

DANILOV'S RESOLUTION AND REPRESENTATIONS OF THE MCKAY QUIVER

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Abstract. We construct a family of McKay quiver representations on the Danilov resolution of the $\frac{1}{r}(1, a, r - a)$ singularity. This allows us to show that the resolution is the normalization of the coherent component of the fine moduli space of θ -stable McKay quiver representations for a suitable stability condition θ . We describe explicitly the corresponding union of chambers of stability conditions for any coprime numbers r, a .

1. Introduction. In [12] King introduced the notion of stability of a quiver representation and via GIT constructed the fine moduli of stable representations. A well-known example of King's moduli is the G -Hilbert scheme for any finite group $G \subset \mathrm{GL}(n, \mathbf{C})$, i.e., the scheme parameterizing all G -invariant 0-dimensional subschemes of \mathbf{C}^n of length equal to the order of G . It is the moduli of representations of the McKay quiver, defined by the inclusion of G in the general linear group and representation theory of G . The G -Hilbert scheme was introduced by Ito and Nakamura in [9], where they proved that it is the minimal (crepant) resolution of \mathbf{C}^2/G for a finite group $G \subset \mathrm{SL}(2, \mathbf{C})$, and the relation of the G -Hilbert scheme with the moduli of representations of the McKay quiver was observed by Ito and Nakajima in [8]. The result from [9] was extended by Bridgeland, King and Reid [1] by showing that the G -Hilbert scheme is a crepant resolution of the singularity \mathbf{C}^3/G for any finite group $G \subset \mathrm{SL}(3, \mathbf{C})$. In particular, the moduli is smooth and irreducible. Moreover, if G is abelian, it turns out that by varying the stability parameter one can get all projective crepant resolutions of \mathbf{C}^3/G (cf. [2]). The above results suggest that it may be possible to interpret some other resolutions of quotient singularities as moduli of the McKay quiver representations.

In the following paper we accomplish this task for the Danilov resolution of 3-dimensional cyclic terminal quotient singularities. The proof relies on Logvinenko's classification of all natural families (called *gnat-families*) of McKay quiver representations on some fixed resolution Y of \mathbf{C}^n/G for any finite abelian group $G \subset \mathrm{GL}(n, \mathbf{C})$ and his characterization of such families as satisfying *the reductor condition* [13]. The most obvious candidate to be isomorphic to the resolution is the coherent component defined by Craw, MacLagan and Thomas as the irreducible component of stable moduli of the McKay quiver representations containing points parameterizing free orbits. In [3], they prove that, for a finite abelian group

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G , it is a possibly non-normal toric variety which admits projective birational morphism to \mathbb{C}^n/G .

Let $G = \langle \text{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a}) \rangle$ be a cyclic group of order r generated by a diagonal matrix, where $\varepsilon = e^{2\pi i/r}$ for fixed coprime numbers a, r such with $r > 1$. The quotient singularity $X = \mathbb{C}^3/G$ is a 3-dimensional cyclic, terminal quotient singularity (cf. [14]). We call this the singularity of type $\frac{1}{r}(1, a, r-a)$. In his proof of the weak factorization theorem for toric threefolds, Danilov [4] introduced a recursively defined resolution of the singularity of type $\frac{1}{r}(1, a, r-a)$, which was subsequently named *Economic resolution* by Reid [15, p. 381].

For $a = \pm 1$, Kędzierski [11] proved that the Danilov resolution is isomorphic to the component of the G -Hilbert scheme that contains the free G -orbits. This cannot hold for $a \neq \pm 1$ as the G -Hilbert scheme is singular. Our main result Theorem 8.2 establishes that, for $a \neq \pm 1$, the Danilov resolution is isomorphic to the normalization of the coherent component of the moduli space of representations of the McKay quiver for a suitably chosen stability parameter. In fact, we describe precisely the appropriate union of GIT chambers of stability conditions. Finally, we conjecture that the coherent component for such stability conditions is normal.

The main idea is to define *a priori* a family of McKay quiver representations on the Danilov resolution and to use the universal property of the moduli space. To show that the resulting map is injective, we prove that any two representations in the given family are non-isomorphic. This is done by exploiting the recursive nature of the resolution. On the level of representations, the recursive step can be seen by grouping the vertices of the McKay quiver in so-called L-bricks and R-bricks. Those bricks can be seen as vertices of smaller McKay quivers.

The paper is organized as follows. In Section 2, we recall the definition of the Danilov resolution. In Section 3, we define effective divisors X_i, Y_i, Z_i and \mathcal{Q} -divisors R_i which will be used in constructing a family of McKay quiver representations on the Danilov resolution. The McKay quiver is defined in Section 4. The family of quiver representations is constructed in Section 5 and we check that any two representations in that family are non-isomorphic. In Section 6, we recall elementary facts on stability of quiver representations. In Section 7, we determine a cone of stability conditions for the constructed family. Finally, in Section 8 we prove the main theorem, and compute explicitly the union of chambers of stability conditions for $\frac{1}{5}(1, 2, 3)$.

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2. Recursive definition of the Danilov resolution. We denote $N_0 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ the free \mathbb{Z} -module with the basis $\{e_1, e_2, e_3\}$, and $M_0 = \text{Hom}_{\mathbb{Z}}(N_0, \mathbb{Z}) = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^* \oplus \mathbb{Z}e_3^*$ with the dual basis $\{e_1^*, e_2^*, e_3^*\}$. The vector $a_1e_1 + a_2e_2 + a_3e_3$ is also denoted by (a_1, a_2, a_3) .

For coprime integers $0 \leq a < r$, we denote by $G(r, a)$ the finite cyclic subgroup of $\text{GL}(3, \mathbf{C})$ generated by the diagonal matrix $\text{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a})$, where $\varepsilon = e^{2\pi i/r}$.

We set

$$N(r, a) = N_0 + \mathbf{Z} \frac{1}{r}(e_1 + ae_2 + (r - a)e_3) = N_0 + \mathbf{Z}(1/r, a/r, (r - a)/r)$$

and $M(r, a) = \text{Hom}_{\mathbf{Z}}(N(r, a), \mathbf{Z})$. The lattice $M(r, a)$ is identified with the sublattice of M_0 consisting of the exponents of $G(r, a)$ -invariant Laurent monomials.

For any elements p_1, \dots, p_n in $N(r, a)$, we denote by $\langle p_1, \dots, p_n \rangle$ the cone generated by these points. For a rational polyhedral cone σ in $N(r, a) \otimes_{\mathbf{Z}} \mathbf{R}$, we denote by U_σ the toric chart $\text{Spec}(\mathbf{C}[\sigma^\vee \cap M(r, a)])$.

For a positive integer t and an integer s , we denote by $\langle s \rangle_t$ the least non-negative integer u such that t divides $s - u$. Sometimes, we write it simply $\langle s \rangle$ when t is obvious.

Let r and a be coprime integers with $0 \leq a < r$. We recall the definition of the Danilov resolution of the singularity $\frac{1}{r}(1, a, r - a)$ (cf. [15, p. 381]). Let $\Delta(r, a) = \langle e_1, e_2, e_3 \rangle$ be the positive octant in $N(r, a) \otimes_{\mathbf{Z}} \mathbf{R}$. There exists a ring isomorphism of $\mathbf{C}[\Delta(r, a)^\vee \cap M(r, a)]$ with the ring of $G(r, a)$ -invariant regular functions on \mathbf{C}^3 . Therefore the quotient $X(r, a) = \mathbf{C}^3/G(r, a)$ is the toric variety given by the cone $\Delta(r, a)$ in the real vector space with underlying lattice $N(r, a)$. Note that $X(1, 0)$ is a smooth variety and for $r \geq 2$ the quotient $X(r, a)$ is singular.

Let b be the inverse of a modulo r . Set

$$p_i = \frac{1}{r}(\langle -ib \rangle_r, r - i, i) \in N(r, a) \quad \text{for } i = 0, \dots, r.$$

Note that $p_{r-a} = 1/r(1, a, r - a)$, which represents the type of the singularity and $p_0 = e_2, p_r = e_3$. The following well-known lemma implies that the toric varieties associated to cones $\langle e_1, e_2, p_{r-a} \rangle, \langle e_1, e_3, p_{r-a} \rangle$ are isomorphic to the quotients of type $1/(r - a)$ ($1, \langle r \rangle_{r-a}, \langle -r \rangle_{r-a}$) and $1/a(1, \langle -r \rangle_a, \langle r \rangle_a)$, respectively.

LEMMA 2.1. *There exist \mathbf{Z} -linear isomorphisms*

$$L(r, a) : N(r - a, \langle r \rangle_{r-a}) \rightarrow N(r, a),$$

$$R(r, a) : N(a, \langle -r \rangle_a) \rightarrow N(r, a),$$

such that $L(r, a)(\Delta(r - a, \langle r \rangle_{r-a})) = \langle e_1, e_2, p_{r-a} \rangle, R(r, a)(\Delta(a, \langle -r \rangle_a)) = \langle e_1, e_3, p_{r-a} \rangle$, and $L(r, a)(e_1) = R(r, a)(e_1) = e_1, L(r, a)(e_3) = R(r, a)(e_2) = p_{r-a}$.

DEFINITION 2.2. By a weighted blow-up of the singularity of type $\frac{1}{r}(1, a, r - a)$ at the point $p_{r-a} = 1/r(1, a, r - a)$, we mean a toric variety $\overline{X(r, a)}$ obtained by the star subdivision of the cone $\Delta(r, a)$ at the point $1/r(1, a, r - a)$.

The weighted blow-up induces a proper, birational morphism $\overline{X(r, a)} \rightarrow X(r, a)$ with exceptional divisor equal to the weighted projective space $\mathbf{P}(1, a, r - a)$.

DEFINITION 2.3. The Danilov resolution of the singularity of type $\frac{1}{r}(1, a, r - a)$ is a resolution obtained by the weighted blow-up of the singularity of type $\frac{1}{r}(1, a, r - a)$ at

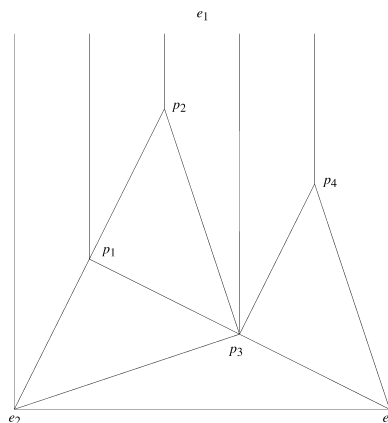


FIGURE 1. The fan of Danilov resolution of $\frac{1}{5}(1, 2, 3)$ cut with hyperplane $e_2^* + e_3^* = 1$.

the point p_{r-a} and, recursively, the Danilov resolutions of the singularities of type $1/(r - a)(1, \langle r \rangle_{r-a}, \langle -r \rangle_{r-a})$ and $1/a(1, \langle -r \rangle_a, \langle r \rangle_a)$.

DEFINITION 2.4. For fixed r and a , we call the resolution of $1/(r - a)(1, \langle r \rangle_{r-a}, \langle -r \rangle_{r-a})$ singularity an L-resolution and the resolution of $1/a(1, \langle -r \rangle_a, \langle r \rangle_a)$ an R-resolution. The 3-dimensional cones of the fan of the Danilov resolution will be called L-cones or R-cones if they are subsets of the cones $\langle e_1, e_2, p_{r-a} \rangle$ or $\langle e_1, e_3, p_{r-a} \rangle$, respectively.

The fan of the Danilov resolution consists of $2r - 1$ simplicial cones of dimension three. Precisely r cones of dimension three contain e_1 . We denote them $\sigma_0, \dots, \sigma_{r-1}$.

DEFINITION 2.5. Set

$$\sigma_i = \langle p_i, p_{i+1}, e_1 \rangle \quad \text{for } i = 0, \dots, r - 1.$$

Note that the resolution can be constructed by $r - 1$ weighted blow-ups at the points p_1, \dots, p_{r-1} with a suitable order.

COROLLARY 2.6. The Danilov resolution of the singularity $\frac{1}{r}(1, 1, r - 1)$ is obtained by the consecutive blow-ups at the points p_{r-1}, \dots, p_1 . In the case of $\frac{1}{r}(1, r - 1, 1)$, the blow-ups are done at the points p_1, \dots, p_{r-1} .

DEFINITION 2.7. For fixed r and a , let $D_i(r, a)$ denote the T -invariant toric divisor associated to the ray generated by the lattice point p_i for $i = 0, \dots, r$. Let E_j denote the T -invariant toric divisor associated to the ray generated by e_j for $j = 1, 2, 3$.

Note that $D_0(r, a) = E_2$ and $D_r(r, a) = E_3$. To simplify notation, we write D_i instead of $D_i(r, a)$.

3. Divisors X_i, Y_i, Z_i, R_i and their properties. In this section we start by defining recursively a permutation $\tau(r, a)$ of $\{0, 1, \dots, r - 1\}$. It will be subsequently used in the construction of divisors X_i, Y_i, Z_i on the Danilov resolution. These divisors will define the structure of a gnat-family in the sense of Logvinenko [13] on the Danilov resolution.

DEFINITION 3.1. If $a \in \{1, r - 1\}$ set $\tau(r, a)(i) = \langle ai - 1 \rangle_r$ for $i = 0, \dots, r - 1$, and otherwise

$$\tau(r, a)(i) = \begin{cases} \tau(r - a, \langle r \rangle_{r-a})(\langle i \rangle_{r-a}) & \text{if } i \geq a, \\ (r - a) + \tau(a, \langle -r \rangle_a)(i) & \text{if } i < a. \end{cases}$$

Observe that $\tau(r, a)(0) = r - 1$. Note the recursive nature of the above definition. The permutation τ will play a crucial role in determining the stability parameters connected with the moduli structure on the Danilov resolution.

EXAMPLE 3.2. In order to compute $\tau(5, 2)$ we need to know $\tau(2, 1) = (0, 1)$ and $\tau(3, 2) = (0, 2)$ (written as cyclic permutations in the standard notation). Then,

$$\tau(5, 2)(i) = \tau(3, 2)(\langle i \rangle_3) \quad \text{for } i \geq 2,$$

$$\tau(5, 2)(i) = 3 + \tau(2, 1)(i) \quad \text{for } i < 2,$$

hence $\tau(5, 2) = (0, 4, 1, 3, 2)$ as a cyclic permutation. In an analogous way, we could check that $\tau(7, 2) = (0, 6, 3, 2)(1, 5, 4)$ which in turn allows us to compute $\tau(12, 7) = (0, 11, 3, 7)(1, 10, 4, 6, 8, 2, 5, 9)$.

DEFINITION 3.3. We call the sequence of numbers $i, i + (r - a), \dots, i + s(r - a)$ an L-brick if

- i) $i < r - a$,
- ii) $i + (s + 1)(r - a) > r$,
- iii) every number in the sequence is strictly smaller than r .

The sequence of numbers $i, i + a, \dots, i + sa$ is called an R-brick if

- 1) $i < a$,
- 2) $i + (s + 1)a > r$,
- 3) every number in the sequence is strictly smaller than r .

The L- and R-bricks connect the characters of a cyclic group of order r with the characters of cyclic groups of order $r - a$ and a , respectively. They can be identified with fibers of the projections $\{0, \dots, r - 1\} \rightarrow \mathbf{Z}/(r - a)\mathbf{Z}$ and $\{0, \dots, r - 1\} \rightarrow \mathbf{Z}/a\mathbf{Z}$, respectively, which shows that there are $r - a$ different L-bricks and a different R-bricks.

DEFINITION 3.4. For fixed r and a , let $Y(r, a)$ denote the Danilov resolution of the $\frac{1}{r}(1, a, r - a)$ singularity.

The rest of this section is devoted to finding effective toric divisors $X_i(r, a), Y_i(r, a), Z_i(r, a)$ on $Y(r, a)$ for $i = 0, \dots, r - 1$. These divisors will yield \mathcal{Q} -divisors $R_i(r, a)$, used in defining the structure of a moduli space on $Y(r, a)$. Note that the addition in the indices of $X_i(r, a), Y_i(r, a), Z_i(r, a)$ is always meant modulo r .

DEFINITION 3.5. For fixed r and a , let

$$Y_{i-a}(r, a) = \sum_{k=0}^{\tau(r,a)(i)} D_k(r, a) \quad \text{for } i = 0, \dots, r-1,$$

$$Z_i(r, a) = \sum_{k=\tau(r,a)(i)+1}^r D_k(r, a) \quad \text{for } i = 0, \dots, r-1,$$

and let X_i be the divisor defined by the equations $X_0 = E_1$ and

$$X_i(r, a) + Z_{i+1}(r, a) = Z_i(r, a) + X_{i-a}(r, a) \quad \text{for } i = 0, \dots, r-1.$$

These conditions ensure that the divisors X_i are uniquely determined. Note that by definition

$$Y_{i-a}(r, a) + Z_i(r, a) = \sum_{i=0}^r D_k(r, a),$$

in particular $Y_{i-a}(r, a) + Z_i(r, a)$ does not depend on i . Moreover,

$$X_i(r, a) + Y_{i+1}(r, a) = Y_i(r, a) + X_{i+a}(r, a).$$

REMARK 3.6. To shorten notation, we write X_i, Y_i, Z_i instead of $X_i(r, a), Y_i(r, a), Z_i(r, a)$ when no confusion can arise.

The divisors X_i, Y_i, Z_i satisfy commutativity relations

$$(3.1) \quad X_i + Y_{i+1} = Y_i + X_{i+a},$$

$$(3.2) \quad X_i + Z_{i+1} = Z_i + X_{i-a},$$

$$(3.3) \quad Y_i + Z_{i+a} = Z_i + Y_{i-a}.$$

LEMMA 3.7. For $a = 1$, we have $Y_i = D_0 + \dots + D_i$ for $i = 0, \dots, r-1$ and $Z_0 = D_r, Z_i = D_i + \dots + D_r$ for $i = 1, \dots, r-1$. Moreover, $X_i = Y_i + E_1 - D_0$ for $i = 0, \dots, r-1$. For $a = r-1$, we have $Y_0 = D_0, Y_i = D_0 + \dots + D_{r-i}$ for $i = 1, \dots, r-1$ and $Z_i = D_{r-i} + \dots + D_r, X_i = Z_i + E_1 - D_r$ for $i = 0, \dots, r-1$.

PROOF. To see this, it is enough to combine Definitions 3.1 and 3.5. □

EXAMPLE 3.8. For $r = 5$ and $a = 2$, we have $\tau(5, 2) = (0, 4, 1, 3, 2)$ (cf. Example 3.2), and hence

$$\begin{array}{ll} Y_0 = D_0, & Z_0 = D_5, \\ Y_1 = D_0 + D_1 + D_2, & Z_1 = D_4 + D_5, \\ Y_2 = D_0 + D_1, & Z_2 = D_1 + D_2 + D_3 + D_4 + D_5, \\ Y_3 = D_0 + D_1 + D_2 + D_3 + D_4, & Z_3 = D_3 + D_4 + D_5, \\ Y_4 = D_0 + D_1 + D_2 + D_3, & Z_4 = D_2 + D_3 + D_4 + D_5. \end{array}$$

By solving the linear equations $X_i + Z_{i+1} = Z_i + X_{i-2}$ in X_i 's, we have

$$\begin{aligned} X_0 &= E_1, \\ X_1 &= E_1 + D_2 + D_4, \\ X_2 &= E_1 + D_1 + D_2, \\ X_3 &= E_1 + D_4, \\ X_4 &= E_1 + D_1 + 2D_2 + D_3 + D_4. \end{aligned}$$

DEFINITION 3.9. For fixed r, a , let $r_L = r - a$ and $a_L = \langle r \rangle_{r-a}$. By $X_i^L(r, a)$, $Y_i^L(r, a)$, $Z_i^L(r, a)$ we mean the pullbacks of the divisors $X_i(r_L, a_L)$, $Y_i(r_L, a_L)$, $Z_i(r_L, a_L)$ by the rational inverse of the inclusion $Y(r_L, a_L) \hookrightarrow Y(r, a)$, respectively. Similarly, let $r_R = a$ and $a_R = \langle -r \rangle_a$. By $X_i^R(r, a)$, $Y_i^R(r, a)$, $Z_i^R(r, a)$ we mean the pullbacks of the divisors $X_i(r_R, a_R)$, $Y_i(r_R, a_R)$, $Z_i(r_R, a_R)$ by the rational inverse of the inclusion $Y(r_R, a_R) \hookrightarrow Y(r, a)$, respectively. In particular,

$$\begin{aligned} Z_i^L(r, a) &= \sum_{k=\tau(r_L, a_L)(i)+1}^{r_L} D_k(r, a) \quad \text{for } i = 0, \dots, r_L - 1, \\ Z_i^R(r, a) &= \sum_{k=\tau(r_R, a_R)(i)+1}^{r_R} D_{k+r_L}(r, a) \quad \text{for } i = 0, \dots, r_R - 1. \end{aligned}$$

When no confusion can arise, we write these divisors by $X_i^L, Y_i^L, Z_i^L, X_i^R, Y_i^R, Z_i^R$ for short. We note that in the definition of Z_i^R there is a shift by r_L in the index of D_i since the divisor $D_{i+r_L}(r, a)$ on the resolution $Y(r, a)$ corresponds to the divisor $D_i(r_R, a_R)$ on the resolution $Y(r_R, a_R)$ for $i = 0, \dots, r_R - 1$.

EXAMPLE 3.10. For $r = 5$ and $a = 2$, by Lemma 3.7, we have

$$\begin{aligned} Z_0^L &= D_3, & X_0^L &= E_1, \\ Z_1^L &= D_2 + D_3, & X_1^L &= E_1 + D_2, \\ Z_2^L &= D_1 + D_2 + D_3, & X_2^L &= E_1 + D_1 + D_2, \\ Z_0^R &= D_5, & X_0^R &= E_1, \\ Z_1^R &= D_4 + D_5, & X_1^R &= E_1 + D_4. \end{aligned}$$

Note the indices in the divisors D_i 's on the R-resolution are shifted by $r_L = 3$. Since the L-resolution is a resolution of the $\frac{1}{3}(1, 2, 1)$ singularity, we have by Lemma 3.7

$$Y_0^L = D_0, \quad Y_1^L = D_0 + D_1, \quad Y_2^L = D_0 + D_1 + D_2.$$

On the other hand, the R-resolution is a resolution of the $\frac{1}{2}(1, 1, 1)$ singularity, hence

$$Y_0^R = D_3, \quad Y_1^R = D_3 + D_4.$$

Following lemmas are useful in later sections.

LEMMA 3.11. *Let $i, \dots, i + s(r - a)$ be an L-brick. The restriction of the divisor $Z_{i+s(r-a)}(r, a)$ to the L-resolution is equal to the divisor $Z_i^L(r, a)$. If $i, \dots, i + sa$ is an R-brick, then the restriction of the divisor $Z_i(r, a)$ to the R-resolution is equal to the divisor $Z_i^R(r, a)$.*

PROOF. Observe that if $i, \dots, i + s(r - a)$ is an L-brick, then $i + s(r - a) \geq a$ and $\langle i + s(r - a) \rangle_{r-a} = i$. Therefore $\tau(r, a)(i + s(r - a)) = \tau(r - a, \langle r \rangle)(i)$. If $i, \dots, i + sa$ is an R-brick, then $i < a$ and $\tau(r, a)(i) = (r - a) + \tau(a, \langle -r \rangle)(i)$. \square

Similar facts hold for the restriction of the divisors X_i .

LEMMA 3.12. *For any $i \leq r - 2$, the divisor $X_i(r, a)$ restricted to the L-resolution is equal to the divisor $X_j^L(r, a)$ for $j = \langle i \rangle_{r-a}$, and the divisor $X_i(r, a)$ restricted to the R-resolution is equal to the divisor $X_j^R(r, a)$ for $j = \langle i \rangle_a$.*

PROOF. We give the proof only in the case of the restriction to the R-resolution. First we show that if $i, \dots, i + sa$ is an R-brick such that $i + sa \neq r - 1$, then the R-restrictions of the divisors X_i, \dots, X_{i+sa} are equal to each other. To see this, observe that the restrictions of the divisors Z_j for $j \geq a$ to the R-resolution are equal to each other (cf. Definitions 3.1 and 3.5). This, combined with the commutativity relations (3.2),

$$Z_j - Z_{j+1} = X_j - X_{j-a}$$

for $j = i + a, i + 2a, \dots, i + sa$, implies that the R-restrictions of the divisors X_i, \dots, X_{i+sa} are equal to each other.

Similarly, if $i, \dots, i + sa$ is an R-brick such that $i + sa = r - 1$, then the restrictions of the divisors $X_i, \dots, X_{i+(s-1)a}$ (i.e., all but the last) to the R-resolution are equal to each other.

Hence, it is enough to prove the lemma assuming $i < a$. Denote by $X_i|_R$ the restriction of the divisor X_i to the R-resolution. Obviously $X_0|_R = X_0^R$ and by Lemma 3.11 we obtain relations

$$X_i|_R + Z_{i+1}^R = Z_i^R + X_{i-a}|_R$$

for $i = 0, \dots, a - 2$. We have proven already that $X_{i-a}|_R = X_{(i+r)_a}|_R$, and so the above relations can be rewritten as

$$X_i|_R + Z_{i+1}^R = Z_i^R + X_{(i+r)_a}|_R.$$

These are exactly the equations (3.2) for $r_R = a$ and $a_R = \langle -r \rangle_a$, and we have

$$X_i|_R = X_i^R \quad \text{for } i = 0, \dots, a - 2.$$

Let j be the last element of an R-brick containing $a - 1$. Then $j \neq r - 1$ so $0 \leq \langle j + a \rangle_r < a - 1$, and the equation (3.2) for $i = j + a$

$$X_{j+a} + Z_{j+a+1} = X_j + Z_{j+a},$$

restricted to the R-resolution becomes

$$X_{j+a-r}^R + Z_{j+a-r+1}^R = X_{a-1}|_R + Z_{j+a-r}^R.$$

This finishes the proof since the above is exactly the equation (3.2) for $r_R = a$, $a_R = \langle -r \rangle_a$ and $i = j + a - r$. \square

EXAMPLE 3.13. For $r = 5$ and $a = 2$, there are 3 different L-bricks $(0, 3)$, $(1, 4)$, (2) and 2 different R-bricks $(0, 2, 4)$, $(1, 3)$. The L-resolution contains divisors D_0, D_1, D_2, D_3, E_1 and the R-resolution contains divisors D_3, D_4, D_5, E_1 . By Examples 3.8 and 3.10, we note that

$$\begin{aligned} Z_3|_L &= Z_0^L, & Z_4|_L &= Z_1^L, & Z_2|_L &= Z_2^L, \\ Z_0|_R &= Z_0^R, & Z_1|_R &= Z_1^R. \end{aligned}$$

Moreover,

$$\begin{aligned} X_0|_L &= X_3^L = X_0^L, & X_0|_R &= X_2|_R = X_0^R, \\ X_1|_L &= X_1^L, & X_1|_R &= X_1^R. \end{aligned}$$

LEMMA 3.14. For a, r coprime, the divisors $X_i - E_1, Y_i - E_2, Z_i - E_3$ are effective for $i = 0, \dots, r - 1$.

PROOF. The result is true for $Y_i - E_2$ and $Z_i - E_3$ by definition, since $E_2 = D_0$ and $E_3 = D_r$. It remains to prove the result for $X_i - E_1$.

Note that for $a \in \{1, r - 1\}$ either $X_i - E_1 = Y_i - E_2$ or $X_i - E_1 = Z_i - E_3$ by Lemma 3.7. For $1 < a < r - 1$, by recursion and Lemma 3.12, the restrictions of $X_i - E_1$ to L- and R-resolution are effective for $i \neq r - 1$. Finally, note that

$$X_{r-1} - E_1 = (X_{r-a-1} - E_1) + (Z_{r-1} - Z_0),$$

where both summands are effective, since $Z_0 = D_r$ for any r, a . \square

DEFINITION 3.15. For fixed r and a , let

$$D_X(r, a) = \frac{1}{r} \operatorname{div}(re_1^*), \quad D_Y(r, a) = \frac{1}{r} \operatorname{div}(re_2^*), \quad D_Z(r, a) = \frac{1}{r} \operatorname{div}(re_3^*)$$

be \mathcal{Q} -divisors on $Y(r, a)$ where $\operatorname{div}(re_i^*)$ denotes the divisor of zeros and poles of the rational function re_i^* . We write them by D_X, D_Y, D_Z for short.

We introduce the \mathcal{Q} -divisors $R_i(r, a)$ which later will define the desired family of McKay quiver representations on $Y(r, a)$.

DEFINITION 3.16. For fixed r and a , define the \mathcal{Q} -divisors $R_i(r, a)$ for $i = 0, \dots, r - 1$ by the equations

$$Z_i(r, a) = D_Z(r, a) + R_i(r, a) - R_{i-a}(r, a), \quad R_0(r, a) = 0.$$

We write R_i for short when no confusion can arise.

The divisors R_i are uniquely determined by the condition $R_0 = 0$ since r, a are coprime and the rank of the matrix determining equations for R_i is equal to $r - 1$. Using the equation $Z_i + Y_{i-a} = D_Y + D_Z$ we get $Y_i = D_Y + R_i - R_{i+a}$.

LEMMA 3.17. *For any coprime r and a , we have*

$$R_1 = D_X - E_1 .$$

PROOF. By definition, we have

$$\begin{aligned} R_1 &= (r - b)D_Z - (Z_0 + Z_{-a} + Z_{-2a} + \cdots + Z_{-(r-b-1)a}) \\ &= (r - b)D_Z - (Z_{a+1} + Z_{2a+1} + Z_{3a+1} + \cdots + Z_0) \end{aligned}$$

since $\langle -(r - b - i)a \rangle_r = ia + 1$. Therefore it is enough to show that

$$E_1 = D_X - (r - b)D_Z - (Z_{a+1} + Z_{2a+1} + \cdots + Z_0) .$$

Assume that $a \notin \{1, r - 1\}$ as otherwise the statement is trivial (cf. Lemma 3.7). We use a recursive argument. Observe that the numbers in the sequence

$$(\star) \quad \langle a + 1 \rangle_r, \langle 2a + 1 \rangle_r, \langle 3a + 1 \rangle_r, \dots, \langle 0 \rangle_r$$

not greater than $a - 1$ (i.e., the first numbers in R-bricks) are equal to the numbers

$$\langle a_R + 1 \rangle_a, \langle 2a_R + 1 \rangle_a, \langle 3a_R + 1 \rangle_a, \dots, \langle 0 \rangle_a ,$$

where $a_R = \langle -r \rangle_a$. Moreover, the numbers in the sequence (\star) greater or equal to a (i.e., the last numbers in L-bricks) are equal modulo $r - a$ to the numbers

$$\langle a_L + 1 \rangle_{r-a}, \langle 2a_L + 1 \rangle_{r-a}, \langle 3a_L + 1 \rangle_{r-a}, \dots, \langle 0 \rangle_{r-a} ,$$

where $a_L = \langle r \rangle_{r-a}$. We omit a proof of this arithmetic fact. Let W^L, W^R denote the pullbacks of the divisors $D_X(r_L, a_L) - (r_L - b_L)D_Z(r_L, a_L)$ and $D_X(r_R, a_R) - (r_R - b_R)D_Z(r_R, a_R)$ (where $a_L b_L = 1$ modulo r_L and $a_R b_R = 1$ modulo r_R) by the rational inverses of the inclusions $Y(r_L, a_L) \hookrightarrow Y(r, a)$ and $Y(r_R, a_R) \hookrightarrow Y(r, a)$, respectively. Note that the first coordinate of the point p_{i+1} is not smaller than the first coordinate of the point p_i if and only if the toric ray dual to the cone $\langle p_i, p_{i+1} \rangle$ is equal to $e_1^* - (r - b)e_3^*$, which gives an intrinsic explanation of the value of $e_1^* - (r - b)e_3^*$ on the generator of a ray. Since $(e_1^* - (r - b)e_3^*)(\frac{1}{r}(1, a, r - a)) = b_L - r_L$, we get $D_X - (r - b)D_Z = W^L + (b_L - r_L)W^R - E_1$, as E_1 is counted twice. To finish the proof, observe that in the sequence (\star) exactly $r_L - b_L$ numbers are greater than or equal to a and use Lemma 3.11. \square

LEMMA 3.18. *For fixed r and a , the divisors R_i satisfy the following equations for $i = 0, \dots, r - 1$.*

$$\begin{aligned} X_i &= D_X + R_i - R_{i+1} , \\ Y_i &= D_Y + R_i - R_{i+a} , \\ Z_i &= D_Z + R_i - R_{i-a} . \end{aligned}$$

PROOF. It is enough to prove the first equality. Set $\widetilde{X}_i = D_X + R_i - R_{i+1}$ and note that $\sum \widetilde{X}_i = rD_X$. Moreover, the divisors \widetilde{X}_i satisfy the commutativity relations (3.2), and hence $X_i - \widetilde{X}_i$ is constant. Since $\widetilde{X}_0 = D_X - R_1 = X_0$ by Lemma 3.17, the constant is equal to 0. \square

4. The McKay quiver. By a quiver we mean a finite directed graph Q . The set of vertices of Q will be denoted by Q_0 and the set of arrows by Q_1 . For any arrow a in Q_1 denote by $t(a)$ the tail of a and by $h(a)$ the head of a . In what follows, we restrict the general definition of quiver representations to the simple case where the dimension vector is equal to $(1, \dots, 1)$. For any $v \in Q_0$, let C_v denote 1-dimensional complex vector space assigned to the vertex v . A representation of the quiver Q is an element of

$$\text{Rep}(Q) = \bigoplus_{a \in Q_1} \text{Hom}_{\mathbb{C}}(C_{t(a)}, C_{h(a)}).$$

By fixing a basis in each C_v we can identify $\text{Rep}(Q)$ with an affine space. With this choice, for any representation $V \in \text{Rep}(Q)$ and $a \in Q_1$ denote by $V(a)$ the constant representing the linear map for the arrow a in V .

A path q in quiver Q is a sequence of arrows a_1, \dots, a_2, a_1 where $h(a_i) = t(a_{i+1})$. For such path q and for any representation V of Q define $V(q) = V(a_l) \cdots V(a_1)$. A linear combination of paths q_i is called an admissible relation, if paths q_i have the same heads and tails. Any set R of admissible relations for a quiver Q defines an affine subscheme of $\text{Rep}(Q)$ cut by the polynomial equations coming from R , i.e.,

$$\text{Rep}(Q, R) := \{V \in \text{Rep}(Q) ; V(c) = 0 \text{ for } c \in R\},$$

where the function $V(c)$ denotes the linear extension of the functions $V(q_i)$.

Two representations of a quiver Q are isomorphic if and only if they lie in the same orbit of the group $\text{GL}(Q, \mathbb{C}) = \bigoplus_{v \in Q_0} \mathbb{C}^*$, acting from the left on the set $\text{Rep}(Q)$ in the way

$$(g \cdot V)(a) = g(h(a))V(a)g(t(a))^{-1} \text{ for any } V \in \text{Rep}(Q).$$

This action leaves $\text{Rep}(Q, R)$ invariant. Dividing by the 1-dimensional subgroup acting trivially, we are left with a faithful action of the group

$$\text{PGL}(Q, \mathbb{C}) = \text{GL}(Q, \mathbb{C})/\mathbb{C}^*(1, \dots, 1).$$

DEFINITION 4.1. By a subquiver $Q' \subset Q$ we mean a subset of vertices $Q'_0 \subset Q_0$ and a subset of arrows $Q'_1 \subset Q_1$ satisfying $t(a), h(a) \in Q'_0$ for any $a \in Q'_1$. Let $V \in \text{Rep}(Q, R)$ be a representation of Q . A subrepresentation V' of V is a representation of a subquiver $Q' \subset Q$ satisfying

$$V'(a) = V(a) \text{ for any } a \in Q'_1.$$

We do not need the general definition of the McKay quiver, so we quote only the specialization to the case of a cyclic group action.

DEFINITION 4.2. (McKay) Let G be a cyclic group $G \subset \text{GL}(3, \mathbb{C})$ of order r , such that the quotient singularity \mathbb{C}^3/G is of type $\frac{1}{r}(1, a, r - a)$. Define the McKay quiver for G as a finite graph with r vertices $0, 1, \dots, r - 1$ and $3r$ arrows $x_0, y_0, z_0, \dots, x_{r-1}, y_{r-1}, z_{r-1}$ such that $t(x_i) = t(y_i) = t(z_i) = i$ and

$$h(x_i) = (i + 1)_r, \quad h(y_i) = (i + a)_r, \quad h(z_i) = (i - a)_r.$$

The vertices of the McKay quiver correspond to the characters of G .

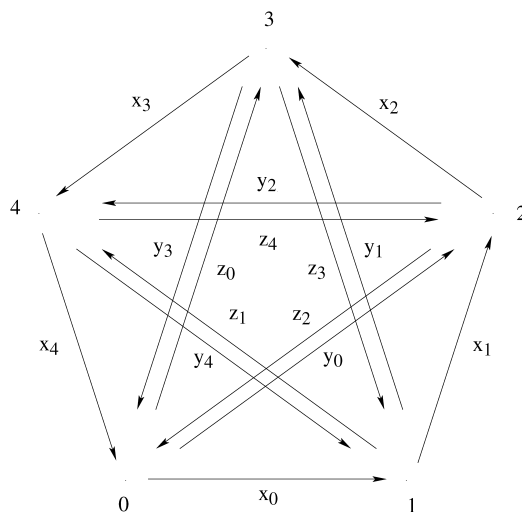


FIGURE 2. The McKay quiver for $r = 5, a = 2$.

DEFINITION 4.3. A representation of the McKay quiver is an element of $\text{Rep}(Q, R)$, where Q is the McKay quiver (for fixed r, a) and R is the set relations

$$R = \{y_{i+1}x_i - x_{i+a}y_i, z_{i+1}x_i - x_{i-a}z_i, y_{i-a}z_i - z_{i+a}y_i ; i = 0, \dots, r - 1\},$$

where all indices are meant modulo r .

5. Family of representations of the McKay quiver. In this section we will define a family of the McKay quiver representations over the Danilov resolution using line bundles determined by the effective divisors X_i, Y_i, Z_i .

DEFINITION 5.1 (King, Logvinenko). Fix coprime r, a and let $Y(r, a)$ be the Danilov resolution. By a family of McKay quiver representations on $Y(r, a)$ for the action of type $\frac{1}{r}(1, a, r - a)$, we mean a set of r \mathcal{Q} -divisors R_i for $i = 0, \dots, r - 1$ on $Y(r, a)$, such that the \mathcal{Q} -divisors

$$X_i = D_X + R_i - R_{i+1}, \quad Y_i = D_Y + R_i - R_{i+a}, \quad Z_i = D_Z + R_i - R_{i-a}$$

are effective divisors for $i = 0, \dots, r - 1$ (where D_X, D_Y, D_Z are as in Definition 3.15).

The above condition is called *the reductor condition* in [13]. It ensures that the family of quiver representations on the resolution $Y(r, a)$ is *natural*, that is, the support of a representation (seen as a finite dimensional $\mathbb{C}[x, y, z]$ -module) parameterized by a point p in the resolution coincides with the G -orbit parameterized by the image of p in the quotient space (cf. [13, Definition 1.4]).

DEFINITION 5.2. For fixed a and r , denote by $\mathcal{F}(r, a)$ (or \mathcal{F} for short) the family given by the \mathcal{Q} -divisors $R_i(r, a)$ in Definition 3.16.

REMARK 5.3. Note that the divisors R_i satisfy the reductor condition by Lemma 3.18 and the divisors X_i, Y_i, Z_i are effective by Lemma 3.14.

One could define the family $\mathcal{F}(r, a)$ by representing the arrows of the McKay quiver, on each affine toric chart, by the corresponding character of the piecewise linear function given by the effective Cartier divisors X_i, Y_i and Z_i from Definition 5.1. Note that in this case every such character is a regular function on the toric chart. Equivalently, define line bundles $L_i = \mathcal{O}_Y(-R_i)$ on Y for $0 \leq i \leq r - 1$. Then, for each $0 \leq i \leq r - 1$, the multiplications by the sections defining the divisors X_i, Y_i and Z_i , respectively, determine morphisms from L_i to L_{i+1} , to L_{i+a} and to L_{i-a} , respectively.

We will show later that there exist stability conditions θ such that every representation in the family $\mathcal{F}(r, a)$ is θ -(semi)stable. In fact, it will turn out that such stability conditions θ are exactly those for which the representations parameterized by T -fixed point of the cones $\sigma_0, \dots, \sigma_{r-1}$ are simultaneously θ -(semi)stable.

DEFINITION 5.4. For the family \mathcal{F} from Definition 5.2 and for any 3-dimensional cone σ in the fan of the Danilov resolution, we call an arrow in the McKay quiver σ -distinguished if the corresponding divisor (that is, X_i, Y_i or Z_i) does not contain the T -fixed point of toric chart U_σ .

Observe, for example, that no z_i -arrow is σ_{r-1} -distinguished since the divisors $Z_i - E_3$ are effective for any i .

LEMMA 5.5. For any 3-dimensional cone σ in the fan of the Danilov resolution, any two vertices of the McKay quiver can be connected by an undirected path of σ -distinguished arrows different from x_{r-1} .

PROOF. The lemma is true for $a \in \{1, r - 1\}$. Note that any two vertices of the McKay quiver lying in the same L-brick can be joined by a sequence of z -arrows for any L-cone σ , and any two vertices lying in the same R-brick can be joined by a sequence of y -arrows if σ is an R-cone. By the inductive step, any two bricks can be joined by a sequence of σ -distinguished arrows. It is enough to consider the cone $\sigma = \langle p_0, p_{r-a}, p_r \rangle$, and we are done since the only σ -distinguished arrows are x_0, \dots, x_{r-2} . \square

LEMMA 5.6. Let $p, p' \in Y$ be two points in the Danilov resolution, belonging to two distinct toric charts isomorphic to \mathbb{C}^3 . Then the representations parameterized by p and p' in \mathcal{F} are not isomorphic.

PROOF. Let σ and σ' be 3-dimensional cones in the fan of the Danilov resolution corresponding to charts containing p and p' , respectively. There are at most two common primitive generators of the cones σ and σ' belonging to the set $\{p_0, \dots, p_r\}$. This implies that at least one y - or z -arrow is σ -distinguished and not σ' -distinguished. Hence the representations parameterized by p and p' in \mathcal{F} are not isomorphic. \square

LEMMA 5.7. *Let $p, p' \in Y$ be distinct points in the Danilov resolution belonging to a single toric chart, isomorphic to \mathbb{C}^3 , on the Danilov resolution. The representations parameterized by p and p' in \mathcal{F} are not isomorphic.*

PROOF. Let $\sigma = \langle p_l, p_m, p_n \rangle$ be the 3-dimensional cone in the fan of Danilov resolution such that $p, p' \in U_\sigma$, where U_σ stands for the toric chart given by σ . We claim that there exist $i, j, k \in \{0, \dots, r-1\}, i \neq r-1$, such that the restrictions of X_i, Y_j, Z_k to the chart U_σ are equal to the restrictions of the divisors $D_l|_{U_\sigma}, D_m|_{U_\sigma}, D_n|_{U_\sigma}$, respectively. This holds for $a \in \{1, r-1\}$ and can be proven for $a \notin \{1, r-1\}$ using recursion and Lemmas 3.11 and 3.12. In the orbit of the group $\text{GL}(Q, \mathbb{C})$, there exists exactly one representation such that all σ -distinguished arrows are represented by the number 1 (by Lemma 5.5). Therefore, in this unique element of the orbit, the arrows x_i, y_j, z_k are represented by toric coordinates on U_σ . The points p, p' have at least one different toric coordinate. Therefore they parameterize non-isomorphic representations. \square

COROLLARY 5.8. *For any two distinct points $p, p' \in Y$ in the Danilov resolution, the representations parameterized by p and p' in \mathcal{F} are not isomorphic.*

DEFINITION 5.9. For fixed coprime r and a , let \mathcal{F}_i denote the representation of the McKay quiver parameterized in the family $\mathcal{F}(r, a)$ by the unique T -fixed point belonging to the toric chart U_{σ_i} for $i = 0, \dots, r-1$ (cf. Definition 2.5).

Since the divisors $X_j - E_1$ are effective, no x_j -arrow is σ_i -distinguished for any i . Moreover, by the definition of the permutation τ , for any i there exists a unique j such that y -arrow and z -arrow joining vertices j and $j+a$ are not σ_i -distinguished. For i and j as above, if $j' \neq j$, then among the y - and z -arrows joining j' and $j'+a$ exactly once is σ_i -distinguished (cf. Example 7.4). Hence the representations \mathcal{F}_i are particularly easy to deal with.

6. Stability of quiver representations. In this section, we recall some facts and definitions concerning θ -stability of quiver representations (see [12]), note that we restrict to the case of dimension vector equal to $(1, \dots, 1)$. We prove that the representations in the family \mathcal{F} (cf. Definition 5.2) on the Danilov resolution are simultaneously θ -(semi)stable if and only if the representation $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$ are θ -(semi)stable.

For any quiver Q , set

$$\text{Wt}(Q) = \left\{ \theta : Q_0 \rightarrow \mathbf{Q} ; \sum_{v \in Q_0} \theta(v) = 0 \right\}.$$

Given a function $\theta \in \text{Wt}(Q)$ for which $\theta(Q_0) \subset \mathbf{Z}$, we obtain a character χ_θ of $\text{PGL}(Q, \mathbb{C})$. Explicitly, for any such θ , the character is

$$\chi_\theta(g) = \prod_{v \in Q_0} g(v)^{\theta(v)},$$

where $g \in \text{PGL}(Q, \mathbb{C})$. Therefore, we will call $\text{Wt}(Q)$ the weight space for Q .

DEFINITION 6.1. (A. King) For any subrepresentation V' of a representation $V \in \text{Rep}(Q)$ and $\theta \in \text{Wt}(Q)$, set

$$\theta(V') = \sum_{v \in Q'_0} \theta(v).$$

A representation V is called θ -semistable if $\theta(V') \geq \theta(V) = 0$ for every proper non-zero subrepresentation $V' \subset V$. A representation V is called θ -stable if an analogous condition with strict inequality holds.

DEFINITION 6.2. We say that a stability parameter $\theta \in \text{Wt}(Q)$ is generic if every θ -semistable representation is θ -stable.

THEOREM 6.3 (King). *Let Q be a quiver, R a set of admissible relations for Q and let $\theta \in \text{Wt}(Q)$. A point in $\text{Rep}(Q, R)$ is χ_θ -(semi)stable under the action of $\text{PGL}(Q, \mathbb{C})$ if and only if the corresponding representation of Q is θ -(semi)stable. Denote by $M_\theta(Q, R)$ the GIT quotient of $\text{Rep}(Q, R)$ by $\text{PGL}(Q, \mathbb{C})$ with respect to the χ_θ -linearization of the trivial bundle over $\text{Rep}(Q, R)$. That is, $M_\theta(Q, R)$ is the scheme constructed from the graded ring of semi-invariants, namely,*

$$M_\theta(Q, R) := \text{Rep}(Q, R) //_{\chi_\theta} \text{PGL}(Q, \mathbb{C}) = \text{Proj} \bigoplus_{k=0}^{\infty} \mathbb{C}[\text{Rep}(Q, R)]^{\chi_\theta^k},$$

where elements of the set $\mathbb{C}[\text{Rep}(Q, R)]^{\chi_\theta^k}$ are regular functions f on the representation space $\text{Rep}(Q, R)$, such that $f(g \cdot v) = \chi(g) f(v)$ for any $g \in \text{PGL}(Q, \mathbb{C})$ and any $v \in \text{Rep}(Q, R)$.

For a generic θ , the scheme $M_\theta(Q, R)$ is a fine moduli space of θ -stable representations.

PROOF. See [12, Propositions 3.1, 5.2, 5.3]. □

We need a fact concerning families of θ -(semi)stable quiver representations on affine toric varieties, which is true in a more general setting.

LEMMA 6.4. *Let U_σ be an affine toric chart in the Danilov resolution Y containing a unique T -fixed point p_σ . Let \mathcal{F} be a family of McKay quiver representations on Y as in Definition 5.1. If the representation parameterized by the point p_σ is θ -(semi)stable, then all representation in \mathcal{F} parameterized by closed points in U_σ are θ -(semi)stable.*

PROOF. The θ -(semi)stability is an open condition and it is invariant under the T -action since the divisors R_i are T -equivariant. Moreover, the T -fixed point lies in the closure of all orbits in U_σ . □

We need also two simple lemmas.

LEMMA 6.5. *Let U_σ be a toric chart in the Danilov resolution, where $\sigma = \langle p_l, p_m, p_n \rangle$ and $l < m < n$. If the arrow y_i is σ -distinguished, then it is σ_j -distinguished for $j \geq l$. If the arrow y_i is not σ -distinguished, then it is not σ_j -distinguished for $j \leq l$. If the arrow z_i is σ -distinguished, then it is σ_j -distinguished for $j < n$. If the arrow z_i is not σ -distinguished, then it is not σ_j -distinguished for $j \geq n - 1$.*

PROOF. It follows directly from Definition 3.5. □

LEMMA 6.6. *Let $i, i + a, \dots, i + sa$ be an R -brick. Then, for $j = i, i + (r - a), \dots, i + (s - 1)(r - a)$, the y_j -arrows are σ -distinguished for any cone σ in the R -resolution. Let $i, i + (r - a), \dots, i + s(r - a)$ be an L -brick. Then, for $j = i + a, \dots, i + s(r - a)$, the z_j -arrows are σ -distinguished for any cone σ in the L -resolution.*

PROOF. Let $i, i + (r - a), \dots, i + s(r - a)$ be an L -brick. Then, $i + k(r - a) < a$ for $k = 0, \dots, s - 1$. Hence, by Definition 3.1, $\tau(r, a)(i + k(r - a)) \geq r - a$. This implies that the supports of divisors $Z_i, Z_{i+(r-a)}, \dots, Z_{i+s(r-a)}$ are disjoint from any torus-fixed point on the L -resolution. The other case follows analogously. □

Using the above tools, we can prove that θ -stability of the representations $\mathcal{F}_1, \dots, \mathcal{F}_{r-1}$ (see Definition 5.9) controls stability of the whole family \mathcal{F} .

LEMMA 6.7. *Let θ be a stability parameter such that $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$ are θ -(semi) stable. Then every representation in the family \mathcal{F} is θ -(semi)stable.*

PROOF. By Lemma 6.4, to conclude, it is enough to prove that every representation in \mathcal{F} , parameterized by a T -fixed point, is θ -(semi)stable.

Let $p_\sigma \in U_\sigma$ be a T -fixed point, where $\sigma = \langle p_l, p_m, p_n \rangle$ is a 3-dimensional cone in the fan of the Danilov resolution and $l < m < n$. Let V be a subrepresentation of the representation parameterized by p_σ in the family \mathcal{F} . The proof is done by induction on r . We will show that there exists $j \in \{0, \dots, r - 1\}$ and a subrepresentation of \mathcal{F}_j supported on the same set of vertices as V .

The theorem is trivial if $a \in \{1, r - a\}$. Assume that $r > 1$ and the theorem is true for any $r' < r$. Let V be a subrepresentation as above. By $S(V) \subset \{0, \dots, r - 1\}$ we mean a subset of the vertices of the McKay quiver, supporting V . Consider a sequence $i, i + (r - a), \dots, i + s(r - a)$ of vertices in the set $S(V)$, such that the vertices $i - (r - a), i + (s + 1)(r - a)$ are not in $S(V)$. The set $S(V)$ is a union of such sequences. There is no loss of generality in assuming that $S(V)$ itself is a single sequence. Note that y_i -arrow and $z_{i+s(r-a)}$ -arrow are not σ -distinguished (V is a subrepresentation, see Definition 4.1).

Suppose that $z_{i-(r-a)}$ -arrow is σ -distinguished or the $y_{i+(s+1)(r-a)}$ -arrow is σ -distinguished. We can assume that $k < r + 1$, otherwise there is nothing to prove. By Lemma 6.5, the vertices $i, \dots, i + s(r - a)$ form a subrepresentation of some of the representations $\mathcal{F}_i, \dots, \mathcal{F}_{k-1}$.

Now we turn to the case when both the $z_{i-(r-a)}$ -arrow and the $y_{i+(s+1)(r-a)}$ -arrow are not σ -distinguished. Assume that σ is an L -cone. Since the y_i -arrow and the $z_{i+s(r-a)}$ -arrow are not σ -distinguished, the sequence $i, \dots, i + s(r - a)$ is concatenated out of some L -bricks by Lemma 6.6. These L -bricks correspond to vertices of the McKay quiver for $(1/(r - a))(1, \langle r \rangle, \langle -r \rangle)$. Moreover, the vertices corresponding to these L -bricks form a subrepresentation of the representation parameterized by p_σ in the family $\mathcal{F}(r - a, \langle r \rangle)$ on the L -resolution. Now we can use the inductive assumption. □

7. Stability of the representations $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$. We proved that every representation in the family \mathcal{F} is θ -(semi)stable if and only if the representations $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$ are simultaneously θ -(semi)stable. We will show how to get such parameters θ using permutation τ .

DEFINITION 7.1. Let $\xi(r, a) = \tau(r, a)^{-1}$ denote the inverse of the permutation τ (see Definition 3.1).

Since no x_i -arrow is σ_j -distinguished, any two vertices of \mathcal{F}_j can be joined by a sequence of z - and y -arrows by Lemma 5.5. Moreover, the arrows $z_{\xi(r,a)(j)}$ and $y_{\xi(r,a)(j)+(r-a)}$ are not σ_j -distinguished. Therefore, the quiver supporting representation \mathcal{F}_j consists of vertices $0, 1, \dots, r-1$ and every two vertices $i, i+(r-a)$ are joined either by a z -arrow or a y -arrow (but not both) unless $i = \xi(r, a)(j)$ (cf. Example 7.4).

DEFINITION 7.2. Let

$$\overline{V}_j = \{i ; \text{vertices } i, i+(r-a), \dots, \xi(r, a)(j) \text{ of } Q \text{ form a subrepresentation of } \mathcal{F}_j\},$$

$$\overline{W}_j = \{i ; \text{vertices } \xi(r, a)(j) + (r-a), \dots, i \text{ of } Q \text{ form a subrepresentation of } \mathcal{F}_j\}.$$

For any $i \in \overline{V}_j$, let $V_{i,j}$ be the subrepresentation of \mathcal{F}_j consisting of vertices $i, i+(r-a), \dots, \xi(r, a)(j)$. For any $i \in \overline{W}_j$, let $W_{i,j}$ be the subrepresentation of \mathcal{F}_j consisting of vertices $\xi(r, a)(j) + (r-a), \dots, i - (r-a), i$.

Note that $i \in \overline{V}_j$ if and only if the y_i -arrow is not σ_j -distinguished, and $i \in \overline{W}_j$ if and only if the z_i -arrow is not σ_j -distinguished.

LEMMA 7.3. Let $\theta \in \text{Wt}(Q)$ be a fixed stability parameter. The representation \mathcal{F}_j is θ -semistable if and only if $\theta(V_{i,j}) \geq 0$ for any $i \in \overline{V}_j$ and $\theta(W_{i,j}) \geq 0$ for any $i \in \overline{W}_j$. It is θ -stable if and only if the above conditions hold with strict inequalities.

PROOF. The ‘only if’ direction is obvious. Let U be a subrepresentation of \mathcal{F}_j . Without loss of generality, assume it is supported on the vertices $i, i+(r-a), \dots, i+s(r-a)$. Then, by Definition 4.1, $i \in \overline{V}_j$ and $i+s(r-a) \in \overline{W}_j$ and hence $V_{i,j}, W_{i+s(r-a),j}$ are subrepresentations of \mathcal{F}_j . The lemma follows since $\theta(U) = \theta(V_{i,j}) + \theta(W_{i+s(r-a),j})$. \square

EXAMPLE 7.4. The following diagram shows the representations $\mathcal{F}_0, \dots, \mathcal{F}_4$, respectively, in the case of $\frac{1}{5}(1, 2, 3)$ (cf. Example 3.8). The solid arrows stand for arrows represented by a non-zero number, i.e., σ_j -distinguished.

$$\begin{aligned} \mathcal{F}_0 : & \quad 0 \xrightarrow{z_0} 3 \xrightarrow{z_3} 1 \xrightarrow{z_1} 4 \xrightarrow{z_4} 2 \\ \mathcal{F}_1 : & \quad 2 \xleftarrow{y_0} 0 \xrightarrow{z_0} 3 \xrightarrow{z_3} 1 \xrightarrow{z_1} 4 \\ \mathcal{F}_2 : & \quad 1 \xrightarrow{z_1} 4 \xleftarrow{y_2} 2 \xleftarrow{y_0} 0 \xrightarrow{z_0} 3 \\ \mathcal{F}_3 : & \quad 4 \xleftarrow{y_2} 2 \xleftarrow{y_0} 0 \xrightarrow{z_0} 3 \xleftarrow{y_1} 1 \\ \mathcal{F}_4 : & \quad 3 \xleftarrow{y_1} 1 \xleftarrow{y_4} 4 \xleftarrow{y_2} 2 \xleftarrow{y_0} 0 \end{aligned}$$

Moreover, $\xi(5, 2)(0)=2, \xi(5, 2)(1)=4, \xi(5, 2)(2)=3, \xi(5, 2)(3)=1, \xi(5, 2)(4)=0$ and, for example $\overline{V}_2 = \{1, 3, 4\}, \overline{W}_2 = \{2, 3, 4\}$. If U is a subrepresentation of \mathcal{F}_2 consisting of vertices 2, 4, then $\theta(U) = \theta(V_{4,2}) + \theta(W_{2,2})$.

DEFINITION 7.5. We set $\varphi(r, a)(j) = \langle \xi(r, a)(j) + (r - a) \rangle_r$ for any coprime a, r and $j = 0, \dots, r - 1$.

Note that since $\xi(r, a)$ is a permutation of the set $\{0, \dots, r - 1\}$, so is $\varphi(r, a)$. Let n_0, \dots, n_{r-1} be some rational numbers. The addition in the indices of n_i 's is modulo r . Observe that there is a linear map $Q^r \ni (n_0, \dots, n_{r-1}) \rightarrow \text{Wt}(Q)$ of rank $r - 1$ sending (n_0, \dots, n_{r-1}) to the function $\theta \in \text{Wt}(Q)$ such that $\theta(i) = n_i - n_{i+(r-a)}$ for $i = 0, \dots, r - 1$ with kernel spanned by the vector $(1, \dots, 1)$.

LEMMA 7.6. For any coprime r, a and any rational numbers n_0, \dots, n_{r-1} , set $\theta(i) = n_i - n_{i+(r-a)}$. The representations $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$ are simultaneously θ -stable if and only if

$$n_{\varphi(r,a)(0)} < n_{\varphi(r,a)(1)} < \dots < n_{\varphi(r,a)(r-1)}.$$

PROOF. Fix j and $i \neq \xi(r, a)(j) + (r - a)$. Then either the $z_{i-(r-a)}$ - or the y_i -arrow is σ_j -distinguished. In the first case, $i \in \overline{V}_j$ and $\theta(V_{i,j}) = n_i - n_{\xi(r,a)(j)+(r-a)} > 0$. Otherwise $i - (r - a) \in \overline{W}_j$ and $\theta(W_{i-(r-a),j}) = -n_i + n_{\xi(r,a)(j)+(r-a)} > 0$. By the definition of the permutation τ and by the definition of the divisors Y_i, Z_i , exactly $r - 1 - j$ of z -arrows are σ_j -distinguished. Moreover, if $y_{j'}$ -arrow is $\sigma_{j'}$ -distinguished, then it is σ_j -distinguished for any $j \geq j'$ (cf. Lemma 6.5). To prove the ‘only if’ direction, note that if \mathcal{F}_0 is θ -stable, then $n_{\varphi(r,a)(0)}$ is smallest in n_i 's. If in addition \mathcal{F}_1 is θ -stable, then $n_{\varphi(r,a)(1)}$ is second smallest, and so on. For the other direction, note that, for any $j = 0, \dots, r - 1$, we have $\theta(V_{i,j}) > 0$ for any $i \in \overline{V}_j$ and $\theta(W_{i,j}) > 0$ for any $i \in \overline{W}_j$. By Lemma 7.3 the representation \mathcal{F}_j is θ -stable for $j = 0, \dots, r - 1$. \square

REMARK 7.7. For any coprime r and a , the set of stability conditions for which the representations $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$ are simultaneously θ -stable is a simplicial and top dimensional cone in the space $\text{Wt}(Q)$, since it is in bijection with the cone given by the conditions $n_{\varphi(r,a)(0)} = 0$ and $0 < n_{\varphi(r,a)(1)} < \dots < n_{\varphi(r,a)(r-1)}$.

8. Main theorem. We have defined a family of pairwise non-isomorphic representations of the McKay quiver Q on the Danilov resolution Y , which are θ -stable with respect to stability parameters θ determined in Lemma 7.6. The universal property of the moduli space $M_\theta(Q, R)$ will ensure that the Danilov resolution dominates one of its components.

DEFINITION 8.1 (Craw, Maclagan, Thomas). Let $\theta \in \text{Wt}(Q)$ be generic. The coherent component Y_θ is the unique irreducible component of the moduli $M_\theta(Q, R)$, containing representations of the McKay quiver with all arrows represented by a non-zero number (cf. [3, Theorem 4.3]).

Note that representations of the McKay quiver with all arrows represented by a non-zero number are θ -stable under any stability condition $\theta \in \text{Wt}(Q)$. The coherent component

is reduced irreducible, not-necessarily-normal, toric variety of dimension 3, projective over $X = \mathbb{C}^3/G$ (see [3, Theorem 4.3]). Denote by π_θ the corresponding projective birational morphism

$$\pi_\theta : Y_\theta \rightarrow X .$$

Denote by π the natural toric morphism given by a sequence of toric weighted blowups

$$\pi : Y \rightarrow X ,$$

where Y denotes the Danilov resolution.

Assume that the generic stability parameter θ is chosen so that all representations of the McKay quiver in the family \mathcal{F} on Y are θ -stable. Since θ is generic, by King [12, Proposition 5.3], there exists a universal family of McKay quiver representations over $M_\theta(Q, R)$ and there exists a unique morphisms

$$\rho : Y \rightarrow M_\theta(Q, R) .$$

Since the family \mathcal{F} is defined by Logvinenko's reductor condition, by [13, Theorem 4.1, Definition 1.4] we see that $\pi_\theta \circ \rho = \pi$.

THEOREM 8.2 (Main Theorem). *For any coprime natural numbers a, r and any rational numbers n_0, \dots, n_{r-1} such that*

$$n_{\varphi(r,a)(0)} < \dots < n_{\varphi(r,a)(r-1)} ,$$

where $\varphi(r, a)(j) = \langle \xi(r, a)(j) - a \rangle_r$ and $\xi(r, a)$ is the inverse of the permutation $\tau(r, a)$ (see Definition 3.1), the Danilov resolution of the singularity of type $\frac{1}{r}(1, a, r - a)$ is the normalization of the coherent component $Y_\theta \subset M_\theta(Q, R)$ for generic $\theta \in \text{Wt}(Q)$, given by the condition

$$\theta(i) = n_i - n_{i+(r-a)} .$$

PROOF. Let θ be a generic stability condition satisfying the above conditions (cf. Remark 7.7). By Lemma 7.6, every representation of the McKay quiver in the family $\mathcal{F}(r, a)$ on the Danilov resolution, constructed in Section 5, is θ -stable. Therefore, there exists a unique morphisms

$$\rho : Y \rightarrow M_\theta(Q, R) ,$$

and the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & Y_\theta \\ \pi \searrow & & \swarrow \pi_\theta \\ & X & \end{array}$$

Morphism ρ is proper since morphisms π and π_θ are proper (see [7, Corollary II.4.8.(e)], π_θ is projective, hence separated). By [7, Exercise II.4.4], the image of ρ in Y_θ is closed and is of dimension 3 (see Corollary 5.8).

By the work of Craw, Maclagan, Thomas [3], the coherent component Y_θ is a not-necessarily-normal toric variety of dimension 3, hence ρ is surjective onto Y_θ . We are done

if ρ is injective, that is, representations of the McKay quiver corresponding to distinct closed points on the Danilov resolution are non-isomorphic. This is the content of Corollary 5.8. \square

DEFINITION 8.3. A chamber of stability conditions is a connected component of the set of generic stability conditions (cf. [5], [16]).

THEOREM 8.4. *The closure of the cone defined by the condition of Theorem 8.2 is a union of closures of chambers of stability conditions for the action of the group $\mathrm{PGL}(Q, \mathbb{C})$ on the space $\mathrm{Rep}(Q, R)$.*

PROOF. By proof of Lemma 7.6, if some of the inequalities in the condition for n_i is not strict, then some representation \mathcal{F}_j is strictly θ -semistable. Conversely, if all inequalities are strict, then all representations \mathcal{F}_j for $j = 0, \dots, r - 1$ are θ -stable. \square

We note that there may be non-generic points on the other components of $\mathrm{Rep}(Q, R)$ which may define walls subdividing the simplicial cone of stability conditions defined in Theorem 8.2.

EXAMPLE 8.5. The permutation $\tau(5, 2)$ is cyclic of length 5, namely

$$\tau(5, 2) = (0, 4, 1, 3, 2).$$

The sequence

$$n_0 < n_2 < n_1 < n_4 < n_3$$

implies that the union of closures of chambers is given by the conditions $\theta_0 + \dots + \theta_4 = 0$ and

$$0 \leq \theta_2 \leq \theta_1 + \theta_2 + \theta_4 \leq \theta_2 + \theta_4 \leq -\theta_0.$$

Using computer algebra packages, the author has checked that, for small values of a and r and stability parameters as in the Main Theorem, the moduli space of representation of the McKay quiver is normal. This suggests that the following holds.

CONJECTURE 8.6. The coherent component is normal in this case, that is, the Danilov resolution is isomorphic to the coherent component Y_θ for any θ in Theorem 8.2.

REFERENCES

- [1] T. BRIDGELAND, A. KING AND M. REID, The McKay correspondence as an equivalence of derived categories, *J. Amer. Math. Soc.* 14 (2001), 535–554.
- [2] A. CRAW AND A. ISHII, Flops of G -Hilb and equivalences of derived categories by variation of GIT quotient, *Duke Math. J.* 124 (2004), 259–307.
- [3] A. CRAW, D. MACLAGAN AND R. R. THOMAS, Moduli of McKay quiver representations. I. The coherent component, *Proc. Lond. Math. Soc.* (3) 95 (2007), 179–198.
- [4] V. I. DANILOV, Birational geometry of three-dimensional toric varieties, *Izv. Akad. Nauk SSSR Ser. Mat.* 46 (1982), 971–982, 1135.
- [5] I. V. DOLGACHEV AND Y. HU, Variation of geometric invariant theory quotients, *Inst. Hautes Études Sci. Publ. Math.* 87(1998), 5–56.

- [6] W. FULTON, Introduction to toric varieties, Ann. of Math. Stud. 131, The William H. Roever Lectures in Geometry, Princeton University Press, Princeton, NJ, 1993.
- [7] R. HARTSHORNE, Algebraic geometry, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977.
- [8] Y. ITO AND H. NAKAJIMA, McKay correspondence and Hilbert schemes in dimension three, Topology 39 (2000), 1155–1191.
- [9] Y. ITO AND I. NAKAMURA, Hilbert schemes and simple singularities, In New trends in algebraic geometry (Warwick, 1996), 151–233, London Math. Soc. Lecture Note Ser. 264, Cambridge Univ. Press, Cambridge, 1999.
- [10] O. KEDZERSKI, Cohomology of the G -Hilbert scheme for $\frac{1}{r}(1, 1, r - 1)$, Serdica Math. J. 30 (2004), 293–302.
- [11] O. KĘDZIERSKI, The G -Hilbert scheme for $\frac{1}{r}(1, a, r-a)$, Glasg. Math. J. 53 (2010), 115–129.
- [12] A. D. KING, Moduli of representations of finite-dimensional algebras, Quart. J. Math. Oxford Ser. (2) 45 (1994), 515–530.
- [13] T. LOGVINENKO, Natural G -constellation families, Doc. Math. 13 (2008), 803–823.
- [14] D. R. MORRISON AND G. STEVENS, Terminal quotient singularities in dimensions three and four, Proc. Amer. Math. Soc. 90 (1984), 15–20.
- [15] M. REID, Young person's guide to canonical singularities, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math. 46, Amer. Math. Soc., Providence, RI, 1987.
- [16] M. THADDEUS, Geometric invariant theory and flips, J. Amer. Math. Soc. 9 (1996), 691–723.

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