A NOTE ON THE CONJECTURE OF BLAIR IN CONTACT RIEMANNIAN GEOMETRY

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Abstract. The conjecture of Blair says that there are no nonflat Riemannian metrics of nonpositive curvature associated with a contact structure. We prove this conjecture for a certain class of contact structures on closed 3-dimensional manifolds and construct a local counterexample.

1. Introduction. In [2, p. 99], the author states the following conjecture:

CONJECTURE 1.1. *There are no nonflat Riemannian metrics of nonpositive curvature that are associated with a contact structure.*

On a 3-torus, standard Euclidean metric is associated with a contact structure given by the kernel of the one-form $\cos(z)dx + \sin(z)dy$. Despite the fact that there exist contact structures on higher dimensional tori [3], as it was shown in [1], flat metric cannot be associated with any contact structure when the dimension of a manifold is greater than three.

Using the result of A. Zeghib [9] on the existence of geodesic flows on closed manifolds, we have that the conjecture of Blair is true for the Riemannian metrics of strictly negative sectional curvature. In [6], it has been shown that the conjecture is true for the homogenous Riemannian metric associated with a homogenous contact structure.

Note that, in view of the results in [5], closed contact metric manifolds of nonpositive curvature would provide a source of examples of tight contact structures.

The main result of the present paper is the proof of Blair's conjecture for contact structures which are sufficiently nontrivial as fibrations. We prove the following theorem.

THEOREM 1.2. *Assume that M is a closed 3-manifold with a contact structure ξ which cannot be decomposed as a sum of two one-dimensional fibrations, i.e., $\xi \neq \eta_1 \oplus \eta_2$. Then the conjecture of Blair is true for (M, ξ).*

We end with a local counterexample to the conjecture. We construct a Riemannian metric associated with a standard contact structure on $\boldsymbol{R}^3$ which has strictly negative curvature in some neighborhood of zero in $\boldsymbol{R}^3$.

2. Contact metric manifolds.

2.1. Associated metrics. Assume that (M, ξ) is a contact 3-manifold. If we fix a one-form α among the conformal class $\{f\alpha'$; positive functions f on $M\}$, which we call the contact one-form associated with the contact structure, then there is a unique vector field N called the Reeb vector field of α such that

$$\alpha(N) = 1, \ L_N\alpha = \iota_N d\alpha = 0 \,.$$

Let J be an almost complex structure on ξ (i.e., $J^2 = -\mathrm{id}$). We may complement it to a linear operator on TM by setting $JN = 0$.

DEFINITION 2.1. A Riemannian metric $\langle \cdot, \cdot \rangle$ is said to be associated with ξ if there is an associated 1-form α and an almost complex structure J such that

$$\langle N, X \rangle = \alpha(X), \ k\langle X, JY \rangle = d\alpha(X, Y) \,,$$

where k is some constant and X and Y are the vector fields on M.

By a contact metric manifold we are going to understand the tuple $(M, \xi, \alpha, \langle \cdot, \cdot \rangle, J)$.

2.2. Second fundamental form. The second fundamental form of a plane field is a symmetric bilinear form which generalizes the corresponding notion for a surface inside Riemannian manifold. The following definition is due to Reinhart [7].

DEFINITION 2.2. The second fundamental form of a plane field ξ is a bilinear form on ξ defined as

$$II(X, Y) = \frac{1}{2}\langle \nabla_X Y + \nabla_Y X, N \rangle \,,$$

where X and Y are in ξ, N is a unit normal vector field to ξ and ∇ is a Levi-Civita connection of $\langle \cdot, \cdot \rangle$.

We are going to call the linear operator A_N which corresponds to II with respect to $\langle \cdot, \cdot \rangle$ a *shape operator* of ξ. Since II is symmetric, the shape operator has two real eigenvalues that we call the principal curvatures of ξ. The eigenvectors of A_N will be called the principal directions of ξ. We also define the *extrinsic curvature K_e* and the *mean curvature H* of ξ as the determinant and the half trace of the *shape operator*, correspondingly. When the plane field ξ is integrable, the second fundamental form of ξ coincides with the second fundamental forms of the integral surfaces. All notions of the classic surface theory extend naturally to the context of plane distributions.

2.3. Extrinsic geometry in associated metric. When M is a contact metric manifold, the contact structure ξ has a very special geometry with respect to the associated metric $\langle \cdot, \cdot \rangle$. We have the following proposition.

PROPOSITION 2.3 ([2]). *With respect to an associated metric, the Reeb vector field N is a unit speed geodesic vector field (i.e., $\nabla_N N = 0$) and the contact structure is minimal (i.e., $H = 0$).*

Since ξ is minimal with respect to an associated metric, its extrinsic curvature is nonpositive. In the case when it is strictly negative, shape operator has two distinct eigenvalues. The

eigenvectors that correspond to these eigenvalues are pointwise linearly independent. In particular, for every $p \in M$, there is a cacnonical decomposition of ξ_p into the one-dimensional eigenspaces of A_N. In this case ξ, viewed as a two-dimensional fibration over M will split as a sum of one-dimensional fibrations. If ξ cannot be split in such a way, there would be a point $p \in M$ such that $K_e = 0$ at this point and p will be an umbilic point.

Below, we are also going to summarize several properties of the contact structures with respect to an associated metric that will be used in the derivation of the curvature tensor.

LEMMA 2.4. *Let $(M, \xi, \alpha, \langle \cdot, \cdot \rangle, J)$ be a contact metric manifold. Then,*

1. *J is a rotation by $\pi/2$ in ξ.*

2. *For every pair of orthonormal vectors X and Y in ξ, the function $\langle [X, Y], N \rangle$ is a constant $\pm k$ with $k > 0$.*

3. *If X and Y are unit orthogonal principal directions of ξ, then*

$$\langle \nabla_X Y, N \rangle = -\langle \nabla_Y X, N \rangle = \pm \frac{k}{2}.$$

4. *If X is a unit principal direction of ξ, then*

$$\nabla_X N = \mp \frac{k}{2} JX - A_N X.$$

PROOF. For every pair of vectors X and Y in ξ, we have

$$k\langle JX, JY \rangle = d\alpha(JX, Y) = -d\alpha(Y, JX) = -k\langle Y, J^2 Y \rangle = k\langle X, Y \rangle.$$

We are left to check that X is orthogonal to JX. This follows from

$$k\langle X, JX \rangle = d\alpha(X, X) = 0.$$

If X and Y are orthonormal, then $Y = \pm JX$. We have

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = -\langle [X, Y], N \rangle.$$

On the other hand

$$d\alpha(X, Y) = \pm k\langle X, X \rangle = \pm k,$$

which proves (2).

Since X and Y are the eigenvectors of A_N, $\frac{1}{2}\langle \nabla_X Y + \nabla_Y X, N \rangle = 0$. From (2),

$$\langle \nabla_X Y, N \rangle = \frac{1}{2}\langle \nabla_X Y + \nabla_Y X, N \rangle + \frac{1}{2}\langle \nabla_X Y - \nabla_Y X, N \rangle = \pm \frac{k}{2}.$$

Finally, since

$$\langle \nabla_X N, X \rangle = -\langle \nabla_X X, N \rangle = -\langle A_N X, X \rangle$$

and

$$\langle \nabla_X N, JX \rangle = -\langle \nabla_X JX, N \rangle = \mp \frac{k}{2},$$

follows (4). $\square$

3. Curvature tensor of the associated metric on a 3-manifold. In this section we are going to compute the matrix of the curvature tensor of an associated metric. Assume that $(M, \xi, \alpha, \langle \cdot, \cdot \rangle, J)$ is a contact metric manifold. Let N be the Reeb vector field of α. Denote by X, Y the (local) orthonormal frame in ξ that consists of the eigenvectors of the shape operator at a given point $p \in M$.

Let λ be a principal curvature that corresponds to a principal direction X. Since ξ is minimal, the mean curvature of ξ vanishes and Y corresponds to the principal curvature $-\lambda$.

LEMMA 3.1. *With respect to a basis of bivectors $X \wedge Y$, $X \wedge N$ and $Y \wedge N$ the matrix of the curvature tensor of $\langle \cdot, \cdot \rangle$ is given by*

$$
\mathcal{R} = \begin{pmatrix}
-3k^2/4 + \lambda^2 + K & Y(\lambda) - 2\lambda\langle\nabla_X X, Y\rangle & X(\lambda) - 2\lambda\langle\nabla_Y Y, X\rangle \\
Y(\lambda) - 2\lambda\langle\nabla_X X, Y\rangle & k^2/4 - \lambda^2 + N(\lambda) & 2\lambda\langle\nabla_N X, Y\rangle \\
X(\lambda) - 2\lambda\langle\nabla_Y Y, X\rangle & 2\lambda\langle\nabla_N X, Y\rangle & k^2/4 - \lambda^2 - N(\lambda)
\end{pmatrix},
$$

where

$$
K = X(\langle\nabla_Y Y, X\rangle) + Y(\langle\nabla_X X, Y\rangle) - \langle\nabla_Y Y, X\rangle^2 - \langle\nabla_X X, Y\rangle^2 - \langle[X, Y], N\rangle\langle[N, Y], X\rangle
$$

is the curvature of a generalized Webster connection (see [8] for the definition) and λ is an eigenvalue of the shape operator which corresponds to X.

PROOF. By replacing X by $-X$ if required we may assume that $\langle[X, Y], N\rangle = k$. *Calculation of $\mathcal{R}_{11} = \langle R(X, Y)Y, X\rangle$.* By the definition, we have

$$
\langle R(X, Y)Y, X\rangle = \langle\nabla_X \nabla_Y Y, X\rangle - \langle\nabla_Y \nabla_X Y, X\rangle - \langle\nabla_{[X,Y]} Y, X\rangle.
$$

The first summand is

$$
\langle\nabla_X \nabla_Y Y, X\rangle = X(\langle\nabla_Y Y, X\rangle) - \langle\nabla_Y Y, \nabla_X X\rangle
$$
$$
= X(\langle\nabla_Y Y, X\rangle) - \langle\nabla_Y Y, N\rangle\langle\nabla_X X, N\rangle = X(\langle\nabla_Y Y, X\rangle) + \lambda^2.
$$

The second summand is

$$
-\langle\nabla_Y \nabla_X Y, X\rangle = -Y(\langle\nabla_X Y, X\rangle) + \langle\nabla_X Y, \nabla_Y X\rangle = Y(\langle\nabla_X X, Y\rangle) - \frac{k^2}{4}
$$

as follows from (3) in Lemma 2.4. The third summand is

$$
-\langle\nabla_{[X,Y]} Y, X\rangle = -\langle[X, Y], X\rangle\langle\nabla_X Y, X\rangle - \langle[X, Y], Y\rangle\langle\nabla_Y Y, X\rangle - \langle[X, Y], N\rangle\langle\nabla_N Y, X\rangle
$$
$$
= -\langle\nabla_X X, Y\rangle^2 - \langle\nabla_Y Y, X\rangle^2 - \langle[X, Y], N\rangle(\langle\nabla_Y N, X\rangle + \langle[N, Y], X\rangle)
$$
$$
= -\langle\nabla_X X, Y\rangle^2 - \langle\nabla_Y Y, X\rangle^2 - \frac{k^2}{2} - \langle[X, Y], N\rangle\langle[N, Y], X\rangle.
$$

Summing these up will give us the desired expression for $\mathcal{R}_{11}$.

Calculation of $\mathcal{R}_{22} = \langle R(X, N)N, X\rangle$. By the definition, we have

$$
\langle R(N, X)X, N\rangle = \langle\nabla_N \nabla_X X, N\rangle - \langle\nabla_X \nabla_N X, N\rangle - \langle\nabla_{[N,X]} X, N\rangle.
$$

The first summand is

$$
\langle\nabla_N \nabla_X X, N\rangle = N(\langle\nabla_X X, N\rangle) - \langle\nabla_X X, \nabla_N N\rangle = N(\lambda).
$$

Here we used that $\nabla_N N = 0$. The second summand is

$$-\langle \nabla_X \nabla_N X, N \rangle = -X\langle \nabla_N X, N \rangle + \langle \nabla_N X, \nabla_X N \rangle$$
$$= -X(N\langle X, N \rangle - \langle X, \nabla_N N \rangle) + \langle \nabla_N X, \nabla_X N \rangle = \langle \nabla_N X, \nabla_X N \rangle.$$

Finally, the last summand is

$$-\langle \nabla_{[N,X]} X, N \rangle = -\langle [N,X], X \rangle \langle \nabla_X X, N \rangle - \langle [N,X], Y \rangle \langle \nabla_Y X, N \rangle$$
$$= -\lambda^2 - \langle [N,X], Y \rangle \langle \nabla_Y X, N \rangle.$$

Summing these expressions we get

$$\mathcal{R}_{22} = N(\lambda) - \lambda^2 - \langle [N,X], Y \rangle \langle \nabla_Y X, N \rangle + \langle \nabla_N X, Y \rangle \langle Y, \nabla_X N \rangle.$$

Using (2) and (3) of Lemma 2.4, we obtain

$$\mathcal{R}_{22} = N(\lambda) - \lambda^2 + \frac{k}{2}(\langle [N,X], Y \rangle - \langle \nabla_N X, Y \rangle) = N(\lambda) - \lambda^2 + \frac{k^2}{4}.$$

Calculation of $\mathcal{R}_{33} = \langle R(Y,N)N, Y \rangle$. By exactly the same calculations replacing X by Y we get

$$\langle R(Y,N)N, Y \rangle = -N(\lambda) - \lambda^2 + \frac{k^2}{4}.$$

Calculation of $\mathcal{R}_{23} = \langle R(X,N)N, Y \rangle$. Analogously,

$$\langle R(X,N)N, Y \rangle = \langle \nabla_X \nabla_N N, Y \rangle - \langle \nabla_N \nabla_X N, Y \rangle - \langle \nabla_{[X,N]} N, Y \rangle.$$

Obviously, since N is geodesic the first summand is zero. Rewrite the second summand,

$$-\langle \nabla_N \nabla_X N, Y \rangle = -N(\langle \nabla_X N, Y \rangle) + \langle \nabla_X N, \nabla_N Y \rangle = \langle \nabla_X N, \nabla_N Y \rangle$$
$$= \langle \nabla_X N, X \rangle \langle X, \nabla_N Y \rangle + \langle \nabla_X N, Y \rangle \langle Y, \nabla_N Y \rangle = -\lambda \langle X, \nabla_N Y \rangle.$$

For the last summand, we have

$$-\langle \nabla_{[X,N]} N, Y \rangle = -\langle [X,N], X \rangle \langle \nabla_X N, Y \rangle - \langle [X,N], Y \rangle \langle \nabla_Y N, Y \rangle.$$

Summing these expressions we get

$$\langle R(Y,N)N, Y \rangle = \lambda \langle \nabla_N X, Y \rangle + \lambda \langle \nabla_X N, Y \rangle - \lambda \langle [X,N], Y \rangle = 2\lambda \langle \nabla_N X, Y \rangle.$$

Calculation of $\mathcal{R}_{13} = \langle R(X,Y)N, Y \rangle$.

$$\langle R(X,Y)N, Y \rangle = -\langle R(X,Y)Y, N \rangle = -\langle \nabla_X \nabla_Y Y, N \rangle + \langle \nabla_Y \nabla_X Y, N \rangle + \langle \nabla_{[X,Y]} Y, N \rangle.$$

The first summand is

$$-\langle \nabla_X \nabla_Y Y, N \rangle = -X\langle \nabla_Y Y, N \rangle + \langle \nabla_Y Y, \nabla_X N \rangle = X(\lambda) + \langle \nabla_Y Y, X \rangle \langle X, \nabla_X N \rangle$$
$$= X(\lambda) - \lambda \langle \nabla_Y Y, X \rangle.$$

The second summand is

$$\langle \nabla_Y \nabla_X Y, N \rangle = Y(\langle \nabla_X Y, N \rangle) - \langle \nabla_X Y, \nabla_Y N \rangle = -\langle \nabla_X Y, X \rangle \langle X, \nabla_Y N \rangle$$
$$= -\langle \nabla_X X, Y \rangle \langle N, \nabla_Y X \rangle.$$

Finally, the last summand,

$$\langle \nabla_{[X,Y]}Y, N \rangle = \langle [X, Y], X \rangle \langle \nabla_X Y, N \rangle + \langle [X, Y], Y \rangle \langle \nabla_Y Y, N \rangle$$
$$= -\langle \nabla_X X, Y \rangle \langle N, \nabla_X Y \rangle - \lambda \langle \nabla_Y Y, X \rangle.$$

Summing these up gives us

$$\langle R(X, Y)N, Y \rangle = X(\lambda) - 2\lambda \langle \nabla_Y Y, X \rangle.$$

Calculation of $\mathcal{R}_{12} = \langle R(X, Y)N, X \rangle$. Analogously, we get

$$\langle R(X, Y)N, X \rangle = Y(\lambda) - 2\lambda \langle \nabla_X X, Y \rangle. \qquad \square$$

PROOF OF THEOREM 1.2.   Under the assumptions of the theorem, for every Riemannian metric g on M, ξ must have an *umbilic point*. At this point we have $\lambda = 0$ and

$$\mathcal{R}_{22} + \mathcal{R}_{33} = \frac{k^2}{2} - 2\lambda^2 = \frac{k^2}{2} > 0$$

Therefore, g cannot have nonpositive curvature. $\square$

4.   Local counterexample to the conjecture of Blair.   On $\mathbf{R}^3$ with cartesian coordinates (x, y, z), consider a standard contact structure ξ given by the kernel of the one-form $\alpha = dz + x\,dy$. We will construct a Riemannian metric which would be associated with ξ and have nonpositive (even strictly negative) curvature in some neighborhood of zero in $\mathbf{R}^3$.

With respect to this metric, the Reeb vector field of α has to be a unit geodesic vector field and ξ has to be a minimal distribution. It is easy to check that in this case the matrix of g should have the form

$$g = \begin{pmatrix} a & b & 0 \\ b & c & x \\ 0 & x & 1 \end{pmatrix},$$

where the functions a, b and c additionally satisfy the condition

$$H = \frac{1}{2}\frac{\partial}{\partial z}(a(c - x^2) - b^2) = 0.$$

This condition will be automatically satisfied if we choose

$$\begin{cases} a = Ae^z, \\ b = 1, \\ c = x^2 + Be^{-z} \end{cases}$$

for some positive numbers A and B ($AB > 1$).

With respect to an orthonormal frame

$$\left(\frac{\partial}{\partial z}, \frac{1}{\sqrt{Ae^z}}\frac{\partial}{\partial x}, \sqrt{\frac{Ae^z}{AB - 1}}\left(-\frac{1}{Ae^z}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \right) \right),$$

the curvature tensor is given by a matrix

$$
\begin{pmatrix}
\frac{1}{4}\frac{AB-3-2x^2Ae^z}{AB-1} & -\frac{1}{2}x\sqrt{\frac{Ae^z}{AB-1}} & \frac{1}{2}\frac{x\sqrt{Ae^z}}{AB-1} \\
-\frac{1}{2}x\sqrt{\frac{Ae^z}{AB-1}} & -\frac{1}{4} & 0 \\
\frac{1}{2}\frac{x\sqrt{Ae^z}}{AB-1} & 0 & -\frac{1}{4}
\end{pmatrix}.
$$

Clearly when $AB \in (1,3)$, the matrix of the curvature tensor is negatively definite in some neighborhood of zero in $\mathbf{R}^3$.

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