \text{for all } ;\, k \geq 0 \ \text{(due to the "concave" term)}. \tag{83}$$

Comparing (82) and (83), we conclude that $y_0 \neq 0$. Therefore $y_0 \in C_0^1(\overline{\Omega})$ is a nodal solution of (1) ($\lambda \in (0, \lambda^*)$). □

Summarizing the situation for problem (1), we have the following multiplicity theorem with precise sign information for all solutions.

THEOREM 4.6. *If hypotheses* H_0' *and* H_1 *hold, then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$ *problem* (1) *has at least five nontrivial smooth solutions*

$$u_0, \widehat{u} \in \operatorname{int} C_+, \quad u_0 \leq \widehat{u}, \quad u_0 \neq \widehat{u}$$
$$v_0, \widehat{v} \in -\operatorname{int} C_+, \quad \widehat{v} \leq v_0, \quad \widehat{v} \neq v_0$$
$$\textit{and} \quad y_0 \in C_0^1(\overline{\Omega}) \ \textit{nodal}.$$

REMARK 4.7. It is interesting to know if the above theorem is still valid for differential operator satisfying hypotheses H. What is missing, is a nonlinear maximum principle for such operators.

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