Tohoku Math. J.
65 (2013), 321–330

HARDY TYPE INEQUALITIES ON BALLS

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(Received February 15, 2012, revised December 4, 2012)

Abstract. Hardy type inequalities are presented on balls with radius R at the origin in $\boldsymbol{R}^n$ with $n = 2$ at least. A special attention is paid on the behavior of functions on the boundary.

1. Introduction. The classical Hardy inequalities in one space dimension are formulated as

$$\int_0^\infty x^{-r-1}\left|\int_0^x f(y)dy\right|^p dx \le \left(\frac{p}{r}\right)^p \int_0^\infty x^{p-r-1}|f(x)|^p dx\,, \tag{1.1}$$

$$\int_0^\infty x^{r-1}\left|\int_x^\infty f(y)dy\right|^p dx \le \left(\frac{p}{r}\right)^p \int_0^\infty x^{p+r-1}|f(x)|^p dx\,, \tag{1.2}$$

where $1 \le p < \infty$ and $r > 0$ (see [6] for instance). For higher space dimensions, there are substitutes for (1.1) and (1.2) which are also known as the Hardy inequalities. For $n \ge 3$, the following inequality holds for all $f \in H^1(\boldsymbol{R}^n)$:

$$\left\|\frac{f}{|x|}\right\|_{L^2(\boldsymbol{R}^n)} \le \frac{2}{n-2}\|\nabla f\|_{L^2(\boldsymbol{R}^n)}\,. \tag{1.3}$$

In [2], (1.3) is regarded as a special case of Pitt's inequality. In [16], (1.3) is called the uncertainty principle lemma. A dilational characterization of this inequality is given in [14]. There is a number of both mathematical and physical applications of Hardy type inequalities. We refer the reader to [1, 2, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18].

For $n = 2$, (1.3) makes no sense and the inequality

$$\left\|\frac{f}{|x|(1+|\log|x||)}\right\|_{L^2(B_1)} \le C\|f\|_{H^1(\boldsymbol{R}^2)} \tag{1.4}$$

holds for all $f \in H^1(\boldsymbol{R}^2)$, where $B_1 = \{x \in \boldsymbol{R}^2; |x| < 1\}$ (see [5]). The inequality (1.4) is equivalent to

$$\left\|\frac{f}{|x|(1+|\log|x||)}\right\|_{L^2(\boldsymbol{R}^2)} \le C\|f\|_{H^1(\boldsymbol{R}^2)} \tag{1.5}$$

2010 *Mathematics Subject Classification.* Primary 26D10; Secondary 46E35.
Key words and phrases. Hardy inequalities.

since

$$\left\| \frac{f}{|x|(1+|\log|x||)} \right\|_{L^2(\boldsymbol{R}^2\setminus B_1)} \leq \|f\|_{L^2(\boldsymbol{R}^2)} .$$

The purpose of this paper is to study Hardy type inequalities on the ball $B_R \equiv \{x \in \boldsymbol{R}^n; |x| < R\}$ with $R > 0$ and $n \geq 2$, with taking into account the behavior of H^1 functions on the boundary $\partial B_R = \{x \in \boldsymbol{R}^n; |x| = R\}$. Corresponding Hardy inequalities outside the balls are easily obtained by the Kelvin transform.

THEOREM 1. *Let $n \geq 3$. For any $R > 0$ and any $f \in H^1(\boldsymbol{R}^n)$ the following inequalities hold:*

$$\left(\int_{B_R} \frac{1}{|x|^2} \left| f(x) - f\left(R\frac{x}{|x|}\right) \right|^2 dx \right)^{1/2} \leq \frac{2}{n-2} \left(\int_{B_R} \left| \frac{x}{|x|} \cdot \nabla f(x) \right|^2 dx \right)^{1/2}, \tag{1.6}$$

$$\begin{aligned} &\left(\int_{B_R} \frac{1}{|x|^2} |f(x)|^2 dx \right)^{1/2} \\ &\quad \leq \left(\frac{n}{n-2} \right)^{1/2} \frac{1}{R} \left(\int_{B_R} |f(x)|^2 dx \right)^{1/2} \\ &\qquad + \frac{2}{n-2} \left(1 + \left(\frac{n}{n-2} \right)^{1/2} \right) \left(\int_{B_R} \left| \frac{x}{|x|} \cdot \nabla f(x) \right|^2 dx \right)^{1/2}. \end{aligned} \tag{1.7}$$

COROLLARY 2. *Let $n \geq 3$ and $R > 0$.*

(1) *The inequality*

$$\left(\int_{B_R} \frac{1}{|x|^2} |f(x)|^2 dx \right)^{1/2} \leq \frac{2}{n-2} \left(\int_{B_R} \left| \frac{x}{|x|} \cdot \nabla f(x) \right|^2 dx \right)^{1/2} \tag{1.8}$$

holds for all $f \in H_0^1(B_R)$ and fails for some $f \in H^1(B_R)$.

(2) *The inequality*

$$\begin{aligned} &\left(\int_{B_R} \frac{1}{|x|^2} |f(x)|^2 dx \right)^{1/2} \\ &\quad \leq \left(\frac{n}{n-2} \right)^{1/2} \frac{1}{R} \left(\int_{B_R} |f(x)|^2 dx \right)^{1/2} \\ &\qquad + \frac{2}{n-2} \left(1 + \left(\frac{n}{n-2} \right)^{1/2} \right) \left(\int_{B_R} \left| \frac{x}{|x|} \cdot \nabla f(x) \right|^2 dx \right)^{1/2} \end{aligned} \tag{1.9}$$

holds for all $f \in H^1(B_R)$.

COROLLARY 3. *Let $n \geq 3$. Then the inequalities*

$$\left\| \frac{f}{|x|} \right\|_{L^2(\boldsymbol{R}^n)} \leq \frac{2}{n-2} \left\| \frac{x}{|x|} \cdot \nabla f \right\|_{L^2(\boldsymbol{R}^n)}, \tag{1.10}$$

$$\left\|\frac{f}{|x|}\right\|_{L^2(\boldsymbol{R}^n)} \le \left(1+\left(\frac{n}{n-2}\right)^{1/2}\right)\left(\|f\|_{L^2(\boldsymbol{R}^n)} + \frac{2}{n-2}\left\|\frac{x}{|x|}\cdot\nabla f\right\|_{L^2(B_1)}\right), \tag{1.11}$$

hold for all $f \in H^1(\boldsymbol{R}^n)$.

REMARK 4. The inequality (1.9) becomes an equality for $f \equiv 1 \in H^1(B_R)$. Similarly, (1.7) becomes an equality for $f \in H^1(\boldsymbol{R}^n)$ with $f \equiv 1$ in a neighborhood of B_R.

THEOREM 5. *Let* $n = 2$. *For any* $R > 0$ *and any* $f \in H^1(\boldsymbol{R}^2)$, *the following inequalities hold:*

$$\left(\int_{B_R} \frac{1}{|x|^2\left|\log\frac{R}{|x|}\right|^2}\left|f(x) - f\left(R\frac{x}{|x|}\right)\right|^2 dx\right)^{1/2} \le 2\left(\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla f(x)\right|^2 dx\right)^{1/2}, \tag{1.12}$$

$$\begin{aligned}&\left(\int_{B_R} \frac{|f(x)|^2}{|x|^2\left(1+\left|\log\frac{R}{|x|}\right|\right)^2}dx\right)^{1/2} \\ &\quad\le \frac{\sqrt{2}}{R}\left(\int_{B_R}|f(x)|^2dx\right)^{1/2} + 2(1+\sqrt{2})\left(\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla f(x)\right|^2 dx\right)^{1/2}.\end{aligned} \tag{1.13}$$

The inequality

$$\left(\int_{B_R} \frac{|f(x)|^2}{(1+|x|)^2(1+|\log|x||)^2}dx\right)^{1/2} \le C\|\nabla f\|_{L^2(\boldsymbol{R}^2)} \tag{1.14}$$

fails for some $f \in H^1(\boldsymbol{R}^2)$.

COROLLARY 6. *Let* $n = 2$ *and* $R > 0$.

(1) *The inequality*

$$\left(\int_{B_R} \frac{|f(x)|^2}{|x|^2\left|\log\frac{R}{|x|}\right|^2}dx\right)^{1/2} \le 2\left(\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla f(x)\right|^2 dx\right)^{1/2} \tag{1.15}$$

holds for all $f \in H_0^1(B_R)$ *and fails for some* $f \in H^1(B_R)$.

(2) *The inequality*

$$\begin{aligned}&\left(\int_{B_R} \frac{|f(x)|^2}{|x|^2\left(1+\left|\log\frac{R}{|x|}\right|\right)^2}dx\right)^{1/2} \\ &\quad\le \frac{\sqrt{2}}{R}\left(\int_{B_R}|f(x)|^2dx\right)^{1/2} + 2(1+\sqrt{2})\left(\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla f(x)\right|^2 dx\right)^{1/2}\end{aligned} \tag{1.16}$$

holds for all $f \in H^1(B_R)$.

(3) *Let* $f \in H^1(B_R)$ *satisfy* $f/(|x|\log(R/|x|)) \in L^2(B_R)$. *Then* $f \in H_0^1(B_R)$.

COROLLARY 7. *Let* $n = 2$. *Then the inequality*

$$\left\|\frac{f}{|x|(1+|\log|x||)}\right\|_{L^2(\boldsymbol{R}^2)} \le (1+\sqrt{2})\left(\|f\|_{L^2(\boldsymbol{R}^2)} + 2\left\|\frac{x}{|x|}\cdot\nabla f\right\|_{L^2(B_1)}\right) \tag{1.17}$$

holds for all $f \in H^1(\boldsymbol{R}^2)$.

REMARK 8. The inequality (1.16) becomes an equality for $f \equiv 1 \in H^1(B_R)$. Similarly, (1.13) becomes an equality for $f \in H^1(\boldsymbol{R}^2)$ with $f \equiv 1$ in a neighborhood of B_R.

REMARK 9. The inequality (1.15) is essentially proved in [10, 11] for smooth functions vanishing on the boundary.

REMARK 10. The inequality same to (1.14) is claimed in [13], where the authors refer [10] for the proof.

REMARK 11. For a result similar to Corollary 6 (3), see [12, Theorem 11.8].

We prove the main theorems in subsequent sections. In Sections 2 and 3, we study the cases $n \geq 3$ and $n = 2$, respectively.

2. The case $n \geq 3$.

PROOF OF THEOREM 1. By a density argument it suffices to prove (1.6) and (1.7) for $f \in C_0^\infty(\boldsymbol{R}^n)$. We introduce polar coordinates $(r, \omega) = (|x|, x/|x|) \in (0, \infty) \times S^{n-1}$ and the Lebesgue measure σ on the unit sphere S^{n-1}. We rewrite the integral on the left-hand side of (1.6) in polar coordinates and then by integration by parts to obtain

$$\begin{aligned}
\int_{B_R} \frac{1}{|x|^2}\left|f(x) - f\left(R\frac{x}{|x|}\right)\right|^2 dx &= \int_0^R r^{n-3} \int_{S^{n-1}} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega) dr \\
&= \left[\frac{1}{n-2} r^{n-2} \int_{S^{n-1}} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega)\right]_{r=0}^{r=R} \\
&\quad - \frac{1}{n-2} \int_0^R r^{n-2} \left(\frac{d}{dr} \int_{S^{n-1}} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega)\right) dr \\
&= -\frac{2}{n-2} \int_0^R r^{n-2} \mathrm{Re} \int_{S^{n-1}} (f(r\omega) - f(R\omega))\omega \cdot \overline{\nabla f(r\omega)} d\sigma(\omega) dr\,.
\end{aligned}$$

By the Schwarz inequality, we have

$$\begin{aligned}
&\int_{B_R} \frac{1}{|x|^2}\left|f(x) - f\left(R\frac{x}{|x|}\right)\right|^2 dx \\
&\quad \leq \frac{2}{n-2}\left(\int_0^R r^{n-3} \int_{S^{n-1}} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega) dr\right)^{1/2} \\
&\qquad \cdot \left(\int_0^R r^{n-1} \int_{S^{n-1}} |\omega \cdot \nabla f(r\omega)|^2 d\sigma(\omega) dr\right)^{1/2} \\
&\quad = \frac{2}{n-2}\left(\int_{B_R} \frac{1}{|x|^2}\left|f(x) - f\left(R\frac{x}{|x|}\right)\right|^2 dx\right)^{1/2} \left(\int_{B_R} \left|\frac{x}{|x|} \cdot \nabla f\right|^2 dx\right)^{1/2},
\end{aligned}$$

from which we have (1.6). The left-hand side of (1.7) is bounded by

$$
\begin{aligned}
&\left(\int_{B_R}\frac{1}{|x|^2}|f(x)|^2dx\right)^{1/2} \\
&\leq\left(\int_{B_R}\frac{1}{|x|^2}\left|f(x)-f\left(R\frac{x}{|x|}\right)\right|^2dx\right)^{1/2}+\left(\int_{B_R}\frac{1}{|x|^2}\left|f\left(R\frac{x}{|x|}\right)\right|^2dx\right)^{1/2}.
\end{aligned}
\tag{2.1}
$$

The second term on the right-hand side of (2.1) is rewritten and estimated as

$$
\begin{aligned}
&\left(\int_{B_R}\frac{1}{|x|^2}\left|f\left(R\frac{x}{|x|}\right)\right|^2dx\right)^{1/2}=\left(\int_0^R r^{n-3}\int_{S^{n-1}}|f(R\omega)|^2d\sigma(\omega)dr\right)^{1/2} \\
&\quad=\left(\frac{R^{n-2}}{n-2}\int_{S^{n-1}}|f(R\omega)|^2d\sigma(\omega)\right)^{1/2} \\
&\quad=\left(\frac{R^{n-2}}{n-2}\frac{n}{R^n}\int_0^R r^{n-1}\int_{S^{n-1}}|f(R\omega)|^2d\sigma(\omega)dr\right)^{1/2} \\
&\quad=\left(\frac{n}{n-2}\right)^{1/2}\frac{1}{R}\left(\int_{B_R}\left|f\left(R\frac{x}{|x|}\right)\right|^2dx\right)^{1/2} \\
&\quad\leq\left(\frac{n}{n-2}\right)^{1/2}\frac{1}{R}\left[\left(\int_{B_R}\left|f\left(R\frac{x}{|x|}\right)-f(x)\right|^2dx\right)^{1/2}+\left(\int_{B_R}|f(x)|^2dx\right)^{1/2}\right] \\
&\quad\leq\left(\frac{n}{n-2}\right)^{1/2}\left(\int_{B_R}\frac{1}{|x|^2}\left|f\left(R\frac{x}{|x|}\right)-f(x)\right|^2dx\right)^{1/2} \\
&\qquad+\left(\frac{n}{n-2}\right)^{1/2}\frac{1}{R}\left(\int_{B_R}|f(x)|^2dx\right)^{1/2}.
\end{aligned}
\tag{2.2}
$$

Combining (2.1), (2.2) and (1.6), we obtain (1.7). This proves Theorem 1. □

PROOF OF COROLLARY 2. We first prove (1.8) for $f \in H_0^1(B_R)$. By a density argument, it suffices to prove (1.8) for $f \in C_0^\infty(B_R)$, which follows from (1.6). The inequality (1.8) fails for $f \equiv 1$ since the right-hand side of (1.8) vanishes while the left-hand side of (1.8) is positive unless $R = 0$. The inequality (1.9) follows from (1.6) by another density argument. □

PROOF OF COROLLARY 3. The inequality (1.10) follows from (1.6) or (1.8) by a density argument and the limiting argument on $R \to \infty$. The inequality (1.11) follows from (1.7) with $R = 1$ and

$$
\left(\int_{\boldsymbol{R}^n\setminus B_1}\frac{1}{|x|^2}|f(x)|^2dx\right)^{1/2}\leq\|f\|_{L^2(\boldsymbol{R}^n)}.
$$

□

3. The case $n = 2$.

PROOF OF THEOREM 5. By a density argument is suffices to prove (1.12) and (1.13) for $f \in C_0^\infty(\boldsymbol{R}^2)$. We rewrite the integral on the left-hand side of (1.12) in polar coordinates and then by integration by parts to obtain

$$
\begin{aligned}
&\int_{B_R} \frac{1}{|x|^2 |\log(R/|x|)|^2} \left| f(x) - f\left(R\frac{x}{|x|}\right)\right|^2 dx \\
&\quad = \int_0^R \frac{1}{r\big(\log(R/r)\big)^2} \int_{S^1} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega) dr \\
&\quad = \left[\frac{1}{\log(R/r)} \int_{S^1} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega)\right]_{r=0}^{r=R} \\
&\qquad - \int_0^R \frac{1}{\log(R/r)} \left(\frac{d}{dr}\int_{S^1} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega)\right) dr \\
&\quad = -2\int_0^R \frac{1}{\log(R/r)} \mathrm{Re} \int_{S^1} (f(r\omega) - f(R\omega))\omega \cdot \overline{\nabla f(r\omega)} d\sigma(\omega) dr,
\end{aligned}
$$

where the boundary value at $r = R$ vanishes since

$$
\log\frac{R}{r} = \log\left(1 + \left(\frac{R}{r} - 1\right)\right) \geq \frac{R}{r} - 1 = \frac{R - r}{r},
$$

$$
|f(r\omega) - f(R\omega)|^2 \leq \|\nabla f\|_{L^\infty}^2 |R - r|^2 .
$$

By the Schwarz inequality, we have

$$
\begin{aligned}
&\int_{B_R} \frac{1}{|x|^2 |\log(R/|x|)|^2} \left| f(x) - f\left(R\frac{x}{|x|}\right)\right|^2 dx \\
&\quad \leq 2\left(\int_0^R \frac{1}{r(\log(R/r))^2} \int_{S^1} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega) dr\right)^{1/2} \\
&\qquad \cdot \left(\int_0^R r \int_{S^1} |\omega \cdot \nabla f(r\omega)|^2 d\sigma(\omega) dr\right)^{1/2} \\
&\quad = 2\left(\int_{B_R} \frac{1}{|x|^2 |\log(R/|x|)|^2} \left| f(x) - f\left(R\frac{x}{|x|}\right)\right|^2 dx\right)^{1/2} \left(\int_{B_R} \left|\frac{x}{|x|} \cdot \nabla f(x)\right|^2 dx\right)^{1/2},
\end{aligned}
$$

from which we have (1.12). The left-hand side of (1.13) is bounded by

$$
\begin{aligned}
&\left(\int_{B_R} \frac{1}{|x|^2 (1 + |\log(R/|x|)|)^2} |f(x)|^2 dx\right)^{1/2} \\
&\quad \leq \left(\int_{B_R} \frac{1}{|x|^2 (1 + |\log(R/|x|)|)^2} \left| f(x) - f\left(R\frac{x}{|x|}\right)\right|^2 dx\right)^{1/2} \\
&\qquad + \left(\int_{B_R} \frac{1}{|x|^2 (1 + |\log(R/|x|)|)^2} \left| f\left(R\frac{x}{|x|}\right)\right|^2 dx\right)^{1/2} .
\end{aligned}
\tag{3.1}
$$

The second term on the right-hand side of (3.1) is rewritten and estimated as

$$
\begin{aligned}
&\left(\int_{B_R} \frac{1}{|x|^2(1+|\log(R/|x|)|)^2}\left|f\left(R\frac{x}{|x|}\right)\right|^2 dx\right)^{1/2} \\
&= \left(\int_0^R \frac{1}{r(1+|\log(R/r)|)^2}\int_{S^1} |f(R\omega)|^2 d\sigma(\omega) dr\right)^{1/2} \\
&= \left(\int_{S^1} |f(R\omega)|^2 d\sigma(\omega)\right)^{1/2} \\
&= \left(\frac{2}{R^2}\int_0^R r\int_{S^1} |f(R\omega)|^2 d\sigma(\omega) dr\right)^{1/2} = \frac{\sqrt{2}}{R}\left(\int_{B_R}\left|f\left(R\frac{x}{|x|}\right)\right|^2 dx\right)^{1/2} \qquad (3.2)\\
&\le \frac{\sqrt{2}}{R}\left[\left(\int_{B_R}\left|f\left(R\frac{x}{|x|}\right)-f(x)\right|^2 dx\right)^{1/2} + \left(\int_{B_R} |f(x)|^2 dx\right)^{1/2}\right] \\
&\le \sqrt{2}\left(\int_{B_R}\frac{1}{|x|^2(1+|\log(R/|x|)|)^2}\left|f\left(R\frac{x}{|x|}\right)-f(x)\right|^2 dx\right)^{1/2} \\
&\quad + \frac{\sqrt{2}}{R}\left(\int_{B_R} |f(x)|^2 dx\right)^{1/2},
\end{aligned}
$$

where we have used

$$\frac{1}{R^2} \le \frac{1}{r^2\big(1+\log(R/r)\big)^2},$$

which follows from

$$\frac{d}{dr}\left(\frac{1}{r^2(1+\log(R/r))^2}\right) \le 0 .$$

Combining (3.1), (3.2) and (1.12), we obtain (1.13).

To prove that (1.14) fails, we define a sequence of functions $\{\varphi_j\}$ on $\boldsymbol{R}$ by

$$
\varphi_j(r) = \begin{cases} 1 & \text{if } |\log r| \le j , \\ 2 - |\log r|/j & \text{if } j < |\log r| < 2j , \\ 0 & \text{if } |\log r| \ge 2j , \end{cases}
$$

and $f_j(x) = \varphi_j(|x|)$ for $x \in \boldsymbol{R}^2$. Then

$$
\begin{aligned}
&\int_{B_1} \frac{1}{(1+|x|)^2(1+|\log|x||)^2}|f_j(x)|^2 dx \\
&\quad = 2\pi\int_0^1 \frac{1}{(1+r)^2(1+|\log r|)^2}|\varphi_j(r)|^2 r dr \\
&\quad = 2\pi\int_0^\infty \frac{1}{e^{2t}(1+e^{-t})^2(1+t)^2}|\varphi_j(e^{-t})|^2 dt
\end{aligned}
$$

$$\geq 2\pi \int_0^1 \frac{1}{(e^t+1)^2(1+t)^2} |\varphi_j(e^{-t})|^2 dt$$
$$\geq \frac{2\pi}{(e+1)^2} \int_0^1 \frac{1}{(1+t)^2} dt = \frac{2\pi}{(e+1)^2} ,$$

while, with $\psi_j(t) = \varphi_j(e^{-t})$,

$$\begin{aligned} \|\nabla f_j\|^2_{L^2(\boldsymbol{R}^2)} &= 2\pi \int_0^\infty |\varphi_j'(r)|^2 r dr = 2\pi \int_{-\infty}^\infty |\varphi_j'(e^{-t})|^2 e^{-2t} dt \\ &= 2\pi \int_{-\infty}^\infty |\psi_j'(t)|^2 dt = 4\pi \int_j^{2j} \frac{1}{j^2} dt = \frac{4\pi}{j} \to 0 \end{aligned}$$

as $j \to \infty$. This is a contradiction to (1.14). This proves Theorem 5. □

PROOF OF COROLLARY 6. Parts (1) and (2) are proved similarly as Corollary 2. We prove Part (3) following the argument of [12, Theorem 11.8]. Let $f \in H^1(B_R)$ satisfy $(|x||\log(R/|x|)|)^{-1} f \in L^2(B_R)$. Then the inequality

$$\log \frac{R}{|x|} = \log \left(\left(\frac{R}{|x|} - 1 \right) + 1 \right) \leq \frac{R}{|x|} - 1 = \frac{R - |x|}{|x|}$$

implies that $(R - |x|)^{-1} f \in L^2(B_R)$. Let ζ be a smooth function on $\boldsymbol{R}$ satisfying $0 \leq \zeta \leq 1$, $\zeta(r) = 0$ for $r \leq 1/2$, $\zeta(r) = 1$ for $r \geq 1$. We define $\rho_j(x) = \zeta(j(1 - |x|/R))$, $x \in \boldsymbol{R}^2$, $j \geq 1$. Then $\rho_j(x) = 1$ for $|x| \leq R(1 - (1/j))$ and $\rho_j(x) = 0$ for $|x| \geq R(1 - (1/2j))$. Moreover, we have

$$(\nabla \rho_j)(x) = -j\zeta'\left(j\left(1 - \frac{|x|}{R}\right)\right) \frac{x}{R|x|} = -\left(j\left(1 - \frac{|x|}{R}\right)\zeta'\left(j\left(1 - \frac{|x|}{R}\right)\right)\right) \frac{1}{R - |x|} \frac{x}{|x|}$$

and therefore

$$|(\nabla \rho_j)(x)| \leq \frac{M}{R - |x|} \chi_{\{y; R(1-(1/j)) < |y| < R(1-(1/2j))\}}(x) ,$$

where $M = \sup\{|r\zeta'(r)|; r \in \boldsymbol{R}\}$ and χ_S is the characteristic function of a set S. Then, $\mathrm{supp}(\rho_j f)$ is compact in B_R and $\rho_j f \to f$, $\rho_j \nabla f \to \nabla f$, $(\nabla \rho_j) f \to 0$ in $L^2(B_R)$ by the Lebesgue dominated convergence theorem. By mollyfying $\rho_j f$, we conclude that f is the $H^1(B_R)$ limit of a sequence of functions in $C_0^\infty(B_R)$, namely $f \in H_0^1(B_R)$. □

PROOF OF COROLLARY 7. The inequality (1.17) follows from (1.13) with $R = 1$ and the inequality

$$\left(\int_{\boldsymbol{R}^2 \setminus B_1} \frac{1}{|x|^2(1 + |\log|x||)^2} |f(x)|^2 dx \right)^{1/2} \leq \|f\|_{L^2(\boldsymbol{R}^2)} .$$

□

REFERENCES

[1] ADIMURTHI, N. CHAUDHURI AND M. RAMASWAMY, An improved Hardy-Sobolev inequality and its application, Proc. Amer. Math. Soc. 130 (2002), 489–505.

[2] W. BECKNER, Pitt's inequality with sharp convolution estimates, Proc. Amer. Math. Soc. 136 (2008), 1871–1885.

[3] J. BERGH AND J. LÖFSTRÖM, Interpolation spaces, An introduction, Grundlehren der Mathematischen Wissenschaften, No. 223, Springer-Verlag, Berlin-New York, 1976.

[4] H. BREZIS AND M. MARCUS, Hardy's inequalities revisited, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), 217–237.

[5] D. E. EDMUNDS AND H. TRIEBEL, Sharp Sobolev embeddings and related Hardy inequalities: the critical case, Math. Nachr. 207 (1999), 79–92.

[6] G. B. FOLLAND, Real analysis, second edition, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1999.

[7] J. P. GARCÍA AZORERO AND I. PERAL ALONSO, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144 (1998), 441–476.

[8] I. W. HERBST, Spectral theory of the operator $(p^2+m^2)^{1/2}-Ze^2/r$, Comm. Math. Phys. 53 (1977), 285–294.

[9] H. KALF AND J. WALTER, Strongly singular potentials and essential self-adjointness of singular elliptic operators in $C_0^{\infty}(\mathbf{R}^n\setminus\{0\})$, J. Functional Analysis 10 (1972), 114–130.

[10] O. A. LADYZHENSKAYA, The mathematical theory of viscous incompressible flow, Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[11] J. LERAY, Etude de diverses équations integrales non linéaires et de quelques problèmes que pose l'hydrodynamique, J. Math. Pures Appl. 12 (1933), 1–82.

[12] J.-L. LIONS AND E. MAGENES, Non-homogeneous boundary value problems and applications. Vol. I, Springer-Verlag, New York-Heidelberg, 1972.

[13] A. MATSUMURA AND N. YAMAGATA, Global weak solutions of the Navier-Stokes equations for multidimensional compressible flow subject to large external potential forces, Osaka J. Math. 38 (2001), 399–418.

[14] T. OZAWA AND H. SASAKI, Inequalities associated with dilations, Commun. Contemp. Math. 11 (2009), 265–277.

[15] L. PICK, Optimal Sobolev embeddings, Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998), 156–199, Acad. Sci. Czech Repub., Prague, 1999.

[16] M. REED AND B. SIMON, Methods of modern mathematical physics. II. Fourier analysis, self-adjointness, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[17] H. TRIEBEL, Sharp Sobolev embeddings and related Hardy inequalities: the sub-critical case, Math. Nachr. 208 (1999), 167–178.

[18] J. ZHANG, Extensions of Hardy inequality, J. Inequal. Appl. (2006), Art. ID 69379.

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