Tohoku Math. J.
66 (2014), 555–562

LIMIT LINEAR SERIES FOR VECTOR BUNDLES

MONTSERRAT TEIXIDOR I BIGAS

(Received March 26, 2013, revised September 12, 2013)

Abstract. We provide a construction of limit linear series for families of curves and justify dimension bounds. We show how to extend the sections of a vector bundle to twists by line bundles and to elementary transformations.

Introduction. Most results on Brill-Noether Theory for line bundles have been proved by degeneration arguments using singular curves. Among the techniques used, perhaps the simplest and most efficient is the one on limit linear series for reducible curves introduced by Eisenbud and Harris (see [EH]). A similar theory for vector bundles was introduced in [T1] and used later in several other contexts [T2], [T3], [T4], [T5] (see [T6] for an overview). The purpose of this note is to present the constructions that were implicit in those papers and justify that the expected dimensions behave as in the case of a non-singular curve. Moreover, we study the behavior of limit linear series under elementary transformations, a situation that does not arise for rank 1.

Moduli spaces of torsion-free sheaves on reducible curves can be constructed up to the choice of a polarization. A universal Poincare sheaf does not necessarily exist, but it exists on a suitable finite cover of these moduli spaces. As we are trying to estimate the dimension of the limit linear series near a given sheaf, we can work locally and we will assume that we have Poincare sheaves. Up to blowing up some of the nodes of the curve and making base changes, we can assume that we are dealing with a vector bundle rather than just a torsion-free sheaf which will also simplify our work.

A limit linear series on a reducible curve gives a vector bundle of high degree on each component and gluing at the nodes. Using one of these vector bundles and suitable twists of the others as well as the gluing, one obtains a vector bundle of the right degree on the whole curve with a space of sections concentrated on the chosen component.

For rank one, Osserman showed that a limit linear series determines spaces of sections for line bundles on the reducible curve that have positive degree on each of the components, not just the ones that have all of the degree on one component and degree zero on the others. We show that his result generalizes to the higher rank vector bundle as well.

In addition to taking twists at the nodes, for rank higher than one, there is a new phenomenon that does not have a separate interpretation in the case of line bundles. One can construct elementary transformations on the vector bundle centered at one of the nodes. This

2010 *Mathematics Subject Classification.* Primary 14H60.
Key words and phrases. Vector bundles, limit linear series.

allows to modify the vector bundles so that the degree distribution is as desired, rather than just modify it by multiples of the rank, as is the case when we tensor with line bundles . We will see that the sections can be preserved in a natural way in the vector bundles obtained by such elementary transformations.

1. Construction of a space of limit linear series and computation of its dimension. We start by setting up the notations that we will be using. We work over an algebraically closed field . Let X be a connected nodal curve of arithmetic genus g with irreducible components $C_t, t = 1, \dots, l$. We glue the point P_j, $j = 1, \dots, s$ on the component $C_{t(j)}$ with the point P'_j on the component $C_{t'(j)}$ to form a node P^j.

The dual graph of X is obtained by taking one vertex for each irreducible component of X and one edge for each node. The edge corresponding to the node P^j joins the vertices corresponding to $C_{t(j)}$ and $C_{t'(j)}$. The curve X is said to be of compact type if X is connected and its dual graph has no loops. The latter condition is equivalent to the Jacobian of the curve being compact. Note that for a curve of compact type, the number of nodes is one fewer than the number of components (that is, $s = l - 1$).

The slope $\mu(E)$ of a vector bundle E on an irreducible non-singular curve is defined as the quotient of the degree by the rank. A vector bundle is semistable (resp. stable) if for each subbundle F, $\mu(F) \leq (<)\mu(E)$. Denoting by $\chi(E)$ the Euler Poincaré characteristic of the vector bundle, this condition can also be written as

$$\frac{\chi(F)}{\text{rank}(F)} \leq (<)\frac{\chi(E)}{\text{rank}(E)} .$$

In order to define semistability and stability for a vector bundle on a reducible nodal curve X, one first chooses a polarization. For our purposes, this is the choice of a weight w_i on each irreducible component so that $0 < w_i < 1, \sum w_i = 1$. A vector bundle of rank r on X is semistable (stable) if for every torsion-free subsheaf F

$$\frac{\chi(F)}{\sum w_i \text{rank}(F_{C_i})} \leq (<)\mu(E) .$$

One can then construct moduli spaces of torsion-free sheaves that are (semi)stable for a given polarization (see [S]).

We need the concept of vanishing sequence. Given a vector bundle E of rank r on a non-singular curve C, a space V of dimension k of sections of E and a point P on C, let $V(-aP)$ denote the subspace of sections of V that vanish at P with multiplicity at least a. The vanishing sequence $a_1 \geq \cdots \geq a_k$ of V at P is the sequence of integers satisfying that a_i is the largest integer such that $\dim V(-a_i P) \geq i$. A fixed integer may appear among the terms in a given vanishing sequence at most r times.

We recall the definition of a limit linear series of rank r degree d and dimension k. The dimension here is the number of independent sections. A limit linear series is obtained from the following data:

- A vector bundle E_t, $t = 1, \dots, l$ on C_t of rank r and degree d_t.

- For each node, gluing isomorphism φ_j of the projectivization of the fibers $E_{t(j),P_j}$ and $E_{t'(j),P'_j}$.
- For every irreducible component C_t of X, spaces of sections $V_t \subset H^0(C_t, E_t)$ of dimension k.
- For every node obtained by gluing $P_j \in C_{t(j)}$ with $P'_j \in C_{t'(j)}$, basis $s_i^j,\ s_i'^j,\ i = 1, \dots, k$ of $V_{t(j)},\ V_{t'(j)}$ respectively.
- A positive integer a

The following conditions must be satisfied:

- $\sum_{t=1}^{l} d_t - ar(l-1) = d$.
- The sections of $H^0(C_{t(j)}, E_{t(j)}(-aP_j)), H^0(C_{t'(j)}, E_{t'(j)}(-aP'_j))$ are completely determined by their value at the points $P_j,\ P'_j$ that glue to form a node.
- The orders of vanishing of the sections at the nodes satisfy $o_{P_j}(s_i^j)+o_{P'_j}(s'^j_{k-i+1}) \geq a$ and s_i^j glues with $s'^j_{k-i+1},\ i = 1, \dots, k$ for every node obtained gluing P_j, P'_j.

We now construct the scheme of limit linear series (see [EH] for the case of rank one).

Let B be an irreducible curve, $\pi : \mathcal{X} \to B$ a flat, proper map such that all the fibers of π but one are non-singular curves and the special fiber X is a curve of compact type as described above. Assume also that there exists a relatively ample divisor D on $\mathcal{X}$ that restricts on each component of X to a divisor of sufficiently large degree. Fix an integer d and a polarization on X. Fix $U = U(r, d)$ a cover of the moduli space of vector bundles of rank r and degree d on the family that are stable by a fixed polarization and such that a universal bundle exists locally. Let $\mathcal{E}$ on $\mathcal{X} \times U(r, d)$ be the universal bundle. If $\mathcal{O}(\sum b_t C_t)$ is a line bundle with support on some of the components of X, then $\mathcal{E} \otimes \mathcal{O}(\sum b_t C_t)$ has degree distributed in a different way on the special fiber.

Assume that $C_t,\ C_{t'}$ intersect at P. Denote with C' the connected component of $X - C_t$ that contains $C_{t'}$ and C the connected component of $X - C_{t'}$ that contains C_t. Then $C + C' = \sum C_t$ is a fiber of the map $\mathcal{X} \to B$ over a point b. Up to replacing B by an open set, we can assume that $\mathcal{O}_B(b) \cong \mathcal{O}_B$. Therefore, $\mathcal{O}_{\mathcal{X}}(C + C') \cong \mathcal{O}_{\mathcal{X}}$. Choose a section $\tau_{t,P}$ that vanishes with multiplicity one on C and a section $\tau_{t',P}$ that vanishes with multiplicity one on C'.

For each component C_t, choose one twist $\mathcal{E}_t$ of the form $\mathcal{O}(\sum b_t C_t)$ so that the degree of $(\mathcal{E}_t)_{|C_i}$ is sufficiently negative for $i \neq t$ and with $\mathcal{E}_{t'} = \mathcal{E}_t(\pi_1^*(\mathcal{O}_{\mathcal{X}}(-aC')))$ for a fixed a. We will write the latter more simply as $\mathcal{E}_t(-aC')$. The choice is possible as

$$\mathcal{E}_{t'} = \mathcal{E}_t(\pi_1^*(\mathcal{O}_{\mathcal{X}}(-aC'))) = \mathcal{E}_{t'}(\pi_1^*(\mathcal{O}_{\mathcal{X}}(-aC' - aC)))$$

and by the isomorphism $\mathcal{O}_{\mathcal{X}}(C + C') \cong \mathcal{O}_{\mathcal{X}}$, the sheaves at the two ends of this chain can be identified.

Let d_D, be the degree of the restriction of D to X. For each t, $\pi_{2*}(\mathcal{E}_t(\pi_1^*(\mathcal{O}(D)))$ is a vector bundle on U of rank $d + rd_D + r(1 - g)$. Let G be the Grassmannian of k planes of $\pi_{2*}(\mathcal{E}_{t_0}(\pi_1^*(\mathcal{O}(D))))$ for a fixed t_0. Using $\tau_{t_0,P},\ \tau_{t'_0,P}$, we identify $\pi_{2*}(\mathcal{E}_{t_0}(\pi_1^*(\mathcal{O}(D))))$ with $\pi_{2*}(\mathcal{E}_{t'_0}(\pi_1^*(\mathcal{O}(D))))$. By the connectedness of X, this allows us to identify G with the Grassmannian G_t of k planes of $\pi_{2*}(\mathcal{E}_t(\pi_1^*(\mathcal{O}(D))))$ for any index t corresponding to a component of X. Therefore, a point in the grassmaniann defines also a space of sections of $\pi_{2*}(\mathcal{E}_t(\pi_1^*(\mathcal{O}(D))))$ for each t. As the dimension of the moduli space of vector bundles is $\dim B + r^2(g-1)+1$, the dimension of G is $\dim B + r^2(g-1)+1+k(d+rd_D+r(1-g)-k)$.

For each node of X obtained by gluing $P_j \in C_{t(j)}$ with $P'_j \in C_{t'(j)}$, let $F_{t,j}$ be the bundle of projective frames (that is, bases for the k-planes defined up to product with a scalar multiple). Once a point in G is fixed, the choice of such a k-frame depends on $k(k-1)$ parameters. Using the identification between G and the Grassmanian G_t of $\pi_{2*}(\mathcal{E}_t(\pi_1^*(\mathcal{O}(D))))$, we will think of $F_{t,j}$ as a frame for a point V_t in G_t.

Let $u : F \to G$ be the product of all the $F_{t,j} \to G$ fibered over U. Then the dimension of F is

$$\dim B + r^2(g-1) + 1 + k(d + r(1-g+d_D) - k) + \sum_{j,t} k(k-1)\,. \tag{1}$$

Denote by V_t the universal subbundle of G_t and by $\sigma_1^{t,j}, \dots, \sigma_k^{t,j}$ the universal basis frame.

For each component C_t and each node on the component corresponding to the index j, fix vanishing sequences $a_1^j, \dots, a_k^j$ such that if $P_j \in C_{t(j)}$ and $P'_j \in C_{t'(j)}$ are identified to produce the node, then $a_i^j + a'^j_{k-i+1} = a$.

The scheme of limit linear series with these fixed vanishing can be obtained as a subscheme of F by imposing the following conditions:

1. The composition of the maps below is zero

$$V \to \pi_{2*}(\mathcal{E}_{t_0}(\pi_1^*(\mathcal{O}(D))) \to \pi_{2*}(\mathcal{E}_{t_0}(\pi_1^*(\mathcal{O}_D(D))))$$

(that is V represents sections of $\pi_{2*}(\mathcal{E}_{t_0})$). Given the identification of G with G_t, the space of sections V_t corresponding to V on the component C_t also vanish at D and therefore give rise to sections of $\pi_{2*}(\mathcal{E}_t)$ for each t.

2. For each node $P_j \in C_{t(j)} \cap C_{t'(j)}$,

$$\sigma_i^j \tau_{t,j}^{a'^j_{k-i+1}} = \sigma'^j_{k-i+1} \tau_{t'j}^{a_i^j}$$

where the identity is understood in V_t.

Note that V is a vector bundle of rank k and $\mathcal{O}_D(D)$ is a syscraper sheaf of rank d_D. Therefore, vanishing condition 1 is given locally by krd_D equations.

Vanishing conditions 2 are equations on the projective bundle V_t of (projective) fiber dimension $k-1$. There are k such equations for every node. So they account for a total of $k(k-1)$ conditions.

We can now give a lower bound for the dimension of each component of the scheme of limit linear series.

We computed the dimension of F in (1). Subtracting the conditions from (1) and (2), every component of the locus of limit linear series has dimension at least

$$\dim B + r^2(g-1) + 1 + (d + r(1-g+d_D) - k)$$
$$+2\sum_j k(k-1) - rkd_D - \sum_j k(k-1)$$
$$= \dim B + r^2(g-1) + 1 - k(k-d+r(g-1))$$
$$+\sum_j k(k-1) = \dim B + \rho + \sum_j k(k-1)\,.$$

The choice of a projective frame at the fiber at one point of a vector bundle of rank k depends on at most $k(k-1)$ parameters. Given a vanishing and a direction at the fiber of a point, there is at most one section (up to constant) with this vanishing at that point and gluing with this given direction. Therefore, when we assume that the orders of vanishing of gluing sections add up to precisely a, once we choose a frame on one component containing a given node, the frame on the other component that glues at the same node is also determined. In particular, when we forget the frames, the dimension goes down by at most $\sum_j k(k-1)$. This implies that any component of the scheme of limit linear series has dimension at least $\dim B + \rho$.

We proved the following:

THEOREM 1.1. *Each component of the scheme of limit linear series of rank r degree d and dimension k on a family with base B has dimension at least* $\dim B + \rho$, *where ρ is the Brill-Noether number $\rho = r^2(g-1) + 1 - k(k-d+r(g-1))$.*

2. Limit linear series and sections of vector bundles of varying degree of restrictions to components. The construction in this paragraph shows that a refined limit linear series of rank r and dimension k on a reducible curve determines also vector spaces of sections on vector bundles on the reducible curve when the degree is not necessarily concentrated in just one component but rather distributed among them. As one would then expect, the sections are non zero in more than one component . The rank one case was dealt with in [O]. For simplicity, we assume that X_0 has only two components. We also assume that all the inequalities in the last condition for a limit linear series are equalities (and we call such a series refined to conform with the notations for rank one).

PROPOSITION 2.1. *Let X_0 be a curve with two components C, C' glued at points $P \in C$, $P' \in C'$. Given a refined limit linear series on X_0 of degree d rank r and dimension k with restrictions to C, C' corresponding to vector bundles E, E', there exists a natural space of sections of dimension k of the vector space obtained from $E(-tP)$, $E'(tP')$, $0 \le t \le a$ with gluing corresponding to the given gluing of the limit series.*

PROOF. We define vector bundles $\mathcal{E}_t$ of degree d on X_0 with restrictions $\bar{E}^t$, $\bar{E}'^t$ of degree $d - tr, d' + tr, 0 \le t \le a$ on C, C' respectively given by $\bar{E}^t = E(-tP)$, $\bar{E}'^t = E'(-(a-t)P')$ glued with the gluing canonically deduced from the given gluing. Consider the spaces of sections of these vector bundles on X_0 given by $\bar{V}^t = V(-tP)$, $\bar{V}'^t = V'(-(a-t)P')$. Note that if s_j, $j = 1, \dots, l(t+1)$ vanish at P to order at least $t+1$, s_j, $j = l(t+1)+1, \dots, l(t)$ vanish at P to order precisely t and s_j, $j = l(t)+1, \dots, k$ vanish at P

to order at most $t-1$, then s'_j, $j=1,\dots,l(t+1)$ vanish at P' to order at most $a-t-1$, s'_j, $j=l(t+1)+1,\dots,l(t)$ vanish at P' to order precisely $a-t$ and s'_j, $j=l(t)+1,\dots,k$ vanish at P' to order at least $a-t+1$. Moreover s_j, $j=l(t+1)+1,\dots,l(t)$ glue with s'_j, $j=l(t+1)+1,\dots,l(t)$. Therefore, we can form sections of $\mathcal{E}_t$ by gluing s_j, $j=1,\dots,l(t+1)$ with the zero section in C', s_j, $j=l(t+1)+1,\dots,l(t)$ with s'_j, $j=l(t+1)+1,\dots,l(t)$ on C' and s'_j, $j=l(t)+1,\dots,k$ with zero on C. □

3. Limit linear series and elementary transformations. The construction in the previous section changes the degree of a vector bundle in multiples of the rank. This is all one needs in rank one. For higher rank though, one may want to also consider elementary transformations that change the vector bundle at just one point and so that the fiber at this point gets modified only in one direction. We will start with the vector bundles $\bar{E}^t$, $\bar{E}'^t$ of the previous section.

PROPOSITION 3.1. *Let X_0 be a curve with two components C, C' glued at points $P\in C$, $P'\in C'$. Given a limit linear series on X_0 of degree d rank r and dimension k with restrictions to C, C' corresponding to vector bundles E, E' of degrees $d-tr, d-(a-t)r$, there exists a natural space of sections of dimension k of the vector bundle obtained from $\hat{E}, \hat{E}'$ where $\hat{E}, \hat{E}'$ are a direct and inverse elementary transformation of $\bar{E}^t$ and $\bar{E}'^t$ respectively compatible with the given gluing of the limit series (see below) and the new gluing is obtained from it.*

PROOF. An elementary transformation $\hat{E}$ of E is the kernel of a projection map

$$0\to\hat{E}\to\bar{E}^t\to \boldsymbol{C}_P\to 0$$

where $\boldsymbol{C}_P$ is a skyscraper sheaf, that is a sheaf supported at P whose fiber at P is one-dimensional.

Similarly, an inverse elementary transformation $\hat{E}'$ of E' is a sheaf that fits in an exact sequence

$$0\to\bar{E}'^t\to\hat{E}'\to \boldsymbol{C}_{P'}\to 0\,.$$

Assume that $\hat{E}$ and $\hat{E}'$ are glued at the point P, P' by an isomorphism φ' (of the projectivization of the fibers $\hat{E}_P$ and $\hat{E}'_{P'}$) satisfying the following: Consider the maps $g:\ \hat{E}\to\bar{E}^t$, $g':\ \hat{E}'\to\bar{E}'^t$. Let W be a subspace of dimension $r-1$ of the fiber at P of $\hat{E}$ such that the restriction of g_P to W is one to one. Then, the restriction to W of the composition $g'_{P'}\varphi\, g_P$ on W should coincide with φ'. Let us check that one can obtain a k-dimensional space of sections of the bundle $\hat{E}_0$ obtained by gluing $\hat{E}$, $\hat{E}'$ using φ'.

Define $\hat{V}$ to be the space of sections of $\bar{V}=V(-tP)$ that map to zero through the map $\bar{E}\to\mathbf{C}_P$ and therefore give rise to sections of $\hat{E}$.

Consider the exact diagram

$$\begin{array}{ccccccccc}
0 & \to & \bar{E}' & \to & \hat{E}' & \to & \mathbf{C}_{P'} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \bar{E}'(P') & \to & \bar{E}'(P') & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
\boldsymbol{C}_{P'} & \to & (\bar{E}'(P'))_{P'} & \to & (\boldsymbol{C}_{P'})^{r-1} & \to & 0\,. & &
\end{array}$$

Define $\hat{V}'$ to be the space of sections of $\bar{V}' = V'(-(a-t)P')$ that map to zero through the map $\bar{E}'(P') \to (\boldsymbol{C}_{P'})^{r-1}$ and therefore give rise to sections of $\hat{E}'$.

We need to show first that the sections of $\hat{V}$, $\hat{V}'$ that do not vanish at P, P' respectively, glue with each other and therefore give rise to sections of $\hat{E}_0$. We also need to see that if we add to these the sections of $\hat{V}$ that vanish at P glued with zero on C' and the sections of $\hat{V}'$ that vanish at P' glued with zero on C, the dimension of the resulting space of sections is k.

We use the notations of the previous section, that is, we write

$$s_j,\ j = l(i+1)+1, \dots, l(i)$$

for the sections in V that vanish at P to order exactly i. We write

$$s'_j,\ j = l(i+1)+1, \dots, l(i)$$

that vanish at P' to order exactly $a-i$ and glue with the previous ones.

The sections of E consist of s_j, $j = 1, \dots, l(t+1)$ glued with zero on C', s_j, $j = l(t+1)+1, \dots, l(t)$ glued with s'_j, $j = l(t+1)+1, \dots, l(t)$ on C' and s'_j, $j = l(t)+1, \dots, k$ glued with zero on C.

Up to a change of basis, we can assume either that

(a) s_j, $j = l(t+1)+1, \dots, l(t)-1$ map to zero on $\boldsymbol{C}_P$ and $s_{l(t)}$ does not or that

(b) s_j, $j = l(t+1)+1, \dots, l(t)$ all map to zero on $\boldsymbol{C}_P$.

Similarly, either

(a′) s_j, $j = l(t+2)+1, \dots, l(t+1)-1$ map to zero on $\boldsymbol{C}_P$ and $s_{l(t+1)}$ does not or that

(b′) s_j, $j = l(t+2)+1, \dots, l(t+1)$ all map to zero on $\boldsymbol{C}_P$.

In case (a), $s_{l(t)}$ is not a section of $\hat{V}$ while s_j, $j = l(t+1)+1, \dots, l(t)-1$ are. The section $s'_{l(t)}$ is a section of $\bar{E}'$ and therefore of $\hat{E}'$ and as a section of $\hat{E}'$ it vanishes at P'. Therefore, it can be glued with zero on C.

Similarly, in case (a′), $s_{l(t+1)}$ gives rise to a section of $\hat{E}$ that is non-zero at P and $s'_{l(t+1)}$ is a section of $\hat{E}'$ that is non- zero at P' and they glue with each other.

In cases (b) and (b′) fewer changes need to be made. The remaining sections essentially carry over from E to $\hat{E}$. □

REFERENCES

[EH] D. EISENBUD AND J. HARRIS, Limit linear series, basic theory, Invent. Math 85 (1986), 337–371.

[O] B. OSSERMAN, A limit linear series moduli scheme, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 4, 1165–1205.

[S] C. S. SESHADRI, Fibrés vectoriels sur les courbes algébriques, Astérisque 96. Société Mathématique de France, Paris, 1982. 209 pp.

[T1] M. TEIXIDOR, Brill-Noether theory for stable vector bundles, Duke Math. J. 62 (1991), 385–400.

[T2] M. TEIXIDOR, Rank two vector bundles with canonical determinant, Math. Nachr. 265 (2004), 100–106.

[T3] M. TEIXIDOR, Petri map for rank two bundles with canonical determinant, Compos. Math. 144 (2008), no. 3, 705–720.

[T4] M. TEIXIDOR, Existence of coherent systems II, Internat. J. Math. 19 (2008), no. 10, 1269–1283.

[T5] M. TEIXIDOR, Existence of vector bundles of rank two with fixed determinant and sections, Proc. Japan Acad. Ser. A Math. Sci. 86 (2010), no. 7, 113–118.

[T6] M. TEIXIDOR, Vector bundles on reducible curves and applications in "Grassmannians, Moduli Spaces and Vector Buncles", Clay Mathematics Proceedings 14 (2011), 169–180.

DEPARTMENT OF MATHEMATICS
TUFTS UNIVERSITY
MEDFORD MA 02155
U.S.A.

E-mail address: mteixido@tufts.edu