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SOME EXAMPLES OF SELF-SIMILAR SOLUTIONS AND TRANSLATING SOLITONS FOR LAGRANGIAN MEAN CURVATURE FLOW

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Abstract. We construct examples of self-similar solutions and translating solitons for Lagrangian mean curvature flow by extending the method of Joyce, Lee and Tsui. Those examples include examples in which the Lagrangian angle is arbitrarily small as the examples of Joyce, Lee and Tsui. The examples are non-smooth zero-Maslov class Lagrangian self-expanders which are asymptotic to a pair of planes intersecting transversely.

1. Introduction. In recent years the Lagrangian mean curvature flow has been extensively studied, as it is a key ingredient in the Strominger-Yau-Zaslow Conjecture [9] and Thomas-Yau Conjecture [10]. Strominger-Yau-Zaslow Conjecture explains Mirror Symmetry of Calabi-Yau 3-folds. In [3], Joyce, Lee and Tsui constructed many examples of self-similar solutions and translating solitons for Lagrangian mean curvature flow. Those Lagrangian submanifolds L are the total space of a 1-parameter family of quadrics Q_s, $s \in I$, where I is an open interval in $\boldsymbol{R}$. In this paper, we construct examples of those Lagrangian submanifolds that associate with the examples of Lagrangian submanifolds given in [2], [3], [4], [5] and so on. To do so we improve theorems in [3] by describing Lagrangian submanifolds of the forms of [3, Ansatz 3.1 and Ansatz 3.3].

Our ambient space is always the complex Euclidean space $\boldsymbol{C}^n$ with coordinates $z_j = x_j + iy_j$ and the standard symplectic form $\omega = \sum_{j=1}^n dx_j \wedge dy_j$. A *Lagrangian submanifold* L is a real n-dimensional submanifold in $\boldsymbol{C}^n$ on which the symplectic form ω vanishes. On L, we can define *Lagrangian angle* $\theta : L \to \boldsymbol{R}$ or $\theta : L \to \boldsymbol{R}/2\pi\boldsymbol{Z}$ by the relation

$$dz_1 \wedge \cdots \wedge dz_n|_L \equiv e^{i\theta}\mathrm{vol}_L\,,$$

and the mean curvature vector H by

$$H = J\nabla\theta\,, \tag{1}$$

where ∇ is the gradient on L and J is the standard complex structure in $\boldsymbol{C}^n$. Equation (1) implies that a Lagrangian submanifold remains Lagrangian under the mean curvature flow, as in Smoczyk [8]. The Maslov class on L is defined by the cohomology class of $d\theta$. Hence L is zero-Maslov class when θ is a single-valued function. A Lagrangian submanifold L is called *Hamiltonian stationary* if the Lagrangian angle θ is harmonic, that is, if $\Delta\theta = 0$, and L is called a *special Lagrangian submanifold* if θ is a constant function. A Hamiltonian stationary

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Lagrangian submanifold is a critical point of the volume functional among all Hamiltonian deformations, and a special Lagrangian is a volume minimizer in its homology class.

DEFINITION 1.1. Let $L \subset \boldsymbol{R}^N$ be a submanifold in $\boldsymbol{R}^N$. L is called *self-similar solution* if $H \equiv \alpha F^{\perp}$ on L for some constant $\alpha \in \boldsymbol{R}$, where $F^{\perp}$ is the orthogonal projection of the position vector F in $\boldsymbol{R}^N$ to the normal bundle of L, and H is the mean curvature vector of L in $\boldsymbol{R}^N$. It is called a *self-shrinker* if $\alpha < 0$ and a *self-expander* if $\alpha > 0$. On the other hand $L \subset \boldsymbol{R}^N$ is called a *translating soliton* if there exists a constant vector T in $\boldsymbol{R}^N$ such that $H \equiv T^{\perp}$, where $T^{\perp}$ is the orthogonal projection of the constant vector T in $\boldsymbol{R}^N$ to the normal bundle of L, and we call T a *translating vector*.

It is well known that if F is a self-similar solution then $F_t = \sqrt{2\alpha t} F$ is moved by the mean curvature flow, and if F is a translating soliton then $F_t = F + tT$ is moved by the mean curvature flow. By Huisken [1], any central blow-up of a finite-time singularity of the mean curvature flow is a self-similar solution.

First we consider self-similar solutions.

THEOREM 1.2. *Let $C, \lambda_1, \dots, \lambda_n \in \boldsymbol{R} \setminus \{0\}$, $\alpha, \psi_1, \dots, \psi_n \in \boldsymbol{R}$, $a_1, \dots, a_n > 0$, and $E > 1$ be constants. Let $I \subset \boldsymbol{R}$ be a connected open neighborhood of $0 \in \boldsymbol{R}$ such that $\inf_{s\in I}(E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s} - 1)$ and $\inf_{s\in I}(1/a_j + \lambda_j s)$ are positive for any $1 \le j \le n$. Define $r_1, \dots, r_n : I \to \boldsymbol{R}$ by*

$$r_j(s) = \sqrt{\frac{1}{a_j} + \lambda_j s}, \quad j = 1, \dots, n, \tag{2}$$

and $\phi_1, \dots, \phi_n : I \to \boldsymbol{R}$ by

$$\phi_j(s) = \psi_j + \frac{\lambda_j}{2}\int_0^s \frac{dt}{(1/a_j + \lambda_j t)\sqrt{E\{\prod_{k=1}^n (1 + a_k\lambda_k t)\}e^{\alpha t} - 1}}, \tag{3}$$

$j = 1, \dots, n$. Then the submanifold L in $\boldsymbol{C}^n$ given by

$$L = \left\{(x_1 r_1(s)e^{i\phi_1(s)}, \dots, x_n r_n(s)e^{i\phi_n(s)}); \sum_{j=1}^n \lambda_j x_j^2 = C, x_j \in \boldsymbol{R}, s \in I\right\}$$

is an immersed Lagrangian submanifold diffeomorphic to $\mathcal{S}^{m-1} \times \boldsymbol{R}^{n-m+1}$, where m is the number of positive λ_j/C, $1 \le j \le n$, and the mean curvature vector H satisfies $CH \equiv \alpha F^{\perp}$ for the position vector F. That is, L is a self-expander when $\alpha/C > 0$ and a self-shrinker when $\alpha/C < 0$. When $\alpha = 0$ the Lagrangian angle θ is constant, so that L is special Lagrangian.

The following Theorem 1.3 is slightly generalized from [3, Theorem C].

THEOREM 1.3. *Let $a_1, \dots, a_n > 0$, $\psi_1, \dots, \psi_n \in \boldsymbol{R}$, $E \ge 1$, and $\alpha \ge 0$ be constants. Define $r_1, \dots, r_n : \boldsymbol{R} \to \boldsymbol{R}$ by*

$$r_j(s) = \sqrt{\frac{1}{a_j} + s^2}, \tag{4}$$

and $\phi_1, \dots, \phi_n : \boldsymbol{R} \to \boldsymbol{R}$ *by*

$$\phi_j(s) = \psi_j + \int_0^s \frac{|t|dt}{(1/a_j + t^2)\sqrt{E\{\prod_{k=1}^n (1 + a_k t^2)\}e^{\alpha t^2} - 1}}. \tag{5}$$

Then the submanifold L *in* $\boldsymbol{C}^n$ *given by*

$$L = \left\{(x_1 r_1(s) e^{i\phi_1(s)}, \dots, x_n r_n(s) e^{i\phi_n(s)}); \sum_{j=1}^n x_j^2 = 1, x_j \in \boldsymbol{R}, s \in \boldsymbol{R},\ s \neq 0\right\} \tag{6}$$

is an embedded Lagrangian diffeomorphic to $(\boldsymbol{R} \setminus \{0\}) \times \mathcal{S}^{n-1}$, *and the mean curvature vector* H *satisfies* $H \equiv \alpha F^{\perp}$, *where* F *is the position vector of* L. *If* $\alpha > 0$, *it is a self-expander, and if* $\alpha = 0$ *it is special Lagrangian. When* $E = 1$ *the construction reduces to that of Joyce, Lee and Tsui* [3, Theorem C]. *So the condition* $s \neq 0$ *on the definition of* L *is not necessary if* $E = 1$.

REMARK 1.3.1. In the situation of Theorem 1.3, define $\bar{\phi}_1, \dots, \bar{\phi}_n > 0$ by

$$\bar{\phi}_j = \int_0^\infty \frac{|t|dt}{(1/a_j + t^2)\sqrt{E\{\prod_{k=1}^n (1 + a_k t^2)\}e^{\alpha t^2} - 1}}.$$

We put $\alpha > 0$ and $E > 0$. From (14), the third equation of (13) and the proof of Theorem 1.3, the Lagrangian angle θ satisfies

$$\begin{aligned} \theta(s) &= \sum_j \phi_j(s) + \arg\left(s + i\frac{|s|}{\sqrt{E\{\prod_{k=1}^n (1 + a_k s^2)\}e^{\alpha s^2} - 1}}\right) \quad \text{and} \\ \dot{\theta}(s) &= \frac{-\alpha|s|}{\sqrt{E\{\prod_{k=1}^n (1 + a_k s^2)\}e^{\alpha s^2} - 1}}. \end{aligned} \tag{7}$$

It follows that θ is strictly decreasing. We define the submanifolds L_1 and L_2 of L so that $s > 0$ on L_1, and $s < 0$ on L_2, respectively. Therefore we have $L = L_1 \cup L_2$. We rewrite θ_1, θ_2 as the Lagrangian angle of L_1, L_2, respectively. Then $\lim_{s \to +\infty} \theta_1(s) < \theta_1(s) < \lim_{s \to +0} \theta_1(s)$ and $\lim_{s \to -0} \theta_2(s) < \theta_2(s) < \lim_{s \to -\infty} \theta_2(s)$ hold. So from the first equation of (7) we have

$$\begin{aligned} &\sum_j \psi_j + \sum_j \bar{\phi}_j < \theta_1(s) < \sum_j \psi_j + \tan^{-1}\frac{1}{\sqrt{E-1}} \quad \text{and} \\ &\sum_j \psi_j + \pi - \tan^{-1}\frac{1}{\sqrt{E-1}} < \theta_2(s) < \sum_j \psi_j + \pi - \sum_j \bar{\phi}_j. \end{aligned} \tag{8}$$

Therefore we can make the oscillations of the Lagrangian angles of L_1 and L_2 arbitrarily small by taking E close to ∞ and hence $\tan^{-1}(1/\sqrt{E-1})$ close to 0. Furthermore, we can prove that the map

$$\Phi : (0, \infty)^n \to \left\{(y_1, \dots, y_n) \in \left(0, \tan^{-1}\frac{1}{\sqrt{E-1}}\right)^n;\ 0 < \sum_{j=1}^n y_j < \tan^{-1}\frac{1}{\sqrt{E-1}}\right\}$$

defined by $\Phi(a_1, \dots, a_n) = (\bar{\phi}_1, \dots, \bar{\phi}_n)$ gives a diffeomorphism similarly to the proof of in [3, Theorem D]. Therefore we also can make the oscillations of the Lagrangian angles of L_1 and L_2 arbitrarily small by taking $\sum_j \bar{\phi}_j$ close to $\tan^{-1}(1/\sqrt{E-1})$.

For understanding Theorem 1.3, we compute

$$\begin{aligned}
\frac{dF}{ds} &= (x_1(\dot{r}_1 + ir_1\dot{\phi}_1)e^{i\phi_1}, \dots, x_n(\dot{r}_n + ir_n\dot{\phi}_n)e^{i\phi_n}) \\
&= \Bigg(x_1 e^{i\phi_1} \Bigg(\frac{s}{\sqrt{1/a_1 + s^2}} + i \frac{|s|}{\sqrt{(1/a_1 + s^2)E\{\prod_{k=1}^n (1 + a_k s^2)\}e^{\alpha s^2} - 1\}}} \Bigg), \dots, \\
&\qquad x_n e^{i\phi_n} \Bigg(\frac{s}{\sqrt{1/a_n + s^2}} + i \frac{|s|}{\sqrt{(1/a_n + s^2)E\{\prod_{k=1}^n (1 + a_k s^2)\}e^{\alpha s^2} - 1\}}} \Bigg)\Bigg) \\
&= \Bigg(s + i \frac{|s|}{\sqrt{E\{\prod_{k=1}^n (1 + a_k s^2)\}e^{\alpha s^2} - 1\}}} \Bigg) \cdot \Bigg(\frac{x_1 e^{i\phi_1}}{\sqrt{1/a_1 + s^2}}, \dots, \frac{x_n e^{i\phi_n}}{\sqrt{1/a_n + s^2}} \Bigg).
\end{aligned}$$

Then we have

$$\left| \frac{dF}{ds} \right| = |s| \sqrt{ \Bigg(1 + \frac{1}{E\{\prod_{k=1}^n (1 + a_k s^2)\}e^{\alpha s^2} - 1} \Bigg) \cdot \sum_j \frac{x_j^2}{1/a_j + s^2} }\,.$$

So we obtain

$$\begin{aligned}
\lim_{s \to +0} \frac{1}{|dF/ds|} \cdot \frac{dF}{ds} &= \Bigg(\frac{1}{\sqrt{1 + 1/(E-1)}} + i \frac{1/\sqrt{E-1}}{\sqrt{1 + 1/(E-1)}} \Bigg) \frac{1}{\sqrt{\sum_j a_j x_j^2}} \\
&\qquad \cdot (x_1 e^{i\psi_1}\sqrt{a_1}, \dots, x_n e^{i\psi_n}\sqrt{a_n})
\end{aligned}$$

and

$$\begin{aligned}
\lim_{s \to -0} \frac{1}{|dF/ds|} \cdot \frac{dF}{ds} &= \Bigg(\frac{-1}{\sqrt{1 + 1/(E-1)}} + i \frac{1/\sqrt{E-1}}{\sqrt{1 + 1/(E-1)}} \Bigg) \frac{1}{\sqrt{\sum_j a_j x_j^2}} \\
&\qquad \cdot (x_1 e^{i\psi_1}\sqrt{a_1}, \dots, x_n e^{i\psi_n}\sqrt{a_n})\,.
\end{aligned}$$

Thus we get

$$\lim_{s \to +0} \frac{1}{|dF/ds|} \cdot \frac{dF}{ds} \neq \lim_{s \to -0} \frac{1}{|dF/ds|} \cdot \frac{dF}{ds}\,.$$

Therefore, if we remove the condition $s \neq 0$ from the definition of L, it is not smooth at any point $s = 0$. In [6], Lotay and Neves proved that smooth zero-Maslov class Lagrangian self-expanders in $\boldsymbol{C}^n$ which are asymptotic to a pair of planes intersecting transversely are locally unique if $n > 2$ and unique if $n = 2$. It is easy to check that L is zero-Maslov class and asymptotic to a pair of planes intersecting transversely. By [3, Theorem C], we can construct a smooth Lagrangian self-expander asymptotic to any pair of Lagrangian planes in $\boldsymbol{C}^n$ which intersect transversely at the origin and have sum of characteristic angles less than π, where the characteristic angle is defined in Lawlor [7]. So Theorem 1.3 shows that, without the smoothness assumption, the uniqueness statement does not hold.

REMARK 1.3.2. In the situation of Theorem 1.3, if we put $E = 1$ and $\alpha = 0$, then changing $0 \mapsto -\infty$ in the integral of (5) gives Joyce's example [2, Example 6.11].

REMARK 1.3.3. In the situation of Theorem 1.2, if we take $C = \lambda_1 = \cdots = \lambda_n = 1$ and $\alpha \geq 0$, then the construction of L reduces to that of Theorem 1.3 where $s > 0$.

Next we turn to translating solitons.

THEOREM 1.4. *Fix $n \geq 2$. Let $\lambda_1, \ldots, \lambda_{n-1} \in \boldsymbol{R} \setminus \{0\}$, $E > 1$, $a_1, \ldots, a_{n-1} > 0$, and α, $\psi_1, \ldots, \psi_{n-1} \in \boldsymbol{R}$ be constants. Let $I \subset \boldsymbol{R}$ be a connected open neighborhood of $0 \in \boldsymbol{R}$ such that $\inf_{s \in I}(E\{\prod_{k=1}^{n-1}(1 + a_k\lambda_k s)\}e^{\alpha s} - 1)$ and $\inf_{s\in I}(1/a_j + \lambda_j s)$ are positive for any $1 \leq j \leq n$. Define $r_1, \ldots, r_{n-1} : I \to \boldsymbol{R}$ by*

$$r_j(s) = \sqrt{\frac{1}{a_j} + \lambda_j s}\,, \quad j = 1, \ldots, n-1, \tag{9}$$

and $\phi_1, \ldots, \phi_{n-1} : I \to \boldsymbol{R}$ by

$$\phi_j(s) = \psi_j + \frac{\lambda_j}{2}\int_0^s \frac{dt}{(1/a_j + \lambda_j t)\sqrt{E\{\prod_{k=1}^{n-1}(1 + a_k\lambda_k t)\}e^{\alpha t} - 1}}, \tag{10}$$

$j = 1, \ldots, n-1$. Then the submanifold L in $\boldsymbol{C}^n$ given by

$$L = \Bigg\{\Bigg(x_1 r_1(s)e^{i\phi_1(s)}, \ldots, x_{n-1}r_{n-1}(s)e^{i\phi_{n-1}(s)}, -\frac{1}{2}\sum_{j=1}^{n-1}\lambda_j x_j^2 + \frac{s}{2} + \frac{i}{2}\int_0^s \frac{dt}{\sqrt{E\{\prod_{k=1}^{n-1}(1 + a_k\lambda_k t)\}e^{\alpha t} - 1}}\Bigg); x_1, \ldots, x_{n-1} \in \boldsymbol{R}, s \in I\Bigg\}$$

is an immersed Lagrangian submanifold diffeomorphic to $\boldsymbol{R}^n$, and the mean curvature vector H satisfies $H \equiv T^\perp$, where $T = (0, \ldots, 0, \alpha) \in \boldsymbol{C}^n$. When $\alpha = 0$ it is special Lagrangian.

The following Theorem 1.5 is slightly generalized from [3, Corollary I].

THEOREM 1.5. *Fix $n \geq 2$. Let $a_1, \ldots, a_{n-1} > 0$, $\psi_1, \ldots, \psi_{n-1} \in \boldsymbol{R}$, $E \geq 1$, and $\alpha \geq 0$ be constants. Define $r_1, \ldots, r_{n-1} : \boldsymbol{R} \to \boldsymbol{R}$ by*

$$r_j(s) = \sqrt{\frac{1}{a_j} + s^2}\,, \quad j = 1, \ldots, n-1\,, \tag{11}$$

and $\phi_1, \ldots, \phi_{n-1} : \boldsymbol{R} \to \boldsymbol{R}$ by

$$\phi_j(s) = \psi_j + \int_0^s \frac{|t|dt}{(1/a_j + t^2)\sqrt{E\{\prod_{k=1}^{n-1}(1 + a_k t^2)\}e^{\alpha t^2} - 1}}\,, \tag{12}$$

$j = 1, \ldots, n-1$. *Then the submanifold L in $\boldsymbol{C}^n$ given by*

$$L = \left\{ \left(x_1 r_1(s) e^{i\phi_1(s)}, \ldots, x_{n-1} r_{n-1}(s) e^{i\phi_{n-1}(s)}, -\frac{1}{2}\sum_{j=1}^{n-1} x_j^2 + \frac{s^2}{2} + i\int_0^s \frac{|t|dt}{\sqrt{E\{\prod_{k=1}^{n-1}(1+a_k t^2)\}e^{\alpha t^2} - 1}} \right); x_1, \ldots, x_{n-1} \in \boldsymbol{R}, s \in \boldsymbol{R}, s \neq 0 \right\}$$

is an embedded Lagrangian submanifold diffeomorphic to $(\boldsymbol{R} \setminus \{0\}) \times \boldsymbol{R}^{n-1}$, *and the mean curvature vector H satisfies $H \equiv T^{\perp}$, where $T = (0, \ldots, 0, \alpha) \in \boldsymbol{C}^n$. When $\alpha = 0$ it is special Lagrangian. When $E = 1$ and $\psi_1 = \cdots = \psi_{n-1} = 0$, the construction reduces to that of Joyce, Lee and Tsui* [3, Corollary I]. *So the condition $s \neq 0$ on the definition of L is not necessary if $E = 1$.*

REMARK 1.5.1. In the situation of Theorem 1.5, we define the submanifolds L_1 and L_2 of L so that $s > 0$ on L_1, and $s < 0$ on L_2, respectively. Similarly to Remark 1.3.1 if we fix $\alpha > 0$, then we can make the oscillations of the Lagrangian angles of L_1 and L_2 arbitrarily small.

REMARK 1.5.2. In the situation of Theorem 1.4, if we put $\lambda_1 = \cdots = \lambda_{n-1} = 1$ and $\alpha \geq 0$, then the construction of L reduces to that of Theorem 1.5 where $s > 0$.

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2. Proofs for self-similar solutions. In order to prove Theorems 1.2 and 1.3, we use the following Lemma 2.1 that is generalized from [3, Theorem B]. The submanifolds in the following Lemma 2.1 are immersed Lagrangian self-similar solutions diffeomorphic to $\mathcal{S}^{m-1} \times \boldsymbol{R}^{n-m+1}$, where $1 \leq m \leq n$.

LEMMA 2.1. *Let I be an open interval in $\boldsymbol{R}$ and D a domain in $\boldsymbol{R}^{n+2}$. Let $\alpha \in \boldsymbol{R}, \lambda_1, \ldots, \lambda_n, C \in \boldsymbol{R} \setminus \{0\}$ and $a_1, \ldots, a_n > 0$ be constants, and $f : I \times D \to \boldsymbol{C} \setminus \{0\}$ a smooth function. Let $u, \phi_1, \ldots, \phi_n, \theta : I \to \boldsymbol{R}$ be smooth functions such that $\{(s, u(s), \phi_1(s), \ldots, \phi_n(s), \theta(s)); s \in I\} \subset I \times D$. Suppose that*

$$\begin{cases} \dfrac{du}{ds} = 2\,\mathrm{Re}(f(s, u, \phi_1, \ldots, \phi_n, \theta))\,, \\ \dfrac{d\phi_j}{ds} = \dfrac{\lambda_j\,\mathrm{Im}\,(f(s, u, \phi_1, \ldots, \phi_n, \theta))}{1/a_j + \lambda_j u(s)}\,, \quad j = 1, \ldots, n\,, \\ \dfrac{d\theta}{ds} = -\alpha\,\mathrm{Im}(f(s, u, \phi_1, \ldots, \phi_n, \theta))\,, \end{cases} \tag{13}$$

hold in I. We also suppose that

$$\inf_{s \in I}(1/a_j + \lambda_j u(s)) > 0\,, \quad j = 1, \ldots, n\,,$$

and

$$\theta(s) = \sum_{j=1}^{n} \phi_j(s) + \arg(f(s, u(s), \phi_1(s), \ldots, \phi_n(s), \theta(s))) \tag{14}$$

hold in I. Then the submanifold L in $\boldsymbol{C}^n$ given by

$$\begin{aligned} L = \Big\{ &\big(x_1\sqrt{1/a_1 + \lambda_1 u(s)}e^{i\phi_1(s)}, \ldots, x_n\sqrt{1/a_n + \lambda_n u(s)}e^{i\phi_n(s)}\big); \\ &\sum_{j=1}^{n} \lambda_j x_j^2 = C, x_j \in \boldsymbol{R}, s \in I \Big\} \end{aligned} \tag{15}$$

is an immersed Lagrangian submanifold diffeomorphic to $\mathcal{S}^{m-1} \times \boldsymbol{R}^{n-m+1}$, where m is the number of positive λ_j/C, $1 \le j \le n$, with Lagrangian angle $\theta(s)$ at

$$(x_1\sqrt{1/a_1 + \lambda_1 u(s)}e^{i\phi_1(s)}, \ldots, x_n\sqrt{1/a_n + \lambda_n u(s)}e^{i\phi_n(s)}) \in L\,,$$

and the mean curvature vector H satisfies $CH \equiv \alpha F^{\perp}$, where F is the position vector of L. Note that $\theta(s)$ is a function depending only on s, and L is a self-expander when $\alpha/C > 0$ and a self-shrinker when $\alpha/C < 0$. When $\alpha = 0$ the Lagrangian angle θ is constant, so that L is special Lagrangian.

REMARK 2.1.1. In the situation of Lemma 2.1, if we set $a_1 = \cdots = a_n = 1$, $\alpha = -\sum_{k=1}^{n} \lambda_k$ and

$$\begin{cases} f(s, y_1, \ldots, y_{n+2}) = i\,, \\ u(s) = 0\,, \\ \phi_j(s) = \lambda_j s, \quad 1 \le j \le n\,, \\ \theta(s) = -\alpha s + \dfrac{\pi}{2} = \left(\displaystyle\sum_{k=1}^{n} \lambda_k\right)s + \dfrac{\pi}{2}\,, \end{cases} \tag{16}$$

then it is easily seen that this setting satisfies the assumptions of Lemma 2.1, and the construction is Hamiltonian stationary in addition to being self-similar and it reduces to that of Lee and Wang [5, Theorem 1.1]. If f is a real valued function, then the submanifold L is an open subset of the special Lagrangian n-plane

$$\{(y_1 e^{i\xi_1}, \ldots, y_n e^{i\xi_n});\, y_j \in \boldsymbol{R},\ 1 \le j \le n\}\,,$$

where $\xi_j \in \boldsymbol{R}$.

PROOF OF LEMMA 2.1. Write

$$\omega_j(s) = \sqrt{1/a_j + \lambda_j u(s)}\, e^{i\phi_j(s)}\,, \quad 1 \le j \le n\,.$$

We compute

$$\frac{d\omega_j}{ds} = \frac{d}{ds}\Big(\sqrt{1/a_j + \lambda_j u(s)}\Big) \cdot e^{i\phi_j(s)} + \sqrt{1/a_j + \lambda_j u(s)} \cdot i\frac{d\phi_j}{ds}e^{i\phi_j(s)}$$

$$= \left(\frac{\lambda_j \mathrm{Re}(f(s,u,\phi_1,\dots,\phi_n,\theta))}{\sqrt{1/a_j+\lambda_j u(s)}} + i\frac{\lambda_j \mathrm{Im}(f(s,u,\phi_1,\dots,\phi_n,\theta))}{\sqrt{1/a_j+\lambda_j u(s)}}\right)e^{i\phi_j(s)}$$
$$= \frac{\lambda_j f(s,u,\phi_1,\dots,\phi_n,\theta)}{\overline{\omega_j}}\,.$$

Thus we obtain

$$\text{(17)}\qquad \begin{cases} \dfrac{d\omega_j}{ds} = \dfrac{\lambda_j f(s,u,\phi_1,\dots,\phi_n,\theta)}{\overline{\omega_j}}, \quad j=1,\dots,n\,, \\ \dfrac{d\theta}{ds} = -\alpha\,\mathrm{Im}(f(s,u,\phi_1,\dots,\phi_n,\theta))\,. \end{cases}$$

By (14) and (17), we can prove this theorem similarly to the proof of [3, Theorem A]. The details are left to the reader. This finishes the proof of Lemma 2.1. □

Now we can show Theorems 1.2 and 1.3.

PROOF OF THEOREM 1.2. Define $\tilde{f} : I \to \boldsymbol{C}\setminus\{0\}$ by

$$\tilde{f}(s) = \frac{1}{2} + \frac{i}{2\sqrt{E\left\{\prod_{k=1}^{n}(1+a_k\lambda_k s)\right\}e^{\alpha s}-1}}$$

and $f : I\times \boldsymbol{R}^{n+2} \to \boldsymbol{C}\setminus\{0\}$ by $f(s,y_1,\dots,y_{n+2}) = \tilde{f}(s)$. Note that f is a function depending only on $s\in I$. We also define $u : I \to \boldsymbol{R}$ by

$$u(s) = 2\int_0^s \mathrm{Re}(\tilde{f}(t))dt = s\,,$$

and $\theta : I \to \boldsymbol{R}$ by

$$\text{(18)}\qquad \theta(s) = \sum_{j=1}^{n}\phi_j(s) + \arg(\tilde{f}(s))\,.$$

Then we get

$$r_j(s) = \sqrt{\frac{1}{a_j} + \lambda_j u(s)}$$

and

$$\text{(19)}\qquad \frac{d\phi_j}{ds} = \frac{\lambda_j\,\mathrm{Im}\,(\tilde{f})}{1/a_j+\lambda_j u}$$

for any $j=1,\dots,n$. By our assumption we have

$$\inf_{s\in I}(1/a_j+\lambda_j u(s)) = \inf_{s\in I}(1/a_j+\lambda_j s) > 0\,, \quad j=1,\dots,n\,.$$

Since

$$\frac{d}{ds}\arg(\tilde{f}) = \frac{d}{ds}\tan^{-1}\left(\frac{\mathrm{Im}(\tilde{f})}{\mathrm{Re}(\tilde{f})}\right)$$
$$= \left(1+\frac{\mathrm{Im}(\tilde{f})^2}{\mathrm{Re}(\tilde{f})^2}\right)^{-1}\cdot\frac{d}{ds}\left(\frac{\mathrm{Im}(\tilde{f})}{\mathrm{Re}(\tilde{f})}\right)$$

$$
\begin{aligned}
&= \left(1+\frac{1}{E\{\prod_{k=1}^n(1+a_k\lambda_k s)\}e^{\alpha s}-1}\right)^{-1} \\
&\quad \cdot \frac{d}{ds}\left(\frac{1}{\sqrt{E\{\prod_{k=1}^n(1+a_k\lambda_k s)\}e^{\alpha s}-1}}\right) \\
&= \frac{E\{\prod_{k=1}^n(1+a_k\lambda_k s)\}e^{\alpha s}-1}{E\{\prod_{k=1}^n(1+a_k\lambda_k s)\}e^{\alpha s}} \cdot \frac{-1}{2\big[E\{\prod_{k=1}^n(1+a_k\lambda_k s)\}e^{\alpha s}-1\big]^{3/2}} \\
&\quad \cdot \left[E\left\{\sum_{l=1}^n \frac{\{\prod_{k=1}^n(1+a_k\lambda_k s)\}a_l\lambda_l}{1+a_l\lambda_l s}\right\}e^{\alpha s} + E\left\{\prod_{k=1}^n(1+a_k\lambda_k s)\right\}\alpha e^{\alpha s}\right] \\
&= \frac{1}{E\{\prod_{k=1}^n(1+a_k\lambda_k s)\}e^{\alpha s}} \cdot \frac{-1}{2\sqrt{E\{\prod_{k=1}^n(1+a_k\lambda_k s)\}e^{\alpha s}-1}} \\
&\quad \cdot E\left\{\prod_{k=1}^n(1+a_k\lambda_k s)\right\}e^{\alpha s}\left(\sum_{l=1}^n \frac{a_l\lambda_l}{1+a_l\lambda_l s}+\alpha\right) \\
&= \frac{-1}{2\sqrt{E\{\prod_{k=1}^n(1+a_k\lambda_k s)\}e^{\alpha s}-1}}\left(\sum_{l=1}^n \frac{\lambda_l}{1/a_l+\lambda_l u}+\alpha\right) \\
&= -\operatorname{Im}(\tilde{f})\left(\sum_{l=1}^n \frac{\lambda_l}{1/a_l+\lambda_l u}+\alpha\right),
\end{aligned}
$$

we obtain

$$
\sum_{j=1}^n \frac{\lambda_j \operatorname{Im}(\tilde{f})}{1/a_j+\lambda_j u} + \frac{d}{ds}\arg(\tilde{f}) = -\alpha \operatorname{Im}(\tilde{f})\,. \tag{20}
$$

From (18), (19) and (20), we get

$$
\frac{d\theta}{ds} = -\alpha \operatorname{Im}(\tilde{f}(s))\,.
$$

Accordingly,

$$
\begin{cases}
\dfrac{du}{ds} = 2\operatorname{Re}(f(s,u,\phi_1,\ldots,\phi_n,\theta))\,, \\
\dfrac{d\phi_j}{ds} = \dfrac{\lambda_j \operatorname{Im}(f(s,u,\phi_1,\ldots,\phi_n,\theta))}{1/a_j+\lambda_j u(s)}\,, \qquad j=1,\ldots,n\,, \\
\dfrac{d\theta}{ds} = -\alpha \operatorname{Im}(f(s,u,\phi_1,\ldots,\phi_n,\theta))\,.
\end{cases}
$$

Therefore we can apply Lemma 2.1 to the data f, u, ϕ_j, θ above. This finishes the proof of Theorem 1.2. □

PROOF OF THEOREM 1.3. We define $\tilde{f} : \boldsymbol{R} \setminus \{0\} \to \boldsymbol{C} \setminus \{0\}$ by

$$\tilde{f}(s) = s + i \frac{|s|}{\sqrt{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2} - 1}}$$

and $f : (\boldsymbol{R} \setminus \{0\}) \times \boldsymbol{R}^{n+2} \to \boldsymbol{C} \setminus \{0\}$ by $f(s, y_1, \dots, y_{n+2}) = \tilde{f}(s)$. We also define $u : \boldsymbol{R} \setminus \{0\} \to \boldsymbol{C} \setminus \{0\}$ by

$$u(s) = 2\int_0^s \mathrm{Re}(\tilde{f}(t))dt = s^2$$

and $\theta : \boldsymbol{R} \setminus \{0\} \to \boldsymbol{R}$ by

$$\theta(s) = \sum_{j=1}^{n} \phi_j(s) + \arg(\tilde{f}(s))\,. \tag{21}$$

Then we get $r_j(s) = \sqrt{1/a_j + u(s)}$ and

$$\frac{d\phi_j}{ds} = \frac{\mathrm{Im}\,(\tilde{f})}{1/a_j + u} \tag{22}$$

for any $j = 1, \dots, n$. It is clear that

$$\inf_{s\in\boldsymbol{R}\setminus\{0\}}(1/a_j + u(s)) = \inf_{s\in\boldsymbol{R}\setminus\{0\}}(1/a_j + s^2) = 1/a_j > 0\,, \quad j = 1, \dots, n\,.$$

Since

$$\begin{aligned}
\frac{d}{ds}\arg(\tilde{f}) &= \frac{d}{ds}\tan^{-1}\left(\frac{\mathrm{Im}(\tilde{f})}{\mathrm{Re}(\tilde{f})}\right) \\
&= \left(1 + \frac{\mathrm{Im}(\tilde{f})^2}{\mathrm{Re}(\tilde{f})^2}\right)^{-1} \cdot \frac{d}{ds}\left(\frac{\mathrm{Im}(\tilde{f})}{\mathrm{Re}(\tilde{f})}\right) \\
&= \left(1 + \frac{1}{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2} - 1}\right)^{-1} \\
&\quad \cdot \frac{d}{ds}\left(\frac{|s|}{s\sqrt{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2} - 1}}\right) \\
&= \frac{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2} - 1}{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2}} \cdot \frac{|s|}{s}\frac{d}{ds}\left(\frac{1}{\sqrt{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2} - 1}}\right) \\
&= \frac{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2} - 1}{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2}} \cdot \frac{|s|}{s} \cdot \frac{-1}{2\bigl[E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2} - 1\bigr]^{3/2}} \\
&\quad \cdot \left[E\left\{\sum_{l=1}^{n}\frac{\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}2a_l s}{1 + a_l s^2}\right\}e^{\alpha s^2} + E\left\{\prod_{k=1}^{n}(1 + a_k s^2)\right\}2\alpha s e^{\alpha s^2}\right] \\
&= \frac{1}{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2}} \cdot \frac{|s|}{s} \cdot \frac{-1}{2\sqrt{E\bigl\{\prod_{k=1}^{n}(1 + a_k s^2)\bigr\}e^{\alpha s^2} - 1}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot 2sE\left\{\prod_{k=1}^{n}(1+a_k s^2)\right\}e^{\alpha s^2}\left(\sum_{l=1}^{n}\frac{a_l}{1+a_l s^2}+\alpha\right)\\
&=\frac{-|s|}{\sqrt{E\{\prod_{k=1}^{n}(1+a_k s^2)\}e^{\alpha s^2}-1}}\left(\sum_{l=1}^{n}\frac{1}{1/a_l+u}+\alpha\right)\\
&=-\mathrm{Im}(\tilde f)\left(\sum_{l=1}^{n}\frac{1}{1/a_l+u}+\alpha\right),
\end{aligned}
$$

we obtain

$$
\sum_{j=1}^{n}\frac{\mathrm{Im}(\tilde f)}{1/a_j+u}+\frac{d}{ds}\arg(\tilde f)=-\alpha\,\mathrm{Im}(\tilde f)\,. \tag{23}
$$

From (21), (22) and (23), we have $d\theta/ds=-\alpha\,\mathrm{Im}(\tilde f(s))$. Thus

$$
\begin{cases}
\dfrac{du}{ds}=2\,\mathrm{Re}(f(s,u,\phi_1,\dots,\phi_n,\theta))\,,\\[2mm]
\dfrac{d\phi_j}{ds}=\dfrac{\mathrm{Im}\,(f(s,u,\phi_1,\dots,\phi_n,\theta))}{1/a_j+u(s)}\,, \quad j=1,\dots,n\,,\\[2mm]
\dfrac{d\theta}{ds}=-\alpha\,\mathrm{Im}(f(s,u,\phi_1,\dots,\phi_n,\theta))\,.
\end{cases}
$$

So we can apply Lemma 2.1 to the data $\lambda_1=\cdots=\lambda_n=1$ and f, u, ϕ_j, θ above. That L is embedded follows from the same argument as the proof of [3, Theorem C]. This completes the proof of Theorem 1.3. □

3. Proofs for translating solitons. This section is analogous to Section 2. In order to prove Theorems 1.4 and 1.5, we use the following Lemma 3.1 that is generalized from [3, Corollary H]. The following Lemma 3.1 sets up the ordinary differential equations for immersed Lagrangian translating soliton diffeomorphic to $\boldsymbol{R}^n$.

LEMMA 3.1. *Fix $n\ge 2$. Let I be an open interval in $\boldsymbol{R}$ and D a domain in $\boldsymbol{R}^{n+1}\times\boldsymbol{C}$. Let $\alpha\in\boldsymbol{R}, \lambda_1,\dots,\lambda_{n-1}, C\in\boldsymbol{R}\setminus\{0\}$ and $a_1,\dots,a_{n-1}>0$ be constants, and $f: I\times D\to\boldsymbol{C}\setminus\{0\}$ a smooth function. Let $u,\phi_1,\dots,\phi_{n-1},\theta: I\to\boldsymbol{R}$ and $\beta: I\to\boldsymbol{C}$ be smooth functions such that $\{(s,u(s),\phi_1(s),\dots,\phi_{n-1}(s),\theta(s),\beta(s)); s\in I\}\subset I\times D$. Suppose that*

$$
\begin{cases}
\dfrac{du}{ds}=2\,\mathrm{Re}(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))\,,\\[2mm]
\dfrac{d\phi_j}{ds}=\dfrac{\lambda_j\,\mathrm{Im}\,(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))}{1/a_j+\lambda_j u(s)}\,, \quad j=1,\dots,n-1\,,\\[2mm]
\dfrac{d\theta}{ds}=-\alpha\,\mathrm{Im}(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))\,,\\[2mm]
\dfrac{d\beta}{ds}=f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta)\,.
\end{cases} \tag{24}
$$

hold in I. We also suppose that

$$\inf_{s\in I}(1/a_j + \lambda_j u(s)) > 0\,, \quad j = 1, \dots, n-1\,,$$

and

$$\theta(s) = \sum_{j=1}^{n-1} \phi_j(s) + \arg(f(s, u(s), \phi_1(s), \dots, \phi_{n-1}(s), \theta(s), \beta(s))) \tag{25}$$

hold in I. Then the submanifold L in $\boldsymbol{C}^n$ given by

$$L = \left\{ \left(x_1 r_1(s) e^{i\phi_1(s)}, \dots, x_{n-1} r_{n-1}(s) e^{i\phi_{n-1}(s)}, -\frac{1}{2}\sum_{j=1}^{n-1} \lambda_j x_j^2 + \beta(s) \right);\; x_1, \dots, x_{n-1} \in \boldsymbol{R}, s \in I \right\}$$

is an immersed Lagrangian submanifold diffeomorphic to $\boldsymbol{R}^n$ with Lagrangian angle $\theta(s)$ at

$$\left(x_1 r_1(s) e^{i\phi_1(s)}, \dots, x_{n-1} r_{n-1}(s) e^{i\phi_{n-1}(s)}, -1/2\sum_{j=1}^{n-1} \lambda_j x_j^2 + \beta(s) \right) \in L\,,$$

and the mean curvature vector H satisfies $H \equiv T^{\perp}$, where $T = (0, \dots, 0, \alpha)$. When $\alpha = 0$ it is special Lagrangian.

REMARK 3.1.1. In the situation of Lemma 3.1, if we set $\alpha = -\sum_{k=1}^{n} a_j \lambda_k$ and

$$\begin{cases} f(s, y_1, \dots, y_{n+1}, z) = i\,, \\ u(s) = 0\,, \\ \phi_j(s) = a_j \lambda_j s\,, \quad 1 \le j \le n-1\,, \\ \theta(s) = -\alpha s + \dfrac{\pi}{2} = \left(\displaystyle\sum_{k=1}^{n-1} a_k \lambda_k \right) s + \dfrac{\pi}{2}\,, \\ \beta(s) = is\,, \end{cases} \tag{26}$$

then it is easy to check that this setting satisfies the assumptions of Lemma 3.1, and the construction is Hamiltonian stationary in addition to being translating solition. If f is a real valued function, then the submanifold L is an open subset of the special Lagrangian n-plane

$$\{(y_1 e^{i\xi_1}, \dots, y_{n-1} e^{i\xi_{n-1}}, y_n);\, y_j \in \boldsymbol{R},\ 1 \le j \le n\}\,,$$

where $\xi_l \in \boldsymbol{R}$, $1 \le l \le n-1$.

PROOF OF LEMMA 3.1. Write

$$\omega_j(s) = \sqrt{1/a_j + \lambda_j u(s)}\, e^{i\phi_j(s)}\,, \quad 1 \le j \le n-1\,.$$

We compute

$$\begin{aligned}\frac{d\omega_j}{ds} &= \frac{d}{ds}\left(\sqrt{1/a_j+\lambda_j u(s)}\right)\cdot e^{i\phi_j(s)} + \sqrt{1/a_j+\lambda_j u(s)}\cdot i\frac{d\phi_j}{ds}e^{i\phi_j(s)} \\ &= \left(\frac{\lambda_j \mathrm{Re}(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))}{\sqrt{1/a_j+\lambda_j u(s)}} + i\frac{\lambda_j \mathrm{Im}(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))}{\sqrt{1/a_j+\lambda_j u(s)}}\right)e^{i\phi_j(s)} \\ &= \frac{\lambda_j f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta)}{\overline{\omega_j}}\,.\end{aligned}$$

Accordingly,

$$\begin{cases}\dfrac{d\omega_j}{ds} = \dfrac{\lambda_j f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta)}{\overline{\omega_j}}, \quad j=1,\dots,n-1\,, \\ \dfrac{d\theta}{ds} = -\alpha\,\mathrm{Im}(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))\,, \\ \dfrac{d\beta}{ds} = f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta)\,.\end{cases} \tag{27}$$

By (25) and (27), we can prove this theorem similarly to the proof of [3, Theorem G]. This finishes the proof, the detailed verification being left to the reader. □

Now we can show Theorems 1.4 and 1.5.

PROOF OF THEOREM 1.4. Define $\tilde{f} : \boldsymbol{I} \to \boldsymbol{C}\setminus\{0\}$ by

$$\tilde{f}(s) = \frac{1}{2} + \frac{i}{2\sqrt{E\{\prod_{k=1}^{n-1}(1+a_k\lambda_k s)\}e^{\alpha s}-1}}$$

and $f : \boldsymbol{I}\times\boldsymbol{R}^{n+1}\times\boldsymbol{C} \to \boldsymbol{C}\setminus\{0\}$ by $f(s,y_1,\dots,y_{n+1},z) = \tilde{f}(s)$. We also define

$$u(s) = 2\int_0^s \mathrm{Re}(\tilde{f}(t))dt = s\,,$$

$$\theta(s) = \sum_{j=1}^{n-1}\phi_j(s) + \arg(\tilde{f}(s))\,,$$

and

$$\beta(s) = \int_0^s \tilde{f}(t)dt = \frac{s}{2} + \frac{i}{2}\int_0^s \frac{dt}{\sqrt{E\{\prod_{k=1}^{n-1}(1+a_k\lambda_k t)\}e^{\alpha t}-1}}.$$

Then we get $r_j(s) = \sqrt{1/a_j+\lambda_j u(s)}$ and

$$\frac{d\phi_j}{ds} = \frac{\lambda_j\,\mathrm{Im}\,(\tilde{f})}{1/a_j+\lambda_j u}$$

for any $j=1,\dots,n-1$. By our assumption we have

$$\inf_{s\in I}(1/a_j+\lambda_j u(s)) = \inf_{s\in I}(1/a_j+\lambda_j s) > 0, \quad j=1,\dots,n-1\,.$$

We can check $d\theta/ds = -\alpha \mathrm{Im}(\tilde{f})$ similarly to the proof of Theorem 1.2. Thus we obtain

$$\begin{cases} \dfrac{du}{ds} = 2\,\mathrm{Re}(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))\,, \\ \dfrac{d\phi_j}{ds} = \dfrac{\lambda_j \,\mathrm{Im}\,(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))}{1/a_j+\lambda_j u(s)}\,, \quad j=1,\dots,n-1\,, \\ \dfrac{d\theta}{ds} = -\alpha\,\mathrm{Im}(f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta))\,, \\ \dfrac{d\beta}{ds} = f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta)\,. \end{cases}$$

Therefore we can apply Lemma 3.1 to the data $f, u, \phi_j, \theta, \beta$ above. This finishes the proof of Theorem 1.4. □

PROOF OF THEOREM 1.5. We define $\tilde{f} : \boldsymbol{R}\setminus\{0\} \to \boldsymbol{C}\setminus\{0\}$ by

$$\tilde{f}(s) = s + i\frac{|s|}{\sqrt{E\left\{\prod_{k=1}^{n-1}(1+a_k s^2)\right\}e^{\alpha s^2}-1}}$$

and $f : (\boldsymbol{R}\setminus\{0\}) \times \boldsymbol{R}^{n+1} \times \boldsymbol{C} \to \boldsymbol{C}\setminus\{0\}$ by $f(s, y_1, \dots, y_{n+1}, z) = \tilde{f}(s)$. We also define

$$u(s) = 2\int_0^s \mathrm{Re}(\tilde{f}(t))dt = s^2\,,$$

$$\theta(s) = \sum_{j=1}^{n-1}\phi_j(s) + \arg(\tilde{f}(s))\,,$$

and

$$\beta(s) = \int_0^s \tilde{f}(t)dt = \frac{s^2}{2} + i\int_0^s \frac{|t|dt}{\sqrt{E\left\{\prod_{k=1}^{n-1}(1+a_k t^2)\right\}e^{\alpha t^2}-1}}\,.$$

Then we have $r_j(s) = \sqrt{1/a_j + u(s)}$ and

$$\frac{d\phi_j}{ds} = \frac{\mathrm{Im}\,(\tilde{f})}{1/a_j+u}$$

for any $j = 1, \dots, n-1$. It is clear that

$$\inf_{s\in\boldsymbol{R}\setminus\{0\}}(1/a_j + u(s)) = \inf_{s\in\boldsymbol{R}\setminus\{0\}}(1/a_j + s^2) = 1/a_j > 0\,, \quad j = 1,\dots,n-1\,.$$

We can check $d\theta/ds = -\alpha\mathrm{Im}(\tilde{f})$ similarly to the proof of Theorem 1.3. Thus we obtain

$$\begin{cases} \dfrac{du}{ds} = 2\,\mathrm{Re}(f(s,u,\phi_1,\dots,\phi_n,\theta))\,, \\ \dfrac{d\phi_j}{ds} = \dfrac{\mathrm{Im}\,(f(s,u,\phi_1,\dots,\phi_n,\theta))}{1/a_j+u(s)}\,, \quad j=1,\dots,n-1\,, \\ \dfrac{d\theta}{ds} = -\alpha\,\mathrm{Im}(f(s,u,\phi_1,\dots,\phi_n,\theta))\,, \\ \dfrac{d\beta}{ds} = f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta)\,. \end{cases}$$

So we can apply Lemma 3.1 to the data $\lambda_1 = \cdots = \lambda_{n-1} = 1$ and $f, u, \phi_j, \theta, \beta$ above. That L is embedded follows from the same argument as the proof of [3, Theorem C]. This completes the proof of Theorem 1.5. □

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