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CONVEXITY OF REFLECTIVE SUBMANIFOLDS IN SYMMETRIC R-SPACES

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Abstract. We show that every reflective submanifold of a symmetric R-space is (geodesically) convex.

Introduction. The main result in this article is the following.

THEOREM 1. *Reflective submanifolds of symmetric R-spaces are (geodesically) convex.*

We organized this article as follows. In Section 1, we define all notions used in Theorem 1. Reflective submanifolds in symmetric R-spaces are described in Section 2. The proof of Theorem 1 can be found in Section 3. In Section 4, we explain why the assumption "symmetric R-space" in Theorem 1 can not be generalized to all compact symmetric spaces.

Symmetric R-spaces, introduced by Takeuchi and Nagano in the 1960s, form a class of compact symmetric spaces that have very peculiar geometric properties and appear in various contexts. For example, symmetric R-spaces arise as certain spaces of shortest geodesics, namely as those centrioles (see [CN88]) that are formed by midpoints of shortest geodesics arcs joining a base point to a pole (see e.g. [MQ12]). Reflective submanifolds in symmetric spaces include among others polars and centrioles (see e.g. [CN88, Na88, Qu11]). An iterative construction involving such centrioles has been used by Bott [Bo59] in the first proof of his famous periodicity result for the homotopy groups of classical Lie groups (see also [Mi69, § 23, 24] and [Mi88, § 7]). For the construction described in [MQ11, Sect. 1.2], it is important that the distance between a base point and a pole in a centriole of certain R-spaces measured in the centriole is the same as the distance measured in the ambient R-space. This follows directly from Theorem 1.

Theorem 1 also provides a conceptional proof of [NS91, Remark 3.2b] in the case where the ambient space is a symmetric R-space.

1. Preliminaries. We first define the terminology used in Theorem 1.

Reflective submanifolds. A *reflective submanifold* M of a Riemannian manifold P is a connected component of the fixed point set of an involutive isometry τ of P, that is τ^2 equals

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the identity. Thus reflective submanifolds are totally geodesic (see e.g. [BCO03, Prop. 8.3.4]). To contain many reflective submanifolds, the ambient Riemannian manifold P should have a large isometry group. An interesting class of ambient manifolds are therefore symmetric spaces. In the series of papers [Le73, Le74, Le79a, Le79b], Leung studied and classified reflective submanifolds in simply connected irreducible symmetric spaces of compact type.

Convexity. We call a connected Riemannian submanifold $M \subset P$ of a Riemannian manifold P (geodesically) *convex*, if the (Riemannian) distance $d_M(m_1, m_2)$ in M between any pair of points $m_1, m_2 \in M$ coincides with the Riemannian distance $d_P(m_1, m_2)$ in the abient space P. In other words, a complete totally geodesic submanifold $M \subset P$ is convex if any shortest geodesic arc in M joining two arbitrarily chosen points m_1 and m_2 in M is still shortest in P (see also [Sa96, pp. 26, 84]).

Symmetric spaces. Before defining the terminology "symmetric R-space", we shortly introduce some useful notions about symmetric spaces. We refer to the Helgason's standard monograph [He78] for proofs and further details about symmetric spaces. Let S be a (Riemannian) *symmetric space,* that is a connected Riemannian manifold such that for any point $p \in S$ there exists an isometry s_p of S that fixes p and whose differential at p is $-\mathrm{Id}$ on T_pS. One can show that symmetric spaces are geodesically complete and homogeneous.

We now fix an origin $o \in S$ and get an involutive Lie group automorphism σ of the isometry group $\mathrm{I}(S)$ of S defined by $\sigma(g) = s_o \circ g \circ s_o$ for any $g \in \mathrm{I}(S)$. Its differential σ_* at the identity is an involutive automorphism of the Lie algebra of $\mathrm{I}(S)$.

The (-1)-eigenspace $\mathfrak{s}$ of σ_* is called the *Lie triple* corresponding to (S, o). It is identified with T_oS by the differential at the identity of the principal bundle $\mathrm{I}(S) \to S,\ g \mapsto g \cdot o$, where $g \cdot o$ denotes the point in S obtained by applying g to the origin o. By this identification, $\mathfrak{s}$ carries a scalar product denoted by $\langle . , . \rangle$ induced from the Riemannian metric on T_oS. The curvature tensor on T_oP is the expressed in $\mathfrak{s}$ by double Lie brackets and the geodesics of P emanating from o are of the form

$$t \mapsto \exp(tX) \cdot o \quad \text{with} \quad X \in \mathfrak{s}\,,$$

where exp is the Lie theoretic exponential map. The linear isotropy action on $\mathfrak{s}$ coincides with the adjoint action restricted to $\mathfrak{s}$.

Orthogonal unit lattices. We choose a maximal abelian subspace $\mathfrak{t} \subset \mathfrak{s}$ in $\mathfrak{s}$. Then $T := \exp(\mathfrak{t}) \cdot o$ is a maximal complete connected totally geodesic flat submanifold of S, a maximal flat torus. For a compact symmetric space S, the *unit lattice*

$$\Gamma(S, \mathfrak{t}) := \{X \in \mathfrak{t}\ ;\ \exp(X) \cdot o = o\}$$

of S is said to be *orthogonal*, if there exists a basis $\{b_1, \ldots, b_r\}$ of $\mathfrak{t}$ with the properties

(i) $\langle b_j, b_k \rangle = 0$, if $j \neq k$,

(ii) $\Gamma(S, \mathfrak{t}) = \mathrm{span}_{\mathbf{Z}}(b_1, \ldots, b_r) = \left\{ \sum_{j=1}^{r} x_j b_j\ ;\ x_j \in \mathbf{Z} \right\}$.

Symmetric R-spaces. Symmetric R-spaces, introduced by Takeuchi and Nagano in the 1960s, form a distinguished subclass of compact Riemannian symmetric spaces. They arise as particular orbits of s-representations, i.e., linear isotropy representations of symmetric spaces of compact type.

Let S be a symmetric space of compact type, that is the universal Riemannian cover of S is still compact, and o an origin in S. Using the notation introduced above, we take a nonzero element $\xi \in \mathfrak{s}$ that satisfies

$$\mathrm{ad}(\xi)^3 = -\mathrm{ad}(\xi)\,.$$

Then the connected isotropy orbit $P := \mathrm{Ad}_{\mathrm{I}(S)}(H)\xi \subset \mathfrak{s}$ is a *symmetric R-space.* Here $H \subset \mathrm{I}(S)$ denotes the identity component of the isotropy group of $o \in S$, which is a compact Lie group. Thus symmetric R-spaces are always compact.

The orbit $P \subset \mathfrak{s}$ is extrinsically symmetric in the Euclidean space $\mathfrak{s}$, that is, P is invariant under the reflections through all its affine normal spaces (see [Fe80]). In particular, symmetric R-spaces are (Riemannian) symmetric spaces (w.r.t. the submanifold metric induced by the scalar product on $\mathfrak{s}$). Ferus [Fe74] (see also [Fe80, EH95]) has shown that the converse also holds. Namely, every full compact extrinsically symmetric submanifold in a Euclidean space is a symmetric R-space.

Irreducible symmetric R-spaces have been first classified by Kobayashi and Nagano in [KN64]. A list of them can also be found in [BCO03, p. 311]. Takeuchi [Ta84] has shown that irreducible symmetric R-spaces are either irreducible hermitian symmetric spaces of compact type or compact connected real forms of them and vice-versa.

THEOREM 2 ([Lo85, Satz 3]). *The unit lattice of a symmetric R-space P is orthogonal.*

Following Loos [Lo85], this property is actually an intrinsic characterization of symmetric R-spaces among compact symmetric spaces.

2. Reflective submanifolds of symmetric R-spaces. Let now $M \subset P$ be a reflective submanifold of a symmetric R-space P and $o \in M$ a chosen origin. Since P is compact and M a closed subset of P, M is also compact. Let G be the transvection group of P, that is the identity component of $\mathrm{I}(P)$. The topology underlying the Lie group structure of G is the compact-open topology (see e.g. [He78, Ch. IV, §2,3]). Thus the identity component L of $\{g \in G\ ;\ g(M) \subset M\}$ is a closed subgroup of the compact Lie group G and therefore a compact Lie group, too. Since M is a totally geodesic submanifold of P, L contains all transvections of P along geodesics of M. Thus L acts transitively (but maybe highly non effectively) on M.

The involution σ of G given by $\sigma(g) = s_o \circ g \circ s_o$ for all $g \in G$ leaves $\{g \in G\ ;\ g(M) \subset M\}$ and therefore also L invariant and induces an involutive automorphism of L which we also denote by σ. We set

$$H := \{l \in L\ ;\ l \cdot o = o\}\,.$$

Since H is a closed subgroup of the compact Lie group L, H is a compact Lie subgroup of L.

OBSERVATION 3. (L, H) *is a compact Riemannian symmetric pair* (*in the sense defined in* [Sa77, p. 137]).

PROOF. We are left to show that $L_e^\sigma \subset H \subset L^\sigma$, where $L^\sigma \subset L$ is the fixed point set of σ in L and L_e^σ its identity component.

Let K be the subgroup of G formed by all transvections of P that leave o fix. It is well known that $G_e^\sigma \subset K \subset G^\sigma$, here G^σ is the fixed point set of σ in G and G_e^σ is its identity component (see e.g. [He78, Ch. IV, §3]). Since $H = L \cap K$, $L^\sigma = L \cap G^\sigma$ and L_e^σ is the identity component of $L \cap G_e^\sigma$, the claims follows, because

$$L_e^\sigma \subset L \cap G_e^\sigma \subset H = L \cap K \subset L \cap G^\sigma = L^\sigma\,. \qquad \square$$

Let $\mathfrak{p}$ be the Lie triple corresponding to (P, o) and $\mathfrak{m} \subset \mathfrak{p}$ the Lie subtriple of $\mathfrak{p}$ corresponding to (M, o) (see [He78, Ch. IV, §7] for further explications). If τ denotes the involutive isometry of P such that M is a connected component of the fixed point set of τ and τ_* denotes the involution on $\mathfrak{p}$ induced by the differential of τ at o, then $\mathfrak{m}$ is the fix point set of τ_* and its orthogonal complement $\mathfrak{m}^\perp$ in $\mathfrak{p}$ is the (-1)-eigenspace of τ_*. Notice that s_o and τ commute (see [Le73, p. 156]). We get an involutive Lie group automorphism

$$\tilde{\tau} : G \to G, \quad g \mapsto \tau \circ g \circ \tau\,.$$

Since the curves $t \mapsto (\tau \circ \exp(tX) \circ \tau) \cdot o$ and $t \mapsto (\tau \circ \exp(tX)) \cdot o$ in P coincide, we see that, on $\mathfrak{p} \cong T_oP$, the differential $\tilde{\tau}_*$ of $\tilde{\tau}$ at the identity coincides with the differential τ_* of τ at o and therefore leaves $\mathfrak{p}$ invariant.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$ and $\mathfrak{t}$ a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$.

OBSERVATION 4 ([TT12, Lemma 3.1]). $\mathfrak{t}$ *is invariant under* τ_*.

PROOF. The arguments given here are similar to the proof of [Lo69II, Prop. 3.2, p. 125]. Take $T \in \mathfrak{t}$, then $T + \tau_*(T)$ lies in $\mathfrak{m}$. Since

$$\begin{aligned}[A, T + \tau_*(T)] &= [A, T] + [A, \tau_*(T)] = [\tau_*(A), \tau_*(T)] \\ &= [\tilde{\tau}_*(A), \tilde{\tau}_*(T)] \qquad = \tilde{\tau}_*([A, T]) \\ &= 0\end{aligned}$$

for all $A \in \mathfrak{a}$ and since $\mathfrak{a}$ is a maximal abelian subset of $\mathfrak{m}$, we see that $T + \tau_*(T) \in \mathfrak{a}$ and hence $\tau_*(T) = (T + \tau_*(T)) - T \in \mathfrak{t}$. $\square$

The space $\mathfrak{t}$ splits as an orthogonal direct sum

$$\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{a}^\perp$$

with $\mathfrak{a}^\perp = \mathfrak{t} \cap \mathfrak{m}^\perp$.

Since τ_* is the differential of an involutive isometry of P that leaves $\mathfrak{t}$ invariant, $\tau_*|_\mathfrak{t}$ is an orthogonal transformation of $\mathfrak{t}$ that squares to the identity and preserves the unit lattice $\Gamma(P, \mathfrak{t}) \subset \mathfrak{t}$. Since the unit lattice of the symmetric R-space P is orthogonal (see Theorem 2 due to Loos [Lo85]), there exists an orthogonal basis $\{b_1, \dots, b_r\}$ of $\mathfrak{t}$ that generates $\Gamma(P, \mathfrak{t})$ over $\mathbf{Z}$.

PROPOSITION 5 ([TT12, Proposition 3.3]). *There exists an orthogonal basis* $\{e_1, \dots, e_r\}$ *of* $\mathfrak{t}$ *with the properties*

(i) $\Gamma(P,\mathfrak{t}) = \mathrm{span}_{\boldsymbol{Z}}(e_1,\dots,e_r) = \left\{ \sum_{j=1}^{r} x_j e_j \ ; \ x_j \in \boldsymbol{Z} \right\}$,

(ii) *there exist integer numbers* p, q *with* $0 \le 2p \le q \le r$ *such that*
- $\tau_*(e_{2j}) = e_{2j-1}$ *for* $1 \le j \le p$,
- $\tau_*(e_j) = e_j$ *for* $2p+1 \le j \le q$,
- $\tau_*(e_j) = -e_j$ *for* $q+1 \le j \le r$.

PROOF. Tanaka and Tasaki presented a differential geometric proof of this result (see [TT12, proof of Prop. 3.3]). In this paper we are inclined to give an elementary linear algebraic construction of the orthogonal basis $\{e_1,\dots,e_r\}$.

Without loss of generality, we may assume that the orthogonal basis $B = \{b_1,\dots,b_r\}$ that generates the unit lattice $\Gamma(P,\mathfrak{t})$ over $\boldsymbol{Z}$ is ordered by length, that is $\|b_1\| \le \|b_2\| \le \cdots \le \|b_r\|$. Let $s \in \{1,\dots,r\}$ be the integer number such that $\|b_1\| = \|b_j\|$ for $j = 1,\cdots,s$ and $\|b_1\| < \|b_{s+1}\|$. If $0 \ne x = \sum_{j=1}^{r} x_j b_j \in \Gamma(P,\mathfrak{t})$, that is $x_j \in \boldsymbol{Z}$, then $\|x\|^2 = \sum_{j=1}^{r} x_j^2 \|b_j\|^2 \ge \|b_1\|^2$, and $\|x\| = \|b_1\|$ holds if and only if $x \in \{\pm b_1,\dots,\pm b_s\}$. Since τ_* is an orthogonal map that preserves $\Gamma(P,\mathfrak{t})$, we conclude that

$$\tau_*(b_j) \in \{\pm b_1,\dots,\pm b_s\} \quad \text{for all} \quad j \in \{1,\dots,s\}\,.$$

Let $V := \mathrm{span}_{\boldsymbol{R}}\{b_1,\dots,b_s\}$, then $V^{\perp} = \mathrm{span}_{\boldsymbol{R}}\{b_{s+1},\dots,b_r\}$. Since the orthogonal endomorphism τ_* leaves V invariant, the same holds for $V^{\perp}$. By applying the above arguments to $\tau_*|_{V^{\perp}}$ and by iterating this scheme, we get

$$\tau_*(b_j) \in \{\pm b_1,\dots,\pm b_r\} \quad \text{for all} \quad j \in \{1,\dots,r\}\,.$$

Since τ_* is involutive, $\tau_*(b_j) = b_k$ implies $\tau_*(b_k) = b_j$ and $\tau_*(b_j) = -b_k$ implies $\tau_*(b_k) = -b_j$.

After renumbering $\{b_1,\dots,b_r\}$ suitably, we can therefore assume that
- $\tau_*(b_{2j}) = \pm b_{2j-1}$ for $1 \le j \le p$,
- $\tau_*(b_j) = b_j$ for $2p+1 \le j \le q$,
- $\tau_*(b_j) = -b_j$ for $q+1 \le j \le r$,

for some integers p, q with $0 \le 2p \le q \le r$. We now choose the desired basis $\{e_1,\dots,e_r\}$ as follows:
- $e_{2j-1} = b_{2j-1}$ for $1 \le j \le p$,
- $e_{2j} = \begin{cases} b_{2j} & \text{if} \quad \tau_*(b_{2j}) = b_{2j-1} \\ -b_{2j} & \text{if} \quad \tau_*(b_{2j}) = -b_{2j-1} \end{cases}$
  for $1 \le j \le p$,
- $e_j = b_j$ for $2p+1 \le j \le r$. □

Since $\mathfrak{a}$ is the fixed point set of τ_* in $\mathfrak{t}$, Proposition 5 implies the following corollary.

COROLLARY 6 ([TT12, Proposition 3.3]). *We have the equalitiy*

$$\mathfrak{a} = \left\{ \sum_{j=1}^{r} x_j e_j \ ; \ x_{2j-1} = x_{2j} \ \textit{for}\ 1 \le j \le p \ \textit{and}\ x_{q+1} = \cdots = x_r = 0 \right\}.$$

3. Proof of the main result, Theorem 1. A reflective submanifold $M \subset P$ in a compact symmetric R-space is itself a compact connected symmetric space and hence complete. The classical theorem of Hopf and Rinow (see e.g. [Sa96, p. 84]) tells us that any two points $m_1, m_2 \in M$ can be joined by a geodesic in M that is shortest within M. If such a shortest geodesic in M is still shortest within P, then M is geodesically convex.

The *tangent cut locus* $\tilde{C}(T_p P)$ of a compact Riemannian manifold P at a point $p \in M$ is the set of all tangent vectors $X \in T_p P$ such that

- $d_P(p, \gamma_X(t)) = t\|X\|$ for $t \in [0, 1]$ and
- $d_P(p, \gamma_X(t)) < t\|X\|$ for $t > 1$,

where d_P denotes the Riemannian distance in P (see e.g. [Sa96, p. 26] for the definition) and γ_X is the geodesic in P that emanates from p in the direction X. We refer to [Sa96, p. 104] for further explication concerning the tangent cut locus.

Thus $M \subset P$ is convex, if

$$\tilde{C}(T_m M) = T_m M \cap \tilde{C}(T_m P) \tag{1}$$

holds for any point $m \in M$. Since M is homogeneous, it suffices to verify Equation (1) at just one point $o \in M$.

Sakai [Sa77, Thm. 2.5] has shown that the tangent cut locus of a compact symmetric space is determined up to the isotropy action by the tangent cut locus of a maximal flat totally geodesic torus. Tasaki [Ta10, Lemma 2.2] adapted Sakai's result to totally geodesic submanifolds. We now state Tasaki's result in a version that is specialized to fit best our needs and set up. We use again the notions established in Sections 1 and 2. Tasaki's assumptions in [Ta10, Lemma 2.2] concerning the symmetric pairs are satisfied by Observation 3.

LEMMA 7 ([Ta10, Lemma 2.2]). *Let M be a reflective submanifold of a symmetric R-space P, o a point in M, $\mathfrak{a}$ an arbitrarily chosen maximal abelian linear subspace of $\mathfrak{m} \cong T_o M$ and $\mathfrak{t}$ a maximal abelian linear subspace of $\mathfrak{p} \cong T_o P$ that contains $\mathfrak{a}$. Let A be the maximal flat torus of M corresponding to $\mathfrak{a}$ and T the maximal flat torus of P corresponding to $\mathfrak{t}$, that is $\mathfrak{a} \cong T_o A$ and $\mathfrak{t} \cong T_o T$. If*

$$\tilde{C}(\mathfrak{a}) = \mathfrak{a} \cap \tilde{C}(\mathfrak{t}) \tag{2}$$

then $\tilde{C}(\mathfrak{m}) = \mathfrak{m} \cap \tilde{C}(\mathfrak{p})$ and M is a (geodesically) convex submanifold of P.

Thus, to prove Theorem 1, we just need to show that Equation (2) is satisfied, that is, A is a convex submanifold of T. We do this by showing the following claim.

CLAIM 8. *For all points $a \in A$ we have*

$$d_A(o, a) = d_T(o, a)\,.$$

PROOF. Both maps

$$\mathfrak{a} \to A,\ X \mapsto \exp(X) \cdot o \quad \text{and} \quad \mathfrak{t} \to T,\ Y \mapsto \exp(Y) \cdot o$$

are Riemannian coverings between flat spaces. Thus they map straight lines in $\mathfrak{a}$ and $\mathfrak{t}$ onto geodesics of A and T, and every geodesic arises in this way.

Let $a \in A$ be an arbitrarily chosen point in A, then $a = \exp(X) \cdot o$ for some $X \in \mathfrak{a}$. In view of Corollary 6, one can write $X = \sum_{j=1}^{r} x_j e_j$ with

- $x_{2j} = x_{2j-1}$ for $1 \leq j \leq p$,
- $x_{q+1} = \cdots = x_r = 0$,

where $\{e_1, \ldots, e_r\}$ is the orthogonal basis of $\mathfrak{t}$ mentioned in Proposition 5. Using Theorem 2, we get

$$\begin{aligned} d_T^2(o, a) &= \min\{\|X + Y\|^2 \ ; \ Y \in \Gamma(P, \mathfrak{t})\} \\ &= \min\left\{\left\| \sum_{j=1}^{r} (x_j + y_j) e_j \right\|^2 \ ; \ y_1, \ldots, y_r \in \mathbf{Z}\right\} \\ &= \min\left\{ \sum_{j=1}^{r} (x_j + y_j)^2 \ \|e_j\|^2; \ y_1, \ldots, y_r \in \mathbf{Z}\right\}. \end{aligned}$$

We now choose integer numbers $z_1, \ldots, z_r \in \mathbf{Z}$ as follows:

- For $1 \leq j \leq p$, we choose z_{2j} such that

  $$(x_{2j} + z_{2j})^2 = \min\{(x_{2j} + y_{2j})^2 \ ; \ y_{2j} \in \mathbf{Z}\}$$

  and set $z_{2j-1} := z_{2j}$. Since $x_{2j} = x_{2j-1}$, we also get

  $$(x_{2j-1} + z_{2j-1})^2 = \min\{(x_{2j-1} + y_{2j-1})^2 \ ; \ y_{2j-1} \in \mathbf{Z}\} .$$

- For $2p + 1 \leq j \leq q$, we choose $z_j \in \mathbf{Z}$ such that

  $$(x_j + z_j)^2 = \min\{(x_j + y_j)^2 \ ; \ y_j \in \mathbf{Z}\} .$$

- $z_{q+1} = \cdots = z_r = 0$.

These choices ensure that

$$\sum_{j=1}^{r} (x_j + z_j)^2 \|e_j\|^2 = d_T^2(o, a) .$$

Moreover the vector $Z = \sum_{j=1}^{r} z_j e_j \in \Gamma(P, \mathfrak{t})$ satisfies

- $z_{2j} = z_{2j-1}$ for $1 \leq j \leq p$,
- $z_{q+1} = \cdots = z_r = 0$,

that is $Z \in \mathfrak{a}$.

Notice that $\exp(X + Z) \cdot o = \exp(X) \exp(Z) \cdot o = \exp(X) \cdot o = a$ and $d_T^2(o, a) = \|X + Z\|^2$. Since $X + Z \in \mathfrak{a}$ and $d_T^2(o, a) \leq d_A^2(o, a)$, we get $d_T^2(o, a) = d_A^2(o, a)$, and Claim 8 follows. □

4. Counterexamples. Though our proof of Theorem 1 relies on Loos' characterization of symmetric R-spaces in terms of orthogonal unit lattices, one may ask if the statement of Theorem 1 is still true for reflective submanifolds in arbitrary compact symmetric spaces. In this section we present two counterexamples for such a statement, that arose in discussion with Jost-Hinrich Eschenburg.

EXAMPLE 9. Take a flat 2-torus P with a non-rectangular rhombic lattice. Then the long diagonal in the rhombic lattice gives a reflective submanifold M of P. The shortest geodesic in P joining the midpoint of a rhombic fundamental domain to a vertex of it follows the short diagonal and therefore does not lie in the reflective submanifold M. Thus M is not convex.

With this picture in mind for a maximal flat torus in a symmetric space, one gets a first counterexample of the statement in Theorem 1, if one replaces "symmetric R-space" by "symmetric space of compact type".

EXAMPLE 10. Consider SU_3 equipped with the bi-invariant metric induced by

$$\langle X, Y\rangle = \operatorname{trace}(XY),\ \ X, Y \in \mathfrak{su}_3\,.$$

The complex conjugation is an involutive isometry of SU_3 whose fixed point set is SO_3. Since the complex conjugation leaves the center

$$C = \left\{I_3,\ e^{2\pi i/3}I_3,\ e^{4\pi i/3}I_3\right\}$$

of SU_3 invariant, it descends to an involutive isometry σ of the irreducible symmetric spaces $P = SU_3/C \cong \mathrm{Ad}(SU_3)$. As SO_3 meets the center of SU_3 only in I_3, the restriction of the Riemannian covering

$$\pi : SU_3 \to SU_3/C,\ \ x \mapsto [x]\,,$$

to SO_3 is an injective map and $M = \pi(SO_3)$ is the fixed point set of σ and therefore a reflective submanifold of P.

A shortest geodesic arc within M that joins $[I_3]$ to the point

$$\left[\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\right] = \left[\begin{pmatrix}e^{\pi i/3} & 0 & 0\\ 0 & e^{\pi i/3} & 0\\ 0 & 0 & e^{-2\pi i/3}\end{pmatrix}\right]$$

is given by

$$\gamma_1 : [0,\pi] \to M \subset P,\ \ t \mapsto \left[e^{tX_1}\right] \text{ with } X_1 = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

But there is a considerably shorter geodesic arc in P joining the given endpoints, namely,

$$\gamma_2 : [0,\pi] \to M \subset P,\ \ t \mapsto \left[e^{tX_2}\right] \text{ with } X_2 = \begin{pmatrix}i/3 & 0 & 0\\ 0 & i/3 & 0\\ 0 & 0 & -2i/3\end{pmatrix}.$$

Notice that $\|X_2\|^2 = 2/3 < 2 = \|X_1\|^2$. This shows that the reflective submanifold M of P is not geodesically convex.

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