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MEASURE OF A 2-COMPONENT LINK

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Abstract. A two-component link produces a torus as the product of the component knots in a two-point configuration space of a three-sphere. This space can be identified with a cotangent bundle and also with an indefinite Grassmannian. We show that the integration of the absolute value of the canonical symplectic form is equal to the area of the torus with respect to the pseudo-Riemannian structure, and that it attains the minimum only at the "best" Hopf links.

Introduction. Since energy of knots was introduced in [14] about twenty years ago, aiming at producing an optimal knot for each knot type as an energy minimizer, a lot of related works have appeared, which form so-called *geometric knot theory* (see, for example, [4, 5, 16]). The present paper deals with the same type of topic. We introduce a functional on the space of 2-component links such that the absolute minimum is attained only at "best" Hopf links, not at trivial links.

Let $C_1 \cup C_2$ be a 2-component link in S^3. The value of our functional $A(C_1, C_2)$ can be interpreted in the following two ways. Observe that the link produces a torus $C_1 \times C_2$ in $S^3 \times S^3 \setminus \Delta$, where Δ is the diagonal set.

First, there is a natural identification between $S^3 \times S^3 \setminus \Delta$ and the total space of the cotangent bundle T^*S^3. The pull-back ω of the canonical symplectic form of T^*S^3 to $S^3 \times S^3 \setminus \Delta$ is the unique 2-form (up to multiplication by a constant) which is invariant under the diagonal action of the Möbius group. The 2-form ω can also be considered as a natural symplectic form on the space of geodesics in a hyperbolic 4-space H^4. As ω is exact, $\int_{C_1 \times C_2} \omega$ vanishes, but $\int_{C_1 \times C_2} |\omega|$ does not, which is $A(C_1, C_2)$. In this sense, it can be considered as an "*absolute symplectic measure*" of the torus $C_1 \cup C_2$ in T^*S^3.

Second, from a Möbius geometric viewpoint, $S^3 \times S^3 \setminus \Delta$ can be identified with the Grassmannian manifold $SO(4,1)/SO(1,1) \times SO(3)$ of oriented time-like 2-dimensional vector subspaces in the 5-dimensional Minkowski space $\boldsymbol{R}^5_1$. By taking a pseudo-orthogonal complement of an oriented time-like 2-dimensional vector subspace, we can identify this space with the Grassmannian manifold of oriented space-like 3-dimensional vector subspaces in $\boldsymbol{R}^5_1$.

It has a natural pseudo-Riemannian structure which is compatible with the action of the Lorentz group, which induces the diagonal action of the Möbius group to $S^3 \times S^3 \setminus \Delta$. Then

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$A(C_1, C_2)$ is equal to the measure (area) of the torus $C_1 \cup C_2$ with respect to the pseudo-Riemannian metric.

The key of the proof is that both the pull-back ω of the canonical symplectic form and the "*imaginary signed area element*" with respect to the pseudo-Riemannian structure coincide with the real part of the *infinitesimal cross ratio*, which is a "complex valued 2-form" on $C_1 \times C_2$ used in the joint paper with Langevin [12]. Geometrically, it can be considered as the cross ratio of x, $x + dx$, y and $y + dy$, where these four points are considered as complex numbers by identifying a sphere through them with the Riemann sphere $\boldsymbol{C} \cup \{\infty\}$.

Some remarks on the result on the energy of links [1], which is another characterization of the "best" Hopf link, will be given in Subsection 4.4.

Throughout the paper, a link means a smooth (or at least of class C^1) 2-component link.

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1. Two structures on $S^3 \times S^3 \setminus \Delta$. We introduce two structures on $S^3 \times S^3 \setminus \Delta$, the symplectic structure and the pseudo-Riemannian structure, both compatible with Möbius transformations. It is easy to see that both can be naturally generalized to $S^n \times S^n \setminus \Delta$ for any n.

1.1. Symplectic structure of $S^3 \times S^3 \setminus \Delta$.

1.1.1. Via hyperbolic space. As S^3 can be considered as the boundary of 4-dimensional hyperbolic space H^4, $S^3 \times S^3 \setminus \Delta$ can be considered as the space of oriented geodesics in H^4, which is denoted by $\mathcal{G}$. The tangent space $T_\gamma \mathcal{G}$ along a geodesic γ is the space of Jacobi fields along γ. Let ∇ denote the Levi-Civita connection. Then, if we put

$$\omega_g(\xi, \eta) = (\xi(t), \nabla_{\dot{\gamma}}\, \eta(t)) - (\eta(t), \nabla_{\dot{\gamma}}\, \xi(t)) \quad (t \in \boldsymbol{R})$$

for $\xi, \eta \in T_\gamma \mathcal{G}$, where (,) denotes the standard inner product on $T_{\gamma(t)} H^4$, then ω_g is an isometry-invariant symplectic form on $\mathcal{G}$ (see [3, 2C], [11, 3.1]). Since an isometry of H^4 induces a Möbius transformation of the boundary sphere S^3, ω_g defines a symplectic form on $S^3 \times S^3 \setminus \Delta$ which is invariant under the diagonal action of the Möbius group.

1.1.2. Via cotangent bundle. It is known that the space $\mathcal{G}$ of geodesics in H^4 is symplectomorphic to the cotangent bundle T^*S^3 [6]. Let us give an identification between $S^3 \times S^3 \setminus \Delta$ and T^*S^3 explicitly.

Assume S^3 is the unit sphere in $\boldsymbol{R}^4$. Let x be a point in S^3 and $p_{\boldsymbol{x}} \colon S^3 \setminus \{x\} \to (\mathrm{Span}\langle x \rangle)^\perp$ be a stereographic projection. By identifying $(\mathrm{Span}\langle x \rangle)^\perp$ with $T_{\boldsymbol{x}} S^3 \cong T_{\boldsymbol{x}}^* S^3$, we obtain a bijection

$$\varphi_{\boldsymbol{x}} \colon S^3 \setminus \{x\} \ni y \mapsto \big(T_{\boldsymbol{x}} S^3 \ni v \mapsto p_{\boldsymbol{x}}(y) \cdot v \in \boldsymbol{R}\big) \in T_{\boldsymbol{x}}^* S^3,$$

where $\cdot$ denotes the standard inner product in $\boldsymbol{R}^4$. It induces a bijection

$$\varphi \colon S^3 \times S^3 \setminus \Delta \ni (x, y) \mapsto (x, \varphi_{\boldsymbol{x}}(y)) \in T^* S^3. \tag{1}$$

Let ω_{S^3} be the canonical symplectic form of the cotangent bundle T^*S^3. Put $\omega = \varphi^*\omega_{S^3}$. In [12], we showed that ω is invariant under the diagonal action of the Möbius group. The converse is also true. Namely, if a 2-form ρ is invariant under the diagonal action of the Möbius group, then $\rho = c\,\omega$ for some $c \in \boldsymbol{R}$ (Proposition 4.1 in Appendix). Therefore, we can see that ω coincides with ω_g mentioned above up to a constant factor.

1.2. Pseudo-Riemannian structure of $S^3 \times S^3 \setminus \Delta$. The *Minkowski space* $\boldsymbol{R}^5_1$ is $\boldsymbol{R}^5$ with the indefinite inner product

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_4y_4 .$$

The set of light-like vectors and the origin $L = \{v \in \boldsymbol{R}^5_1\,;\ \langle v, v\rangle = 0\}$ is called the *light cone*. The 3-sphere can be considered as the projectivization $\boldsymbol{P}L$ of the light cone. It can also be identified isometrically with the intersection of the light cone and a hyperplane given by $\{x\,;\ \langle x, n\rangle = -1\}$, where n is a unit time-like vector. A 2-dimensional vector subspace Π of $\boldsymbol{R}^5_1$ is said to be time-like if $\langle\ ,\ \rangle|_\Pi$ is non-degenerate and indefinite, namely, if Π intersects the light cone transversely.

A pair of points in S^3 can be considered as the intersection of S^3 and a 2-dimensional time-like subspace of $\boldsymbol{R}^5_1$. Therefore, if we also take the order of the points into account, $S^3 \times S^3 \setminus \Delta$ can be identified with the Grassmannian manifold $\widetilde{\mathrm{Gr}}_-(2; \boldsymbol{R}^5_1)$ of oriented 2-dimensional time-like subspaces of $\boldsymbol{R}^5_1$, i.e., a homogeneous space $SO(4,1)/SO(3) \times SO(1,1)$.

Let Π be an oriented time-like 2-dimensional plane spanned by an ordered basis $\{u, v\}$. Then Π corresponds to a pure 2-vector $u \wedge v \in \bigwedge^2 \boldsymbol{R}^5_1$, which is determined by Π up to a positive factor. As is stated on page 280 of [9], $u \wedge v$ is time-like, i.e., $\langle u\wedge v, u\wedge v\rangle < 0$, where the indefinite inner product on $\bigwedge^2 \boldsymbol{R}^5_1$ is given by

$$\langle u^1 \wedge u^2, v^1 \wedge v^2\rangle = \det\big(\langle u^i, v^j\rangle\big) .$$

On the other hand, it is known that a pure 2-vector determines a 2-plane. Thus the Grassmannian manifold $\widetilde{\mathrm{Gr}}_-(2; \boldsymbol{R}^5_1)$ of oriented 2-dimensional time-like subspaces of $\boldsymbol{R}^5_1$ can be identified with the set of unit time-like pure 2-vectors in $\bigwedge^2 \boldsymbol{R}^5_1$, where the norm of $\bigwedge^2 \boldsymbol{R}^5_1$ is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$. It is a 6-dimensional pseudo-Riemannian manifold with index 3. By taking a pseudo-orthogonal complement of an oriented time-like 2-dimensional vector subspace, we can identify $\widetilde{\mathrm{Gr}}_-(2; \boldsymbol{R}^5_1)$ with the Grassmannian manifold $\widetilde{\mathrm{Gr}}_+(3; \boldsymbol{R}^5_1)$ of oriented space-like 3-dimensional vector subspaces in $\boldsymbol{R}^5_1$, which, in turn, can be identified with the set $\Theta(0,3)$ of unit space-like pure 3-vectors in $\bigwedge^3 \boldsymbol{R}^5_1$.

Through the identifications mentioned above, the bijection from $\widetilde{\mathrm{Gr}}_-(2; \boldsymbol{R}^5_1)$ to $\widetilde{\mathrm{Gr}}_+(3; \boldsymbol{R}^5_1)$ is equal to the minus of the restriction of the Hodge $\star$ which is an isomorphism from $\bigwedge^2 \boldsymbol{R}^5_1$ to $\bigwedge^3 \boldsymbol{R}^5_1$ given by

$$a \wedge \star b = \langle a, b\rangle\, e_0 \wedge e_1 \wedge \cdots \wedge e_4 \quad \Big(a, b \in \bigwedge^2 \boldsymbol{R}^5_1\Big)$$

(see [9, p. 288]).

Let u and v be light-like vectors in $\boldsymbol{R}^5_1$. Put $u \times v = -\star(u \wedge v) \in \bigwedge^3 \boldsymbol{R}^5_1$. Since the Hodge $\star$ satisfies $\langle \star a, \star b \rangle = -\langle a, b \rangle$, where $a, b \in \bigwedge^2 \boldsymbol{R}^5_1$, we have

$$\langle u^1 \times u^2, v^1 \times v^2 \rangle = -\det\left(\langle u^i, v^j \rangle\right). \tag{2}$$

Thus we have a bijection

$$\psi : S^3 \times S^3 \setminus \varDelta \ni (x, y) \mapsto \frac{x \times y}{\|x \times y\|} \in \varTheta(0,3). \tag{3}$$

Since the indefinite inner product in (2) is invariant under the action of the Lorentz group $O(4,1)$, the pseudo-Riemannian structure on $S^3 \times S^3 \setminus \varDelta$ induced by ψ is invariant under the diagonal action of the Möbius group.

2. Measure of a 2-component link. All the pairs of points $\{(x, y)\,;\ x \in C_1, y \in C_2\}$ form a torus in $S^3 \times S^3 \setminus \varDelta$. Let us call it the *product torus* of a 2-component link $L = C_1 \cup C_2$.

2.1. Area of the product torus of a link. Let σ be the composite of maps:

$$\sigma : C_1 \times C_2 \overset{\iota}{\hookrightarrow} S^3 \times S^3 \setminus \varDelta \xrightarrow[\psi]{\cong} \varTheta(0,3).$$

We identify $\sigma(C_1 \times C_2)$ with $C_1 \times C_2$ in what follows. The *area element* dv of $C_1 \times C_2$ associated with the pseudo-Riemannian structure of $\varTheta(0,3)$ is given by

$$dv = \sqrt{\left|\det\begin{pmatrix} \langle \sigma_x, \sigma_x \rangle & \langle \sigma_x, \sigma_y \rangle \\ \langle \sigma_y, \sigma_x \rangle & \langle \sigma_y, \sigma_y \rangle \end{pmatrix}\right|}\, dx \wedge dy\,,$$

where σ_x and σ_y denote $\partial\sigma/\partial x(x, y)$ and $\partial\sigma/\partial y(x, y)$ in $T_{\sigma(x,y)}\varTheta(0,3)$, respectively.

DEFINITION 2.1. Define the *measure* of a 2-component link $L = C_1 \cup C_2$ by the area of the product torus

$$A(C_1, C_2) = \int_{C_1 \times C_2} dv = \int_{C_1 \times C_2} \sqrt{\left|\det\begin{pmatrix} \langle \sigma_x, \sigma_x \rangle & \langle \sigma_x, \sigma_y \rangle \\ \langle \sigma_y, \sigma_x \rangle & \langle \sigma_y, \sigma_y \rangle \end{pmatrix}\right|}\, dx \wedge dy\,.$$

2.2. Main Theorem.

THEOREM 2.2. (i) *The measure of a* 2*-component link satisfies*

$$A(C_1, C_2) = \int_{C_1 \times C_2} |\iota^* \omega|\,, \tag{4}$$

where ι is the inclusion from $C_1 \times C_2$ into $S^3 \times S^3 \setminus \varDelta$ and ω is the pull-back of the canonical symplectic form of T^*S^3 to $S^3 \times S^3 \setminus \varDelta$.

(ii) *The measure of a* 2*-component link takes its minimum value* 0 *if and only if L is the image of the "best" Hopf link*

$$\{(z, w) \in \boldsymbol{C}^2;\ |z| = 1, w = 0\} \cup \{(z, w) \in \boldsymbol{C}^2;\ z = 0, |w| = 1\} \subset S^3 \tag{5}$$

by a Möbius transformation.

The equation (4) implies that the area of the product torus can also be called the "*absolute symplectic measure*" of it.

We prove the theorem in the next section.

2.3. Area element of a product torus in $S^3 \times S^3 \setminus \Delta$.

LEMMA 2.3. *Both σ_x and σ_x are null vectors, i.e., $\langle \sigma_x, \sigma_x \rangle = \langle \sigma_y, \sigma_y \rangle = 0$. Therefore the area element dv is given by $\sigma^* dv = |\langle \sigma_x, \sigma_y \rangle| \, dx \wedge dy$.*

PROOF. Suppose S^3 is embedded in $\boldsymbol{R}^5_1$, and points in C_1 and C_2 are expressed by $\bar{x}(s)$ and $\bar{y}(t)$, respectively. Put $p(s,t) = \bar{x}(s) \times \bar{y}(t)$ and $\tilde{\sigma}(s,t) = \sigma(\bar{x}(s), \bar{y}(t))$. Then it is given by

$$\tilde{\sigma}(s,t) = \frac{p(s,t)}{\langle p(s,t), p(s,t) \rangle^{1/2}}\,.$$

Since $\bar{x}$ and $\bar{y}$ are light-like vectors, the formula (2) implies

$$\langle p, p \rangle = \langle \bar{x}, \bar{y} \rangle^2\,, \quad \langle p, p_s \rangle = \langle \bar{x}, \bar{y} \rangle \langle \bar{x}_s, \bar{y} \rangle\,, \quad \langle p_s, p_s \rangle = \langle \bar{x}_s, \bar{y} \rangle^2\,.$$

Therefore

$$\langle \tilde{\sigma}_s, \tilde{\sigma}_s \rangle = \frac{\langle p, p \rangle \langle p_s, p_s \rangle - \langle p, p_s \rangle^2}{\langle p, p \rangle^2} = 0\,.$$

□

We also put geometric explanation in Subsection 4.2 in Appendix.

Let us call $\langle \sigma_x, \sigma_y \rangle \, dx \wedge dy$ the *imaginary signed area element* of a product torus $C_1 \times C_2$.

3. Proof of the main theorem.

3.1. The infinitesimal cross ratio. We assume that both components C_1 and C_2 are oriented. Suppose $x \in C_1$ and $y \in C_2$. Let $\Gamma(x,x,y)$ be the circle which is tangent to C_1 at x that passes through y, oriented by the tangent vectors to C_1 at x. Let θ $(0 \le \theta \le \pi)$ be the angle between $\Gamma(x,x,y)$ and the tangent vector to C_2 at y. We call it the *conformal angle* between x and y and denote it by $\theta_L(x,y)$. It was introduced by Doyle and Schramm.

Let Ω_L be a complex valued 2-form on $C_1 \times C_2$ given by

$$\Omega_L(x,y) = e^{i\theta_L(x,y)} \frac{dx \wedge dy}{|x-y|^2} \tag{6}$$

(see [12]). As both the conformal angle θ_L and the 2-form $dxdy/|x-y|^2$ are equivariant under the diagonal action of a Möbius transformation T, so is Ω_L, namely, $(T \times T)^* \Omega_{T(L)} = \Omega_L$ [12].

Let us give a geometric interpretation of Ω_L. Let $\Sigma_L(x,y)$ be a sphere that passes through four points x, $x+dx$, y and $y+dy$, i.e., a sphere which is tangent to C_1 at x and to C_2 at y. Let p be a stereographic projection from $\Sigma_L(x,y)$ to $\boldsymbol{C} \cup \{\infty\}$ and $\tilde{x}$, $\tilde{x} + \widetilde{dx}$, $\tilde{y}$ and $\tilde{y} + \widetilde{dy}$ the images by p of the four points x, $x+dx$, y and $y+dy$, respectively. Then $\Omega_L(x,y)$ is equal to the cross ratio $(\tilde{x} + \widetilde{dx}, \tilde{y}; \tilde{x}, \tilde{y} + \widetilde{dy})$:

$$\Omega_L(x,y) = \frac{\widetilde{dx}\widetilde{dy}}{(\tilde{x}-\tilde{y})^2} \sim \frac{(\tilde{x}+\widetilde{dx}) - \tilde{x}}{(\tilde{x}+\widetilde{dx}) - (\tilde{y}+\widetilde{dy})} : \frac{\tilde{y}-\tilde{x}}{\tilde{y} - (\tilde{y}+\widetilde{dy})}\,. \tag{7}$$

This is why we call Ω_L the *infinitesimal cross ratio*. We remark that the cross ratio does not depend on the stereographic projection p.

REMARK 3.1. The form $dzdw/(z-w)^2$ on $\boldsymbol{C} \times \boldsymbol{C} \setminus \Delta$, which has been used in complex analysis, can also be obtained as the cross ratio of $w, w+dw, z$ and $z+dz$, as was mentioned by Rob Kusner, for example. In this sense, the infinitesimal cross ratio can be considered as generalization of $dzdw/(z-w)^2$ to a complex valued 2-form on $C_1 \times C_2$, or in general, $C \times C \setminus \Delta$, where C is a union of space curves. In fact, when C is a *plane* curve, the infinitesimal cross ratio coincides up to complex conjugacy with the 2-form that is obtained by restricting $dzdw/(z-w)^2$ to $C \times C \setminus \Delta$, which was used by Hélein [8] to show the isoperimetric inequality.

However, there is difficulty for space curves. First, $dzdw/(z-w)^2$ cannot be generalized to a 2-form on the ambient space $S^3 \times S^3 \setminus \Delta$, so the restriction which works for the planar case does not work. To be precise, while the real part of $dzdw/(z-w)^2$ can be generalized to a 2-form on $S^n \times S^n \setminus \Delta$ as we will see in the next subsection, the imaginary part cannot when $n \geq 3$ as we will see in Proposition 4.1.

Secondly, even if we try to use the cross ratio to define the 2-form, the cross ratio of four points in $\boldsymbol{R}^n$ $(n \geq 3)$ is not so well-behaved as in the planar case. This might be a reason why Ahlfors studied only the absolute cross ratio for the points in $\boldsymbol{R}^n$ $(n \geq 3)$ [2]. When we want to define the cross ratio of (ordered) four points in $\boldsymbol{R}^3$, we need the orientation of the sphere through the four points to avoid the ambiguity of complex conjugacy. There is a way to assign continuously the orientations to all the spheres given by the sets of ordered four points in $\boldsymbol{R}^3$, i.e., there is a continuous map from $(\boldsymbol{R}^3)^4 \setminus \Delta$, where Δ is a big diagonal set, to the set of oriented 2-speres in $\boldsymbol{R}^3$, which can be identified with the de Sitter space in 5-dimensional Minkowski space $\boldsymbol{R}^5_1$. However, according to this method, the imaginary part of the cross ratio of any four points in $\boldsymbol{R}^3$ is always non-negative (or, always non-positive according to the choice of a continuous map from $(\boldsymbol{R}^3)^4 \setminus \Delta$ to the de Sitter space). The reader is referred to [15] for the details. As a result, the imaginary part of the infinitesimal cross ratio may have singularity where it vanishes, just like that of the absolute value of a smooth function. Anyway, we do not use the imaginary part in this paper.

3.2. The real part of the infinitesimal cross ratio. In [12] we showed that the pull-back of the canonical symplectic form of T^*S^3 to $C_1 \times C_2$ coincides with the real part of the infinitesimal cross ratio up to a constant;

$$\iota^*\omega = \iota^*\varphi^*\omega_{S^3} = -2\,\mathfrak{Re}\,\Omega_L = -2\,\frac{\cos\theta_L(x,y)\,dx \wedge dy}{|x-y|^2}\,. \tag{8}$$

It seems that this fact in the case of S^2 is well known in symplectic geometry.

LEMMA 3.2. *The imaginary signed area element of $C_1 \times C_2$ with respect to the pseudo-Riemannian structure coincides with the real part of the infinitesimal cross ratio up to a constant;*

$$\langle \sigma_x, \sigma_y \rangle\, dx \wedge dy = 2\,\mathfrak{Re}\,\Omega_L\,.$$

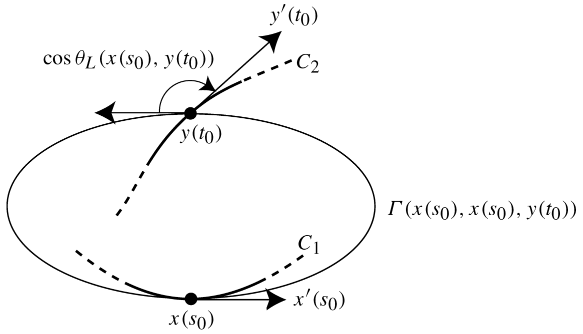

FIGURE 1.

PROOF. Suppose points in C_1 and C_2 are expressed as $x(s)$ and $y(t)$. Suppose S^3 is embedded in $\boldsymbol{R}^5_1$ as the intersection of the light cone and a level hyperplane $\{x_0 = 1\}$. Let $\bar{x}$ and $\bar{y}$ be points in $\boldsymbol{R}^5_1$ corresponding to $x(s)$ and $y(t)$, i.e., $\bar{x}(s) = (1, x(s))$ and $\bar{y}(t) = (1, y(t))$. Put $\tilde{\sigma}(s,t) = \sigma(\bar{x}(s), \bar{y}(t))$ as before.

The pull-back of the real part of the infinitesimal cross ratio is given by

$$\big((x \times y)^* \Re\mathfrak{e}\, \Omega_L\big)(s,t) = \frac{\cos\theta_L(x(s), y(t))}{|x(s) - y(t)|^2} |x'(s)||y'(t)|\, ds \wedge dt\,. \tag{9}$$

On the other hand, the pull-back of the imaginary signed area element is given by

$$\big((x \times y)^* (\langle \sigma_x, \sigma_y \rangle\, dx \wedge dy)\big)(s,t) = \langle \tilde{\sigma}_s, \tilde{\sigma}_t \rangle (s,t)\, ds \wedge dt\,. \tag{10}$$

Fix any (s_0, t_0). The Möbius invariance of the both sides allows us to assume that $x(s_0)$ and $y(t_0)$ are antipodal. Then, at (s_0, t_0),

$$\langle \bar{x}, \bar{x} \rangle = \langle \bar{y}, \bar{y} \rangle = 0\,, \quad \langle \bar{x}, \bar{y} \rangle = -2\,.$$

Therefore, by the formula (2), at (s_0, t_0) there holds

$$\langle p, p \rangle = 4\,, \quad \langle p, p_s \rangle = 0\,, \quad \langle p_s, p_t \rangle = -2x'(s_0) \cdot y'(t_0)\,,$$

which implies

$$\langle \tilde{\sigma}_s, \tilde{\sigma}_t \rangle = \frac{\langle p, p \rangle \langle p_s, p_t \rangle - \langle p, p_s \rangle \langle p, p_t \rangle}{\langle p, p \rangle^2} = -\frac{1}{2} x'(s_0) \cdot y'(t_0)\,.$$

Since $x_0 = x(s_0)$ and $y_0 = y(t_0)$ are antipodal, we have (Figure 1)

$$\theta_L(x(s_0), y(t_0)) = \pi - \angle x'(s_0) \cdot y'(t_0)\,.$$

It follows that

$$\langle \tilde{\sigma}_s, \tilde{\sigma}_t \rangle (s_0, t_0) = -\frac{1}{2} x'(s_0) \cdot y'(t_0) = 2 \frac{|x'(s_0)||y'(t_0)|}{|x(s_0) - y(t_0)|^2} \cos\theta_L(x(s_0), y(t_0))\,,$$

which implies that the right-hand sides of (9) and (10) coincide. $\square$

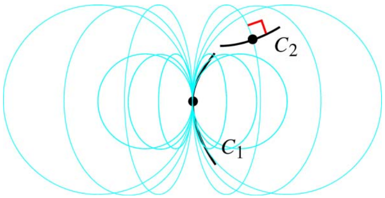

FIGURE 2. The circles of $\mathcal{C}_x$.

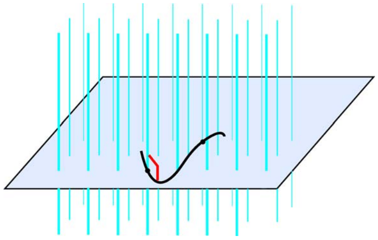

FIGURE 3. The image by π.

We remark that an alternative geometric proof can be obtained if we use pseudo-orthonormal basis of $S^3 \times S^3 \setminus \Delta$ illustrated in Figure 10. This is because

$$\langle \tilde{\sigma}_s + \tilde{\sigma}_t, \tilde{\sigma}_s + \tilde{\sigma}_t \rangle (s_0, t_0) = -x'(s_0) \cdot y'(t_0)$$

implies $\langle \tilde{\sigma}_s, \tilde{\sigma}_t \rangle (s_0, t_0) = -(1/2)\, x'(s_0) \cdot y'(t_0)$.

COROLLARY 3.3. *The imaginary signed area element of a product torus $C_1 \times C_2$ with respect to the pseudo-Riemannian structure is equal to minus the pull-back of the canonical symplectic form:*

$$\langle \sigma_x, \sigma_y \rangle \, dx \wedge dy = -\iota^* \varphi^* \omega_{S^3} .$$

This completes the proof of Theorem 2.2 (i).

We remark that a statement similar to that of the above corollary does not hold for a general surface in $S^3 \times S^3 \setminus \Delta$ as we will see in Subsection 4.3 in Appendix.

3.3. Proof of Theorem 2.2 (ii). As

$$A(C_1, C_2) = 2 \int_{C_1 \times C_2} \frac{|\cos \theta_L(x, y)|}{|x - y|^2} \, dx \, dy \,, \tag{11}$$

it is equal to 0 if and only if the conformal angle $\theta_L(x, y)$ is equal to $\pi/2$ for any $x \in C_1$ and $y \in C_2$.

Suppose $A(C_1, C_2) = 0$. Let x be a point in C_1. Let $\mathcal{C}_x$ be the set of the circles which are tangent to C_1 at x. Then C_2 can intersect circles in $\mathcal{C}_x$ only at a right angle (Figure 2). Consider a stereographic projection π from $S^3 \setminus \{x\}$ to $\boldsymbol{R}^3$. It maps $\mathcal{C}_x$ to the set of parallel lines. Since $\pi(C_2)$ can intersect lines of $\pi(\mathcal{C}_x)$ only at a right angle, $\pi(C_2)$ is contained in a 2-plane which is orthogonal to the lines in $\pi(\mathcal{C}_x)$ (Figure 3). Therefore, C_2 is contained in a sphere Σ_x which intersects C_1 at a right angle at x (Figure 4).

Let x' be a point of C_1 close to x. As C_1 intersects Σ_x orthogonally at x, we can take x' outside Σ_x. Therefore, $\Sigma_x \neq \Sigma_{x'}$. Since C_2 is contained in the intersection $\Sigma_x \cap \Sigma_{x'}$, C_2 must be a circle (Figure 5). The same argument shows that C_1 is also a circle.

Consider the stereographic projection π again. Since C_1 is a circle, $\pi(C_1)$ is a line. Then $\pi(C_2)$ is the intersection of two spheres which intersect the line $\pi(C_1)$ at a right angle (Figure 6). Therefore, $\pi(C_2)$ is symmetric in the line $\pi(C_1)$. It follows that $\pi(C_1) \cup \pi(C_2)$ is an image of the standard Hopf link (Figure 7).

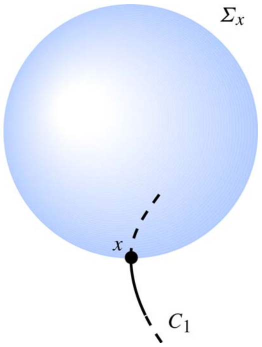

FIGURE 4.

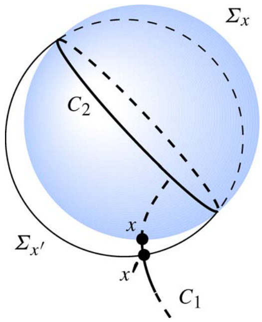

FIGURE 5.

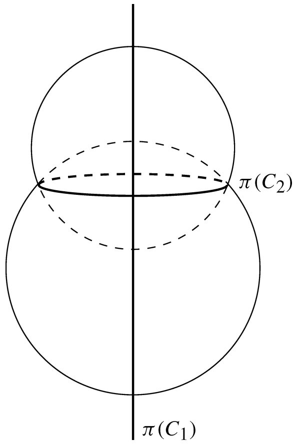

FIGURE 6.

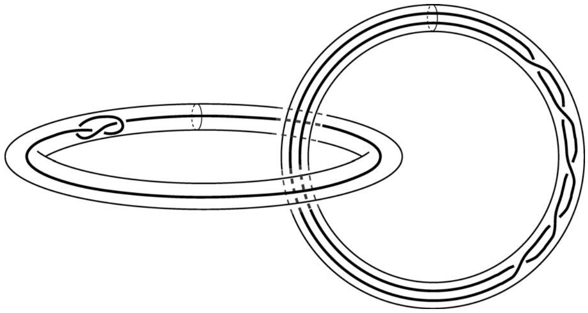

FIGURE 7. A satellite link of a Hopf link.

This completes the proof of Theorem 2.2 (ii).

3.4. Corollary and Conjecture. Let $[L]$ denote an isotopy class of a link L. Define

$$A([L]) = \inf_{C_1' \cup C_2' \in [L]} A(C_1', C_2') .$$

COROLLARY 3.4. *If L is a separable link or a satellite link of a Hopf link, then* Area $([L]) = 0$.

PROOF. Suppose $L = C_1 \cup C_2$ is a separable link in $\boldsymbol{R}^3$. We can make $|x - y|$ $(x \in C_1, y \in C_2)$ as big as we like. Now the conclusion follows from the formula (11).

Suppose $L = C_1 \cup C_2$ is a satellite link of a Hopf link. Then, after an ambient isotopy, it can be contained in a very thin tubular neighbourhood of the standard Hopf link given by (5). Furthermore, for any positive constants δ_1 and δ_2, the link can be placed so that, outside a small region of $C_1 \times C_2$ whose measure is $\delta_1 Length(C_1) \cdot Length(C_2)$, the conformal angle satisfies $|\theta_L - \pi/2| \leq \delta_2$. Then the formula (11) implies the assertion of the corollary since $|x-y|$ ($x \in C_1, y \in C_2$) is bounded below. □

CONJECTURE 3.5. We conjecture that $A([L])$ does not always vanish. For example, if $L = C_1 \cup C_2$ is a hyperbolic link each component of which is a non-trivial knot, then there is no solid torus H_1 so that C_1 is contained in H_1 and C_2 in $\boldsymbol{R}^2 \setminus H_1$. We conjecture that $A([L])$ is positive for such a link type.

4. Appendix.

4.1. Diagonal Möbius invariance characterizes ω.

PROPOSITION 4.1. *Suppose ρ is a 2-form on $S^n \times S^n \setminus \Delta$ which is invariant under the diagonal action of orientation preserving Möbius transformations. Then $\rho = c\,\omega$ for some constant c if $n \neq 1$, where ω is the pull-back of the canonical symplectic form of T^*S^n by the bijection from $S^n \times S^n \setminus \Delta$ to T^*S^n given by* (1), *and $\rho = c_1\omega + c_2\mathfrak{Im}(dz \wedge dw/(w-z)^2)$ under the identification $S^2 \cong \boldsymbol{C} \cup \{\infty\}$ for some $c_1, c_2 \in \boldsymbol{R}$ if $n = 2$.*

This fact has been mentioned in [10] in a more general form (see § 3.2). We put the proof here since the author could not find it in the literature.

PROOF. Since $S^n \times S^n \setminus \Delta$ is a homogeneous space of the Möbius group, it suffices to show the statement for ρ restricted to a point $(x, -x)$ and the action of its isotropy group $H_{(\boldsymbol{x},-\boldsymbol{x})}$.

We may assume, without loss of generality, that x and $-x$ correspond to $(1, 1, 0, \dots, 0)$ and $-x = (1, -1, 0, \dots, 0)$ in the Minkowski space $\boldsymbol{R}^{n+2}_1$, respectively. Suppose an orientation preserving Möbius transformation $T_A \in H_{(\boldsymbol{x},-\boldsymbol{x})}$ is given by $A \in SO(n+1, 1)$. Then, since A keeps both $\mathrm{Span}\langle e_0, e_1\rangle$ and $\mathrm{Span}\langle e_2, \dots, e_{n+1}\rangle$ invariant, A can be expressed as

$$A = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix},$$

where $A_1 \in SO(1,1)$ and $A_2 \in SO(n)$, or $A_1 \in O(1,1) \setminus SO(1,1)$ and $A_2 \in O(n) \setminus SO(n)$. However, in the latter case we have $T_A(x) = -x$, which does not fit the assumption that T_A belongs to the isotropy group $H_{(\boldsymbol{x},-\boldsymbol{x})}$. It follows that the action of an element of $H_{(\boldsymbol{x},-\boldsymbol{x})}$ on

$$T_{(\boldsymbol{x},-\boldsymbol{x})}(S^n \times S^n \setminus \Delta) = T_{\boldsymbol{x}}S^n \oplus T_{-\boldsymbol{x}}S^n \cong \boldsymbol{R}^n \oplus \boldsymbol{R}^n$$

is generated by

$$\begin{aligned} &(u, v) \mapsto (cu, c^{-1}v) \qquad (c \in \boldsymbol{R}^\times), \\ &(u, v) \mapsto (gu, gv) \qquad (g \in SO(n)). \end{aligned} \tag{12}$$

Thus, it is enough to give $\rho(e_2, v)$ ($e_2 \in T_{\boldsymbol{x}}S^n$, $v \in T_{-\boldsymbol{x}}S^n$) to determine ρ.

(i) Suppse $n \geq 3$. Let R be a 180 degree rotation of $\boldsymbol{R}^n = \mathrm{Span}\langle e_2, \dots, e_{n+1}\rangle$ around x_2-axis. Then, since $\omega(e_2, v) = (1/2)\, v \cdot e_2$, we have

$$\rho(e_2, v) = \frac{1}{2}\left(\rho(e_2, v) + \rho(e_2, Rv)\right) = \rho\left(e_2, \frac{v + Rv}{2}\right)$$

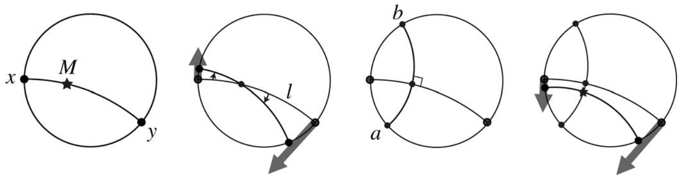

FIGURE 8. Pseudo-orthogonal basis of a tangent space of $S^1 \times S^1 \setminus \varDelta$. A picture in Poincaré disc model.

$$
\begin{aligned}
&= \rho(e_2, (v\cdot e_2)e_2) = (v\cdot e_2)\rho(e_2, e_2) \\
&= 2\rho(e_2, e_2)\, \omega_{S^n}(e_2, v),
\end{aligned} \tag{13}
$$

which implies $\rho = 2\rho(e_2, e_2)\omega$.

(ii) When $n = 1$, the equation (13) also holds, which implies $\rho = 2\rho(e_2, e_2)\,\omega$.

(iii) Suppose $n = 2$. Then

$$
\begin{aligned}
\rho(e_2, v) &= (v\cdot e_2)\rho(e_2, e_2) + (v\cdot e_3)\rho(e_2, e_3) \\
&= 2\rho(e_2, e_2)\,\omega(e_2, v) + 4\rho(e_2, e_3)\,\Im\mathfrak{m}\left(dz\wedge dw/(w-z)^2\right)(e_2, v)\,,
\end{aligned}
$$

which implies $\rho = 2\rho(e_2, e_2)\,\omega + 4\rho(e_2, e_3)\,\Im\mathfrak{m}\left(dz\wedge dw/(w-z)^2\right)$.

□

It follows that the imaginary part of $dz\wedge dw/(w-z)^2$ cannot be generalized to $S^n\times S^n\setminus\varDelta$ when $n\geq 3$. In fact, it can naturally be generalized to a Kähler form on $SO(n+1,1)/SO(2)\times SO(n-1,1)$, which is the space of oriented codimension 2 spheres in S^n.

4.2. Pseudo-orthogonal basis of $S^3\times S^3\setminus\varDelta$. Let us start with a baby case $S^1\times S^1\setminus\varDelta$. It can be identified with the set of oriented time-like planes in the 3-dimensional Minkowski space $\boldsymbol{R}^3_1$. By taking a positive unit normal vector to each of these planes, $S^1\times S^1\setminus\varDelta$ can be identified with the 2-dimensional de Sitter space $\varLambda = \{x\in\boldsymbol{R}^3_1;\ \langle x, x\rangle = 1\}$. Let $\varSigma = \{x, y\}$ be a pair of points in $S^1\cong\partial\boldsymbol{H}^2$. Let l denote the geodesic in $\boldsymbol{H}^2$ which joins x and y. Take a point M on l (Figure 8), then it determines two pencils as follows.

Let a and b be the "end points" of the geodesic in $\boldsymbol{H}^2$ which is orthogonal to l at point M (the third of Figure 8). Let $\mathcal{P}_+$ be a pencil obtained by rotating the geodesic l around M and $\mathcal{P}_-$ the Poncelet pencil with limit points a and b. Then $\mathcal{P}_+$ and $\mathcal{P}_-$ can be considered as geodesics in $\varLambda$, namely, the intersections with $\varLambda$ and space-like and time-like 2-planes $\varPi_\pm$. A pair of the unit tangent vectors to $\mathcal{P}_+$ and $\mathcal{P}_-$ at σ can serve as a pseudo-orthonormal basis of $T_\sigma\varLambda$, where σ is a point in $\varLambda$ that corresponds to $\varSigma$. These vectors can be obtained in $\varPi_\pm$ by rotation and Lorentz boost (hyperbolic rotation) of σ. The corresponding vectors in $S^1\times S^1\setminus\varDelta$ are illustrated as the second and the last of Figure 8

Suppose $\{u, v\}$ is a pseudo-orthonormal basis of $T_\sigma\varLambda$. Then we have another basis, $\{(u+v)/\sqrt{2}, (u-v)/\sqrt{2}\}$ consisting of two light-like vectors (Figure 9). This illustrates why σ_x and σ_y in Subsection 2.3 are null vectors.

FIGURE 9. Light-like basis of a tangent space of $S^1 \times S^1 \setminus \Delta$. A picture in Poincaré disc model.

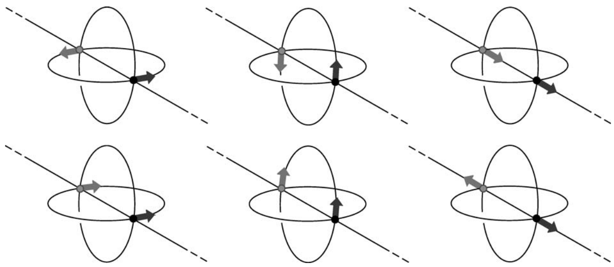

FIGURE 10. Pseudo-orthogonal basis of a tangent space of $S^3 \times S^3 \setminus \Delta$. Space-like vectors above and time-like vectors below. A picture in $\boldsymbol{R}^3$ obtained through a stereographic projection.

The pseudo-orthonormal basis of $S^3 \times S^3 \setminus \Delta$ can be given by that of $S^1 \times S^1 \setminus \Delta$. In fact, we can consider three mutually orthogonal circles through a given pair of points, and take a pseudo-orthonormal basis in each circle as illustrated in Figure 10.

4.3. The imaginary signed area element and the symplectic form. Corollary 3.3 does not necessarily hold for a surface in $S^3 \times S^3 \setminus \Delta$ which is not the product of two curves in S^3. Let us show it in $\boldsymbol{R}^3 \times \boldsymbol{R}^3 \setminus \Delta$, fixing a stereographic projection p from S^3 to $\boldsymbol{R}^3 \cup \{\infty\}$.

Suppose a pair of points in $\boldsymbol{R}^3$ are expressed by $X(s,t)$ and $Y(s,t)$. Let M be a surface $\{(X(s,t), Y(s,t))\}_{(s,t)\in D}$ in $\boldsymbol{R}^3 \times \boldsymbol{R}^3 \setminus \Delta$, where D is a domain in $\boldsymbol{R}^2$. Put $X_s = \partial X/\partial s$, $X_t = \partial X/\partial t$, and

$$\widetilde{X}_s = 2\left(X_s, \frac{X-Y}{|X-Y|}\right)\frac{X-Y}{|X-Y|} - X_s, \ \widetilde{X}_t = 2\left(X_t, \frac{X-Y}{|X-Y|}\right)\frac{X-Y}{|X-Y|} - X_t .$$

Then $\widetilde{X}_s$ is the tangent vector at Y to a circle which is tangent to X_s at X that passes through Y with $\left|\widetilde{X}_s\right| = |X_s|$. The same interpretation also holds for $\widetilde{X}_t$.

Note that the pull-back $\omega_{\boldsymbol{R}^3}$ of the symplectic form ω introduced in Subsection 1.1 by a map $p^{-1} \times p^{-1}$ from $\boldsymbol{R}^3 \times \boldsymbol{R}^3 \setminus \Delta$ to $S^3 \times S^3 \setminus \Delta$, where p is a stereographic projection from

S^3 minus one point to $\boldsymbol{R}^3$, is given by

$$\omega_{\boldsymbol{R}^3} = 2\left(\frac{\sum_{i=1}^3 dX_i \wedge dY_i}{|X-Y|^2} - 2\frac{\left(\sum_{i=1}^3 (X_i - Y_i)\, dX_i\right) \wedge \left(\sum_{j=1}^3 (X_j - Y_j)\, dY_j\right)}{|X-Y|^4} \right)$$

(see [12]). The pull-back of $\omega_{\boldsymbol{R}^3}$ by the map $X \times Y$ from D to $\boldsymbol{R}^3 \times \boldsymbol{R}^3 \setminus \Delta$ is given by

$$(X \times Y)^* \omega_{\boldsymbol{R}^3} = -2\big(\widetilde{X}_s \cdot Y_t - \widetilde{X}_t \cdot Y_s\big) \frac{ds \wedge dt}{|X-Y|^2}\,.$$

This can be verified by showing that the both sides coincide when X and Y are located on specific positions, say $X(s_0, t_0) = (1, 0, 0)$ and $Y(s_0, t_0) = (-1, 0, 0)$ because the both sides are equivariant under the diagonal action of Möbius transformations.

On the other hand, the "*signed area element*" α_M of M associated with the pseudo-Riemannian structure of $\Theta(0, 3)$ can be given as follows. Let $\hat{\sigma}$ be the composite

$$\hat{\sigma} : D \xrightarrow{X \times Y} M \hookrightarrow \boldsymbol{R}^3 \times \boldsymbol{R}^3 \setminus \Delta \xrightarrow{p^{-1} \times p^{-1}} S^3 \times S^3 \setminus \Delta \xrightarrow[\psi]{\cong} \Theta(0, 3)\,.$$

Using the pseudo-orthonormal basis illustrated in Figure 10 and the Möbius invariance, we have

$$\begin{aligned}(X \times Y)^* \alpha_M &= \sqrt{\det \begin{pmatrix} \langle \hat{\sigma}_s, \hat{\sigma}_s \rangle & \langle \hat{\sigma}_s, \hat{\sigma}_t \rangle \\ \langle \hat{\sigma}_t, \hat{\sigma}_s \rangle & \langle \hat{\sigma}_t, \hat{\sigma}_t \rangle \end{pmatrix}}\, ds \wedge dt \\ &= 2\sqrt{\det \begin{pmatrix} 2\widetilde{X}_s \cdot Y_s & \widetilde{X}_s \cdot Y_t + \widetilde{X}_t \cdot Y_s \\ \widetilde{X}_s \cdot Y_t + \widetilde{X}_t \cdot Y_s & 2\widetilde{X}_t \cdot Y_t \end{pmatrix}}\, \frac{ds \wedge dt}{|X-Y|^2}\,.\end{aligned}$$

Therefore, the imaginary signed area element $\sqrt{-1}\,\alpha_M$ coincides with the pull-back of the canonical symplectic form $\omega_{\boldsymbol{R}^3}\big|_{C_1 \times C_2}$ up to sign if and only if $(\widetilde{X}_s \cdot Y_t)(\widetilde{X}_t \cdot Y_s) = (\widetilde{X}_s \cdot Y_s)(\widetilde{X}_t \cdot Y_t)$, which holds if and only if $\widetilde{X}_s \times \widetilde{X}_t \perp Y_s \times Y_t$. It does not always hold in general.

We remark that this condition does not necessarily imply that the surface is a product of two curves. We also remark that the above condition is always satisfied for a surface in $S^1 \times S^1 \setminus \Delta$.

4.4. Remark on energy minimizing Hopf links. There is another variational characterization of the "best" Hopf link.

The *Möbius cross energy* [7] of a 2-component link $C_1 \cup C_2$, which is generalzation of the energy for knots defined by the author [14], is given by

$$E(C_1, C_2) = \int_{C_1 \times C_2} \frac{dx dy}{|x-y|^2}\,.$$

This energy is also invariant under Möbius transformations. Recently, Agol, Marques and Neves proved Freedman-He-Wang's conjecture, namely, they showed that if the linking number of C_1 and C_2 is equal to ± 1, then $E(C_1, C_2) \geq 2\pi^2$, and that the equality holds if and only if $C_1 \cup C_2$ is an image of the "best" Hopf link by a Möbius transformation. This is a much more difficult problem, and was proved using min-max theory which has also been used in the proof of the Willmore conjecture [13].

The formula (11) implies $E(C_1, C_2) \geq (1/2)A(C_1, C_2)$. To be more precise, the equality does not occur since the conformal angle between different components of a link cannot be identically zero. It might be interesting to point out that the infimum of $A(C_1, C_2)$ over all the 2-component links is attained not at trivial links, but at the "best" Hopf link and the conformal image of it, whereas the infimum of $E(C_1, C_2)$ over all the 2-component links is not attained, as $E(C_1, C_2)$ tends to $+0$ as the distance between C_1 and C_2 tends to $+\infty$.

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