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REGULARIZED PERIODS OF AUTOMORPHIC FORMS ON $GL(2)$

SHINGO SUGIYAMA

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Abstract. In this paper, we study regularized periods of cusp forms and Eisenstein series on $GL(2)$ introduced by Masao Tsuzuki.

Introduction. In the theory of automorphic forms and automorphic representations, it is an important problem to study the periods of automorphic forms, because central values of automorphic L-functions appear in explicit formulas of the periods of automorphic forms. So far, many works on the periods of automorphic forms have been done. For a remarkable example, we should mention the paper [10], in which Waldspurger studied the central values of several kinds of automorphic L-functions in connection with the toral period of cusp forms.

Let F be an algebraic number field and $\boldsymbol{A}$ its adele ring. In [9], Tsuzuki introduced a notion of regularized periods of functions on $GL(2, \boldsymbol{A})$ in the following way. For $C > 0$, let $\mathcal{B}(C)$ be the space of all holomorphic functions β on $\{z \in \boldsymbol{C}; |\operatorname{Re}(z)| < C\}$ satisfying that

(1) the equality $\beta(z) = \beta(-z)$ holds,

(2) the estimate
$$|\beta(\sigma + it)| \prec (1 + |t|)^{-l}, \quad \sigma \in [a, b]$$
holds for any $[a, b] \subset (-C, C)$ and any $l > 0$.

Let $\mathcal{B}$ be the space of all entire functions β on $\boldsymbol{C}$ such that the restriction of β to $\{z \in \boldsymbol{C}; |\operatorname{Re}(z)| < C\}$ is contained in $\mathcal{B}(C)$ for any $C > 0$. For $\beta \in \mathcal{B}$, $t > 0$ and $\lambda \in \boldsymbol{C}$, we consider
$$\hat{\beta}_\lambda(t) := \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\beta(z)}{z + \lambda} t^z dz, \quad (\sigma > -\operatorname{Re}(\lambda)).$$
For a function $\varphi : GL(2, F)\backslash GL(2, \boldsymbol{A}) \to \boldsymbol{C}$, $\beta \in \mathcal{B}$, $\lambda \in \boldsymbol{C}$ and a unitary character η of $\boldsymbol{A}^\times/F^\times$, we consider
$$P^{\eta}_{\beta,\lambda}(\varphi) := \int_{F^\times\backslash \boldsymbol{A}^\times} \{\hat{\beta}_\lambda(|t|_{\boldsymbol{A}}) + \hat{\beta}_\lambda(|t|_{\boldsymbol{A}}^{-1})\}\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x_\eta \\ 0 & 1\end{pmatrix}\right)\eta(t)\eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}})d^\times t,$$
where $x_\eta = (x_{\eta,v})_{v \in \Sigma_F} \in \boldsymbol{A}$ is the adele which will be defined in §4, $x_{\eta,\mathrm{fin}}$ is the projection of x_η to the finite adele ring $\boldsymbol{A}_{\mathrm{fin}}$ of F and η_{fin} is the restriction of η to $\boldsymbol{A}_{\mathrm{fin}}^\times$. For the function φ, we assume the following:

- For any $\beta \in \mathcal{B}$ there exists a constant $C \in \boldsymbol{R}$ such that if $\operatorname{Re}(\lambda) > C$ the integral $P^{\eta}_{\beta,\lambda}(\varphi)$ converges.

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- For any $\beta \in \mathcal{B}$, the function $\{z \in \boldsymbol{C};\, \mathrm{Re}(z) > C\} \ni \lambda \mapsto P^{\eta}_{\beta,\lambda}(\varphi)$ has a meromorphic continuation to a neighborhood of $\lambda = 0$.
- The constant term $\mathrm{CT}_{\lambda=0}P^{\eta}_{\beta,\lambda}(\varphi)$ of the Laurent expansion of $P^{\eta}_{\beta,\lambda}(\varphi)$ at $\lambda = 0$ is proportional to the Dirac delta distribution supported at 0 as a linear functional of $\mathcal{B}$.

Then, the proportionality constant $P^{\eta}_{\mathrm{reg}}(\varphi)$ is called *the regularized η-period of φ*, i.e.,

$$\mathrm{CT}_{\lambda=0}P^{\eta}_{\beta,\lambda}(\varphi) = P^{\eta}_{\mathrm{reg}}(\varphi)\beta(0)$$

for all $\beta \in \mathcal{B}$.

When F is totally real, Tsuzuki obtained the following results in [9].

(1) The regularized periods of cusp forms which are associated with cuspidal automorphic representations of $GL(2)$ with square free conductor are explicitly described in terms of central L-values.

(2) The regularized periods of Eisenstein series constructed by induced representations from unramified characters of $\boldsymbol{A}^{\times}/F^{\times}\boldsymbol{R}_{>0}$ are described in terms of the Hecke L-functions.

In this paper, we generalize the above results (1) and (2). First we explain our result on cusp forms. Let (π, V_π) be a $\mathbf{K}_\infty$-spherical cuspidal automorphic representation of $GL(2, \boldsymbol{A})$ with trivial central character. We denote the conductor of π by $\mathfrak{f}_\pi$. Let $\mathfrak{n}$ be an ideal of the integer ring $\mathfrak{o}_F$ of F which is divided by $\mathfrak{f}_\pi$. Tsuzuki explicitly computed the regularized periods of cusp forms in $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ in the case where F is totally real, assuming $\mathfrak{n}$ is square free (cf. [9, Lemma 7.4]). Here $\mathbf{K}_0(\mathfrak{n})$ is the congruence subgroup defined at the end of this section.

In this paper, we explicitly compute the regularized periods of cusp forms in the invariant subspace $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ when the field F is an arbitrary number field and the ideal $\mathfrak{n}$ is not necessarily square free.

Let Σ_F, $\Sigma_{\boldsymbol{R}}$, $\Sigma_{\boldsymbol{C}}$ and Σ_{fin} be the set of all places of F, the set of all real places of F, the set of all complex places of F and the set of all finite places of F, respectively. For an ideal $\mathfrak{a}$ of $\mathfrak{o}_F$ we denote by $S(\mathfrak{a})$ the set of $v \in \Sigma_{\mathrm{fin}}$ such that v divides $\mathfrak{a}$ and denote by $S_k(\mathfrak{a})$ the set of $v \in S(\mathfrak{a})$ such that the order $\mathrm{ord}_v(\mathfrak{a})$ of $\mathfrak{a}$ equals k for any $k \in \boldsymbol{N}$. Let

$$\left\{\varphi_{\pi,\rho}\ ;\ \rho \in \prod_{k=1}^{n} \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}), \{0, \dots, k\})\right\}$$

be an orthogonal basis of $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ which will be constructed in §4, where n is the maximal nonnegative integer k such that $S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}) \neq \emptyset$ and $\mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}), \{0, \dots, k\})$ is the set of all mappings from $S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ to $\{0, \dots, k\}$. We fix a family $\{\pi_v\}_{v\in\Sigma_F}$ consisting of unitarizable irreducible admissible representations such that $\pi \cong \bigotimes_{v\in\Sigma_F}\pi_v$. Let η be a unitary character of $\boldsymbol{A}^{\times}/F^{\times}\boldsymbol{R}_{>0}$ satisfying the following conditions:

$$(\star)\ \begin{cases} v \in \Sigma_{\boldsymbol{R}} \cup \Sigma_{\boldsymbol{C}} \Rightarrow \eta_v = |\cdot|_v^{t_v} \text{ for some } t_v \in i\boldsymbol{R}\,, \\ \mathfrak{f}_\eta \text{ is relatively prime to } \mathfrak{n}\,, \end{cases}$$

where $\mathfrak{f}_\eta$ is the conductor of η. We denote the Gauss sum associated with η by $\mathcal{G}(\eta)$, which will be defined in §1. Then, we prove the following theorem.

MAIN THEOREM A. *Let η be a character of $\boldsymbol{A}^\times/F^\times\boldsymbol{R}_{>0}$ satisfying the condition ($\star$). Then, for any $\rho=(\rho_1,\dots,\rho_n)\in\prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}),\{0,\dots,k\})$, there exist explicitly computable polynomials $Q^{\pi_v}_{\rho_k(v),v}(\eta_v,X)\in\boldsymbol{C}[X]$ for any $v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{f}_\eta)$ and $k\in\{0,\dots,n\}$ such that*

$$P^\eta_{\mathrm{reg}}(\varphi_{\pi,\rho})=\mathcal{G}(\eta)\left\{\prod_{k=1}^n\prod_{v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}Q^{\pi_v}_{\rho_k(v),v}(\eta_v,1)\right\}L(1/2,\pi\otimes\eta)\,.$$

Here $L(s,\pi\otimes\eta)$ denotes the standard L-function of $\pi\otimes\eta$. Indeed, $Q^{\pi_v}_{k,v}(\eta_v,1)$ is given as the following.

- *If $c(\pi_v)=0$ and (α,α^{-1}) is the Satake parameter of π_v, then*

$$\begin{aligned}&Q^{\pi_v}_{k,v}(\eta_v,1)\\&=\begin{cases}1 & (\text{if } k=0)\,,\\ \eta_v(\varpi_v)-\dfrac{\alpha+\alpha^{-1}}{q_v^{1/2}+q_v^{-1/2}} & (\text{if } k=1)\,,\\ q_v^{-1}\eta_v(\varpi_v)^{k-2}(\alpha q_v^{1/2}\eta_v(\varpi_v)-1)(\alpha^{-1}q_v^{1/2}\eta_v(\varpi_v)-1) & (\text{if } k\ge 2)\,.\end{cases}\end{aligned}$$

- *If $c(\pi_v)=1$ and π_v is isomorphic to $\sigma(\chi_v|\cdot|_v^{1/2},\chi_v|\cdot|_v^{-1/2})$, then*

$$Q^{\pi_v}_{k,v}(\eta_v,1)=\begin{cases}1 & (\text{if } k=0)\,,\\ \eta_v(\varpi_v)^{k-1}(\eta_v(\varpi_v)-q_v^{-1}\chi_v(\varpi_v)^{-1}) & (\text{if } k\ge 1)\,.\end{cases}$$

- *If $c(\pi_v)\ge 2$, then $Q^{\pi_v}_{k,v}(\eta_v,1)=\eta_v(\varpi_v)^k$ for any $k\in\boldsymbol{N}_0$.*

Next we explain our result on Eisenstein series. Let χ be a character of $\boldsymbol{A}^\times/F^\times\boldsymbol{R}_{>0}$. Let $\mathbf{K}$ be the standard maximal compact subgroup of $GL(2,\boldsymbol{A})$. For $\nu\in\boldsymbol{C}$, we denote by $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})$ the space of all smooth functions $f:GL(2,\boldsymbol{A})\to\boldsymbol{C}$ which are $\mathbf{K}$-finite and satisfy the condition

$$f\left(\begin{pmatrix}a&b\\0&d\end{pmatrix}g\right)=\chi(a/d)|a/d|_{\boldsymbol{A}}^{(\nu+1)/2}f(g)$$

for all $\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\in\left(\begin{smallmatrix}\boldsymbol{A}^\times&\boldsymbol{A}\\0&\boldsymbol{A}^\times\end{smallmatrix}\right)$ and $g\in GL(2,\boldsymbol{A})$. For $f^{(\nu)}\in I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})$, $E(f^{(\nu)},g)$ denotes the Eisenstein series for $f^{(\nu)}$. Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}_F$ divided by $\mathfrak{f}_\chi^2$. Tsuzuki explicitly computed regularized periods of $E(f^{(\nu)},g)$ for $f^{(\nu)}\in I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ in the case where F is totally real, χ is unramified and $\mathfrak{n}$ is square free (cf. [9, Lemma 7.5]).

In this paper we explicitly compute regularized periods of $E(f^{(\nu)},g)$ for $f^{(\nu)}\in I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ when the field F is an arbitrary number field, χ is an arbitrary character and the ideal $\mathfrak{n}$ is not necessarily square free. Let

$$\left\{f^{(\nu)}_{\chi,\rho}\ ;\ \rho\in\prod_{k=1}^n\mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}),\{0,\dots,k\})\right\}$$

be the subset of $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ constructed in §9, which is an orthonormal basis of $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ if $\nu \in i\boldsymbol{R}$. Here n is the maximal nonnegative integer k such that $S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}) \neq \emptyset$. We write $E_{\chi,\rho}(\nu, g)$ for $E(f_{\chi,\rho}^{(\nu)}, g)$. Let $\mathrm{N}(\mathfrak{f}_\chi)$ and D_F denote the absolute norm of $\mathfrak{f}_\chi$ and that of the global different of $F/\boldsymbol{Q}$, respectively. Then we prove the following theorem (see Theorem 37 in detail).

MAIN THEOREM B. *Let η be a character of $\boldsymbol{A}^\times/F^\times\boldsymbol{R}_{>0}$ satisfying the condition* $(\star)$. *We assume $\nu \in i\boldsymbol{R}$ if $S(\mathfrak{f}_\chi) = \emptyset$. Then, for any $\rho = (\rho_1, \ldots, \rho_n) \in \prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}), \{0, \ldots, k\})$, there exists an explicitly computable meromorphic function $B_{\chi,\rho}^\eta(s,\nu)$ on $\boldsymbol{C}\times\boldsymbol{C}$ such that*

$$P_{\mathrm{reg}}^\eta(E_{\chi,\rho}(\nu,-)) = (2\pi)^{\#\Sigma_C}\mathcal{G}(\eta)D_F^{-\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2-\nu} \times B_{\chi,\rho}^\eta(1/2,\nu)\frac{L((1+\nu)/2,\chi\eta)L((1-\nu)/2,\chi^{-1}\eta)}{L(1+\nu,\chi^2)}.$$

We explain the structure of this paper. In §1, we introduce notations for fundamental objects and review notions of spherical functions and local new forms on $GL(2,F_v)$ for $v \in \Sigma_F$, where F_v denotes the completion of F at v. In §2, for $v \in \Sigma_{\mathrm{fin}}$, we construct a basis of $I(\chi_v)^{\mathbf{K}_0(\mathfrak{p}_v^n)}$ which is orthogonal for any $GL(2,F_v)$-invariant hermitian inner product, where $I(\chi_v)$ is a unitarizable spherical principal series representation of $GL(2,F_v)$ with trivial central character and $\mathfrak{p}_v$ denotes the maximal ideal of the integer ring of F_v. In §3, for $v \in \Sigma_{\mathrm{fin}}$, we construct an orthogonal basis of $V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)+n})}$, where (π_v, V_{π_v}) is an infinite dimensional unitarizable irreducible nonspherical representation of $GL(2,F_v)$ with trivial central character and $c(\pi_v)$ denotes the exponent of the conductor of π_v. In §4, we construct a basis of $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ and explicitly compute modified global zeta integrals of cusp forms in $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$. Moreover we construct polynomials $Q_{k,v}^{\pi_v}(\eta_v, X)$. In §5, we recall regularized periods defined by Tsuzuki and prove Main Theorem A.

From §6 to §10, we consider regularized periods of Eisenstein series. In §6, we review notions of induced representations of $GL(2,\boldsymbol{A})$ and Eisenstein series on $GL(2,\boldsymbol{A})$. In §7 and 8, we construct an orthonormal basis of $I(\chi_v|\cdot|_v^{\nu/2})^{\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)+n})}$ if $\nu \in i\boldsymbol{R}$, where $I(\chi_v|\cdot|_v^{\nu/2})$ denotes an induced representation from a ramified character $\chi_v|\cdot|_v^{\nu/2}$ of $F_v^\times$ and $f(\chi_v)$ denotes the exponent of the conductor of χ_v. In §9, we construct an orthonormal basis of $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ if $\nu \in i\boldsymbol{R}$ and compute constant terms $E_{\chi,\rho}^\circ(\nu,g)$ of $E_{\chi,\rho}(\nu,g)$ and modified global zeta integrals of $E_{\chi,\rho}(\nu,g) - E_{\chi,\rho}^\circ(\nu,g)$. In §10, we prove Main Theorem B and compute regularized periods of the residue $\mathfrak{e}_{\chi,\rho,-1}(g)$ and the constant term $\mathfrak{e}_{\chi,\rho,0}(g)$ at $\nu = 1$ of $E_{\chi,\rho}(\nu,g)$.

NOTATION. Let $\boldsymbol{N}$ be the set of natural numbers not including the number 0 and put $\boldsymbol{N}_0 := \boldsymbol{N}\cup\{0\}$. For any sets A and B, we denote by $\mathrm{Map}(A,B)$ the set of all mappings from A to B.

For any set X and two nonnegative functions $f : X \to \boldsymbol{R}_{\geq 0}$ and $g : X \to \boldsymbol{R}_{\geq 0}$, we write $f(x) \prec g(x)$ if there exists $C > 0$ such that $f(x) \leq Cg(x)$, for all $x \in X$. For any set X and its subset A, we denote the characteristic function of A by ch_A. For any condition P, the

Kronecker symbol $\delta(P)$ is defined by

$$\delta(P) := \begin{cases} 1 & (\text{if } P \text{ is true}), \\ 0 & (\text{if } P \text{ is false}). \end{cases}$$

Let F be an algebraic number field of degree d_F and $\mathfrak{o}_F$ its ring of integers. Let Σ_F, Σ_∞, $\Sigma_{\boldsymbol{R}}$, $\Sigma_{\boldsymbol{C}}$ and Σ_{fin} be the set of places of F, the set of infinite places of F, the set of real places of F, the set of complex places of F and the set of finite places of F, respectively. The completion of F at a place $v \in \Sigma_F$ is denoted by F_v. If v is a finite place of F, the field F_v is a non-archimedean local field, whose ring of integers is denoted by $\mathfrak{o}_v$. We fix a uniformizer ϖ_v of $\mathfrak{o}_v$ once and for all, and denote by q_v the cardinality of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$, where $\mathfrak{p}_v = \varpi_v\mathfrak{o}_v$ is the maximal ideal of $\mathfrak{o}_v$. For any $v \in \Sigma_F$, we write $|\cdot|_v$ for the normalized valuation of F_v. Let $\boldsymbol{A}$ and $\boldsymbol{A}_{\mathrm{fin}}$ be the adele ring of F and the finite adele ring of F, respectively.

For an ideal $\mathfrak{a}$ of $\mathfrak{o}_F$, let $S(\mathfrak{a})$ denote the set of all $v \in \Sigma_{\mathrm{fin}}$ such that $\mathfrak{a}\mathfrak{o}_v \subset \mathfrak{p}_v$. For all $k \in \boldsymbol{N}$, let $S_k(\mathfrak{a})$ denote the set of all $v \in S(\mathfrak{a})$ such that $\mathfrak{a}\mathfrak{o}_v = \mathfrak{p}_v^k$. Then, $S(\mathfrak{a}) = \coprod_{k=1}^n S_k(\mathfrak{a})$, where n is the maximal nonnegative integer m such that $S_m(\mathfrak{a}) \neq \emptyset$. We denote the absolute norm of $\mathfrak{a}$ by $\mathrm{N}(\mathfrak{a})$.

For any algebraic subgroup M defined over F of $G = GL(2)$ and $v \in \Sigma_F$, groups of F_v-rational points, F-rational points and $\boldsymbol{A}$-rational points are denoted by M_v, M_F and $M_{\boldsymbol{A}}$, respectively. We denote the unit element of G by e. Let B be the Borel subgroup of G consisting of all upper triangular matrices and Z the center of G. For $v \in \Sigma_F$, we put

$$\mathbf{K}_v := \begin{cases} O(2, \boldsymbol{R}) & (\text{if } v \in \Sigma_{\boldsymbol{R}}), \\ U(2, \boldsymbol{C}) & (\text{if } v \in \Sigma_{\boldsymbol{C}}), \\ GL(2, \mathfrak{o}_v) & (\text{if } v \in \Sigma_{\mathrm{fin}}). \end{cases}$$

Then, $\mathbf{K} := \prod_{v\in\Sigma_F} \mathbf{K}_v$ is a maximal compact subgroup of $G_{\boldsymbol{A}}$. We set $\mathbf{K}_\infty := \prod_{v\in\Sigma_\infty} \mathbf{K}_v$ and $\mathbf{K}_0(\mathfrak{p}_v^n) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{K}_v;\, c \equiv 0 \pmod{\mathfrak{p}_v^n}\right\}$ for $n \in \boldsymbol{N}_0$. For an ideal $\mathfrak{a}$ of $\mathfrak{o}_F$, we put $\mathbf{K}_0(\mathfrak{a}) := \prod_{v\in\Sigma_{\mathrm{fin}}} \mathbf{K}_0(\mathfrak{a}\mathfrak{o}_v)$.

1. Preliminaries.

1.1. Local and global differents. For $v \in \Sigma_{\mathrm{fin}}$, let $\mathfrak{p}_v^{d_v}$ be the local different of F_v. Let D_F be the discriminant of $F/\boldsymbol{Q}$, which is defined as the absolute norm of the global different of $F/\boldsymbol{Q}$. Then, D_F equals $\prod_{v\in\Sigma_{\mathrm{fin}}} q_v^{d_v}$.

Let $\boldsymbol{A}_{\boldsymbol{Q}}$ be the adele ring of $\boldsymbol{Q}$ and $\psi_{\boldsymbol{Q}}$ the additive character of $\boldsymbol{A}_{\boldsymbol{Q}}/\boldsymbol{Q}$ with archimedean component $\boldsymbol{R} \ni x \mapsto \exp(2\pi i x)$. Then, $\psi_F := \psi \circ \mathrm{tr}_{F/\boldsymbol{Q}}$ is a nontrivial additive character of $\boldsymbol{A}/F$ and decomposed into a product of local additive characters ψ_{F_v} $(v \in \Sigma_F)$. Moreover, $\mathfrak{p}_v^{-d_v}$ equals the maximal fractional ideal of $\mathfrak{o}_v$ contained in $\mathrm{Ker}\psi_{F_v}$ for any $v \in \Sigma_{\mathrm{fin}}$.

1.2. Haar measures and Gauss sums. For $v \in \Sigma_F$, let dx_v be the self-dual Haar measure of F_v with respect to ψ_{F_v}. Then, the equalities $\mathrm{vol}([0, 1]) = 1$, $\mathrm{vol}(\{\sigma + it;\, \sigma, t \in [0, 1]\}) = 2$ and $\mathrm{vol}(\mathfrak{o}_v) = q_v^{-d_v/2}$ hold for $v \in \Sigma_{\boldsymbol{R}}$, $v \in \Sigma_{\boldsymbol{C}}$ and $v \in \Sigma_{\mathrm{fin}}$, respectively. We denote the Haar measure $c_v dx_v/|x_v|_v$ of $F_v^\times$ by $d^\times x_v$, where $c_v = 1$ for $v \in \Sigma_\infty$ and

$c_v = (1 - q_v^{-1})^{-1}$ for $v \in \Sigma_{\mathrm{fin}}$. We denote the Haar measure $\prod_{v\in\Sigma_F} d^\times x_v$ of $\boldsymbol{A}^\times$ by $d^\times x$. We fix Haar measures dg_v on G_v and dk_v on $\mathbf{K}_v$ such that $\mathrm{vol}(\mathbf{K}_v, dg_v) = \mathrm{vol}(\mathbf{K}_v, dk_v) = 1$ for $v \in \Sigma_F$, respectively. We denote the Haar measure $\prod_{v\in\Sigma_F} dk_v$ of $\mathbf{K}$ by dk.

Let $|\cdot|_{\boldsymbol{A}} = \prod_{v\in\Sigma_F} |\cdot|_v$ be the idele norm of $\boldsymbol{A}^\times$ and $\boldsymbol{A}^1 = \{x \in \boldsymbol{A}^\times; |x|_{\boldsymbol{A}} = 1\}$ the norm one subgroup of $\boldsymbol{A}^\times$. For $y \in \boldsymbol{R}_{>0}$, $\underline{y}$ denotes the idele whose v-component satisfies

$$\underline{y}_v = \begin{cases} y^{1/d_F} & (v \in \Sigma_\infty)\,, \\ 1 & (v \in \Sigma_{\mathrm{fin}})\,. \end{cases}$$

Then, $\boldsymbol{A}^\times$ is isomorphic to $\boldsymbol{R}_{>0} \times \boldsymbol{A}^1$ by the map $\boldsymbol{A}^\times \ni x \mapsto (\underline{|x|_{\boldsymbol{A}}}, \underline{|x|_{\boldsymbol{A}}}^{-1}x) \in \boldsymbol{R}_{>0} \times \boldsymbol{A}^1$. Set $G^1_{\boldsymbol{A}} = \{g \in G_{\boldsymbol{A}}; |\det g|_{\boldsymbol{A}} = 1\}$ and $\mathfrak{A} = \left\{ \left(\begin{smallmatrix} \underline{y} & 0 \\ 0 & \underline{y} \end{smallmatrix}\right); y > 0 \right\}$. Then, we have $G_{\boldsymbol{A}} = \mathfrak{A} G^1_{\boldsymbol{A}}$.

For $v \in \Sigma_{\mathrm{fin}}$ and a quasi character χ_v of $F_v^\times$, the number $f(\chi_v)$ stands for the minimal nonnegative integer f such that the restriction of χ_v to $1 + \mathfrak{p}_v^f$ equals identically one. The ideal $\mathfrak{p}_v^{f(\chi_v)}$ is called the conductor of χ_v. We define the Gauss sum associated with χ_v by

$$\mathcal{G}(\chi_v) := \int_{\mathfrak{o}_v^\times} \chi_v(u\varpi_v^{-d_v - f(\chi_v)}) \psi_{F_v}(u\varpi_v^{-d_v - f(\chi_v)}) d^\times u\,.$$

Then, $\mathcal{G}(\chi_v)$ equals $\chi_v(\varpi_v^{-d_v}) q_v^{-d_v/2}$ for any unramified quasi character χ_v of $F_v^\times$. For any quasi character $\chi = \prod_{v\in\Sigma_F} \chi_v$ of $\boldsymbol{A}^\times/F^\times$, we define the conductor of χ by the ideal $\mathfrak{f}_\chi$ of $\mathfrak{o}_F$ such that $\mathfrak{f}_\chi \mathfrak{o}_v = \mathfrak{p}_v^{f(\chi_v)}$ for all $v \in \Sigma_{\mathrm{fin}}$. We write χ_{fin} for $\prod_{v\in\Sigma_{\mathrm{fin}}} \chi_v$. The Gauss sum associated with χ is defined by

$$\mathcal{G}(\chi) := \prod_{v\in\Sigma_{\mathrm{fin}}} \mathcal{G}(\chi_v)\,.$$

For $v \in \Sigma_F$, we denote the trivial character of $F_v^\times$ by $\mathbf{1}_v$, and the trivial character of $\boldsymbol{A}^\times$ by $\mathbf{1}$. Throughout this paper, whenever we consider a quasi character χ of $\boldsymbol{A}^\times/F^\times$, we assume that $\chi(\underline{y}) = 1$ for all $y \in \boldsymbol{R}_{>0}$. Such a quasi character is a character.

1.3. Induced representations. For $v \in \Sigma_F$ and any quasi character χ_v of $F_v^\times$, $I(\chi_v)$ denotes the space of all smooth functions $f : G_v \to \boldsymbol{C}$ which are $\mathbf{K}_v$-finite and satisfy the condition

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_v(a/d)|a/d|_v^{1/2} f(g)$$

for all $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B_v$ and $g \in G_v$. Then, $I(\chi_v)$ is a $(\mathfrak{g}_v, \mathbf{K}_v)$-module if $v \in \Sigma_\infty$, where $\mathfrak{g}_v$ is the complexification of the Lie algebra of G_v.

1.4. Spherical functions on $GL(2, \boldsymbol{R})$ and $GL(2, \boldsymbol{C})$. For $v \in \Sigma_\infty$, let π_v be a $\mathbf{K}_v$-spherical unitarizable irreducible admissible $(\mathfrak{g}_v, \mathbf{K}_v)$-module with trivial central character. Then, π_v is isomorphic to $I(|\cdot|_v^\nu)$ for some $\nu \in \boldsymbol{C}$. The Whittaker model of π_v with respect to ψ_{F_v} is denoted by V_{π_v}. Let $f_{0,v}^{\pi_v}$ be the spherical vector in $I(|\cdot|_v^\nu)$ normalized so that $f_{0,v}^{\pi_v}(e)$ equals one. For $v \in \Sigma_{\boldsymbol{R}}$ (resp. $v \in \Sigma_{\boldsymbol{C}}$), we denote $\phi_{0,v}$ the spherical Whittaker function in

V_{π_v} which corresponds to $\Gamma_{\boldsymbol{R}}(1+2\nu)f_{0,v}^{\pi_v}$ (resp. $(2\pi)^{-1}\Gamma_{\boldsymbol{C}}(1+2\nu)f_{0,v}^{\pi_v}$) by the isomorphism

$$I(|\cdot|_v^{\nu}) \ni f \mapsto W_f(g) := \int_{F_v} f\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\psi_{F_v}(-x)dx \in V_{\pi_v},$$

where $w_0 = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, $\Gamma_{\boldsymbol{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\boldsymbol{C}}(s) = (2\pi)^{-s}\Gamma(s)$.

We define the local zeta integral by

$$Z(s, \eta_v, \phi) = \int_{F_v^{\times}} \phi\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\eta_v(t)|t|_v^{s-1/2}d^{\times}t$$

for any quasi character η_v of $F_v^{\times}$ and $\phi \in V_{\pi_v}$. The defining integral converges absolutely for $\mathrm{Re}(s) \gg 0$ and $Z(s, \eta_v, \phi)$ has a meromorphic continuation to $\boldsymbol{C}$ as a function in s. If η_v is of the form $|\cdot|_v^{t_v}$ for some $t_v \in \boldsymbol{C}$, then

$$Z(s, \eta_v, \phi_{0,v}) = L(s, \pi_v \otimes \eta_v) = \begin{cases}\Gamma_{\boldsymbol{R}}(s+\nu+t_v)\Gamma_{\boldsymbol{R}}(s-\nu+t_v) & (\text{if } v \in \Sigma_{\boldsymbol{R}})\\ \Gamma_{\boldsymbol{C}}(s+\nu+t_v)\Gamma_{\boldsymbol{C}}(s-\nu+t_v) & (\text{if } v \in \Sigma_{\boldsymbol{C}})\end{cases}$$

holds (cf. [3, Proposition 3.4.6] and [11, Proposition (2.3.14)]).

1.5. Local new forms. For $v \in \Sigma_{\mathrm{fin}}$, let π_v be an infinite dimensional irreducible admissible representation of G_v with trivial central character. The Whittaker model of π_v with respect to ψ_{F_v} is denoted by V_{π_v}. The local zeta integral $Z(s, \eta_v, \phi)$ for any quasi character η_v of $F_v^{\times}$ and $\phi \in V_{\pi_v}$ is defined in the same way as the archimedean case.

We consider a compact open subgroup $\mathbf{K}_0(\mathfrak{p}_v^n)$ of $\mathbf{K}_v$ for $n \in \boldsymbol{N}_0$. Then, $\{\mathbf{K}_0(\mathfrak{p}_v^n);\, n \in \boldsymbol{N}_0\}$ gives a decreasing filtration of G_v. The invariant subspace $V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^n)}$ is nonzero for some n. We put $c(\pi_v) := \min\{n \in \boldsymbol{N}_0;\, V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^n)} \neq 0\}$. By the theory of local new forms for $GL(2)$, we have the following proposition (cf. [6, p. 3], [7], and [8, Theorem 11.13]).

PROPOSITION 1. *The dimension of $V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)})}$ equals one. For any $n \in \boldsymbol{N}_0$, we have*

$$V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)+n})} = \bigoplus_{k=0}^{n} \pi_v\begin{pmatrix}\varpi_v^{-k} & 0\\ 0 & 1\end{pmatrix} V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)})}.$$

There exists a unique element $\phi_{0,v} \in V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)})}$ such that

$$Z(s, \eta_v, \phi_{0,v}) = \mathrm{vol}(\mathfrak{o}_v^{\times}, d^{\times}t)\eta_v(\varpi_v)^{-d_v}q_v^{d_v(s-1/2)}L(s, \pi_v \otimes \eta_v)$$

for any unramified quasi character η_v of $F_v^{\times}$.

REMARK 2. In fact, $\phi_{0,v}$ is given by the following:

- If $c(\pi_v) = 0$, then we have

$$\phi_{0,v}\begin{pmatrix}\varpi_v^m & 0\\ 0 & 1\end{pmatrix} = q_v^{-(m+d_v)/2}\frac{\alpha_1^{m+d_v+1} - \alpha_2^{m+d_v+1}}{\alpha_1 - \alpha_2}\delta(m \geq -d_v)$$

for any $m \in \boldsymbol{Z}$, where (α_1, α_2) is the Satake parameter of π_v.

- If $c(\pi_v)=1$, then π_v is isomorphic to the special representation $\sigma(\chi_v|\cdot|_v^{1/2}, \chi_v|\cdot|_v^{-1/2})$ for some unramified character χ_v of $F_v^\times$ satisfying $\chi_v^2=\mathbf{1}_v$, and we have
$$\phi_{0,v}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \chi_v(\varpi_v^{d_v}t)|\varpi_v^{d_v}t|_v \mathrm{ch}_{\mathfrak{p}_v^{-d_v}}(t)$$
for any $t\in F_v^\times$.

- If $c(\pi_v)\geq 2$, then we have
$$\phi_{0,v}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{ch}_{\varpi_v^{-d_v}\mathfrak{o}_v^\times}(t)$$
for any $t\in F_v^\times$.

In addition, we assume that π_v is unitarizable. Then, the G_v-invariant hermitian inner product on the Whittaker model V_{π_v} is given by
$$\langle W_1|W_2\rangle = \int_{F_v^\times} W_1\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\overline{W_2\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}}d^\times t$$
for any $W_1, W_2\in V_{\pi_v}$ (cf. [2, Theorem 12]). By a direct computation, we have the following.

LEMMA 3. *We have*

$$\langle\phi_{0,v}|\phi_{0,v}\rangle = \begin{cases} q_v^{-d_v/2}\dfrac{1-q_v^{-2}|\alpha_1\alpha_2|^2}{(1-q_v^{-1}|\alpha_1|^2)(1-q_v^{-1}|\alpha_2|^2)|1-q_v^{-1}\alpha_1\overline{\alpha_2}|^2} & (\textit{if } c(\pi_v)=0), \\ q_v^{-d_v/2}(1-q_v^{-2})^{-1} & (\textit{if } c(\pi_v)=1), \\ q_v^{-d_v/2} & (\textit{if } c(\pi_v)\geq 2). \end{cases}$$

2. A basis of the $\mathbf{K}_0(\mathfrak{p}_v^n)$-invariant subspace in a spherical representation. For $v\in\Sigma_{\mathrm{fin}}$, let π_v be an infinite dimensional unitarizable irreducible spherical representation of G_v with trivial central character. Then, π_v is isomorphic to $I(\chi_v)$ for some unramified quasi character χ_v of $F_v^\times$. We denote by $f_{0,v}^{\pi_v}$ the spherical vector in $I(\chi_v)$ normalized so that $f_{0,v}^{\pi_v}(e)$ equals one.

Let $\mathcal{H}(G_v,\mathbf{K}_v)$ be the vector space of all $\mathbf{K}_v$-biinvariant functions $f: G_v\to \boldsymbol{C}$ with compact support. The space $\mathcal{H}(G_v,\mathbf{K}_v)$ is called the spherical Hecke algebra of G_v. Let $M(2,\mathfrak{o}_v)$ be the set of all 2×2 matrices with coefficients in $\mathfrak{o}_v$. Set
$$T(\mathfrak{p}_v^k) := \mathrm{ch}_{\{g\in M(2,\mathfrak{o}_v);\,(\det g)\mathfrak{o}_v=\mathfrak{p}_v^k\}}$$
for any $k\in \boldsymbol{N}_0$. We denote $\left(\begin{smallmatrix} \varpi_v^k & 0 \\ 0 & \varpi_v^l \end{smallmatrix}\right)$ by $\delta_v(k,l)$ for any $k,l\in\boldsymbol{Z}$ and set
$$R(\mathfrak{p}_v) := \mathrm{ch}_{\mathbf{K}_v\delta_v(1,1)\mathbf{K}_v} = \mathrm{ch}_{\delta_v(1,1)\mathbf{K}_v}.$$
Then, $T(\mathfrak{p}_v)$, $R(\mathfrak{p}_v)$ and $R(\mathfrak{p}_v)^{-1}$ generate $\mathcal{H}(G_v,\mathbf{K}_v)$ as an algebra. For simplicity we write $f_{0,v}$ for $f_{0,v}^{\pi_v}$ in the proofs in this section.

LEMMA 4. *For every $k \in \boldsymbol{N}_0$, there exists $b_{\pi_v}(k) \in \boldsymbol{R}$ such that* $\pi_v(T(\mathfrak{p}_v^k)) f_{0,v}^{\pi_v} = b_{\pi_v}(k) f_{0,v}^{\pi_v}$.

PROOF. We prove this assertion by induction on k. For simplicity, we write b_k for $b_{\pi_v}(k)$. The assertion holds for $k = 0$ because we can take $b_0 = 1$. By the theory of the spherical Hecke algebra, we have $\pi_v(T(\mathfrak{p}_v)) f_{0,v} = q_v^{1/2}(\alpha_1 + \alpha_2) f_{0,v}$, where $(\alpha_1, \alpha_2) = (\chi_v(\varpi_v), \chi_v(\varpi_v)^{-1})$ is the Satake parameter of π_v. Since π_v is unitarizable, $\alpha_1 + \alpha_2$ must be real. Therefore the assertion holds for $k = 1$ because we can take $b_1 = q_v^{1/2}(\alpha_1 + \alpha_2)$. Next, suppose $k \geq 0$ and that there exists b_j for every $j \in \{0, \dots, k+1\}$. By the theory of the spherical Hecke algebra, the equality $T(\mathfrak{p}_v)T(\mathfrak{p}_v^{k+1}) = T(\mathfrak{p}_v^{k+2}) + q_v R(\mathfrak{p}_v) T(\mathfrak{p}_v^k)$ holds (cf. [1, Proposition 4.6.4]). By $\pi_v(T(\mathfrak{p}_v^k)) f_{0,v} = b_k f_{0,v}$ and $\pi_v(T(\mathfrak{p}_v^{k+1})) f_{0,v} = b_{k+1} f_{0,v}$, we have

$$\begin{aligned}\pi_v(T(\mathfrak{p}_v^{k+2})) f_{0,v} &= \pi_v(T(\mathfrak{p}_v))\pi_v(T(\mathfrak{p}_v^{k+1})) f_{0,v} - q_v R(\mathfrak{p}_v) T(\mathfrak{p}_v^k) f_{0,v} \\ &= q_v^{\frac{1}{2}}(\alpha_1 + \alpha_2) b_{k+1} f_{0,v} - q_v \alpha_1 \alpha_2 b_k f_{0,v} \\ &= (q_v^{\frac{1}{2}}(\alpha_1 + \alpha_2) b_{k+1} - q_v b_k) f_{0,v}\,.\end{aligned}$$

Thus we can take $b_{k+2} = q_v^{1/2}(\alpha_1 + \alpha_2) b_{k+1} - q_v b_k$. □

REMARK 5. By solving the recurrence relation of b_k, we obtain

$$b_k = q_v^{k/2} \frac{\alpha_1^{k+1} - \alpha_2^{k+1}}{\alpha_1 - \alpha_2}$$

for any $k \in \boldsymbol{N}_0$.

LEMMA 6. *For every $k \in \boldsymbol{N}_0$, there exists $a_{\pi_v}(k) \in \boldsymbol{R}$ such that*

$$\pi_v\left(\mathrm{ch}_{\mathbf{K}_v \delta_v(k,1) \mathbf{K}_v}\right) f_{0,v}^{\pi_v} = a_{\pi_v}(k) f_{0,v}^{\pi_v}\,.$$

PROOF. We prove this assertion by induction on k. For simplicity, we write a_k for $a_{\pi_v}(k)$. The assertion holds for $k = 0$ because we can take $a_0 = 1$. Next, suppose $k \geq 1$. By definition, we have

$$\begin{aligned}T(\mathfrak{p}_v^k) &= \mathrm{ch}_{\{g \in M(2,\mathfrak{o}_v)\,;\,(\det g)\mathfrak{o}_v = \mathfrak{p}_v^k\}} = \mathrm{ch}_{\bigcup_{r=0}^{k} \mathbf{K}_v \delta_v(k-r,r) \mathbf{K}_v} \\ &= \mathrm{ch}_{\coprod_{r=0}^{\lfloor k/2 \rfloor} \mathbf{K}_v \delta_v(k-r,r) \mathbf{K}_v} = \sum_{r=0}^{\lfloor k/2 \rfloor} \mathrm{ch}_{\mathbf{K}_v \delta_v(k-r,r) \mathbf{K}_v}\,.\end{aligned}$$

Since the central character of π_v is trivial and $\alpha_1 \alpha_2$ equals one,

$$\begin{aligned}b_k f_{0,v} = \pi_v(T(\mathfrak{p}_v^k)) f_{0,v} &= \sum_{r=0}^{\lfloor k/2 \rfloor} \pi_v\left(\mathrm{ch}_{\mathbf{K}_v \delta_v(k-r,r) \mathbf{K}_v}\right) f_{0,v} \\ &= \sum_{r=0}^{\lfloor k/2 \rfloor} (\alpha_1 \alpha_2)^r \pi_v\left(\mathrm{ch}_{\mathbf{K}_v \delta_v(k-2r,0) \mathbf{K}_v}\right) f_{0,v}\end{aligned}$$

$$= \pi_v \left(\mathrm{ch}_{\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v}\right) f_{0,v} + \sum_{r=1}^{\lfloor k/2 \rfloor} a_{k-2r} f_{0,v} .$$

Thus we have $a_k = b_k - \sum_{r=1}^{\lfloor k/2 \rfloor} a_{k-2r} \in \boldsymbol{R}$. □

REMARK 7. By solving the recurrence relation of a_k, we obtain

$$a_k = \begin{cases} b_k & (k = 0, 1) , \\ b_k - b_{k-2} & (k \geq 2) . \end{cases}$$

For any G_v-invariant hermitian inner product $(\cdot|\cdot)_v$ on $I(\chi_v)$, we put $\|f\|_v = \sqrt{(f|f)_v}$ for any $f \in I(\chi_v)$.

LEMMA 8. *For every $k \in \boldsymbol{N}_0$, there exists $\lambda_{\pi_v}(k) \in \boldsymbol{R}$ such that*

$$\left(\pi_v(\delta_v(-k,0)) f_{0,v}^{\pi_v} \middle| f_{0,v}^{\pi_v}\right)_v = \lambda_{\pi_v}(k) \| f_{0,v}^{\pi_v} \|_v^2$$

for any G_v-invariant hermitian inner product $(\cdot|\cdot)_v$ on $I(\chi_v)$. Here $\{\lambda_{\pi_v}(k)\}_{k \in \boldsymbol{N}_0}$ is independent of the choice of a G_v-invariant hermitian inner product.

PROOF. For simplicity, we write λ_k for $\lambda_{\pi_v}(k)$. Let $(\cdot|\cdot)$ be a G_v-invariant hermitian inner product on $I(\chi_v)$. We have

$$\left(\pi_v(\delta_v(-k,0)) f_{0,v} \middle| f_{0,v}\right) = \left(f_{0,v} \middle| \pi_v(k_1 \delta_v(k,0) k_2) f_{0,v}\right)$$

for any $k_1, k_2 \in \mathbf{K}_v$. Hence, we have

$$\begin{aligned} &\left(\pi_v(\delta_v(-k,0)) f_{0,v} \middle| f_{0,v}\right) \\ &\quad = \frac{1}{\mathrm{vol}\,(\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v)} \int_{\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v} \left(f_{0,v} \middle| \pi_v(g) f_{0,v}\right) dg \\ &\quad = \frac{1}{\mathrm{vol}\,(\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v)} \left(f_{0,v} \middle| \int_{\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v} \pi_v(g) f_{0,v} dg\right) \\ &\quad = \frac{1}{\mathrm{vol}\,(\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v)} \left(f_{0,v} \middle| \pi_v\left(\mathrm{ch}_{\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v}\right) f_{0,v}\right) . \end{aligned}$$

Therefore we can take

$$\lambda_k = \frac{a_k}{\mathrm{vol}\,(\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v)} .$$

□

Here, we can explicitly compute $\mathrm{vol}\,(\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v)$ by the following lemma.

LEMMA 9. *For $k \in \boldsymbol{N}_0$, we have*

$$\mathrm{vol}\,(\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v) = \begin{cases} 1 & (k = 0), \\ (q_v + 1) q_v^{k-1} & (k \geq 1). \end{cases}$$

PROOF. This assertion is obvious for $k=0$. For $k \geq 1$, by the Iwasawa decomposition, we have

$$\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v = \coprod_{r=1}^{k-1}\left\{\coprod_{b\in(\mathfrak{o}_v/\mathfrak{p}_v^r)^\times}\begin{pmatrix}\varpi_v^r & b\\ 0 & \varpi_v^{k-r}\end{pmatrix}\mathbf{K}_v\right\}$$
$$\coprod\left\{\coprod_{b\in\mathfrak{o}_v/\mathfrak{p}_v^k}\begin{pmatrix}\varpi_v^k & b\\ 0 & 1\end{pmatrix}\mathbf{K}_v\right\}\coprod\begin{pmatrix}1 & 0\\ 0 & \varpi_v^k\end{pmatrix}\mathbf{K}_v\,.$$

Thus, we have

$$\mathrm{vol}\,(\mathbf{K}_v\delta_v(k,0)\mathbf{K}_v) = \left\{\sum_{r=1}^{k-1}(q_v-1)q_v^{r-1}+q_v^k+1\right\}\mathrm{vol}(\mathbf{K}_v).$$

Since $\mathrm{vol}(\mathbf{K}_v)$ equals one, we obtain the assertion. □

PROPOSITION 10. *For $n\in \boldsymbol{N}$, there exists a unique finite set $\{f_{1,v}^{\pi_v},\dots,f_{n,v}^{\pi_v}\}$ of $I(\chi_v)^{\mathbf{K}_0(\mathfrak{p}_v^n)}$ satisfying the following conditions* (1) *and* (2):

(1) *There exists a finite sequence $\{c_{\pi_v}(k,j)\}_{1\leq k\leq n, 0\leq j\leq k-1}$ of real numbers such that*
$$f_{k,v}^{\pi_v} = \pi_v(\delta_v(-k,0))f_{0,v}^{\pi_v} - \sum_{j=0}^{k-1}c_{\pi_v}(k,j)f_{j,v}^{\pi_v}\ \text{ for all } k\in\{1,\dots,n\}.$$
(2) *For any G_v-invariant hermitian inner product $(\cdot|\cdot)_v$ on $I(\chi_v)$, the set $\{f_{0,v}^{\pi_v},\dots,f_{n,v}^{\pi_v}\}$ is an orthogonal basis of $I(\chi_v)^{\mathbf{K}_0(\mathfrak{p}_v^n)}$.*

Moreover, there exists a unique family $\{\tau_{\pi_v}(k,j)\}_{0\leq k\leq n, 0\leq j\leq k}$ of real numbers such that

$$\left(\pi_v(\delta_v(-k,0))f_{0,v}^{\pi_v}\middle| f_{j,v}^{\pi_v}\right)_v = \tau_{\pi_v}(k,j)\|f_{0,v}^{\pi_v}\|_v^2$$

for all $k,j\in\{0,\dots,n\}$ and for any G_v-invariant hermitian inner product $(\cdot|\cdot)_v$ on $I(\chi_v)$.

PROOF. For simplicity, we write $c_{k,j}$ and $\tau_{k,j}$ for $c_{\pi_v}(k,j)$ and $\tau_{\pi_v}(k,j)$, respectively. By Proposition 1, the finite set

$$\{f_{0,v}\}\cup\left\{f_{k,v}=\pi_v(\delta_v(-k,0))f_{0,v}-\sum_{j=0}^{k-1}c_{k,j}f_{j,v}\ ;\ k\in\{1,\dots,n\}\right\}$$

is a basis of $I(\chi_v)^{\mathbf{K}_0(\mathfrak{p}_v^n)}$ for any $\{c_{k,j}\}_{1\leq k\leq n,0\leq j\leq k-1}\subset \boldsymbol{R}$. We will prove the proposition by induction on n. The assertion holds for $n=1$ since we can take

$$c_{1,0}=\lambda_1=\frac{\chi_v(\varpi_v)+\chi_v^{-1}(\varpi_v)}{q_v^{1/2}+q_v^{-1/2}}\in \boldsymbol{R}\,,$$

$\tau_{0,0}=1$, $\tau_{1,0}=\lambda_{1,0}$, and $\tau_{1,1}=1-c_{1,0}\tau_{1,0}$, respectively (cf. [9, Lemma 6]).

Suppose that $n\geq 2$ and that both $\{c_{k,j}\}_{1\leq k\leq n-1,0\leq j\leq k-1}$ and $\{\tau_{k,j}\}_{0\leq k\leq n-1,0\leq j\leq k}$ have been determined. Let $(\cdot|\cdot)$ be a G_v-invariant hermitian inner product on $I(\chi_v)$. By assumption and a direct computation, we obtain $\| f_{k,v}\|^2 = \tau_{k,k}\| f_{0,v}\|^2$ for all $k \in \{0, \dots, n-1\}$. Thus we have $\tau_{k,k} > 0$ for all $k \in \{0, \dots, n-1\}$.

Now we will show the existence of $\{c_{n,j}\}_{0\le j\le n-1}$. We can take $\tau_{n,0} = \lambda_n$ obviously. Assume $k \in \{1, \dots, n-1\}$. If there exists $\{\tau_{n,j}\}_{0\le j\le k-1}$, we have

$$\begin{aligned}
&\left(\pi_v(\delta_v(-n,0)) f_{0,v} \middle| f_{k,v}\right) \\
&\quad = \left(\pi_v(\delta_v(-n,0)) f_{0,v} \middle| \pi_v(\delta_v(-k,0)) f_{0,v} - \sum_{j=0}^{k-1} c_{k,j} f_{j,v}\right) \\
&\quad = \left(\pi_v(\delta_v(-(n-k),0)) f_{0,v} \middle| f_{0,v}\right) - \sum_{j=0}^{k-1} c_{k,j} \left(\pi_v(\delta_v(-n,0)) f_{0,v} \middle| f_{j,v}\right) \\
&\quad = \left(\lambda_{n-k} - \sum_{j=0}^{k-1} c_{k,j}\tau_{n,j}\right) \| f_{0,v}\|^2 .
\end{aligned}$$

Hence we can take

$$\tau_{n,k} := \lambda_{n-k} - \sum_{j=0}^{k-1} c_{k,j}\tau_{n,j} \in \boldsymbol{R} .$$

Therefore, we can construct $\tau_{n,k}$ for all $k \in \{0, \dots, n-1\}$ inductively.

Next, assume $k \in \{0, \dots, n-1\}$ and we put

$$c_{n,k} := \frac{\tau_{n,k}}{\tau_{k,k}} \in \boldsymbol{R} .$$

Then, we have

$$\begin{aligned}
(f_{n,v} | f_{k,v}) &= \left(\pi_v(\delta_v(-n,0)) f_{0,v} \middle| f_{k,v}\right) - \sum_{j=0}^{n-1} c_{n,j} (f_{j,v} | f_{k,v}) \\
&= \tau_{n,k}\| f_{0,v}\|^2 - c_{n,k}\| f_{k,v}\|^2 \\
&= (\tau_{n,k} - c_{n,k}\tau_{k,k})\| f_{0,v}\|^2 \\
&= 0 .
\end{aligned}$$

Thus we obtain $\{c_{n,j}\}_{0\le j\le n-1}$.

Finally, we show the existence of $\tau_{n,n}$. In the same way as the above computation, we have

$$\left(\pi_v(\delta_v(-n,0)) f_{0,v} | f_{n,v}\right) = \left(\lambda_0 - \sum_{j=0}^{n-1} c_{n,j}\tau_{n,j}\right) \| f_{0,v}\|^2 .$$

Hence we can take

$$\tau_{n,n} := \lambda_0 - \sum_{j=0}^{n-1} c_{n,j}\tau_{n,j} \in \boldsymbol{R} .$$

□

By the argument in the proof of Proposition 10, the family $\{c_{n,k}\}$ can be computed by the following recurrence relations.

COROLLARY 11. *For any $n \in \boldsymbol{N}$, we have*

- $\tau_{n,k} = \lambda_{n-k} - \sum_{j=0}^{k-1} c_{k,j}\tau_{n,j}$ *for all* $k \in \{1, \dots, n\}$,
- $\tau_{n,0} = \lambda_n, \qquad \tau_{0,0} = 1$,
- $c_{n,k} = \dfrac{\tau_{n,k}}{\tau_{k,k}}$ *for all* $k \in \{0, \dots, n-1\}$.

By induction on n and a direct computation, we can prove the following.

COROLLARY 12. *We set $\alpha := \alpha_1$. For $n \in \boldsymbol{N}$, we have the following:*

- $$\tau_{n,k} = \begin{cases} \dfrac{q_v \sum_{j=0}^{n} \alpha^{2j} - \sum_{j=1}^{n-1} \alpha^{2j}}{\alpha^n q_v^{n/2}(1+q_v)} & (\textit{if } k = 0), \\[2ex] \dfrac{\sum_{j=0}^{n-1} \alpha^{2j}(q_v - \alpha^2)(\alpha^2 q_v - 1)}{\alpha^{n+1} q_v^{(n-1)/2}(1+q_v)^2} & (\textit{if } k = 1), \\[2ex] \dfrac{(q_v - \alpha^2)(\alpha^2 q_v - 1)(q_v - 1)\sum_{j=0}^{n-k} \alpha^{2j}}{\alpha^{n-k+2} q_v^{(n-k+4)/2}(1+q_v)} & (\textit{if } n \geq 2, 2 \leq k \leq n), \end{cases}$$
- $$c_{n,k} = \begin{cases} \dfrac{q_v \sum_{j=0}^{n} \alpha^{2j} - \sum_{j=1}^{n-1} \alpha^{2j}}{\alpha^n q_v^{n/2}(1+q_v)} & (\textit{if } k = 0), \\[2ex] \dfrac{\sum_{j=0}^{n-k} \alpha^{2j}}{\alpha^{n-k} q_v^{(n-k)/2}} & (\textit{if } 1 \leq k \leq n-1). \end{cases}$$

3. A basis of the $\mathbf{K}_0(\mathfrak{p}_v^n)$-invariant subspace in a nonspherical representation. For $v \in \Sigma_{\text{fin}}$, let π_v be an infinite dimensional unitarizable irreducible admissible representation with trivial central character. In this section, assume $c(\pi_v) \geq 1$.

Fix $n \in \boldsymbol{N}_0$ and assume $c(\pi_v) \geq 2$. In this case we define the inner product $(\cdot|\cdot)_v$ by $q_v^{d_v/2}\langle\cdot|\cdot\rangle$. For $k \in \{0, \dots, n\}$, we denote $\pi_v(\delta_v(-k, 0))\phi_{0,v}$ by $\phi_{k,v}$. Then, by Lemma 3 and a direct computation, we have the following proposition.

PROPOSITION 13. *$\{\phi_{0,v}, \dots, \phi_{n,v}\}$ is an orthogonal basis on $\left(V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)+n})}, (\cdot|\cdot)_v\right)$ and we have $\|\phi_{0,v}\|_v := \sqrt{(\phi_{0,v}|\phi_{0,v})_v} = 1$.*

Next assume $c(\pi_v) = 1$. Then we have $\pi_v = \sigma(\chi_v|\cdot|_v^{1/2}, \chi_v|\cdot|_v^{-1/2})$ for some unramified character χ_v of $F_v^\times$ satisfying $\chi_v^2 = \mathbf{1}_v$. In this case we define the inner product $(\cdot|\cdot)_v$ by $q_v^{d_v/2}(1 - q_v^{-2})\langle\cdot|\cdot\rangle$. By Lemma 3, we have $\|\phi_{0,v}\|_v = 1$.

LEMMA 14. *For every $k \in \boldsymbol{N}_0$, there exists $\lambda_{\pi_v}(k) \in \boldsymbol{R}^\times$ such that*

$$\left(\pi_v(\delta_v(-k,0))\phi_{0,v}\middle|\phi_{0,v}\right)_v = \lambda_{\pi_v}(k)\,.$$

PROOF. We have

$$\begin{aligned}
\langle \pi_v(\delta_v(-k,0))\phi_{0,v}|\phi_{0,v}\rangle &= \int_{F_v^\times} \chi_v(\varpi_v^{d_v})q_v^{-d_v}\chi_v(\varpi_v^{-k}t)|\varpi_v^{-k}t|_v \mathrm{ch}_{\mathfrak{p}_v^{-d_v}}(\varpi_v^{-k}t) \\
&\quad \times \overline{\chi_v(\varpi_v^{d_v})q_v^{-d_v}\chi_v(t)|t|_v \mathrm{ch}_{\mathfrak{p}_v^{-d_v}}(t)}d^\times t \\
&= q_v^{-2d_v+k}\chi_v(\varpi_v^{-k})\int_{F_v^\times}|t|_v^2 \mathrm{ch}_{\mathfrak{p}_v^{k-d_v}}(t)d^\times t \\
&= q_v^{-2d_v+k}\chi_v(\varpi_v^{-k})\sum_{n=k-d_v}^{\infty}\int_{\mathfrak{o}_v^\times}|\varpi_v^n u|_v^2 d^\times u \\
&= q_v^{-2d_v+k}\chi_v(\varpi_v^{-k})\frac{q_v^{2d_v-2k}}{1-q_v^{-2}}\mathrm{vol}(\mathfrak{o}_v^\times, d^\times t) \\
&= \frac{q_v^{-k}\chi_v(\varpi_v^{-k})}{1-q_v^{-2}}\mathrm{vol}(\mathfrak{o}_v^\times, d^\times t)\,.
\end{aligned}$$

Therefore we can take $\lambda_{\pi_v}(k) = q_v^{-k}\chi_v(\varpi_v^{-k})$ and $\lambda_{\pi_v}(k)$ must be real by $\chi_v(\varpi_v) \in \{\pm 1\}$. □

The proof of the following proposition is the same as that of Proposition 10.

PROPOSITION 15. *There exists a unique finite set $\{\phi_{1,v},\ldots,\phi_{n,v}\}$ of $V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)+n})}$ satisfying the following conditions* (1) *and* (2):

(1) *There exists a finite sequence of real numbers $\{c_{\pi_v}(k,j)\}_{1\le k\le n, 0\le j\le k-1}$ such that*

$$\phi_{k,v} = \pi_v(\delta_v(-k,0))\phi_{0,v} - \sum_{j=0}^{k-1} c_{\pi_v}(k,j)\phi_{j,v} \ \textit{for all } k \in \{1,\ldots,n\}\,.$$

(2) *The set $\{\phi_{0,v},\ldots,\phi_{n,v}\}$ is an orthogonal basis of $\left(V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)+n})}, (\cdot|\cdot)_v\right)$.*

By induction on n and a direct computation, we can prove the following.

COROLLARY 16. *With the notation in the previous Proposition, set*

$$\left(\pi_v(\delta_v(-k,0))\phi_{0,v}\middle|\phi_{j,v}\right)_v = \tau_{\pi_v}(k,j)$$

for all $k, j \in \{0,\ldots,n\}$. Then, for $n \in \boldsymbol{N}$ we have the following:

- $$\tau_{\pi_v}(n,k) = \begin{cases} \dfrac{1}{q_v^n\chi_v(\varpi_v)^n} & (\textit{if } k=0)\,, \\[2ex] \dfrac{q_v^2\chi_v(\varpi_v)^2-1}{q_v^{n-k+2}\chi_v(\varpi_v)^{n-k+2}} & (\textit{if } 1\le k\le n)\,, \end{cases}$$

- $c_{\pi_v}(n,k) = \dfrac{1}{q_v^{n-k}\chi_v(\varpi_v)^{n-k}} \quad (\textit{if } 0 \le k \le n-1)\,.$

4. Zeta integrals of cusp forms on $GL(2)$. Let π be a $\mathbf{K}_\infty$-spherical cuspidal automorphic representation of $G_{\boldsymbol{A}}$ with trivial central character, where the representation space V_π is contained in the space of cusp forms $\mathcal{A}_0(G_F\backslash G_{\boldsymbol{A}}, \mathbf{1})$. For any quasi character η of $\boldsymbol{A}^\times/F^\times$ and $\varphi \in V_\pi$, we define the global zeta integral by

$$Z(s,\eta,\varphi) := \int_{F^\times\backslash \boldsymbol{A}^\times} \varphi\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix} \eta(t)|t|_{\boldsymbol{A}}^{s-1/2} d^\times t\,, \quad s\in \boldsymbol{C}\,.$$

The defining integral converges absolutely for any $s \in \boldsymbol{C}$, and hence $Z(s,\eta,\varphi)$ is an entire function in s.

We fix a family $\{\pi_v\}_{v\in\Sigma_F}$ consisting of unitarizable irreducible admissible representations such that $\pi \cong \bigotimes_{v\in\Sigma_F}\pi_v$. The conductor of π is defined to be the ideal $\mathfrak{f}_\pi$ of $\mathfrak{o}_F$ such that $\mathfrak{f}_\pi\mathfrak{o}_v = \mathfrak{p}_v^{c(\pi_v)}$ for all $v\in\Sigma_{\mathrm{fin}}$. Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}_F$ which is divided by $\mathfrak{f}_\pi$. We construct a basis of $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$.

For $v\in\Sigma_{\mathrm{fin}}$ satisfying the isomorphism $\pi_v\cong I(\chi_v)$ for some unramified quasi character χ_v, we denote by $\phi_{k,v}\in V_{\pi_v}$ the Whittaker function corresponding to $\chi_v(\varpi_v)^{-d_v}(1-\chi_v^2(\varpi_v)q_v^{-1})^{-1}f_{k,v}^{\pi_v}$ by the isomorphism

$$I(\chi_v) \ni f \mapsto W_f(g) := \int_{F_v} f\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi_{F_v}(-x)dx \in V_{\pi_v},$$

for all $k\in\{0,\ldots,n\}$. Then, the function $\phi_{0,v}$ coincides with the local new form which appears in Proposition 1.

Let n be the maximal nonnegative integer m such that $S_m(\mathfrak{n}\mathfrak{f}_\pi^{-1}) \neq \emptyset$. For $\rho = (\rho_k)_{1\le k\le n} \in \prod_{k=1}^n \mathrm{Map}\left(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}), \{0,\ldots,k\}\right)$, let us denote by $\varphi_{\pi,\rho}$ the cusp form in $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ corresponding to

$$\bigotimes_{v\in\Sigma_\infty}\phi_{0,v}\otimes \bigotimes_{v\in S_1(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\phi_{\rho_1(v),v}\otimes\cdots\otimes \bigotimes_{v\in S_n(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\phi_{\rho_n(v),v}\otimes \bigotimes_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\phi_{0,v}$$

by the isomorphism $V_\pi\cong\bigotimes_{v\in\Sigma_F}V_{\pi_v}$.

For $v\in S(\mathfrak{f}_\pi)$, let $(\cdot|\cdot)_v$ be the G_v-invariant hermitian inner product on V_{π_v} defined in §3. For $v\in\Sigma_\infty\cup(\Sigma_{\mathrm{fin}}-S(\mathfrak{f}_\pi))$, we take a G_v-invariant hermitian inner product $(\cdot|\cdot)_v$ on V_{π_v} such that $\|\phi_{0,v}\|_v = 1$. We obtain the following by the same proof as [9, Lemma 2.4].

PROPOSITION 17. *The finite set*

$$\left\{\varphi_{\pi,\rho}\ ;\ \rho\in\prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}),\{0,\ldots,k\})\right\}$$

is an orthogonal basis of $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$. Here $V_\pi\subset L^2(Z_{\boldsymbol{A}}G_F\backslash G_{\boldsymbol{A}})$ is equipped with the L^2-inner product.

LEMMA 18. *For any unramified quasi character η_v of $F_v^\times$ and any $k \in \boldsymbol{N}$, we define polynomials $Q_{k,v}^{\pi_v}(\eta_v, X) \in \boldsymbol{C}[X]$ by the following recurrence relation.*

$$Q_{k,v}^{\pi_v}(\eta_v, X) = \eta_v(\varpi_v)^k X^k - \sum_{j=0}^{k-1} c_{\pi_v}(k, j) Q_{j,v}^{\pi_v}(\eta_v, X)\,,$$
$$Q_{0,v}^{\pi_v}(\eta_v, X) = 1\,,$$

where we put $c_{\pi_v}(k, j) = 0$ for all $k \in \boldsymbol{N}$ and $j \in \{0, \dots, k-1\}$, if $v \in \Sigma_{\mathrm{fin}}$ satisfies $c(\pi_v) \geq 2$. Then, for any $v \in \Sigma_{\mathrm{fin}}$, $k \in \{0, \dots, n\}$ and any unramified quasi character η_v of $F_v^\times$, we have

$$Z(s, \eta_v, \phi_{k,v}) = Q_{k,v}^{\pi_v}(\eta_v, q_v^{1/2-s}) Z(s, \eta_v, \phi_{0,v})\,.$$

PROOF. Suppose $c(\pi_v) \geq 2$. We have

$$\begin{aligned} Z(s, \eta_v, \phi_{k,v}) &= Z\left(s, \eta_v, \pi_v(\delta_v(-k, 0))\phi_{0,v}\right) \\ &= \eta_v(\varpi_v)^k q_v^{k(1/2-s)} Z(s, \eta_v, \phi_{0,v})\,. \end{aligned}$$

Hence we can take $Q_{k,v}^{\pi_v}(\eta_v, X) = \eta_v(\varpi_v)^k X^k$.

Suppose $c(\pi_v) \in \{0, 1\}$. We prove the assertion by induction on k. The assertion is obvious for $k = 0$. Indeed, we can take $Q_{0,v}^{\pi_v}(\eta_v, X) = 1$. Suppose $k \geq 1$. We have

$$\begin{aligned} &Z(s, \eta_v, \phi_{k,v}) \\ &\quad = Z\left(s, \eta_v, \pi_v(\delta_v(-k, 0))\phi_{0,v} - \sum_{j=0}^{k-1} c_{\pi_v}(k, j)\phi_{j,v}\right) \\ &\quad = Z(s, \eta_v, \pi_v(\delta_v(-k, 0))\phi_{0,v}) - \sum_{j=0}^{k-1} c_{\pi_v}(k, j) Z(s, \eta_v, \phi_{j,v}) \\ &\quad = \eta_v(\varpi_v^k) q_v^{k(1/2-s)} Z(s, \eta_v, \phi_{0,v}) - \sum_{j=0}^{k-1} c_{\pi_v}(k, j) Q_{j,v}^{\pi_v}(\eta_v, q_v^{1/2-s}) Z(s, \eta_v, \phi_{0,v})\,. \end{aligned}$$

Thus we can take

$$Q_{k,v}^{\pi_v}(\eta_v, X) = \eta_v(\varpi_v)^k X^k - \sum_{j=0}^{k-1} c_{\pi_v}(k, j) Q_{j,v}^{\pi_v}(\eta_v, X).$$

□

By induction on k and a direct computation, we have the following.

COROLLARY 19. *We have the following:*

- *If $c(\pi_v) = 0$ and (α, α^{-1}) is the Satake parameter of π_v, then we have*

$Q_{k,v}^{\pi_v}(\eta_v, X)$

$$
= \begin{cases} 1 & (\textit{if } k = 0)\,, \\ \eta_v(\varpi_v)X - \dfrac{\alpha + \alpha^{-1}}{q_v^{1/2} + q_v^{-1/2}} & (\textit{if } k = 1)\,, \\ q_v^{-1}\eta_v(\varpi_v)^{k-2}X^{k-2}(\alpha q_v^{1/2}\eta_v(\varpi_v)X - 1)(\alpha^{-1}q_v^{1/2}\eta_v(\varpi_v)X - 1) & (\textit{if } k \geq 2)\,. \end{cases}
$$

- *If $c(\pi_v) = 1$ and π_v is isomorphic to $\sigma(\chi_v|\cdot|_v^{1/2}, \chi_v|\cdot|_v^{-1/2})$, then we have*

$$
Q_{k,v}^{\pi_v}(\eta_v, X) = \begin{cases} 1 & (\textit{if } k = 0)\,, \\ \eta_v(\varpi_v)^{k-1}X^{k-1}(\eta_v(\varpi_v)X - q_v^{-1}\chi_v(\varpi_v)^{-1}) & (\textit{if } k \geq 1)\,. \end{cases}
$$

- *If $c(\pi_v) \geq 2$, then we have*

$$
Q_{k,v}^{\pi_v}(\eta_v, X) = \eta_v(\varpi_v)^k X^k
$$

for any $k \in \boldsymbol{N}_0$.

We consider a character η of $\boldsymbol{A}^\times/F^\times$ satisfying

$$
(\star)\ \begin{cases} v \in \Sigma_\infty \Rightarrow \eta_v = |\cdot|_v^{t_v} \quad \text{for some } t_v \in i\boldsymbol{R}\,, \\ \mathfrak{f}_\eta \text{ is relatively prime to } \mathfrak{n}\,. \end{cases}
$$

For such η and $\varphi \in V_\pi^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$, we define the modified global zeta integral by

$$
Z^*(s, \eta, \varphi) = \eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}})Z\left(s, \eta, \pi\begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix}\varphi\right), \quad s \in \boldsymbol{C}\,.
$$

Here $x_\eta = (x_{\eta,v})_{v \in \Sigma_F} \in \boldsymbol{A}$ is the adele whose v-component satisfies

$$
x_{\eta,v} = \begin{cases} 0 & (v \in \Sigma_\infty) \\ \varpi_v^{-f(\eta_v)} & (v \in \Sigma_{\mathrm{fin}}) \end{cases}
$$

and $x_{\eta,\mathrm{fin}}$ is the projection of x_η to $\boldsymbol{A}_{\mathrm{fin}}$.

PROPOSITION 20. *For any $\rho = (\rho_1, \ldots, \rho_n) \in \prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}), \{0, \ldots, k\})$ and η satisfying $(\star)$, we have*

$$
Z^*(s, \eta, \varphi_{\pi,\rho}) = D_F^{s-1/2}\mathcal{G}(\eta)\left\{\prod_{k=1}^n \prod_{v \in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})} Q_{\rho_k(v),v}^{\pi_v}(\eta_v, q_v^{1/2-s})\right\} L(s, \pi \otimes \eta)\,.
$$

PROOF. We give a proof in the same way as [9, Lemma 2.5]. By definition, we have

$$
\begin{aligned}
&Z^*(s, \eta, \varphi_{\pi,\rho}) \\
&\quad = \prod_{v \in \Sigma_\infty} Z\left(s, \eta_v, \pi_v\begin{pmatrix} 1 & x_{\eta,v} \\ 0 & 1 \end{pmatrix}\phi_{0,v}\right) \\
&\quad \times \prod_{k=1}^n \prod_{v \in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \eta_v(x_{\eta,v})Z\left(s, \eta_v, \pi_v\begin{pmatrix} 1 & x_{\eta,v} \\ 0 & 1 \end{pmatrix}\phi_{\rho_k(v),v}\right)
\end{aligned}
$$

$$\times \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \eta_v(x_{\eta,v}) Z\left(s, \eta_v, \pi_v\begin{pmatrix}1 & x_{\eta,v}\\ 0 & 1\end{pmatrix}\phi_{0,v}\right).$$

For $v\in\Sigma_\infty$, we have $Z\big(s,\eta_v,\pi_v\big(\begin{smallmatrix}1 & x_{\eta,v}\\ 0 & 1\end{smallmatrix}\big)\phi_{0,v}\big) = L(s,\pi_v\otimes\eta_v)$. Next we consider the case of $v\in\Sigma_{\mathrm{fin}}$. Since $\mathfrak{f}_\eta$ is relatively prime to $\mathfrak{n}$, the character η_v is unramified if $v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$. Therefore, for $v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$, by Proposition 1 we have

$$\begin{aligned}
&\eta_v(x_{\eta,v})Z\left(s,\eta_v,\pi_v\begin{pmatrix}1 & x_{\eta,v}\\ 0 & 1\end{pmatrix}\phi_{\rho_k(v),v}\right)\\
&\quad = Z(s,\eta_v,\phi_{\rho_k(v),v})\\
&\quad = Q^{\pi_v}_{\rho_k(v),v}(\eta_v, q_v^{1/2-s})Z(s,\eta_v,\phi_{0,v})\\
&\quad = Q^{\pi_v}_{\rho_k(v),v}(\eta_v, q_v^{1/2-s})\mathrm{vol}(\mathfrak{o}_v^\times, d^\times t)\eta_v(\varpi_v)^{-d_v} q_v^{d_v(s-1/2)} L(s,\pi_v\otimes\eta_v).
\end{aligned}$$

For $v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$, if η_v is unramified, by Proposition 1 we have

$$\begin{aligned}
&\eta_v(x_{\eta,v})Z\left(s,\eta_v,\pi_v\begin{pmatrix}1 & x_{\eta,v}\\ 0 & 1\end{pmatrix}\phi_{0,v}\right)\\
&\qquad = Z(s,\eta_v,\phi_{0,v}) = \mathrm{vol}(\mathfrak{o}_v^\times, d^\times t)\eta_v(\varpi_v)^{-d_v} q_v^{d_v(s-1/2)} L(s,\pi_v\otimes\eta_v).
\end{aligned}$$

Suppose that η_v is ramified. We notice that $L(s,\pi_v\otimes\eta_v)$ is identically one. If $c(\pi_v)=0$, by the definition of $\phi_{0,v}$, we have

$$\phi_{0,v}\begin{pmatrix}\varpi_v^m & 0\\ 0 & 1\end{pmatrix} = q_v^{-(m+d_v)/2}\frac{\alpha_1^{m+d_v+1}-\alpha_2^{m+d_v+1}}{\alpha_1-\alpha_2}\delta(m\ge -d_v)$$

for any $m\in\mathbf{Z}$. By

$$\phi_{0,v}\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \varpi_v^{-f(\eta_v)}\\ 0 & 1\end{pmatrix}\right) = \psi_{F_v}(t\varpi_v^{-f(\eta_v)})\phi_{0,v}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}, \quad (t\in F_v^\times),$$

the equality

$$\eta_v(x_{\eta,v})Z\left(s,\eta_v,\pi_v\begin{pmatrix}1 & x_{\eta,v}\\ 0 & 1\end{pmatrix}\phi_{0,v}\right) = q_v^{d_v(s-1/2)}\mathcal{G}(\eta_v)L(s,\pi_v\otimes\eta_v)$$

holds (cf. [9, Lemma 2.5]). If $c(\pi_v)=1$, then we have

$$\begin{aligned}
&\eta_v(x_{\eta,v})Z\left(s,\eta_v,\pi_v\begin{pmatrix}1 & x_{\eta,v}\\ 0 & 1\end{pmatrix}\phi_{0,v}\right)\\
&= \int_{F_v^\times}\psi_{F_v}(t\varpi_v^{-f(\eta_v)})\chi_v(\varpi_v^{d_v}t)|\varpi_v^{d_v}t|_v \mathrm{ch}_{\mathfrak{o}_v}(\varpi_v^{d_v}t)\eta_v(t\varpi_v^{-f(\eta_v)})|t|_v^{s-1/2}d^\times t\\
&= q_v^{d_v(s-1/2)}\int_{\mathfrak{o}_v-\{0\}}\psi_{F_v}(t\varpi_v^{-d_v-f(\eta_v)})\eta_v(t\varpi_v^{-d_v-f(\eta_v)})\chi_v(t)|t|_v^{s+1/2}d^\times t\\
&= q_v^{d_v(s-1/2)}\sum_{n=0}^{\infty}\int_{\varpi_v^n\mathfrak{o}_v^\times}\psi_{F_v}(t\varpi_v^{-d_v-f(\eta_v)})\eta_v(t\varpi_v^{-d_v-f(\eta_v)})\chi_v(t)|t|_v^{s+1/2}d^\times t
\end{aligned}$$

$$= q_v^{d_v(s-1/2)} \sum_{n=0}^{\infty} (\chi_v(\varpi_v) q_v^{-s-1/2})^n \int_{\mathfrak{o}_v^\times} \psi_{F_v}(t\varpi_v^{-d_v-f(\eta_v)+n}) \eta_v(t\varpi_v^{-d_v-f(\eta_v)+n}) d^\times t$$
$$= q_v^{d_v(s-1/2)} \mathcal{G}(\eta_v) L(s, \pi_v \otimes \eta_v)$$

since $\int_{\mathfrak{o}_v^\times} \psi_{F_v}(t\varpi_v^{-d_v-f(\eta_v)+n}) \eta_v(t\varpi_v^{-d_v-f(\eta_v)+n}) d^\times t$ vanishes if and only if $n \neq 0$.

If $c(\pi_v) \geq 2$, we have

$$\begin{aligned}
&\eta_v(x_{\eta,v}) Z\left(s, \eta_v, \pi_v \begin{pmatrix} 1 & x_{\eta,v} \\ 0 & 1 \end{pmatrix} \phi_{0,v}\right) \\
&\quad = \int_{F_v^\times} \psi_{F_v}(t\varpi_v^{-f(\eta_v)}) \mathrm{ch}_{\varpi_v^{-d_v}\mathfrak{o}_v^\times}(t) \eta_v(t\varpi_v^{-f(\eta_v)}) |t|_v^{s-1/2} d^\times t \\
&\quad = q_v^{d_v(s-1/2)} \mathcal{G}(\eta_v) L(s, \pi_v \otimes \eta_v) .
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&Z^*(s, \eta, \varphi_{\pi,\rho_1,\dots,\rho_n}) \\
&= \prod_{v \in \Sigma_\infty} L(s, \pi_v \otimes \eta_v) \\
&\quad \times \prod_{k=1}^{n} \prod_{v \in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})} Q_{\rho_k(v),v}^{\pi_v}(\eta_v, q_v^{1/2-s}) \mathrm{vol}(\mathfrak{o}_v^\times, d^\times t) \eta_v(\varpi_v)^{-d_v} q_v^{d_v(s-1/2)} L(s, \pi_v \otimes \eta_v) \\
&\quad \times \prod_{v \in \Sigma_{\mathrm{fin}} - (S(\mathfrak{n}\mathfrak{f}_\pi^{-1}) \cup S(\mathfrak{f}_\eta))} \mathrm{vol}(\mathfrak{o}_v^\times, d^\times t) \eta_v(\varpi_v)^{-d_v} q_v^{d_v(s-1/2)} L(s, \pi_v \otimes \eta_v) \\
&\quad \times \prod_{v \in S(\mathfrak{f}_\eta) - S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} q_v^{d_v(s-1/2)} \mathcal{G}(\eta_v) L(s, \pi_v \otimes \eta_v).
\end{aligned}$$

This completes the proof. □

5. Regularized periods of cusp forms. In this section we prove Main Theorem A. We fix a relatively compact set $\omega \subset \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) ; a, d \in \boldsymbol{A}^1, b \in \boldsymbol{A} \right\}$ such that $B_F \omega = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) ; a, d \in \boldsymbol{A}^1, b \in \boldsymbol{A} \right\}$. For any $t > 0$, set

$$\mathfrak{S}(t) := \omega \left\{ \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} ; y_1, y_2 > 0, y_1/y_2 > t \right\} \mathbf{K} .$$

The set $\mathfrak{S}(t)$ is called a Siegel set of $G_{\boldsymbol{A}}$. There exists $t_0 > 0$ such that $G_{\boldsymbol{A}} = G_F \mathfrak{S}(t_0)$. We take such t_0 once and for all and we put $\mathfrak{S} = \mathfrak{S}(t_0)$ (cf. [4, §10]).

For $C > 0$, let $\mathcal{B}(C)$ be the space of all holomorphic functions β on $\{z \in \boldsymbol{C} ; |\mathrm{Re}(z)| < C\}$ satisfying that

(1) the equality $\beta(z) = \beta(-z)$ holds,
(2) the estimate
$$|\beta(\sigma + it)| \prec (1 + |t|)^{-l} , \quad \sigma \in [a, b]$$
holds for any $[a, b] \subset (-C, C)$ and any $l > 0$.

Let $\mathcal{B}$ be the space of all entire functions β on $\boldsymbol{C}$ such that the restriction of β to $\{z \in \boldsymbol{C}; |\operatorname{Re}(z)| < C\}$ is contained in $\mathcal{B}(C)$ for any $C > 0$. For $\beta \in \mathcal{B}$, $t > 0$ and $\lambda \in \boldsymbol{C}$, we consider

$$\hat{\beta}_\lambda(t) := \frac{1}{2\pi i}\int_{L_\sigma} \frac{\beta(z)}{z+\lambda} t^z dz\,, \quad (\sigma > -\operatorname{Re}(\lambda))\,.$$

Here we write L_σ for $\{z \in \boldsymbol{C}; \operatorname{Re}(z) = \sigma\}$ and L_σ is equipped with the direction of increasing imaginary part.

For $\beta \in \mathcal{B}$, $\lambda \in \boldsymbol{C}$, a character η of $\boldsymbol{A}^\times/F^\times$ satisfying $(\star)$ and a function $\varphi : \mathfrak{A}G_F\backslash G_{\boldsymbol{A}} \to \boldsymbol{C}$, we consider

$$P^\eta_{\beta,\lambda}(\varphi) := \int_{F^\times\backslash \boldsymbol{A}^\times} \{\hat{\beta}_\lambda(|t|_{\boldsymbol{A}}) + \hat{\beta}_\lambda(|t|_{\boldsymbol{A}}^{-1})\}\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x_\eta \\ 0 & 1\end{pmatrix}\right)\eta(t)\eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}})d^\times t.$$

For the function $\varphi : \mathfrak{A}G_F\backslash G_{\boldsymbol{A}} \to \boldsymbol{C}$, we assume the following:

- For any $\beta \in \mathcal{B}$, there exists a constant $C \in \boldsymbol{R}$ such that if $\operatorname{Re}(\lambda) > C$ the integral $P^\eta_{\beta,\lambda}(\varphi)$ converges.
- For any $\beta \in \mathcal{B}$, $\{z \in \boldsymbol{C}; \operatorname{Re}(z) > C\} \ni \lambda \mapsto P^\eta_{\beta,\lambda}(\varphi)$ has a meromorphic continuation to a neighborhood of $\lambda = 0$.
- the constant term $\mathrm{CT}_{\lambda=0}P^\eta_{\beta,\lambda}(\varphi)$ of the Laurent expansion of $P^\eta_{\beta,\lambda}(\varphi)$ at $\lambda = 0$ is proportional to the Dirac delta distribution supported at 0 as a linear functional of $\mathcal{B}$.

Then, the proportionality constant $P^\eta_{\mathrm{reg}}(\varphi)$ is called *the regularized η-period of φ*, i.e.,

$$\mathrm{CT}_{\lambda=0}P^\eta_{\beta,\lambda}(\varphi) = P^\eta_{\mathrm{reg}}(\varphi)\beta(0)$$

for all $\beta \in \mathcal{B}$.

In this case, it was proved by Tsuzuki [9, Lemma 7.3] that if φ is rapidly decreasing on $\mathfrak{S} \cap G^1_{\boldsymbol{A}}$, then $P^\eta_{\beta,\lambda}(\varphi)$ converges absolutely for any $(\beta,\lambda) \in \mathcal{B}\times\boldsymbol{C}$, $P^\eta_{\mathrm{reg}}(\varphi)$ can be defined, and $P^\eta_{\mathrm{reg}}(\varphi) = Z^*(1/2,\eta,\varphi)$.

THE PROOF OF MAIN THEOREM A. For any $\rho = (\rho_1,\dots,\rho_n) \in \prod_{k=1}^n \operatorname{Map}(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}), \{0,\dots,k\})$, by Proposition 20, we have

$$\begin{aligned} P^\eta_{\mathrm{reg}}(\varphi_{\pi,\rho}) &= Z^*(1/2,\eta,\varphi) \\ &= \mathcal{G}(\eta)\left\{\prod_{k=1}^n \prod_{v \in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})} Q^{\pi_v}_{\rho_k(v),v}(\eta_v,1)\right\}L(1/2,\pi\otimes\eta)\,. \end{aligned}$$

Therefore, we obtain the formula in Main Theorem A by Corollary 19. □

6. Preliminaries for regularized periods of Eisenstein series. We fix a character $\chi = \prod_{v\in\Sigma_F}\chi_v$ of $\boldsymbol{A}^\times/F^\times$. For $\nu \in \boldsymbol{C}$, we denote by $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})$ the space of all smooth functions $f : G_{\boldsymbol{A}} \to \boldsymbol{C}$ which are $\mathbf{K}$-finite and satisfy

$$f\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} g\right) = \chi(a/d)|a/d|_{\boldsymbol{A}}^{(\nu+1)/2}f(g)$$

for all $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B_{\boldsymbol{A}}$ and $g \in G_{\boldsymbol{A}}$. Here a function $f : G_{\boldsymbol{A}} \to \boldsymbol{C}$ is said to be smooth if a function $GL(2, F \otimes_{\boldsymbol{Q}} \boldsymbol{R}) \ni g_\infty \mapsto f(g_\infty g_{\mathrm{fin}})$ is C^∞ for any $g_{\mathrm{fin}} \in GL(2, \boldsymbol{A}_{\mathrm{fin}})$ and a function $GL(2, \boldsymbol{A}_{\mathrm{fin}}) \ni g_{\mathrm{fin}} \mapsto f(g_\infty g_{\mathrm{fin}})$ is locally constant for any $g_\infty \in GL(2, F \otimes_{\boldsymbol{Q}} \boldsymbol{R})$.

If $\nu \in i\boldsymbol{R}$, then the space $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})$ is unitarizable and a $G_{\boldsymbol{A}}$-invariant hermitian inner product is given by

$$(f_1|f_2) = \int_{\mathbf{K}} f_1(k)\overline{f_2(k)}dk$$

for any $f_1, f_2 \in I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})$. We denote the norm $\sqrt{(f|f)}$ of $f \in I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})$ by $\|f\|$. Similarly, we define a G_v-invariant hermitian inner product $(\cdot|\cdot)_v$ on $I(\chi_v|\cdot|_v^{\nu/2})$ and the norm $\|\cdot\|_v$ of $I(\chi_v|\cdot|_v^{\nu/2})$ for $\nu \in i\boldsymbol{R}$ by integration on $\mathbf{K}_v$.

For $\nu \in \boldsymbol{C}$, we take $f^{(\nu)} \in I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})$. The family $\{f^{(\nu)}\}_{\nu\in\boldsymbol{C}}$ is called a flat section if the restriction of $f^{(\nu)}$ to $\mathbf{K}$ is independent of $\nu \in \boldsymbol{C}$. We define the Eisenstein series for $f^{(\nu)}$ by

$$E(f^{(\nu)}, g) = \sum_{\gamma \in B_F \backslash G_F} f^{(\nu)}(\gamma g)$$

for $g \in G_{\boldsymbol{A}}$ and $\nu \in \boldsymbol{C}$. If $\mathrm{Re}(\nu) > 1$, the defining series converges absolutely. If $\{f^{(\nu)}\}_{\nu\in\boldsymbol{C}}$ is a flat section, $E(f^{(\nu)}, g)$ has a meromorphic continuation to $\boldsymbol{C}$ as a function in ν. The function $E(f^{(\nu)}, g)$ is holomorphic on $i\boldsymbol{R}$ and has the only possible pole at $\nu = 1$ on the half plane $\mathrm{Re}(\nu) > 0$, which occurs only when $\chi^2 = \mathbf{1}$.

Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}_F$. From this section, we assume the following:

- $v \in \Sigma_\infty \Rightarrow \chi_v = |\cdot|_v^{t_v}$ for some $t_v \in i\boldsymbol{R}$,
- $\mathfrak{n}$ is divided by $\mathfrak{f}_\chi^2$.

These conditions are equivalent to $\dim I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})} \geq 1$.

For $v \in \Sigma_\infty$, we denote by $f_{0,\chi_v}^{(\nu)}$ the spherical vector in $I(\chi_v|\cdot|_v^{\nu/2})$ normalized so that $f_{0,\chi_v}^{(\nu)}(e)$ equals one.

7. Local new forms for ramified induced representations. In this section, we assume $v \in S(\mathfrak{f}_\chi)$. By [7, Proposition 2.1.2], we have the following.

PROPOSITION 21. *The invariant subspace $I(\chi_v|\cdot|_v^{\nu/2})^{\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)})}$ is of dimension one. A nonzero vector in $I(\chi_v|\cdot|_v^{\nu/2})^{\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)})}$ is given by*

$$f_{0,\chi_v}^{(\nu)}(g) = \begin{cases} \chi_v(\varpi_v^{-f(\chi_v)}) q_v^{f(\chi_v)\nu/2} \chi_v(a/d)|a/d|_v^{(\nu+1)/2} \\ \qquad\qquad (\text{if } \; g \in \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \gamma_{f(\chi_v)+1} \mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)}),\, a, d \in F_v^\times), \\ 0 \qquad\qquad (\text{if } \; g \notin B_F \gamma_{f(\chi_v)+1} \mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)})), \end{cases}$$

where we put

$$\gamma_i = \begin{cases} \begin{pmatrix} 1 & 0 \\ \varpi_v^{i-1} & 1 \end{pmatrix} & (\textit{if} \ \ i \in \boldsymbol{N}), \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\textit{if} \ \ i = 0). \end{cases}$$

Moreover we have

$$f_{0,\chi_v}^{(\nu)} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \chi_v^{-1}(x) q_v^{f(\chi_v)\nu/2} \mathrm{ch}_{\varpi_v^{f(\chi_v)} \mathfrak{o}_v^\times}(x)$$

for any $x \in F_v^\times$.

For $k \in \boldsymbol{N}_0$, set

$$\tilde{f}_{k,\chi_v}^{(\nu)} := q_v^{-(k+f(\chi_v))\nu/2} T(\chi_v) \pi_v(\delta_v(-k, 0)) f_{0,\chi_v}^{(\nu)},$$

where $B(\mathfrak{o}_v) := B_v \cap \mathbf{K}_v$ and $T(\chi_v) := \mathrm{vol}(B(\mathfrak{o}_v)\gamma_{f(\chi_v)+1}\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)}))^{-1/2}$.

LEMMA 22. *If* $\nu \in i\boldsymbol{R}$, *then* $I(\chi_v|\cdot|_v^{\nu/2})$ *is irreducible and we have* $\|\tilde{f}_{k,\chi_v}^{(\nu)}\|_v = 1$ *for any* $k \in \boldsymbol{N}_0$.

PROOF. Assume $\nu \in i\boldsymbol{R}$. By [7, Lemma 2.1.1] we have

$$\mathbf{K}_v = \coprod_{i=0}^{2f(\chi_v)} B(\mathfrak{o}_v)\gamma_i \mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)}).$$

Therefore we obtain

$$\begin{aligned} \|f_{0,\chi_v}^{(\nu)}\|_v^2 &= \int_{\mathbf{K}_v} |f_{0,\chi_v}^{(\nu)}(k)|^2 dk \\ &= \sum_{i=0}^{2f(\chi_v)} \int_{B(\mathfrak{o}_v)\gamma_i \mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)})} |f_{0,\chi_v}^{(\nu)}(k)|^2 dk \\ &= \int_{B(\mathfrak{o}_v)\gamma_{f(\chi_v)+1} \mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)})} |f_{0,\chi_v}^{(\nu)}(k)|^2 dk \\ &= \mathrm{vol}(B(\mathfrak{o}_v)\gamma_{f(\chi_v)+1}\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)})). \end{aligned}$$

Hence $\|\tilde{f}_{0,\chi_v}^{(\nu)}\|_v = 1$. By definition, we obtain $\|\tilde{f}_{k,\chi_v}^{(\nu)}\|_v = \|\tilde{f}_{0,\chi_v}^{(\nu)}\|_v = 1$ for any $k \in \boldsymbol{N}_0$. □

Here $T(\chi_v)$ is explicitly computed by the following lemma.

LEMMA 23. *We have*

$$\begin{cases} \mathrm{vol}(B(\mathfrak{o}_v)\gamma_{f(\chi_v)+1}\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)})) = q_v^{-f(\chi_v)}(1 - q_v^{-1}), \\ T(\chi_v) = q_v^{f(\chi_v)/2}(1 - q_v^{-1})^{-1/2}. \end{cases}$$

PROOF. Assume $\nu \in i\boldsymbol{R}$. We note that the equality

$$\int_{\mathbf{K}_v} |f(k)|^2 dk = q_v^{d_v/2} \int_{F_v} \left| f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\right|^2 dx$$

holds for any $f \in I(\chi_v|\cdot|_v^{\nu/2})$. By the G_v-invariance of the integration on $\mathbf{K}_v$, we obtain

$$\begin{aligned}\|f_{0,\chi_v}^{(\nu)}\|_v^2 &= \int_{\mathbf{K}_v} |f_{0,\chi_v}^{(\nu)}(kw_0^{-1})|^2 dk = q_v^{d_v/2} \int_{F_v} \left| f_{0,\chi_v}^{(\nu)} \left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w_0^{-1} \right) \right|^2 dx \\ &= q_v^{d_v/2} \int_{F_v} \left| f_{0,\chi_v}^{(\nu)} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \right|^2 dx \\ &= q_v^{d_v/2} \int_{F_v^\times} |\chi_v^{-1}(x) q_v^{f(\chi_v)\nu/2} \mathrm{ch}_{\varpi_v^{f(\chi_v)}\mathfrak{o}_v^\times}(x)|^2 dx \\ &= q_v^{d_v/2} \int_{\varpi_v^{f(\chi_v)}\mathfrak{o}_v^\times} dx.\end{aligned}$$

By the proof of Lemma 22, the equality $\|f_{0,\chi_v}^{(\nu)}\|_v^2 = \mathrm{vol}(B(\mathfrak{o}_v)\gamma_{f(\chi_v)+1}\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)}))$ holds. This completes the proof. □

LEMMA 24. *For $k \in \boldsymbol{N}_0$, we have*

$$W_{\tilde{f}_{k,\chi_v}^{(\nu)}} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = q_v^{(1-\nu)d_v/2 - k\nu/2 + (1/2-\nu)f(\chi_v)} (1-q_v^{-1})^{1/2} \overline{\mathcal{G}(\chi_v)} \mathrm{ch}_{\varpi_v^{-d_v}\mathfrak{o}_v^\times}(\varpi_v^{-k}t)$$

for any $t \in F_v^\times$.

PROOF. By [7, §2.4], we have

$$W_{f_{0,\chi_v}^{(\nu)}} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \chi_v(-1) q_v^{d_v/2} \varepsilon(1, \chi_v|\cdot|_v^{\nu/2}, \psi_{F_v}) \mathrm{ch}_{\varpi_v^{-d_v}\mathfrak{o}_v^\times}(t), \quad t \in F_v^\times .$$

Since χ_v is ramified, we obtain

$$\begin{aligned}\chi_v(-1)\varepsilon(1, \chi_v|\cdot|_v^{\nu/2}, \psi_{F_v}) &= \chi_v(-1) q_v^{-(f(\chi_v)+d_v)\nu/2} (1-q_v^{-1}) \mathcal{G}(\chi_v^{-1}) \\ &= q_v^{-(f(\chi_v)+d_v)\nu/2}(1-q_v^{-1})\overline{\mathcal{G}(\chi_v)} .\end{aligned}$$

This completes the proof. □

LEMMA 25. *For $k \in \boldsymbol{N}_0$ we have the following:*

$$\tilde{f}_{k,\chi_v}^{(\nu)}(\gamma_i) = \begin{cases} \chi_v(\varpi)^{-k-f(\chi_v)} T(\chi_v) & (i = f(\chi_v)+k+1) , \\ 0 & (0 \le i \le 2f(\chi_v)+k , \quad i \neq f(\chi_v)+k+1) . \end{cases}$$

PROOF. This assertion is obvious for $i = 0$ since $f_{0,\chi_v}^{(\nu)}(e) = 0$. When $1 \le i \le 2f(\chi_v)+k$, we have

$$\begin{aligned}& f_{0,\chi_v}^{(\nu)} \left(\begin{pmatrix} 1 & 0 \\ \varpi_v^{i-1} & 1 \end{pmatrix} \begin{pmatrix} \varpi_v^{-k} & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= f_{0,\chi_v}^{(\nu)} \left(\begin{pmatrix} \varpi_v^{-k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi_v^{i-k-1} & 1 \end{pmatrix} \right) \\ &= \chi_v(\varpi_v)^{-k} q_v^{k\nu/2} \times \chi_v^{-1}(\varpi_v^{i-k-1}) q_v^{f(\chi_v)\nu/2} \mathrm{ch}_{\varpi_v^{f(\chi_v)}\mathfrak{o}_v^\times}(\varpi_v^{i-k-1}).\end{aligned}$$

This completes the proof. □

PROPOSITION 26. *For any $k \in \boldsymbol{N}_0$, the restriction of $\tilde{f}^{(\nu)}_{k,\chi_v}$ to $\mathbf{K}_v$ is independent of $\nu \in \boldsymbol{C}$. Fix $n \in \boldsymbol{N}_0$. If $\nu \in i\boldsymbol{R}$, the set $\{\tilde{f}^{(\nu)}_{k,\chi_v}; k \in \{0,\dots,n\}\}$ is an orthonormal basis of $I(\chi_v|\cdot|_v^{\nu/2})^{\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)+n})}$.*

PROOF. By Lemma 25, the first assertion is obvious. We assume $\nu \in i\boldsymbol{R}$. The set $\{\tilde{f}^{(\nu)}_{k,\chi_v}; k \in \{0,\dots,n\}\}$ is a basis of $I(\chi_v|\cdot|_v^{\nu/2})^{\mathbf{K}_0(\mathfrak{p}_v^{2f(\chi_v)+n})}$ by Proposition 1. We show the orthogonality of $\{\tilde{f}^{(\nu)}_{k,\chi_v}; k \in \{0,\dots,n\}\}$. By Lemma 22, we have $\|\tilde{f}^{(\nu)}_{k,\chi_v}\|_v = 1$ for any $k \in \{0,\dots,n\}$. For any $l, m \in \{0,\dots,n\}$ such that $l \neq m$, we have

$$\left(\pi_v(\delta_v(-l,0))f^{(\nu)}_{0,\chi_v}\,\middle|\,\pi_v(\delta_v(-m,0))f^{(\nu)}_{0,\chi_v}\right)$$
$$= \int_{\mathbf{K}_v} \pi_v(\delta_v(m-l,0))f^{(\nu)}_{0,\chi_v}(k)\overline{f^{(\nu)}_{0,\chi_v}(k)}dk$$
$$= q_v^{d_v/2}\int_{F_v} \pi_v(\delta_v(m-l,0))f^{(\nu)}_{0,\chi_v}\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\overline{f^{(\nu)}_{0,\chi_v}\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)}dx\,.$$

Put

$$\Phi_1(x) := \pi_v(\delta_v(m-l,0))f^{(\nu)}_{0,\chi_v}\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$$

and

$$\Phi_2(x) := f^{(\nu)}_{0,\chi_v}\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\,.$$

By the Plancherel theorem, we have

$$\int_{F_v}\Phi_1(x)\overline{\Phi_2(x)}dx = \int_{F_v}\widehat{\Phi_1}(x)\overline{\widehat{\Phi_2}(x)}dx\,.$$

Here $\widehat{\Phi_1}(x)$ and $\widehat{\Phi_2}(x)$ are Fourier transforms of Φ_1 and Φ_2 with respect to ψ_{F_v}, respectively. Hence, by the equalities

$$\pi_v(\delta_v(m-l,0))W_{f^{(\nu)}_{0,\chi_v}}\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} = \chi_v^{-1}(x)|x|_v^{\nu/2}\times|x|_v^{1/2}\widehat{\Phi_1}(x)$$

and

$$W_{f^{(\nu)}_{0,\chi_v}}\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} = \chi_v^{-1}(x)|x|_v^{\nu/2}\times|x|_v^{1/2}\widehat{\Phi_2}(x)\,,$$

we obtain

$$\left(\pi_v(\delta_v(-l,0))f^{(\nu)}_{0,\chi_v}\,\middle|\,\pi_v(\delta_v(-m,0))f^{(\nu)}_{0,\chi_v}\right) = q_v^{d_v/2}\int_{F_v}\widehat{\Phi_1}(x)\overline{\widehat{\Phi_2}(x)}dx$$
$$= q_v^{d_v/2}(1-q_v^{-1})\int_{F_v^\times}\pi_v(\delta_v(m-l,0))W_{f^{(\nu)}_{0,\chi_v}}\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\overline{W_{f^{(\nu)}_{0,\chi_v}}\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}}d^\times x\,.$$

This equals zero by Lemma 24. $\square$

We denote by $M_v(\nu) : I(\chi_v|\cdot|_v^{\nu/2}) \to I(\chi_v^{-1}|\cdot|_v^{-\nu/2})$ the intertwining operator defined by the integral

$$(M_v(\nu)f)(g) = \int_{F_v} f\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)dx$$

if it converges.

LEMMA 27. *For any $k \in \boldsymbol{N}_0$, we have*

$$M_v(\nu)\tilde{f}^{(\nu)}_{k,\chi_v} = q_v^{-(k+f(\chi_v))\nu}\frac{\varepsilon(1-\nu,\chi_v^{-2},\psi_{F_v})\varepsilon(1+\nu/2,\chi_v,\psi_{F_v})}{\varepsilon(1-\nu/2,\chi_v^{-1},\psi_{F_v})}$$
$$\times\frac{L(\nu,\chi_v^2)}{L(1-\nu,\chi_v^{-2})}\tilde{f}^{(-\nu)}_{k,\chi_v^{-1}}.$$

PROOF. If $\mathrm{Re}(\nu) \neq \pm 1$, the representations $I(\chi_v|\cdot|_v^{\nu/2})$ and $I(\chi_v^{-1}|\cdot|_v^{-\nu/2})$ are irreducible. Hence, we obtain this assertion by [7, Proposition 2.2.2]. By meromorphic continuation, we obtain the assertion for ν such that $\mathrm{Re}(\nu) = \pm 1$. $\square$

8. Local new forms for unramified induced representations. In this section we assume $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\chi)$. We denote by $f^{(\nu)}_{0,\chi_v}$ the spherical vector in $I(\chi_v|\cdot|_v^{\nu/2})$ normalized so that $f^{(\nu)}_{0,\chi_v}(e)$ equals one. We set

$$f^{(\nu)}_{k,\chi_v} := \pi_v(\delta_v(-k,0))f^{(\nu)}_{0,\chi_v} - \sum_{j=0}^{k-1}c^{(\nu)}_{\chi_v}(k,j)f^{(\nu)}_{j,\chi_v}$$

for $k \in \boldsymbol{N}_0$. Here the sequence $\{c^{(\nu)}_{\chi_v}(k,j)\}_{1\le k\le n, 0\le j\le k-1}$ is given as follows:

$$c^{(\nu)}_{\chi_v}(k,j) = \begin{cases} \dfrac{q_v\sum_{l=0}^{k}a^{2l} - \sum_{l=1}^{k-1}a^{2l}}{a^k q_v^{k/2}(1+q_v)} & (\text{if } j=0),\\[2ex] \dfrac{\sum_{l=0}^{k-j}a^{2l}}{a^{k-j}q_v^{(k-j)/2}} & (\text{if } 1\le j\le k-1),\end{cases}$$

where we put $a := \chi_v(\varpi_v)q_v^{-\nu/2}$. We note that if $\nu \in i\boldsymbol{R}$, the set $\{f^{(\nu)}_{k,\chi_v}; k \in \{0,\ldots,n\}\}$ is an orthogonal basis of $I(\chi_v|\cdot|_v^{\nu/2})^{\mathbf{K}_0(\mathfrak{p}_v^n)}$ by Proposition 10 and Corollary 12.

LEMMA 28. *For any $k \in \boldsymbol{N}_0$, we have*

$$\pi_v(\delta_v(-k,0))f^{(\nu)}_{0,\chi_v}(\gamma_i) = \begin{cases} a^{-k}q_v^{k/2} & (\textit{if } i=0),\\ a^{k+2-2i}q_v^{i-k/2-1} & (\textit{if } 1\le i\le k).\end{cases}$$

PROOF. The assertion is obvious for $i = 0$. For $i \ge 1$, we have

$$\begin{pmatrix}1 & 0\\ \varpi_v^{i-1} & 1\end{pmatrix}\begin{pmatrix}\varpi_v^{-k} & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\varpi_v^{1-i} & \varpi_v^{-k}\\ 0 & \varpi_v^{i-k-1}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & \varpi_v^{k+1-i}\end{pmatrix}.$$

We note $\begin{pmatrix} 0 & -1 \\ 1 & \varpi_v^{k+1-i} \end{pmatrix} \in \mathbf{K}_v$. Therefore we obtain

$$\pi_v(\delta_v(-k,0))f(\gamma_i) = a^{k+2-2i} q_v^{i-k/2-1}$$

for $i \geq 1$. □

LEMMA 29. *We have the following:*

- *The equalities* $f^{(\nu)}_{0,\chi_v}(\gamma_0) = 1$, $f^{(\nu)}_{1,\chi_v}(\gamma_0) = \dfrac{q_v^{1/2}(q_v - a^2)}{a(1+q_v)}$ *and* $f^{(\nu)}_{1,\chi_v}(\gamma_1) = \dfrac{a^2 - q_v}{a q_v^{1/2}(1+q_v)}$ *hold.*
- *For* $n \geq 2$, *we have*

$$f^{(\nu)}_{n,\chi_v}(\gamma_k) = \begin{cases} (q_v - 1)(q_v - a^2)a^{-n} q_v^{(n-4)/2} & (\textit{if } k = 0)\,, \\ 0 & (\textit{if } 1 \leq k \leq n-1)\,, \\ (a^2 - q_v)a^{-n} q_v^{(n-4)/2} & (\textit{if } k = n)\,. \end{cases}$$

PROOF. We note that

$$f^{(\nu)}_{k,\chi_v}(\gamma_i) = \pi_v(\delta_v(-k,0)) f^{(\nu)}_{0,\chi_v}(\gamma_i) - \sum_{j=0}^{i-1} c^{(\nu)}_{\chi_v}(k,j) f^{(\nu)}_{j,\chi_v}(\gamma_0) - \sum_{j=i}^{k-1} c^{(\nu)}_{\chi_v}(k,j) f^{(\nu)}_{j,\chi_v}(\gamma_i)\,.$$

By induction on n, Lemma 28 and a direct computation, we obtain the assertion. □

For $k \in \boldsymbol{N}_0$, we set

$$\tilde{f}^{(\nu)}_{k,\chi_v} := \begin{cases} f^{(\nu)}_{0,\chi_v} & (\text{if } k = 0)\,, \\ (1 + q_v^{-1}) q_v^{-\nu/2} L(1+\nu, \chi_v^2) f^{(\nu)}_{1,\chi_v} & (\text{if } k = 1)\,, \\ \left(\dfrac{q_v + 1}{q_v - 1}\right)^{1/2} q_v^{-k\nu/2} L(1+\nu, \chi_v^2) f^{(\nu)}_{k,\chi_v} & (\text{if } 2 \leq k \leq n)\,. \end{cases}$$

PROPOSITION 30. *The restriction of* $\tilde{f}^{(\nu)}_{k,\chi_v}$ *to* $\mathbf{K}_v$ *is independent of* $\nu \in \boldsymbol{C}$ *for any* $k \in \boldsymbol{N}_0$. *Fix* $n \in \boldsymbol{N}_0$. *If* $\nu \in i\boldsymbol{R}$ *then the set* $\{\tilde{f}^{(\nu)}_{k,\chi_v}\,;\, k \in \{0,\ldots,n\}\}$ *is an orthonormal basis of* $I(\chi_v|\cdot|_v^{\nu/2})^{\mathbf{K}_0(\mathfrak{p}_v^n)}$.

PROOF. By a direct computation we have $a^{-k}(q_v - a^2) = q_v \chi_v(\varpi_v)^{-k} q_v^{k\nu/2} L(1+\nu, \chi_v^2)^{-1}$. Combining this and Lemma 29, we obtain the first assertion.

Assume $\nu \in i\boldsymbol{R}$. By the definition of $f^{(\nu)}_{k,\chi_v}$, we have the following equality (cf. Corollary 12):

$$\|f^{(\nu)}_{k,\chi_v}\|_v^2 = \begin{cases} 1 & (\text{if } k = 0)\,, \\ \dfrac{1}{(1+q_v^{-1})^2} \dfrac{1}{L(1+\nu, \chi_v^2) L(1-\nu, \chi_v^{-2})} & (\text{if } k = 1)\,, \\ \dfrac{q_v - 1}{q_v + 1} \dfrac{1}{L(1+\nu, \chi_v^2) L(1-\nu, \chi_v^{-2})} & (\text{if } k \geq 2)\,. \end{cases}$$

This completes the proof. $\square$

LEMMA 31. *For any $k \in \boldsymbol{N}_0$, we have*

$$M_v(\nu)\tilde{f}^{(\nu)}_{k,\chi_v} = \begin{cases} q_v^{-d_v/2}\dfrac{L(\nu,\chi_v^2)}{L(1+\nu,\chi_v^2)}\tilde{f}^{(-\nu)}_{0,\chi_v^{-1}} & (\textit{if } k=0), \\ q_v^{-d_v/2-k\nu}\dfrac{L(\nu,\chi_v^2)}{L(1-\nu,\chi_v^{-2})}\tilde{f}^{(-\nu)}_{k,\chi_v^{-1}} & (\textit{if } 1 \le k \le n). \end{cases}$$

PROOF. Applying [1, Proposition 4.6.7], we have

$$f^{(\nu)}_{0,\chi_v} = q_v^{-d_v/2}\frac{L(\nu,\chi_v^2)}{L(1+\nu,\chi_v^2)}f^{(-\nu)}_{0,\chi_v^{-1}}.$$

Combining this and the definition of $\tilde{f}^{(\nu)}_{k,\chi_v}$ for $k \in \{0,\dots,n\}$, we obtain this assertion. $\square$

REMARK 32. For $k \ge 1$, we have

$$M_v(\nu)\tilde{f}^{(\nu)}_{k,\chi_v} = q_v^{-k\nu}\frac{\varepsilon(1-\nu,\chi_v^{-2},\psi_{F_v})\varepsilon(1+\nu/2,\chi_v,\psi_{F_v})}{\varepsilon(1-\nu/2,\chi_v^{-1},\psi_{F_v})}\frac{L(\nu,\chi_v^2)}{L(1-\nu,\chi_v^{-2})}\tilde{f}^{(-\nu)}_{k,\chi_v^{-1}}.$$

9. Constant terms and zeta integrals of Eisenstein series. We consider the invariant subspace $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$. Let n be the maximal nonnegative integer m such that $S_m(\mathfrak{n}\mathfrak{f}_\chi^{-2}) \ne \emptyset$. For $\rho = (\rho_k)_{1\le k\le n} \in \prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}),\{0,\dots,k\})$, let us denote by $f^{(\nu)}_{\chi,\rho}$ the vector in $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})$ corresponding to

$$\bigotimes_{v\in\Sigma_\infty} f^{(\nu)}_{0,\chi_v} \otimes \bigotimes_{v\in S_1(\mathfrak{n}\mathfrak{f}_\chi^{-2})} \tilde{f}^{(\nu)}_{\rho_1(v),\chi_v} \otimes\cdots\otimes \bigotimes_{v\in S_n(\mathfrak{n}\mathfrak{f}_\chi^{-2})} \tilde{f}^{(\nu)}_{\rho_n(v),\chi_v} \otimes \bigotimes_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{n}\mathfrak{f}_\chi^{-2})} \tilde{f}^{(\nu)}_{0,\chi_v}$$

by the isomorphism $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2}) \cong \bigotimes_{v\in\Sigma_F} I(\chi_v|\cdot|_v^{\nu/2})$. By Propositions 26 and 30, we obtain the following.

PROPOSITION 33. *For any $\rho = (\rho_k)_{1\le k\le n} \in \prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}),\{0,\dots,k\})$, the family $\{f^{(\nu)}_{\chi,\rho}\}_{\nu\in\boldsymbol{C}}$ is a flat section. If $\nu \in i\boldsymbol{R}$, the finite set*

$$\left\{ f^{(\nu)}_{\chi,\rho}\ ;\ \rho \in \prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}),\{0,\dots,k\}) \right\}$$

is an orthonormal basis of $I(\chi|\cdot|_{\boldsymbol{A}}^{\nu/2})^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$.

Fix $\rho \in \prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}),\{0,\dots,k\})$. We write $E_{\chi,\rho}(\nu,g)$ for $E(f^{(\nu)}_{\chi,\rho},g)$ and put

$$E^\circ_{\chi,\rho}(\nu,g) := \int_{F\backslash\boldsymbol{A}} E_{\chi,\rho}\left(\nu, \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) dx.$$

The term $E^\circ_{\chi,\rho}(\nu,g)$ is called the constant term of $E_{\chi,\rho}(\nu,g)$. For $k \in \{1,\dots,n\}$, the sets $U_k(\rho)$, $R_k(\rho)$ and $R_0(\rho)$ are defined as follows:

$$U_k(\rho) := \bigcup_{m=k}^n \rho_m^{-1}(k) - S(\mathfrak{f}_\chi), \qquad R_k(\rho) := \bigcup_{m=k}^n \rho_m^{-1}(k) \cap S(\mathfrak{f}_\chi),$$

$$R_0(\rho) := \left(\bigcup_{m=0}^{n} \rho_m^{-1}(0) \cap S(\mathfrak{f}_\chi) \right) \bigcup (S(\mathfrak{f}_\chi) - S(\mathfrak{n}\mathfrak{f}_\chi^{-2})) .$$

For $k \geq 0$, set

$$S_k(\rho) := \begin{cases} R_0(\rho) & (\text{if } \ k = 0) , \\ U_k(\rho) \cup R_k(\rho) & (\text{if } \ k \geq 1) , \end{cases}$$

$R(\rho) := \bigcup_{k=0}^{n} R_k(\rho)$ and $S(\rho) := \bigcup_{k=0}^{n} S_k(\rho)$.

PROPOSITION 34. *We have*

$$E_{\chi,\rho}^{\circ}(\nu, g) = f_{\chi,\rho}^{(\nu)}(g) + D_F^{-1/2} A_{\chi,\rho}(\nu) \frac{L(\nu, \chi^2)}{L(1+\nu, \chi^2)} f_{\chi^{-1},\rho}^{(-\nu)}(g) ,$$

where

$$A_{\chi,\rho}(\nu) = \mathrm{N}(\mathfrak{f}_\chi)^{-\nu} \prod_{k=0}^{n} \prod_{v \in S_k(\rho)} \left\{ q_v^{d_v/2} q_v^{-k\nu} \frac{\varepsilon(1-\nu, \chi_v^{-2}, \psi_{F_v}) \varepsilon(1+\nu/2, \chi_v, \psi_{F_v})}{\varepsilon(1-\nu/2, \chi_v^{-1}, \psi_{F_v})} \times \frac{L(1+\nu, \chi_v^2)}{L(1-\nu, \chi_v^{-2})} \right\} .$$

PROOF. By the same computation as [1, p. 352–354], we have

$$E_{\chi,\rho}^{\circ}(\nu, g) = f_{\chi,\rho}^{(\nu)}(g) + \prod_{v \in \Sigma_\infty} (M_v(\nu) f_{0,\chi_v}^{(\nu)})(g_v) \prod_{k=1}^{n} \prod_{v \in U_k(\rho)} (M_v(\nu) \tilde{f}_{k,\chi_v}^{(\nu)})(g_v) \times \prod_{k=0}^{n} \prod_{v \in R_k(\rho)} (M_v(\nu) \tilde{\tilde{f}}_{k,\chi_v}^{(\nu)})(g_v) \prod_{v \in \Sigma_{\mathrm{fin}} - S(\rho)} (M_v(\nu) f_{0,\chi_v}^{(\nu)})(g_v) .$$

For $v \in \Sigma_\infty$, we have

$$(M_v(\nu) f_{0,\chi_v}^{(\nu)})(g_v) = \frac{L(\nu, \chi_v^2)}{L(1+\nu, \chi_v^2)} f_{0,\chi_v^{-1}}^{(-\nu)}(g_v) ,$$

where $M_v(\nu)$ for $v \in \Sigma_\infty$ is the intertwining operator defined in the same way as the non-archimedean case. Combining this with Lemma 27 and Remark 32, we obtain the assertion. □

We fix a character η of $\mathbf{A}^\times / F^\times$ satisfying ($\star$) in §4. For any $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta)$ and $k \in \mathbf{N}_0$, polynomials $Q_{k,\chi_v}^{(\nu)}(\eta_v, X)$ are defined as follows (cf. Corollary 19):

- For $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\chi)$, set

$$Q_{k,\chi_v}^{(\nu)}(\eta_v, X) := \begin{cases} 1 & (\text{if } \ k = 0) , \\ \eta_v(\varpi_v) X - \dfrac{\chi_v(\varpi_v) q_v^{-\nu/2} + \chi_v(\varpi_v)^{-1} q_v^{\nu/2}}{q_v^{1/2} + q_v^{-1/2}} & (\text{if } \ k = 1) , \\ q_v^{-1} \eta_v(\varpi_v)^{k-2} X^{k-2} \\ \times (\chi_v(\varpi_v) q_v^{(1-\nu)/2} \eta_v(\varpi_v) X - 1)(\chi_v(\varpi_v)^{-1} q_v^{(1+\nu)/2} \eta_v(\varpi_v) X - 1) & (\text{if } \ k \geq 2) . \end{cases}$$

- For $v \in S(\mathfrak{f}_\chi)$, set

$$Q^{(\nu)}_{k,\chi_v}(\eta_v, X) := \eta_v(\varpi_v)^k X^k$$

for any $k \in \boldsymbol{N}_0$.

PROPOSITION 35. *We put* $E^\natural_{\chi,\rho}(\nu, g) := E_{\chi,\rho}(\nu, g) - E^\circ_{\chi,\rho}(\nu, g)$. *Then* $E^\natural_{\chi,\rho}(\nu, -)$ *is left* B_F*-invariant. We have*

$$Z^*(s, \eta, E^\natural_{\chi,\rho}(\nu, -)) = (2\pi)^{\#\Sigma_{\boldsymbol{C}}}\mathcal{G}(\eta) D_F^{-\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2-\nu} \times B^\eta_{\chi,\rho}(s, \nu)\frac{L(s+\nu/2, \chi\eta)L(s-\nu/2, \chi^{-1}\eta)}{L(1+\nu, \chi^2)},$$

where

$$\begin{aligned} B^\eta_{\chi,\rho}(s, \nu) = D_F^{s-1/2}&\left\{\prod_{k=0}^{n}\prod_{v\in S_k(\rho)} Q^{(\nu)}_{k,\chi_v}(\eta_v, q_v^{1/2-s})L(1+\nu, \chi_v^2)\right\} \\ &\times \prod_{v\in U_1(\rho)}(1+q_v^{-1})q_v^{-\nu/2}\prod_{k=2}^{n}\prod_{v\in U_k(\rho)}\left(\frac{q_v+1}{q_v-1}\right)^{1/2} q_v^{-k\nu/2} \\ &\times\left\{\prod_{k=0}^{n}\prod_{v\in R_k(\rho)} q_v^{d_v/2-k\nu/2}(1-q_v^{-1})^{1/2}\overline{\mathcal{G}(\chi_v)}\right\}\prod_{v\in\Sigma_{\mathrm{fin}}-R(\rho)}\chi_v(\varpi_v)^{d_v}. \end{aligned}$$

PROOF. The assertion follows from the following facts (cf. Proposition 20 and [9, Lemma 2.11]).

- We have

$$Z(s, \eta_v, W_{f^{(\nu)}_{0,\chi_v}}) = \begin{cases} \dfrac{L(s+\nu/2, \chi_v\eta_v)L(s-\nu/2, \chi_v^{-1}\eta_v)}{L(1+\nu, \chi_v^2)} & (\text{if } v \in \Sigma_{\boldsymbol{R}}), \\ 2\pi\dfrac{L(s+\nu/2, \chi_v\eta_v)L(s-\nu/2, \chi_v^{-1}\eta_v)}{L(1+\nu, \chi_v^2)} & (\text{if } v\in\Sigma_{\boldsymbol{C}}). \end{cases}$$

- For $v \in U_1(\rho)$, we have

$$Z(s, \eta_v, W_{\tilde{f}^{(\nu)}_{1,\chi_v}}) = (1+q_v^{-1})q_v^{-\nu/2}L(1+\nu, \chi_v^2)Q^{(\nu)}_{1,\chi_v}(\eta_v, q_v^{1/2-s})Z(s, \eta_v, W_{f^{(\nu)}_{0,\chi_v}}).$$

- For $k \geq 2$ and $v \in U_k(\rho)$, we have

$$Z(s, \eta_v, W_{\tilde{f}^{(\nu)}_{k,\chi_v}}) = \left(\frac{q_v+1}{q_v-1}\right)^{1/2} q_v^{-k\nu/2}L(1+\nu, \chi_v^2)Q^{(\nu)}_{k,\chi_v}(\eta_v, q_v^{1/2-s})Z(s, \eta_v, W_{f^{(\nu)}_{0,\chi_v}}).$$

- For $k \geq 0$ and $v \in R_k(\rho)$, we have

$$\begin{aligned} &Z\left(s, \eta_v, \pi_v\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}W_{\tilde{f}^{(\nu)}_{k,\chi_v}}\right) \\ =&q_v^{(1-\nu)d_v/2-k\nu/2+(1/2-\nu)f(\chi_v)}(1-q_v^{-1})^{1/2}\overline{\mathcal{G}(\chi_v)}Q^{(\nu)}_{k,\chi_v}(\eta_v, q_v^{1/2-s})q_v^{d_v(s-1/2)}\mathcal{G}(\eta_v). \end{aligned}$$

- For $v \in \Sigma_{\mathrm{fin}} - R(\rho)$, we have

$$\begin{aligned}&Z(s, \eta_v, W_{f^{(\nu)}_{0,\chi_v}})\\ =&\chi_v(\varpi_v)^{d_v} q_v^{-d_v\nu/2} q_v^{d_v(s-1/2)} \mathcal{G}(\eta_v) \frac{L(s+\nu/2, \chi_v\eta_v)L(s-\nu/2, \chi_v^{-1}\eta_v)}{L(1+\nu, \chi_v^2)}.\end{aligned}$$

□

10. Regularized periods of Eisenstein series. In this section, we compute regularized η-periods of $E_{\chi,\rho}(\nu, -)$. For any characters χ_1 and χ_2 of $\boldsymbol{A}^\times/F^\times$, we put $\delta_{\chi_1,\chi_2} := \delta(\chi_1 = \chi_2)$. The following lemma is needed in order to compute regularized periods of Eisenstein series.

LEMMA 36 (9, Lemma 7.6). *Let ξ be a character of $\boldsymbol{A}^\times/F^\times$. Let λ and w be complex numbers such that* $\mathrm{Re}(w) < \mathrm{Re}(\lambda)$. *Then, for $\varepsilon \in \{0, 1\}$, we have*

$$\begin{aligned}&\int_{F^\times\backslash \boldsymbol{A}^\times} \hat{\beta}_\lambda(|t|_{\boldsymbol{A}})\xi(t)(\log|t|_{\boldsymbol{A}})^\varepsilon |t|_{\boldsymbol{A}}^w d^\times t\\ &= \begin{cases} \delta_{\xi,\mathbf{1}} \operatorname{vol}(F^\times\backslash \boldsymbol{A}^1) \dfrac{\beta(-w)}{\lambda - w} & (\textit{if } \varepsilon = 0),\\ \delta_{\xi,\mathbf{1}} \operatorname{vol}(F^\times\backslash \boldsymbol{A}^1) \dfrac{\beta(-w) - \beta'(-w)(\lambda - w)}{(\lambda - w)^2} & (\textit{if } \varepsilon = 1),\end{cases}\end{aligned}$$

where the integral on the left-hand side converges absolutely.

THEOREM 37. *Assume $\nu \in i\boldsymbol{R}$ if $S(\mathfrak{f}_\chi) = \emptyset$. Then the integral $P^\eta_{\beta,\lambda}(E_{\chi,\rho}(\nu, -))$ converges absolutely for any $(\beta, \lambda) \in \mathcal{B} \times \boldsymbol{C}$ such that* $\mathrm{Re}(\lambda) > 1$. *If $S(\mathfrak{f}_\chi) = \emptyset$, then $P^\eta_{\beta,\lambda}(E_{\chi,\rho}(\nu, -))$ converges absolutely for any $(\beta, \lambda) \in \mathcal{B} \times \boldsymbol{C}$. Moreover $P^\eta_{\mathrm{reg}}(E_{\chi,\rho}(\nu, -))$ can be defined and we have*

$$\begin{aligned}P^\eta_{\mathrm{reg}}(E_{\chi,\rho}(\nu, -)) = (2\pi)^{\#\Sigma_{\boldsymbol{C}}} \mathcal{G}(\eta) D_F^{-\nu/2} \mathrm{N}(\mathfrak{f}_\chi)^{1/2-\nu}&\\ \times B^\eta_{\chi,\rho}(1/2, \nu) \frac{L((1+\nu)/2, \chi\eta)L((1-\nu)/2, \chi^{-1}\eta)}{L(1+\nu, \chi^2)}&.\end{aligned}$$

PROOF. Suppose $S(\mathfrak{f}_\chi) \neq \emptyset$. For $t \in \boldsymbol{A}^\times/F^\times$, we have

$$\begin{aligned}&E^\circ_{\chi,\rho}\left(\nu, \begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x_\eta\\ 0 & 1\end{pmatrix}\right)\\ &= f^{(\nu)}_{\chi,\rho}\begin{pmatrix} t & tx_\eta\\ 0 & 1\end{pmatrix} + D_F^{-1/2} A_{\chi,\rho}(\nu) \frac{L(\nu, \chi^2)}{L(1+\nu, \chi^2)} f^{(-\nu)}_{\chi^{-1},\rho}\begin{pmatrix} t & tx_\eta\\ 0 & 1\end{pmatrix} = 0.\end{aligned}$$

We notice that $\tilde{f}^{(\nu)}_{0,\chi_v}(e) = \tilde{f}^{(-\nu)}_{0,\chi_v^{-1}}(e) = 0$ for $v \in S(\mathfrak{f}_\chi)$. Thus $P^\eta_{\beta,\lambda}(E^\circ_{\chi,\rho}(\nu, -)) = 0$ holds for any $(\beta, \lambda) \in \mathcal{B} \times \boldsymbol{C}$. We put $f^\eta_{\chi,\rho}(z, \nu) := Z^*(z+1/2, \eta, E^\natural_{\chi,\rho}(\nu, -))$ and note that $f^\eta_{\chi,\rho}(z, \nu)$ is entire on the whole z-plane by Proposition 35 and that $S(\mathfrak{f}_\chi) \neq \emptyset$. By exchanging the order of integrals, we have

$$P^\eta_{\beta,\lambda}(E_{\chi,\rho}(\nu, -)) = P^\eta_{\beta,\lambda}(E^\natural_{\chi,\rho}(\nu, -))$$

$$= \int_{F^\times\backslash \boldsymbol{A}^\times} \{\hat{\beta}_\lambda(|t|_{\boldsymbol{A}}) + \hat{\beta}_\lambda(|t|_{\boldsymbol{A}}^{-1})\} E^\natural_{\chi,\rho}\left(\nu, \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix}\right) \eta(t)\eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}}) d^\times t$$
$$= \frac{1}{2\pi i}\int_{L_\sigma} \{f^\eta_{\chi,\rho}(z,\nu) + f^\eta_{\chi,\rho}(-z,\nu)\}\frac{\beta(z)}{\lambda+z}dz$$

where $\sigma > -\mathrm{Re}(\lambda)$. This is justified by Proposition 35. By this we obtain both the convergence of $P^\eta_{\beta,\lambda}(E_{\chi,\rho}(\nu,-))$ for $(\beta,\lambda) \in \mathcal{B}\times \boldsymbol{C}$ and the entireness of $P^\eta_{\beta,\lambda}(E_{\chi,\rho}(\nu,-))$ as a function in λ. By the residue theorem, we have

$$\begin{aligned}
\mathrm{CT}_{\lambda=0}P^\eta_{\beta,\lambda}(E_{\chi,\rho}(\nu,-)) &= \frac{1}{2\pi i}\int_{L_\sigma}\{f^\eta_{\chi,\rho}(z,\nu) + f^\eta_{\chi,\rho}(-z,\nu)\}\frac{\beta(z)}{z}dz \\
&= \frac{1}{2\pi i}\left(\int_{L_\sigma} - \int_{L_{-\sigma}}\right) f^\eta_{\chi,\rho}(z,\nu)\frac{\beta(z)}{z}dz \\
&= \mathrm{Res}_{z=0}\left\{f^\eta_{\chi,\rho}(z,\nu)\frac{\beta(z)}{z}\right\} \\
&= f^\eta_{\chi,\rho}(0,\nu)\beta(0)\,.
\end{aligned}$$

By Proposition 35 we obtain the second assertion when $S(\mathfrak{f}_\chi) \neq \emptyset$.

Assume $\nu \in i\boldsymbol{R}$ and $S(\mathfrak{f}_\chi) = \emptyset$. Then the first assertion is obtained in the same way as [9, Lemma 7.5]. We give a proof of the second assertion in the same way as [9, Lemma 51]. Assume $\mathrm{Re}(\lambda) > 1$. Then, by Proposition 34, we have

$$\begin{aligned}
P^\eta_{\beta,\lambda}(E_{\chi,\rho}(\nu,-)) = P_\chi(\lambda,\nu) &+ D_F^{-1/2}A_{\chi,\rho}(\nu)\frac{L(\nu,\chi^2)}{L(1+\nu,\chi^2)}P_{\chi^{-1}}(\lambda,-\nu) \\
&+ Q^+_{\chi,\rho}(\eta,\lambda,\nu) + Q^-_{\chi,\rho}(\eta,\lambda,\nu)\,,
\end{aligned}$$

where

$$\begin{aligned}
&P_{\chi^{\pm 1}}(\lambda,\pm\nu) \\
&:= \int_{F^\times\backslash \boldsymbol{A}^\times} f^{(\pm\nu)}_{\chi^{\pm 1},\rho}(e)\chi^{\pm}(t)|t|_{\boldsymbol{A}}^{(1\pm\nu)/2}\eta(t)\eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}})\{\hat{\beta}_\lambda(|t|_{\boldsymbol{A}}) + \hat{\beta}_\lambda(|t|_{\boldsymbol{A}}^{-1})\}d^\times t
\end{aligned}$$

and

$$Q^{\pm}_{\chi,\rho}(\eta,\lambda,\nu) := \int_{F^\times\backslash \boldsymbol{A}^\times} E^\natural_{\chi,\rho}\left(\nu, \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix}\right)\eta(t)\eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}})\hat{\beta}_\lambda(|t|_{\boldsymbol{A}}^{\pm 1})d^\times t\,.$$

For $\mathrm{Re}(\lambda) > 1$, by Lemma 36, the integral $P_{\chi^{\pm 1}}(\lambda,\pm\nu)$ converges absolutely and we have

$$P_\chi(\lambda,\nu) = f^{(\nu)}_{\chi,\rho}(e)\delta_{\chi\eta,\mathbf{1}}\mathrm{vol}(F^\times\backslash \boldsymbol{A}^1)\left\{\frac{\beta((-\nu-1)/2)}{\lambda-(\nu+1)/2} + \frac{\beta((\nu+1)/2)}{\lambda+(\nu+1)/2}\right\}$$

and

$$P_{\chi^{-1}}(\lambda,-\nu) = f^{(-\nu)}_{\chi^{-1},\rho}(e)\delta_{\chi,\eta}\mathrm{vol}(F^\times\backslash \boldsymbol{A}^1)\left\{\frac{\beta((\nu-1)/2)}{\lambda-(-\nu+1)/2} + \frac{\beta((-\nu+1)/2)}{\lambda+(-\nu+1)/2}\right\}\,.$$

We put $f^{\eta}_{\chi,\rho}(z,\nu) := Z^*(z+1/2, \eta, E^{\natural}_{\chi,\rho}(\nu,-))$. By exchanging the order of integrals, we have

$$Q^{+}_{\chi,\rho}(\eta,\lambda,\nu) = \frac{1}{2\pi i}\int_{L_\sigma} f^{\eta}_{\chi,\rho}(z,\nu)\frac{\beta(z)}{\lambda+z}dz\,,$$

where $\sigma > 1/2$. This is justified by Proposition 35. Hence the integral $Q^{+}_{\chi,\rho}(\eta,\lambda,\nu)$ is holomorphic on $\mathrm{Re}(\lambda) > -\sigma$, and has an analytic continuation to the whole λ-plane. In a similar fashion, we have

$$Q^{-}_{\chi,\rho}(\eta,\lambda,\nu) = \frac{1}{2\pi i}\int_{L_{-\sigma}} f^{\eta}_{\chi,\rho}(-z,\nu)\frac{\beta(z)}{\lambda+z}dz$$

for $\mathrm{Re}(\lambda) > \sigma > 1/2$. Furthermore, the residue theorem gives us the equality

$$\begin{aligned}
&Q^{-}_{\chi,\rho}(\eta,\lambda,\nu)\\
&= \frac{1}{2\pi i}\int_{L_\sigma} f^{\eta}_{\chi,\rho}(-z,\nu)\frac{\beta(z)}{\lambda+z}dz - \left\{\frac{\beta((\nu+1)/2)}{\lambda+(\nu+1)/2}\mathrm{Res}_{z=(\nu+1)/2}\right.\\
&\quad + \frac{\beta((-\nu+1)/2)}{\lambda+(-\nu+1)/2}\mathrm{Res}_{z=(-\nu+1)/2} + \frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2}\mathrm{Res}_{z=(\nu-1)/2}\\
&\quad \left. + \frac{\beta((-\nu-1)/2)}{\lambda+(-\nu-1)/2}\mathrm{Res}_{z=(-\nu-1)/2}\right\} f^{\eta}_{\chi,\rho}(-z,\nu)\,.
\end{aligned}$$

Therefore, as a function in λ, the integral $P^{\eta}_{\beta,\lambda}(E_{\chi,\rho}(\nu,-))$ has a meromorphic continuation to $\boldsymbol{C}$. Moreover $P^{\eta}_{\beta,\lambda}(E_{\chi,\rho}(\nu,-))$ is holomorphic at $\lambda = 0$ by $\nu \in i\boldsymbol{R}$. By virtue of the residue theorem, we obtain

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}P^{\eta}_{\beta,\lambda}(E_{\chi,\rho}(\nu,-))\\
&= \frac{1}{2\pi i}\int_{L_\sigma} \{f^{\eta}_{\chi,\rho}(z,\nu) + f^{\eta}_{\chi,\rho}(-z,\nu)\}\frac{\beta(z)}{z}dz\\
&\quad - \left\{\frac{\beta((\nu+1)/2)}{(\nu+1)/2}\mathrm{Res}_{z=(\nu+1)/2} + \frac{\beta((-\nu+1)/2)}{(-\nu+1)/2}\mathrm{Res}_{z=(-\nu+1)/2}\right.\\
&\quad \left. + \frac{\beta((\nu-1)/2)}{(\nu-1)/2}\mathrm{Res}_{z=(\nu-1)/2} + \frac{\beta((-\nu-1)/2)}{(-\nu-1)/2}\mathrm{Res}_{z=(-\nu-1)/2}\right\} f^{\eta}_{\chi,\rho}(-z,\nu)\\
&= \mathrm{Res}_{z=0}\left\{f^{\eta}_{\chi,\rho}(-z,\nu)\frac{\beta(z)}{z}\right\} = Z^*(1/2,\eta,E^{\natural}_{\chi,\rho}(\nu,-))\beta(0)\,.
\end{aligned}$$

By this and Proposition 35, we obtain the second assertion when $S(\mathfrak{f}_\chi) = \emptyset$. □

We define two functions $\mathfrak{e}_{\chi,\rho,-1}$ and $\mathfrak{e}_{\chi,\rho,0}$ by the relation

$$E_{\chi,\rho}(\nu,g) = \frac{\mathfrak{e}_{\chi,\rho,-1}(g)}{\nu-1} + \mathfrak{e}_{\chi,\rho,0}(g) + \mathcal{O}(\nu-1)\,, \quad (\nu\to 1)\,.$$

We compute regularized η-periods of $\mathfrak{e}_{\chi,\rho,-1}$ and that of $\mathfrak{e}_{\chi,\rho,0}$. The complete Dedekind zeta function is denoted by $\zeta_F(s)$. We put $R_F := (2\pi)^{\#\Sigma_{\boldsymbol{C}}}\,\mathrm{Res}_{s=1}\,\zeta_F(s) = \mathrm{vol}(F^\times\backslash\boldsymbol{A}^1)$.

LEMMA 38. *We have*

$$\mathfrak{e}_{\chi,\rho,-1}(g) = \delta\big(\chi^2 = \mathbf{1}, \mathfrak{f}_\chi = \mathfrak{o}_F, S(\rho) = \emptyset\big)\frac{(2\pi)^{-\#\Sigma_C} D_F^{-1/2} R_F}{\zeta_F(2)}\chi^{-1}(\det g)$$

for any $g \in G_{\boldsymbol{A}}$.

PROOF. It is sufficient to examine poles of $E^\circ_{\chi,\rho}(\nu, g)$ in order to obtain the information of poles of $E_{\chi,\rho}(\nu, g)$. By Proposition 34 it is sufficient to examine poles of $D_F^{-1/2}A_{\chi,\rho}(\nu)L(\nu, \chi^2)L(1+\nu, \chi^2)^{-1}$. The L-function $L(\nu, \chi^2)$ has a possible simple pole at $\nu = 1$ on the half plane $\mathrm{Re}(\nu) > 0$ if and only if $\chi^2 = \mathbf{1}$. When $\chi^2 = \mathbf{1}$, the function $A_{\chi,\rho}(\nu)$ equals zero at $\nu = 1$ unless $\chi^2 = \mathbf{1}$, $\mathfrak{f}_\chi = \mathfrak{o}_F$ and $\bigcup_{k=1}^n \mathrm{Im}(\rho_k) = \{0\}$ hold. Therefore the assertion follows from the same argument as [9, Lemma 2.13]. □

We obtain the following by the same proof of [9, Lemma 7.7].

THEOREM 39. *For* $\lambda \in \boldsymbol{C}$ *such that* $\mathrm{Re}(\lambda) > 0$, *we have*

$$P^\eta_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,-1}) = \delta\big(\chi^2 = 1, \chi = \eta, \mathfrak{f}_\chi = \mathfrak{o}_F, S(\rho) = \emptyset\big)\frac{2(2\pi)^{-\#\Sigma_C} D_F^{-1/2} R_F^2}{\zeta_F(2)}\frac{\beta(0)}{\lambda}.$$

Moreover, we have $P^\eta_{\mathrm{reg}}(\mathfrak{e}_{\chi,\rho,-1}) = 0$.

For any character ξ of $\boldsymbol{A}^\times/F^\times$, we define $R(\xi)$, $C_0(\xi)$, and $C_1(\xi)$ by the relation

$$L(s, \xi) = \frac{R(\xi)}{s-1} + C_0(\xi) + C_1(\xi)(s-1) + \mathcal{O}((s-1)^2), \quad (s \to 1).$$

We note $R(\xi) = \delta_{\xi,\mathbf{1}}(2\pi)^{-\#\Sigma_C} R_F$ for any character ξ of $\boldsymbol{A}^\times/F^\times$.

THEOREM 40. *Assume* $S(\mathfrak{f}_\chi) = \emptyset$. *Then the integral* $P^\eta_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,0})$ *converges absolutely for any* $(\beta, \lambda) \in \mathcal{B} \times \boldsymbol{C}$ *such that* $\mathrm{Re}(\lambda) > 1$. *There exists an entire function* $f(\lambda)$ *on* $\boldsymbol{C}$ *such that*

$$\begin{aligned}
&P^\eta_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,0})\\
&= \delta_{\chi\eta,\mathbf{1}} R_F f^{(1)}_{\chi,\rho}(e)\left\{\frac{1}{\lambda-1} + \frac{1}{\lambda+1}\right\}\beta(1) + 2\delta_{\chi,\eta} R_F \frac{D_F^{-1/2} f^{(-1)}_{\chi^{-1},\rho}(e)}{L(2,\chi^2)}\\
&\quad\times\left\{\delta_{\chi^2,\mathbf{1}}(2\pi)^{-\#\Sigma_C} R_F\left(-\frac{\zeta_F'(2)}{\zeta_F(2)}A_{\chi,\rho}(1) + A'_{\chi,\rho}(1)\right) + C_0(\chi^2)A_{\chi,\rho}(1)\right\}\frac{\beta(0)}{\lambda} + f(\lambda)\\
&\quad - \mathcal{G}(\eta)D_F^{-1/2}R_F\left\{-\frac{\delta_{\chi\eta,\mathbf{1}}\tilde{B}^\eta_{\chi,\rho}(1)}{\lambda+1} + \frac{\delta_{\chi,\eta}\tilde{B}^\eta_{\chi,\rho}(-1)}{\lambda-1}\right\}\beta(1)\\
&\quad - \frac{\mathcal{G}(\eta)D_F^{-1/2}}{L(2,\chi^2)}\left\{-\delta(\chi = \eta = \eta^{-1})(\tilde{B}^\eta_{\chi,\rho})'(0)(2\pi)^{-\#\Sigma_C}R_F^2\frac{\beta(0)}{\lambda}\right.\\
&\quad - \delta_{\chi\eta,\mathbf{1}}\tilde{B}^\eta_{\chi,\rho}(0)R_F C_0(\chi^2)\frac{\beta(0)}{\lambda} + \delta_{\chi,\eta}\tilde{B}^\eta_{\chi,\rho}(0)R_F C_0(\chi^2)\frac{\beta(0)}{\lambda}\\
&\quad \left. + \delta(\chi = \eta = \eta^{-1})\tilde{B}^\eta_{\chi,\rho}(0)(2\pi)^{-\#\Sigma_C}R_F^2\frac{\beta(0)}{\lambda^2}\right\},
\end{aligned}$$

where $\tilde{B}^{\eta}_{\chi,\rho}(z) := \varepsilon(-z, \chi^{-1}\eta)B^{\eta}_{\chi,\rho}(-z+1/2, 1)$. *Moreover we have*

$$\mathrm{CT}_{\lambda=0} P^{\eta}_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,0}) = \frac{\mathcal{G}(\eta) D_F^{-1/2} \mathrm{N}(\mathfrak{f}_\chi)^{-1/2}}{L(2,\chi^2)} \times \left\{ -\frac{1}{2}\delta(\chi=\eta=\eta^{-1})\tilde{B}^{\eta}_{\chi,\rho}(0)(2\pi)^{-\#\Sigma_{\mathbf{C}}} R_F^2 \beta''(0) + (2\pi)^{\#\Sigma_{\mathbf{C}}} a^{\eta}_{\chi,\rho}(0)\beta(0) \right\},$$

where

$$\begin{aligned} a^{\eta}_{\chi,\rho}(0) :=& -\frac{1}{2}(\tilde{B}^{\eta}_{\chi,\rho})''(0)\delta(\chi=\eta=\eta^{-1})(2\pi)^{-2\#\Sigma_{\mathbf{C}}} R_F^2 \\ &+ (\tilde{B}^{\eta}_{\chi,\rho})'(0)(2\pi)^{-\#\Sigma_{\mathbf{C}}} R_F C_0(\chi^2)(\delta_{\chi,\eta} - \delta_{\chi\eta,\mathbf{1}}) \\ &- \tilde{B}^{\eta}_{\chi,\rho}(0)(2\pi)^{-\#\Sigma_{\mathbf{C}}} R_F C_1(\chi^2)(\delta_{\chi,\eta} + \delta_{\chi\eta,\mathbf{1}}) + \tilde{B}^{\eta}_{\chi,\rho}(0) C_0(\chi\eta) C_0(\chi\eta^{-1}). \end{aligned}$$

PROOF. We give a proof in the same way as [9, Lemma 7.8]. Assume $\mathrm{Re}(\lambda) > 1$. We have the expansion

$$\mathfrak{e}_{\chi,\rho,0}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi(t)|t|_{\mathbf{A}} f^{(1)}_{\chi,\rho}(e) + \mathfrak{e}^1_0(t) + \mathfrak{e}^2_0(t)$$

where

$$\mathfrak{e}^1_0(t) := D_F^{-1/2}\chi^{-1}(t) f^{(-1)}_{\chi^{-1},\rho}(e)\mathrm{CT}_{\nu=1}\left(|t|_{\mathbf{A}}^{(-\nu+1)/2} A_{\chi,\rho}(\nu)\frac{L(\nu,\chi^2)}{L(1+\nu,\chi^2)}\right)$$

and

$$\mathfrak{e}^2_0(t) := E^{\natural}_{\chi,\rho}\left(1, \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).$$

By Lemma 36, we obtain

$$\begin{aligned} &\int_{F^\times\backslash \mathbf{A}^\times} \chi(t)|t|_{\mathbf{A}} f^{(1)}_{\chi,\rho}(e)\{\hat{\beta}_\lambda(|t|_{\mathbf{A}}) + \hat{\beta}_\lambda(|t|_{\mathbf{A}}^{-1})\}\eta(t)\eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}})d^\times t \\ &\qquad = \delta_{\chi\eta,\mathbf{1}} R_F f^{(1)}_{\chi,\rho}(e)\left\{\frac{\beta(-1)}{\lambda-1} + \frac{\beta(1)}{\lambda+1}\right\}. \end{aligned}$$

By a direct computation, we have

$$\begin{aligned} \mathfrak{e}^1_0(t) = \frac{D_F^{-1/2}\chi^{-1}(t) f^{(-1)}_{\chi^{-1},\rho}(e)}{L(2,\chi^2)}\Bigg\{ & R(\chi^2)\left(-\frac{L'(2,\chi^2)}{L(2,\chi^2)} A_{\chi,\rho}(1) + A'_{\chi,\rho}(1)\right) \\ & + C_0(\chi^2) A_{\chi,\rho}(1) - \frac{1}{2} R(\chi^2) A_{\chi,\rho}(1)\log|t|_{\mathbf{A}}\Bigg\}. \end{aligned}$$

Thus, by Lemma 36, we obtain

$$\begin{aligned} &\int_{F^\times\backslash \mathbf{A}^\times} \mathfrak{e}^1_0(t)\{\hat{\beta}_\lambda(|t|_{\mathbf{A}}) + \hat{\beta}_\lambda(|t|_{\mathbf{A}}^{-1})\}\eta(t)\eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}})d^\times t \\ &\quad = 2\delta_{\chi,\eta} R_F \frac{D_F^{-1/2}\chi^{-1}(t) f^{(-1)}_{\chi^{-1},\rho}(e)}{L(2,\chi^2)}\Bigg\{ R(\chi^2)\left(-\frac{L'(2,\chi^2)}{L(2,\chi^2)} A_{\chi,\rho}(1) + A'_{\chi,\rho}(1)\right) \end{aligned}$$

$$+ C_0(\chi^2)A_{\chi,\rho}(1)\bigg\}\frac{\beta(0)}{\lambda}.$$

Further, by the residue theorem, we have

$$\int_{F^\times\backslash\mathbf{A}^\times} \mathfrak{e}_0^2(t)\{\hat{\beta}_\lambda(|t|_{\mathbf{A}}) + \hat{\beta}_\lambda(|t|_{\mathbf{A}}^{-1})\}\eta(t)\eta_{\mathrm{fin}}(x_{\eta,\mathrm{fin}})d^\times t$$
$$= f(\lambda) - (\mathrm{Res}_{z=-1} + \mathrm{Res}_{z=0} + \mathrm{Res}_{z=1})\left\{ f^{\eta}_{\chi,\rho}(-z,1)\frac{\beta(z)}{z+\lambda}\right\},$$

where

$$f(\lambda) := \frac{1}{2\pi i}\int_{L_\sigma}\{f^{\eta}_{\chi,\rho}(z,1) + f^{\eta}_{\chi,\rho}(-z,1)\}\frac{\beta(z)}{z+\lambda}dz$$

and $\sigma > 1$. Here we have the following:

$$\mathrm{Res}_{z=1}\left\{ f^{\eta}_{\chi,\rho}(-z,\nu)\frac{\beta(z)}{z+\lambda}\right\} = \frac{(2\pi)^{\#\Sigma_{\mathbf{C}}}\mathcal{G}(\eta)D_F^{-1/2}\mathrm{N}(\mathfrak{f}_\chi)^{-1/2}}{L(2,\chi^2)}$$
$$\times \tilde{B}^{\eta}_{\chi,\rho}(1)L(2,\chi\eta^{-1})R(\chi\eta)\frac{-\beta(1)}{\lambda+1},$$

$$\mathrm{Res}_{z=-1}\left\{ f^{\eta}_{\chi,\rho}(-z,1)\frac{\beta(z)}{z+\lambda}\right\} = \frac{(2\pi)^{\#\Sigma_{\mathbf{C}}}\mathcal{G}(\eta)D_F^{-1/2}\mathrm{N}(\mathfrak{f}_\chi)^{-1/2}}{L(2,\chi^2)}$$
$$\times \tilde{B}^{\eta}_{\chi,\rho}(-1)L(2,\chi\eta)R(\chi\eta^{-1})\frac{\beta(-1)}{\lambda-1},$$

$$\mathrm{Res}_{z=0}\left\{ f^{\eta}_{\chi,\rho}(-z,1)\frac{\beta(z)}{z+\lambda}\right\}$$
$$= \frac{(2\pi)^{\#\Sigma_{\mathbf{C}}}\mathcal{G}(\eta)D_F^{-1/2}\mathrm{N}(\mathfrak{f}_\chi)^{-1/2}}{L(2,\chi^2)}\bigg\{ -\delta(\chi=\eta=\eta^{-1})(\tilde{B}^{\eta}_{\chi,\rho})'(0)(2\pi)^{-2\#\Sigma_{\mathbf{C}}}R_F^2\frac{\beta(0)}{\lambda}$$
$$-\delta_{\chi\eta,\mathbf{1}}\tilde{B}^{\eta}_{\chi,\rho}(0)C_0(\chi\eta^{-1})(2\pi)^{-\#\Sigma_{\mathbf{C}}}R_F\frac{\beta(0)}{\lambda} + \delta_{\chi,\eta}\tilde{B}^{\eta}_{\chi,\rho}(0)C_0(\chi\eta)(2\pi)^{-\#\Sigma_{\mathbf{C}}}R_F\frac{\beta(0)}{\lambda}$$
$$+\delta(\chi=\eta=\eta^{-1})\tilde{B}^{\eta}_{\chi,\rho}(0)(2\pi)^{-2\#\Sigma_{\mathbf{C}}}R_F^2\frac{\beta(0)}{\lambda^2}\bigg\}.$$

This completes the proof of the first assertion.

By virtue of the residue theorem, we obtain

$$\mathrm{CT}_{\lambda=0}P^{\eta}_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,0}) = \frac{1}{2\pi i}\int_{L_\sigma}\{f^{\eta}_{\chi,\rho}(z,1) + f^{\eta}_{\chi,\rho}(-z,1)\}\frac{\beta(z)}{z}dz$$
$$-(\mathrm{Res}_{z=-1} + \mathrm{Res}_{z=1})\left\{ f^{\eta}_{\chi,\rho}(-z,1)\frac{\beta(z)}{z}\right\}$$
$$= \mathrm{Res}_{z=0}\left\{ f^{\eta}_{\chi,\rho}(z,1)\frac{\beta(z)}{z}\right\}$$
$$= \frac{(2\pi)^{\#\Sigma_{\mathbf{C}}}\mathcal{G}(\eta)D_F^{-1/2}\mathrm{N}(\mathfrak{f}_\chi)^{-1/2}}{L(2,\chi^2)}\left\{\frac{1}{2}a^{\eta}_{\chi,\rho}(-2)\beta''(0) + a^{\eta}_{\chi,\rho}(0)\beta(0)\right\},$$

where $a^{\eta}_{\chi,\rho}(-2)$ and $a^{\eta}_{\chi,\rho}(0)$ are defined as

$$\tilde{B}^{\eta}_{\chi,\rho}(z)L(1-z,\chi\eta)L(1+z,\chi\eta^{-1}) = \frac{a^{\eta}_{\chi,\rho}(-2)}{z^2} + \frac{a^{\eta}_{\chi,\rho}(-1)}{z} + a^{\eta}_{\chi,\rho}(0) + \mathcal{O}(z),\ \ (z\to 0)\,.$$

This completes the proof of the second assertion. □

When χ is unramified, the regularized η-period of $\mathfrak{e}_{\chi,\rho,0}$ can not be defined generally. However, it can be defined in a ramified case.

COROLLARY 41. *We assume $S(\mathfrak{f}_\chi)\neq\emptyset$. Then $P^{\eta}_{\mathrm{reg}}(\mathfrak{e}_{\chi,\rho,0})$ can be defined and*

$$P^{\eta}_{\mathrm{reg}}(\mathfrak{e}_{\chi,\rho,0}) = (2\pi)^{\#\Sigma_{\mathbb{C}}}\mathcal{G}(\eta)D_F^{-1/2}\mathrm{N}(\mathfrak{f}_\chi)^{-1/2}B^{\eta}_{\chi,\rho}(1/2,1)\frac{L(1,\chi\eta)L(0,\chi^{-1}\eta)}{L(2,\chi^2)}\,.$$

We note that both $L(s,\chi\eta)$ and $L(s,\chi^{-1}\eta)$ are entire functions by the condition $(\star)$.

PROOF. By assumption and Lemma 38, the function $E_{\chi,\rho}(\nu,g)$ is holomorphic at $\nu=1$. Therefore we have

$$\mathfrak{e}_{\chi,\rho,0}\left(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x_\eta\\ 0 & 1\end{pmatrix}\right) = E_{\chi,\rho}\left(1,\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x_\eta\\ 0 & 1\end{pmatrix}\right)\,.$$

We note that $f^{\eta}_{\chi,\rho}(z,1)$ is entire on the z-plane. By applying Theorem 37, we obtain

$$\mathrm{CT}_{\lambda=0}P^{\eta}_{\beta,\lambda}(E_{\chi,\rho}(\nu,-)) = f^{\eta}_{\chi,\rho}(0,1)\beta(0)\,.$$

This completes the proof. □

COROLLARY 42. *We assume $S(\mathfrak{f}_\chi)\neq\emptyset$. For η satisfying both $(\star)$ and $\eta^2=\mathbf{1}$, the value $P^{\eta}_{\mathrm{reg}}(\mathfrak{e}_{\chi,\rho,0})$ does not vanish if and only if $\eta_v(\varpi_v)\neq\chi_v(\varpi_v)$ holds for any $v\in\bigcup_{k=1}^{n}U_k(\rho)$.*

PROOF. The value $L(1,\xi)$ is nonzero for any nontrivial character ξ of $\boldsymbol{A}^\times/F^\times$ by [5, Theorem 7-28]. Thus the values $L(1,\chi\eta)$ and $L(0,\chi^{-1}\eta)=\varepsilon(1,\chi\eta)^{-1}L(1,\chi\eta)$ are nonzero. For $v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{f}_\chi\mathfrak{f}_\eta)$, by a direct computation we have

$$Q^{(1)}_{k,\chi_v}(\eta_v,1) := \begin{cases} 1 & (\text{if } k=0)\,,\\ \dfrac{\{\eta_v(\varpi_v)-\chi_v(\varpi_v)\}q_v^{-1/2}+\{\eta_v^{-1}(\varpi_v)-\chi_v^{-1}(\varpi_v)\}q_v^{1/2}}{q_v^{1/2}+q_v^{-1/2}} & (\text{if } k=1)\,,\\ q_v^{-1}\eta_v(\varpi_v)^{k-2}(\chi_v(\varpi_v)\eta_v(\varpi_v)-1)(q_v\chi_v(\varpi_v)^{-1}\eta_v(\varpi_v)-1) & (\text{if } k\geq 2)\,.\end{cases}$$

Therefore $B^{\eta}_{\chi,\rho}(1/2,1)$ vanishes if and only if there exists $v\in\bigcup_{k=1}^{n}U_k(\rho)$ such that $\eta(\varpi_v)=\chi_v(\varpi_v)$. This completes the proof. □

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DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY
TOYONAKA, OSAKA 560–0043
JAPAN

E-mail address: s-sugiyama@cr.math.sci.osaka-u.ac.jp