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BIHARMONIC INTEGRAL $\mathcal{C}$-PARALLEL SUBMANIFOLDS IN 7-DIMENSIONAL SASAKIAN SPACE FORMS

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Abstract. We find the characterization of maximum dimensional proper-biharmonic integral $\mathcal{C}$-parallel submanifolds of a Sasakian space form and then we classify such submanifolds in a 7-dimensional Sasakian space form. Working in the sphere $\boldsymbol{S}^7$ we explicitly find all 3-dimensional proper-biharmonic integral $\mathcal{C}$-parallel submanifolds. We also determine the proper-biharmonic parallel Lagrangian submanifolds of $\boldsymbol{C}P^3$.

1. Introduction. As suggested in 1964 by Eells and Sampson in their famous paper [17], the *biharmonic maps* $\psi : (M, g) \to (N, h)$ between Riemannian manifolds are a natural generalization of *harmonic maps*. The harmonic maps are critical points of the *energy functional*

$$E(\psi) = \frac{1}{2}\int_M |d\psi|^2 \, v_g \ ,$$

while the biharmonic maps are critical points of the *bienergy functional*

$$E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 \, v_g \ ,$$

where $\tau(\psi) = \operatorname{trace} \nabla d\psi$ is the tension field that vanishes for harmonic maps. The Euler-Lagrange equation for the bienergy functional was derived by Jiang in 1986 (see [25]):

$$\begin{aligned}\tau_2(\psi) &= -\Delta\tau(\psi) - \operatorname{trace}\ R^N(d\psi, \tau(\psi))d\psi \\ &= 0\end{aligned}$$

where $\tau_2(\psi)$ is the *bitension field* of ψ. Since any harmonic map is biharmonic, we are interested in non-harmonic biharmonic maps, which are called *proper-biharmonic*.

An important case of biharmonic maps is represented by the biharmonic Riemannian immersions, or biharmonic submanifolds, i.e., submanifolds for which the inclusion map is biharmonic. In Euclidean spaces the biharmonic submanifolds are the same as those defined by Chen in [13], as they are characterized by the equation $\Delta H = 0$, where H is the mean curvature vector field and Δ is the rough Laplacian.

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Pursuing the founding of proper-biharmonic submanifolds in Riemannian manifolds the attention was first focused on space forms, and classification results in this context were obtained, for example, in [8, 11, 13, 16]. More recently such results were also found in spaces of non-constant sectional curvature (see, for example, [12, 23, 28, 29, 33]).

A different and active research direction is the study of proper-biharmonic submanifolds in pseudo-Riemannian manifolds (see, for example, [2, 3, 14]).

During the efforts of studying the biharmonic submanifolds in space forms, the Euclidean spheres proved to be a very giving environment for obtaining examples and classification results (see [7] for detailed proofs). Then, the fact that odd-dimensional spheres can be thought as a class of Sasakian space forms (which do not have constant sectional curvature, in general) led to the idea that the next step would be the study of biharmonic submanifolds in Sasakian space forms. Following this direction, the proper-biharmonic Legendre curves and Hopf cylinders in a 3-dimensional Sasakian space form were classified in [24], whilst in [19] their parametric equations were found. In [20] all proper-biharmonic Legendre curves in any dimensional Sasakian space forms were classified, and it was provided a method to obtain proper-biharmonic anti-invariant submanifolds from proper-biharmonic integral submanifolds. Also, classification results for proper-biharmonic hypersurfaces were obtained in [21].

The goals of our paper are to characterize the maximum dimensional proper-biharmonic integral, and integral $\mathcal{C}$-parallel, submanifolds in a Sasakian space form, and then to use these results in order to obtain the 3-dimensional proper-biharmonic integral $\mathcal{C}$-parallel submanifolds of a 7-dimensional Sasakian space form. The paper is organized as follows. In Section 2 we briefly recall some general facts on Sasakian space forms with a special emphasis on the notion of integral $\mathcal{C}$-parallel submanifolds, and also present some old and new results concerning the proper-biharmonic submanifolds in odd-dimensional spheres. Section 3 is devoted to the study of the biharmonicity of maximum dimensional integral submanifolds in a Sasakian space form. We obtain the necessary and sufficient conditions for such a submanifold to be biharmonic, prove some non-existence results and find the characterization of proper-biharmonic integral $\mathcal{C}$-parallel submanifolds of maximum dimension. In Section 4 we classify all 3-dimensional proper-biharmonic integral $\mathcal{C}$-parallel submanifolds in a 7-dimensional Sasakian space form, whilst in Section 5 we find these submanifolds in the 7-sphere endowed with its canonical and deformed Sasakian structures introduced by Tanno in [30]. In the last section we classify the proper-biharmonic parallel Lagrangian submanifolds of $\boldsymbol{C}P^3$ by determining their horizontal lifts, with respect to the Hopf fibration, in $\boldsymbol{S}^7(1)$.

For a general account of biharmonic maps see [26] and *The Bibliography of Biharmonic Maps* (http://people.unica.it/ biharmonic/).

CONVENTIONS. We work in the C^∞ category, that means manifolds, metrics, connections and maps are smooth. The Lie algebra of vector fields on M is denoted by $C^\infty(TM)$. The manifold M is always assumed to be connected.

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2. Preliminaries.

2.1. Integral $\mathcal{C}$-parallel submanifolds of a Sasakian manifold.

A *contact metric structure* on an odd-dimensional manifold N^{2n+1} is given by (φ, ξ, η, g), where φ is a tensor field of type $(1, 1)$ on N, ξ is a vector field, η is a 1-form and g is a Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi\,, \quad \eta(\xi) = 1$$

and

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad g(U, \varphi V) = d\eta(U, V) \quad \text{for all} \quad U, V \in C^\infty(TN)\,.$$

A contact metric structure (φ, ξ, η, g) is called *normal* if

$$N_\varphi + 2d\eta \otimes \xi = 0\,,$$

where

$$N_\varphi(U, V) = [\varphi U, \varphi V] - \varphi[\varphi U, V] - \varphi[U, \varphi V] + \varphi^2[U, V] \quad \text{for all} \quad U, V \in C^\infty(TN)\,,$$

is the Nijenhuis tensor field of φ.

A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is *regular* if for any point $p \in N$ there exists a cubic neighborhood such that any integral curve of ξ passes through it at most once; and it is *strictly regular* if all integral curves of ξ are homeomorphic to each other.

A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is a *Sasakian manifold* if it is normal or, equivalently, if

$$(\nabla^N_U \varphi)(V) = g(U, V)\xi - \eta(V)U \quad \text{for all} \quad U, V \in C^\infty(TN)\,,$$

where ∇^N is the Levi-Civita connection on (N, g). We shall often use in our paper the formula $\nabla^N_U \xi = -\varphi U$, which holds on a Sasakian manifold.

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a 2-plane generated by U and φU, where U is a unit vector orthogonal to ξ, is called *φ-sectional curvature* determined by U. A Sasakian manifold with constant φ-sectional curvature c is called a *Sasakian space form* and is denoted by $N(c)$. The curvature tensor field of a Sasakian space form $N(c)$ is given by

$$\begin{aligned} R^N(U, V)W = {} & ((c+3)/4)\{g(W, V)U - g(W, U)V\} + ((c-1)/4)\{\eta(W)\eta(U)V \\ & -\eta(W)\eta(V)U + g(W, U)\eta(V)\xi - g(W, V)\eta(U)\xi \\ & +g(W, \varphi V)\varphi U - g(W, \varphi U)\varphi V + 2g(U, \varphi V)\varphi W\}\,. \end{aligned}$$

The classification of the complete, simply connected Sasakian space forms $N(c)$ was given in [30]. Thus, if $c = 1$ then $N(1)$ is isometric to the unit sphere $\mathbb{S}^{2n+1}$ endowed with its canonical Sasakian structure, and if $c + 3 > 0$ then $N(c)$ is isometric to $\mathbb{S}^{2n+1}$ endowed with the deformed Sasakian structure introduced by Tanno in [30], which we present below.

Let $\boldsymbol{S}^{2n+1} = \{z \in \boldsymbol{C}^{n+1}; |z| = 1\}$ be the unit $(2n+1)$-dimensional Euclidean sphere. Consider the following structure tensor fields on $\boldsymbol{S}^{2n+1}$: $\xi_0(z) = -\mathcal{J}z$ for each $z \in \boldsymbol{S}^{2n+1}$, where $\mathcal{J}$ is the usual complex structure on $\boldsymbol{C}^{n+1}$ defined by

$$\mathcal{J}z = (-y^1, \dots, -y^{n+1}, x^1, \dots, x^{n+1})$$

for $z = (x^1, \dots, x^{n+1}, y^1, \dots, y^{n+1})$, and $\varphi_0 = s \circ \mathcal{J}$, where $s : T_z\boldsymbol{C}^{n+1} \to T_z\boldsymbol{S}^{2n+1}$ denotes the orthogonal projection. Equipped with these tensors and the standard metric g_0, the sphere $\boldsymbol{S}^{2n+1}$ becomes a Sasakian space form with φ_0-sectional curvature equal to 1, denoted by $\boldsymbol{S}^{2n+1}(1)$.

Now, consider the deformed Sasakian structure on $\boldsymbol{S}^{2n+1}$

$$\eta = a\eta_0\,, \quad \xi = \frac{1}{a}\xi_0\,, \quad \varphi = \varphi_0\,, \quad g = ag_0 + a(a-1)\eta_0 \otimes \eta_0\,,$$

where a is a positive constant. The structure (φ, ξ, η, g) is still a Sasakian structure and $(\boldsymbol{S}^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant φ-sectional curvature $c = 4/a-3 > -3$, denoted by $\boldsymbol{S}^{2n+1}(c)$ (see also [10]).

A submanifold M^m of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$ is called an *integral submanifold* if $\eta(X) = 0$ for any vector field X tangent to M. We have $\varphi(TM) \subset NM$ and $m \leq n$, where TM and NM are the tangent bundle and the normal bundle of M, respectively. Moreover, for $m = n$, one gets $\varphi(NM) = TM$. If we denote by B the second fundamental form of M then, by a straightforward computation, one obtains the relation

$$g(\varphi Z, B(X, Y)) = g(\varphi Y, B(X, Z))$$

for any vector fields X, Y and Z tangent to M (see also [6]). We also note that $A_\xi = 0$, where A is the shape operator of M (see [10]).

A submanifold $\tilde{M}$ of N is said to be *anti-invariant* if ξ is tangent to $\tilde{M}$ and φ maps the tangent bundle to $\tilde{M}$ into its normal bundle.

Next, we shall recall the notion of an integral $\mathcal{C}$-parallel submanifold of a Sasakian manifold (see, for example, [6]). Let M^m be an integral submanifold of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$. Then M is said to be *integral $\mathcal{C}$-parallel* if $\nabla^\perp B$ is parallel to the characteristic vector field ξ, where $\nabla^\perp B$ is given by

$$(\nabla^\perp B)(X, Y, Z) = \nabla^\perp_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for any vector fields X, Y, Z tangent to M, $\nabla^\perp$ and ∇ being the normal connection and the Levi-Civita connection on M, respectively. Thus, M^m is an integral $\mathcal{C}$-parallel submanifold if $(\nabla^\perp B)(X, Y, Z) = S(X, Y, Z)\xi$ for any vector fields X, Y, Z tangent to M, where $S(X, Y, Z) = g(\varphi X, B(Y, Z))$ is a totally symmetric tensor field of type $(0, 3)$ on M. It is not difficult to check that, when $m = n$, $\nabla^\perp B = 0$ if and only if $B = 0$, i.e., M^n is totally geodesic.

Now, let M^m be an integral submanifold of a Sasakian manifold N^{2n+1}, and denote by H its mean curvature vector field. We say that H is *$\mathcal{C}$-parallel* if $\nabla^\perp H$ is parallel to ξ, i.e., $\nabla^\perp_X H = \theta(X)\xi$, where θ is a 1-form on M. As we shall see, $\theta(X) = g(H, \varphi X)$ for any vector field X tangent to M.

In general, a Riemannian submanifold M of N is called *parallel* if $\nabla^{\perp}B = 0$, and we say that H is *parallel* if $\nabla^{\perp}H = 0$.

The following two results shall be used later in this paper and, for the sake of completeness, we also provide their proofs.

PROPOSITION 2.1. *If the mean curvature vector field H of an integral submanifold M^n of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$ is parallel then M^n is minimal.*

PROOF. Let X, Y be two vector fields tangent to M. Since

$$g(B(X, Y), \xi) = g(\nabla^N_X Y, \xi) = -g(Y, \nabla^N_X \xi) = g(Y, \varphi X) = 0$$

we have $B(X, Y) \in \varphi(TM)$ and, in particular, $H \in \varphi(TM)$. Then

$$g(\nabla^{\perp}_X H, \xi) = g(\nabla^N_X H, \xi) = -g(H, \nabla^N_X \xi) = g(H, \varphi X)\,.$$

Thus, if $\nabla^{\perp}H = 0$ it follows that $g(H, \varphi X) = 0$ for any vector field X tangent to M, and this means $H = 0$, since M has maximal dimension. □

PROPOSITION 2.2. *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold and M^m be an integral $\mathcal{C}$-parallel submanifold with mean curvature vector field H. The following hold*:

(1) $\nabla^{\perp}_X H = g(H, \varphi X)\xi$, *for any vector field X tangent to M, i.e., H is $\mathcal{C}$-parallel*;
(2) *the mean curvature $|H|$ is constant*;
(3) *if $m = n$, then $\Delta^{\perp}H = H$.*

PROOF. In order to prove (1), we consider $\{X_i\}_{i=1}^m$ to be a local geodesic frame at $p \in M$. Then we have at p

$$(\nabla^{\perp}B)(X_i, X_j, X_j) = \nabla^{\perp}_{X_i} B(X_j, X_j) = g(B(X_j, X_j), \varphi X_i)\xi$$

and, by summing for $j = 1, \dots, m$, we obtain $\nabla^{\perp}_{X_i} H = g(H, \varphi X_i)\xi$. Then, for (2), we have

$$X(|H|^2) = 2g(H, \nabla^{\perp}_X H) = 2g(H, \varphi X)g(H, \xi) = 0$$

for any vector field X tangent to M, i.e., $|H|$ is constant.

For the last item, we assume that $m = n$. As $\nabla^N_X \xi = -\varphi X$, from the Weingarten equation, we get $A_\xi = 0$, where A_ξ is the shape operator of M corresponding to ξ, and $\nabla^{\perp}_X \xi = \nabla^N_X \xi = -\varphi X$. Thus

$$\begin{aligned}
\Delta^{\perp}H &= -\sum_{i=1}^n \nabla^{\perp}_{X_i}\nabla^{\perp}_{X_i} H = -\sum_{i=1}^n \nabla^{\perp}_{X_i}(g(H, \varphi X_i)\xi) \\
&= -\sum_{i=1}^n X_i(g(H, \varphi X_i))\xi - \sum_{i=1}^n (g(H, \varphi X_i))\nabla^N_{X_i}\xi \\
&= -\sum_{i=1}^n X_i(g(H, \varphi X_i))\xi + \sum_{i=1}^n (g(H, \varphi X_i))\varphi X_i \\
&= -\sum_{i=1}^n X_i(g(H, \varphi X_i))\xi + H\,.
\end{aligned}$$

But, since $\nabla^N_{X_i}\varphi X_i = \varphi\nabla^N_{X_i}X_i + \xi$, it results

$$\begin{aligned} X_i(g(H,\varphi X_i)) &= g(\nabla^N_{X_i}H,\varphi X_i) + g(H,\varphi\nabla^N_{X_i}X_i + \xi) \\ &= g(-A_H X_i + \nabla^\perp_{X_i}H,\varphi X_i) + g(H,\varphi B(X_i,X_i)) \\ &= 0\,. \end{aligned}$$

We have just proved that $\Delta^\perp H = H$. □

2.2. Biharmonic submanifolds in $\boldsymbol{S}^{2n+1}(1)$. We shall first recall the notion of Frenet curve of osculating order r as it is presented, for example, in [27]. Let (M^m, g) be a Riemannian manifold and $\gamma : I \to M$ a curve parametrized by arc length, that is $|\gamma'| = 1$. Then γ is called a *Frenet curve of osculating order* r, $1 \le r \le m$, if for all $s \in I$ its higher order derivatives

$$\gamma'(s) = (\nabla^0_{\gamma'}\gamma')(s)\,, \quad (\nabla_{\gamma'}\gamma')(s)\,, \quad \ldots\,, \quad (\nabla^{r-1}_{\gamma'}\gamma')(s)$$

are linearly independent but

$$\gamma'(s) = (\nabla^0_{\gamma'}\gamma')(s)\,, \quad (\nabla_{\gamma'}\gamma')(s)\,, \quad \ldots\,, \quad (\nabla^{r-1}_{\gamma'}\gamma')(s)\,, \quad (\nabla^{r}_{\gamma'}\gamma')(s)$$

are linearly dependent in $T_{\gamma(s)}M$. Then there exist unique orthonormal vector fields $E_1, E_2, \ldots, E_r$ along γ such that

$$\nabla_T E_1 = \kappa_1 E_2\,, \quad \nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \quad \ldots\,, \quad \nabla_T E_r = -\kappa_{r-1}E_{r-1}\,,$$

where $E_1 = \gamma' = T$ and $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on I.

REMARK 2.3. A geodesic is a Frenet curve of osculating order 1; a *circle* is a Frenet curve of osculating order 2 with κ_1 constant; a *helix of order* r, $r \ge 3$, is a Frenet curve of osculating order r with $\kappa_1, \ldots, \kappa_{r-1}$ constants; a helix of order 3 is simply called a *helix*.

In [24] Inoguchi proved that there are no proper-biharmonic Legendre curves in $\boldsymbol{S}^3(1)$ whilst in [20] we found the parametric equations of all proper-biharmonic Legendre curves in $\boldsymbol{S}^{2n+1}(1)$, $n \ge 2$. These curves are given by the following theorem.

THEOREM 2.4 ([20]). *Let $\gamma : I \to (\boldsymbol{S}^{2n+1}, \varphi_0, \xi_0, \eta_0, g_0)$, $n \ge 2$, be a proper-biharmonic Legendre curve parametrized by arc length. Then the parametric equation of γ in the Euclidean space $(\boldsymbol{R}^{2n+2}, \langle,\rangle)$ is either*

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(\sqrt{2}s)e_1 + \frac{1}{\sqrt{2}}\sin(\sqrt{2}s)e_2 + \frac{1}{\sqrt{2}}e_3\,,$$

where $\{e_i, \mathcal{J}e_j\}_{i,j=1}^3$ are constant unit vectors orthogonal to one another, or

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(As)e_1 + \frac{1}{\sqrt{2}}\sin(As)e_2 + \frac{1}{\sqrt{2}}\cos(Bs)e_3 + \frac{1}{\sqrt{2}}\sin(Bs)e_4\,,$$

where

$$A = \sqrt{1+\kappa_1}\,, \quad B = \sqrt{1-\kappa_1}\,, \quad \kappa_1 \in (0,1)$$

and $\{e_i\}_{i=1}^4$ are constant unit vectors orthogonal to one another, satisfying

$$\langle e_1, \mathcal{J}e_3\rangle = \langle e_1, \mathcal{J}e_4\rangle = \langle e_2, \mathcal{J}e_3\rangle = \langle e_2, \mathcal{J}e_4\rangle = 0\,, \quad A\langle e_1, \mathcal{J}e_2\rangle + B\langle e_3, \mathcal{J}e_4\rangle = 0\,.$$

REMARK 2.5. We note that if γ is a proper-biharmonic Legendre circle, then $E_2 \perp \varphi T$ and $n \geq 3$. If γ is a proper-biharmonic Legendre helix, then $g_0(E_2, \varphi T) = -A\langle e_1, \mathcal{J}e_2\rangle$ and we have two cases: either $E_2 \perp \varphi T$ and then $\{e_i, \mathcal{J}e_j\}_{i,j=1}^4$ is an orthonormal system in $\boldsymbol{R}^{2n+2}$, so $n \geq 3$, or $g_0(E_2, \varphi T) \neq 0$ and, in this case, $g_0(E_2, \varphi T) \in (-1, 1) \setminus \{0\}$. We also observe that φT cannot be parallel to E_2. When $g_0(E_2, \varphi T) \neq 0$ and $n \geq 3$ the first four vectors (for example) in the canonical basis of the Euclidean space $\boldsymbol{R}^{2n+2}$ satisfy the conditions of Theorem 2.4, whilst for $n = 2$ we can obtain four vectors $\{e_1, e_2, e_3, e_4\}$ satisfying these conditions in the following way. We consider constant unit vectors e_1, e_3 and f in $\boldsymbol{R}^6$ such that $\{e_1, e_3, f, \mathcal{J}e_1, \mathcal{J}e_3, \mathcal{J}f\}$ is a $\mathcal{J}$-basis. Then, by a straightforward computation, it follows that the vectors e_2 and e_4 have to be given by

$$e_2 = \mp\frac{B}{A}\mathcal{J}e_1 + \alpha_1 f + \alpha_2 \mathcal{J}f\,, \quad e_4 = \pm\mathcal{J}e_3\,,$$

where α_1 and α_2 are constants such that $\alpha_1^2 + \alpha_2^2 = 1 - B^2/A^2 = 2\kappa_1/A^2$. As a concrete example, we can start with the following vectors in $\boldsymbol{R}^6$:

$$e_1 = (1, 0, 0, 0, 0, 0)\,, \quad e_3 = (0, 0, 1, 0, 0, 0)\,, \quad f = (0, 1, 0, 0, 0, 0)$$

and obtain

$$e_2 = \left(0, \alpha_1, 0, -\frac{B}{A}, \alpha_2, 0\right), \quad e_4 = (0, 0, 0, 0, 0, 1)\,,$$

where $\alpha_1^2 + \alpha_2^2 = 1 - B^2/A^2$.

The classification of all proper-biharmonic Legendre curves in a Sasakian space form $N^{2n+1}(c)$ was given in [20]. This classification is invariant under an isometry Ψ of N which preserves ξ (or, equivalently, Ψ is φ-holomorphic).

In order to find higher dimensional proper-biharmonic submanifolds in a Sasakian space form we gave the following theorem.

THEOREM 2.6 ([20]). *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian space form with constant φ-sectional curvature c and let $\mathbf{i} : M \to N$ be an m-dimensional integral submanifold of N, $1 \leq m \leq n$. Consider the cylinder*

$$F : \tilde{M} = I \times M \to N\,, \quad F(t, p) = \phi_t(p) = \phi_p(t)\,,$$

where $I = \boldsymbol{S}^1$ or $I = \boldsymbol{R}$ and $\{\phi_t\}_{t\in I}$ is the flow of the vector field ξ. Then $F : (\tilde{M}, \tilde{g} = dt^2 + \mathbf{i}^* g) \to N$ is an anti-invariant Riemannian immersion, and is proper-biharmonic if and only if M is a proper-biharmonic submanifold of N.

Working with anti-invariant submanifolds rather than with cylinders, we can state the following (known) result.

PROPOSITION 2.7. *Let $\tilde{M}^{m+1}$ be an anti-invariant submanifold of the strictly regular Sasakian space form $N^{2n+1}(c)$, $1 \leq m \leq n$, invariant under the flow-action of the characteristic vector field ξ. Then $\tilde{M}$ is locally isometric to $I \times M^m$, where M^m is an integral submanifold of N. Moreover, we have*

(1) *$\tilde{M}$ is proper-biharmonic if and only if M is proper-biharmonic in N;*

(2) *if $m = n$, then $\tilde{M}$ is parallel if and only if M is $\mathcal{C}$-parallel*;

(3) *if $m = n$, then the mean curvature vector field of $\tilde{M}$ is parallel if and only if the mean curvature vector field of M is $\mathcal{C}$-parallel.*

PROOF. The restriction $\xi_{/\tilde{M}}$ of the characteristic vector field ξ to $\tilde{M}$ is a Killing vector field tangent to $\tilde{M}$. Since $\tilde{M}$ is anti-invariant, the horizontal distribution defined on $\tilde{M}$ is integrable. Let $p \in \tilde{M}$ be an arbitrary point and M a small enough integral submanifold of the horizontal distribution on $\tilde{M}$ such that $p \in M$. Then $F : I \times M \to F(I \times M) \subset \tilde{M}$, $F(t, p) = \phi_t(p)$, is an isometry. As M is an integral submanifold of the horizontal distribution on $\tilde{M}$, it is an integral submanifold of N.

The item (1) follows immediately from Theorem 2.6, and (2) and (3) are known and can be checked by straightforward computations. □

As a surface in a strictly regular Sasakian space form which is invariant under the flow-action of the characteristic vector field is also anti-invariant, we have the following corollary.

COROLLARY 2.8. *Let $\tilde{M}^2$ be a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then $\tilde{M}$ is locally isometric to $I \times \gamma$, where γ is a Legendre curve in N and, moreover, $\tilde{M}$ is proper-biharmonic if and only if γ is proper-biharmonic in N.*

Now, consider $\tilde{M}^2$ a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ and let $T = \gamma'$ and E_2 be the first two vector fields defined by the Frenet equations of the above Legendre curve γ. As $\nabla^F_{\partial/\partial t}\tau(F) = -\varphi(\tau(F))$, where ∇^F is the pull-back connection determined by the Levi-Civita connection on N, we can prove the following proposition.

PROPOSITION 2.9. *Let $\tilde{M}^2$ be a proper-biharmonic surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then $\tilde{M}$ has parallel mean curvature vector field if and only if $c > 1$ and $\varphi T = \pm E_2$.*

COROLLARY 2.10. *The proper-biharmonic surfaces of $\boldsymbol{S}^{2n+1}(1)$ invariant under the flow-action of the characteristic vector field ξ_0 are not of parallel mean curvature vector field.*

We shall see that we do have examples of maximum dimensional proper-biharmonic anti-invariant submanifolds of $\boldsymbol{S}^{2n+1}(1)$, invariant under the flow-action of ξ_0, which have parallel mean curvature vector field.

In [31] the parametric equations of all proper-biharmonic integral surfaces in $\boldsymbol{S}^5(1)$ were obtained. Up to an isometry of $\boldsymbol{S}^5(1)$ which preserves ξ_0, we have only one proper-biharmonic integral surface given by

$$x(u, v) = \frac{1}{\sqrt{2}}(\exp(\mathrm{i}u), \mathrm{i}\exp(-\mathrm{i}u)\sin(\sqrt{2}v), \mathrm{i}\exp(-\mathrm{i}u)\cos(\sqrt{2}v)) .$$

The map x induces a proper-biharmonic Riemannian embedding from the 2-dimensional torus $\mathcal{T}^2 = \boldsymbol{R}^2/\Lambda$ into $\boldsymbol{S}^5$, where Λ is the lattice generated by the vectors $(2\pi, 0)$ and $(0, \sqrt{2}\pi)$.

REMARK 2.11. We recall that an isometric immersion $x : M \to \boldsymbol{R}^{n+1}$ of a compact manifold is said to be of *k-type* if its spectral decomposition contains exactly k non-constant terms excepting the center of mass $x_0 = (\mathrm{Vol}(M))^{-1} \int_M x\, v_g$. When $x_0 = 0$, the submanifold is called *mass-symmetric* (see [13]). It was proved in [8, 9] that a proper-biharmonic compact constant mean curvature submanifold M^m of $\boldsymbol{S}^n$ is either a 1-type submanifold of $\boldsymbol{R}^{n+1}$ with center of mass of norm equal to $1/\sqrt{2}$, or a mass-symmetric 2-type submanifold of $\boldsymbol{R}^{n+1}$. Now, using [4, Theorem 3.5], where all mass-symmetric 2-type integral surfaces in $\boldsymbol{S}^5(1)$ were determined, and [11, Proposition 4.1], the result in [31] can be (partially) reobtained.

Further, we consider the cylinder over x and we recover the result in [1]: up to an isometry of $\boldsymbol{S}^5(1)$ which preserves ξ_0, we have only one 3-dimensional proper-biharmonic anti-invariant submanifold of $\boldsymbol{S}^5(1)$ invariant under the flow-action of ξ_0,

$$y(t, u, v) = \exp(-\mathrm{i}t)x(u, v)\,.$$

The map y is a proper-biharmonic Riemannian immersion with parallel mean curvature vector field and it induces a proper-biharmonic Riemannian immersion from the 3-dimensional torus $\mathcal{T}^3 = \boldsymbol{R}^3/\Lambda$ into $\boldsymbol{S}^5$, where Λ is the lattice generated by the vectors $(2\pi, 0, 0)$, $(0, 2\pi, 0)$ and $(0, 0, \sqrt{2}\pi)$. Moreover, a closer look shows that y factorizes to a proper-biharmonic Riemannian embedding in $\boldsymbol{S}^5$, and its image is the Riemannian product between three Euclidean circles, one of radius $1/\sqrt{2}$ and each of the other two of radius $1/2$. Indeed, we may consider the orthogonal transformation of $\boldsymbol{R}^3$ given by

$$T(t, u, v) = \left(\frac{-t+u}{\sqrt{2}}, \frac{-t-u}{\sqrt{2}}, v\right) = (t', u', v')$$

and the map y becomes

$$y_1(t', u', v') = \frac{1}{\sqrt{2}}(\exp(\mathrm{i}\sqrt{2}t'), \mathrm{i}\exp(\mathrm{i}\sqrt{2}u')\sin(\sqrt{2}v'), \mathrm{i}\exp(\mathrm{i}\sqrt{2}u')\cos(\sqrt{2}v'))\,.$$

Then, acting with an appropriate holomorphic isometry of $\boldsymbol{C}^4$, y_1 becomes

$$y_2(t', u', v') = \left(\frac{1}{\sqrt{2}}\exp(\mathrm{i}\sqrt{2}t'), \frac{1}{2}\exp(\mathrm{i}(u'-v'))\,, \frac{1}{2}\exp(\mathrm{i}(u'+v'))\right)$$

and, further, an obvious orthogonal transformation of the domain leads to the desired results.

3. Biharmonic integral submanifolds of maximum dimension in Sasakian space forms. Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c, and M^n an n-dimensional integral submanifold of N. We recall that this means $\eta(X) = 0$ for any vector field X tangent to M. We shall denote by B, A and H the second fundamental form of M in N, the shape operator and the mean curvature vector field, respectively. By $\nabla^\perp$ and $\Delta^\perp$ we shall denote the connection and the Laplacian in the normal bundle. We have the following theorem.

THEOREM 3.1. *The integral submanifold* $\mathbf{i} : M^n \to N^{2n+1}$ *is biharmonic if and only if*

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) - \dfrac{c(n+3)+3n-3}{4} H = 0\,, \\ 4 \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) + n \operatorname{grad}(|H|^2) = 0\,. \end{cases} \tag{3.1}$$

PROOF. Let us denote by ∇^N, ∇ the Levi-Civita connections on N and M, respectively. Consider $\{X_i\}_{i=1}^n$ to be a local geodesic frame at $p \in M$. Then, since $\tau(\mathbf{i}) = nH$, we have at p

$$\tau_2(\mathbf{i}) = -\Delta\tau(\mathbf{i}) - \operatorname{trace} R^N(d\mathbf{i}, \tau(\mathbf{i}))d\mathbf{i} = n\left\{ \sum_{i=1}^n \nabla^N_{X_i} \nabla^N_{X_i} H - \sum_{i=1}^n R^N(X_i, H)X_i \right\}. \tag{3.2}$$

Using the Weingarten equation,

$$\nabla^N_{X_i} H = \nabla^\perp_{X_i} H - A_H(X_i)$$

and the Gauss equation, we get around p

$$\nabla^N_{X_i} \nabla^N_{X_i} H = \nabla^\perp_{X_i} \nabla^\perp_{X_i} H - A_{\nabla^\perp_{X_i} H}(X_i) - \nabla_{X_i} A_H(X_i) - B(X_i, A_H(X_i))\,.$$

Thus, at p, one obtains

$$\begin{aligned} -\frac{1}{n}\Delta\tau(\mathbf{i}) &= \sum_{i=1}^n \nabla^N_{X_i} \nabla^N_{X_i} H \\ &= -\Delta^\perp H - \operatorname{trace} B(\cdot, A_H \cdot) - \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) - \operatorname{trace} \nabla A_H(\cdot, \cdot)\,. \end{aligned} \tag{3.3}$$

The next step is to compute $\operatorname{trace} \nabla A_H(\cdot, \cdot)$. We obtain, at p,

$$\begin{aligned} \operatorname{trace} \nabla A_H(\cdot, \cdot) &= \sum_{i=1}^n \nabla_{X_i} A_H(X_i) = \sum_{i,j=1}^n \nabla_{X_i}(g(A_H(X_i), X_j)X_j) \\ &= \sum_{i,j=1}^n X_i(g(A_H(X_i), X_j))X_j = \sum_{i,j=1}^n X_i(g(B(X_j, X_i), H))X_j \\ &= \sum_{i,j=1}^n X_i(g(\nabla^N_{X_j} X_i, H))X_j\,, \end{aligned}$$

and then

$$\begin{aligned}\operatorname{trace} \nabla A_H(\cdot,\cdot) &= \sum_{i,j=1}^{n} \{g(\nabla^N_{X_i}\nabla^N_{X_j}X_i, H) + g(\nabla^N_{X_j}X_i, \nabla^N_{X_i}H)\}X_j \\ &= \sum_{i,j=1}^{n} g(\nabla^N_{X_i}\nabla^N_{X_j}X_i, H)X_j + \sum_{i,j=1}^{n} g(B(X_j, X_i), \nabla^\perp_{X_i}H)X_j \\ &= \sum_{i,j=1}^{n} g(\nabla^N_{X_i}\nabla^N_{X_j}X_i, H)X_j + \sum_{i,j=1}^{n} g(A_{\nabla^\perp_{X_i}H}(X_i), X_j)X_j \\ &= \sum_{i,j=1}^{n} g(\nabla^N_{X_i}\nabla^N_{X_j}X_i, H)X_j + \operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot)\,.\end{aligned}$$

Further, using the expression of the curvature tensor field R^N, we have

$$\begin{aligned}\operatorname{trace} \nabla A_H(\cdot,\cdot) &= \sum_{i,j=1}^{n} g(\nabla^N_{X_j}\nabla^N_{X_i}X_i + R^N(X_i, X_j)X_i + \nabla^N_{[X_i,X_j]}X_i, H)X_j \\ &\quad + \operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) \\ &= \sum_{i,j=1}^{n} g(\nabla^N_{X_j}\nabla^N_{X_i}X_i, H)X_j + \sum_{i,j=1}^{n} g(R^N(X_i, X_j)X_i, H)X_j \\ &\quad + \operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot)\,.\end{aligned} \tag{3.4}$$

But

$$\begin{aligned}&\sum_{i,j=1}^{n} g(\nabla^N_{X_j}\nabla^N_{X_i}X_i, H)X_j \\ &\quad = \sum_{i,j=1}^{n} g(\nabla^N_{X_j}B(X_i, X_i), H)X_j + \sum_{i,j=1}^{n} g(\nabla^N_{X_j}\nabla_{X_i}X_i, H)X_j \\ &\quad = n\sum_{j=1}^{n} g(\nabla^N_{X_j}H, H)X_j + \sum_{i,j=1}^{n} g(\nabla_{X_j}\nabla_{X_i}X_i + B(X_j, \nabla_{X_i}X_i), H)X_j \\ &\quad = \frac{n}{2}\operatorname{grad}(|H|^2)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\sum_{i,j=1}^{n} g(R^N(X_i, X_j)X_i, H)X_j \\ &\quad = \sum_{i,j=1}^{n} g(R^N(X_i, H)X_i, X_j)X_j = (\operatorname{trace} R^N(d\mathbf{i}, H)d\mathbf{i})^\top\,.\end{aligned} \tag{3.6}$$

Replacing (3.5) and (3.6) into (3.4), we have

$$\operatorname{trace} \nabla A_H(\cdot,\cdot) = \frac{n}{2}\operatorname{grad}(|H|^2) + (\operatorname{trace} R^N(d\mathbf{i}, H)d\mathbf{i})^\top + \operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot)$$

and therefore

$$\begin{aligned}\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) + \operatorname{trace}\nabla A_H(\cdot,\cdot) =& 2\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) + \frac{n}{2}\operatorname{grad}(|H|^2)\\ &+ (\operatorname{trace} R^N(d\mathbf{i}, H)d\mathbf{i})^\top .\end{aligned} \tag{3.7}$$

Now, let $\{X_i\}_{i=1}^n$ be a local orthonormal frame on M. Then $\{X_i, \varphi X_j, \xi\}_{i,j=1}^n$ is a local orthonormal frame on N. By using the expression of the curvature tensor field and $H \in \operatorname{span}\{\varphi X_i; i = 1, \dots, n\}$ one obtains, after a straightforward computation,

$$R^N(X_i, H)X_i = -\frac{c+3}{4}H + \frac{3(c-1)}{4}g(\varphi H, X_i)\varphi X_i .$$

Hence

$$\begin{aligned}\operatorname{trace} R^N(d\mathbf{i}, H)d\mathbf{i} &= \sum_{i=1}^n R^N(X_i, H)X_i\\ &= -\frac{(c+3)n}{4}H + \sum_{i=1}^n \frac{3(c-1)}{4}g(\varphi H, X_i)\varphi X_i\\ &= -\frac{(c+3)n}{4}H - \frac{3(c-1)}{4}H\\ &= -\frac{c(n+3)+3n-3}{4}H ,\end{aligned} \tag{3.8}$$

which implies $(\operatorname{trace} R^N(d\mathbf{i}, H)d\mathbf{i})^\top = 0$.

From (3.2), (3.3), (3.7) and (3.8) we have

$$\begin{aligned}\frac{1}{n}\tau_2(\mathbf{i}) =& -\Delta^\perp H - \operatorname{trace} B(\cdot, A_H\cdot) + \frac{c(n+3)+3n-3}{4}H\\ &- 2\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) - \frac{n}{2}\operatorname{grad}(|H|^2) ,\end{aligned}$$

and we come to the conclusion. □

COROLLARY 3.2. *Let $N^{2n+1}(c)$ be a Sasakian space form with constant φ-sectional curvature $c \le (3-3n)/(n+3)$. Then an integral submanifold M^n with constant mean curvature $|H|$ in $N^{2n+1}(c)$ is biharmonic if and only if it is minimal.*

PROOF. Assume that M^n is a biharmonic integral submanifold with constant mean curvature $|H|$ in $N^{2n+1}(c)$. It follows, from Theorem 3.1, that

$$\begin{aligned} g(\Delta^{\perp} H, H) &= -g(\operatorname{trace} B(\cdot, A_H \cdot), H) + \frac{c(n+3)+3n-3}{4}|H|^2 \\ &= \frac{c(n+3)+3n-3}{4}|H|^2 - \sum_{i=1}^{n} g(B(X_i, A_H X_i), H) \\ &= \frac{c(n+3)+3n-3}{4}|H|^2 - \sum_{i=1}^{n} g(A_H X_i, A_H X_i) \\ &= \frac{c(n+3)+3n-3}{4}|H|^2 - |A_H|^2 . \end{aligned}$$

Thus, from the Weitzenböck formula

$$\frac{1}{2}\Delta|H|^2 = g(\Delta^{\perp} H, H) - |\nabla^{\perp} H|^2 ,$$

one obtains

$$\frac{c(n+3)+3n-3}{4}|H|^2 - |A_H|^2 - |\nabla^{\perp} H|^2 = 0 . \tag{3.9}$$

If $c < (3-3n)/(n+3)$, relation (3.9) is equivalent to $H = 0$. Now, assume that $c = (3-3n)/(n+3)$. As for integral submanifolds $\nabla^{\perp} H = 0$ is equivalent to $H = 0$, again (3.9) is equivalent to $H = 0$. □

COROLLARY 3.3. *Let $N^{2n+1}(c)$ be a Sasakian space form with constant φ-sectional curvature $c \leq (3-3n)/(n+3)$. Then a compact integral submanifold M^n is biharmonic if and only if it is minimal.*

PROOF. Assume that M^n is a biharmonic compact integral submanifold. As in the proof of Corollary 3.2 we have

$$g(\Delta^{\perp} H, H) = \frac{c(n+3)+3n-3}{4}|H|^2 - |A_H|^2$$

and so $\Delta|H|^2 \leq 0$, which implies that $|H|^2$ is constant. Therefore we obtain that M is minimal in this case too. □

REMARK 3.4. From Corollaries 3.2 and 3.3 it is easy to see that in a Sasakian space form $N^{2n+1}(c)$ with constant φ-sectional curvature $c + 3 \leq 0$ a biharmonic compact integral submanifold, or a biharmonic integral submanifold M^n with constant mean curvature, is minimal whatever the dimension of N is.

PROPOSITION 3.5. *Let $N^{2n+1}(c)$ be a Sasakian space form and $\mathbf{i} : M^n \to N^{2n+1}$ be an integral $\mathcal{C}$-parallel submanifold. Then $(\tau_2(\mathbf{i}))^{\top} = 0$.*

PROOF. Indeed, from Proposition 2.2 we have $|H|$ is constant and $\nabla^{\perp} H$ is parallel to ξ, which implies that $A_{\nabla^{\perp}_X H} = 0$ for any vector field X tangent to M, since $A_{\xi} = 0$. Thus we conclude the proof. □

PROPOSITION 3.6. *A non-minimal integral $\mathcal{C}$-parallel submanifold M^n of a Sasakian space form $N^{2n+1}(c)$ is proper-biharmonic if and only if $c > (7-3n)/(n+3)$ and*

$$\operatorname{trace} B(\cdot, A_H \cdot) = \frac{c(n+3)+3n-7}{4} H .$$

PROOF. We know, from Proposition 2.2, that $\Delta^{\perp} H = H$. Hence, from Theorem 3.1 and the above proposition, it follows that M^n is biharmonic if and only if

$$\operatorname{trace} B(\cdot, A_H \cdot) = \frac{c(n+3)+3n-7}{4} H .$$

Next, if M^n verifies the above condition, we contract with H and get

$$|A_H|^2 = \frac{c(n+3)+3n-7}{4} |H|^2 .$$

Since A_H and H do not vanish it follows that $c > (7-3n)/(n+3)$. □

Now, let $\{X_i\}_{i=1}^n$ be an arbitrary orthonormal local frame field on the integral $\mathcal{C}$-parallel submanifold M^n of a Sasakian space form $N^{2n+1}(c)$, and let $A_i = A_{\varphi X_i}$, $i = 1, \dots, n$, be the corresponding shape operators. Then, from Proposition 3.6, we obtain

PROPOSITION 3.7. *A non-minimal integral $\mathcal{C}$-parallel submanifold M^n of a Sasakian space form $N^{2n+1}(c)$, $c > (7-3n)/(n+3)$, is proper-biharmonic if and only if*

$$\begin{pmatrix} g(A_1, A_1) & \dots & g(A_1, A_n) \\ \vdots & \vdots & \vdots \\ g(A_n, A_1) & \dots & g(A_n, A_n) \end{pmatrix} \begin{pmatrix} \operatorname{trace} A_1 \\ \vdots \\ \operatorname{trace} A_n \end{pmatrix} = k \begin{pmatrix} \operatorname{trace} A_1 \\ \vdots \\ \operatorname{trace} A_n \end{pmatrix},$$

where $k = (c(n+3)+3n-7)/4$.

4. 3-dimensional biharmonic integral $\mathcal{C}$-parallel submanifolds of a Sasakian space form $N^7(c)$. In [6] Baikoussis, Blair and Koufogiorgios classified the 3-dimensional integral $\mathcal{C}$-parallel submanifolds in a Sasakian space form $(N^7(c), \varphi, \xi, \eta, g)$. In order to obtain the classification, they worked with a special local orthonormal basis (see also [15]). Here we shall briefly recall how this basis is constructed.

Let $\mathbf{i} : M^3 \to N^7(c)$ be an integral submanifold of non-zero constant mean curvature. Let p be an arbitrary point of M, and consider the function $f_p : U_pM \to \boldsymbol{R}$ given by

$$f_p(u) = g(B(u,u), \varphi u) ,$$

where $U_pM = \{u \in T_pM; g(u,u) = 1\}$ is the unit sphere in the tangent space T_pM. If $f_p(u) = 0$ for all $u \in U_pM$, then, for any $v_1, v_2 \in U_pM$ such that $g(v_1, v_2) = 0$ we have that

$$g(B(v_1,v_1), \varphi v_1) = 0 , \quad g(B(v_1,v_1), \varphi v_2) = 0 , \quad g(B(v_1,v_2), \varphi v_1) = 0 .$$

Now, if $\{X_1, X_2, X_3\}$ is an arbitrary orthonormal basis at p, it follows that $\operatorname{trace} A_{\varphi X_i} = 0$, for any $i \in \{1,2,3\}$, and therefore $H(p) = 0$. Consequently, the function f_p does not vanish identically.

Since U_pM is compact, f_p attains an absolute maximum at a unit vector X_1. It follows that

$$\begin{cases} g(B(X_1, X_1), \varphi X_1) > 0\,, & g(B(X_1, X_1), \varphi X_1) \geq |g(B(w, w), \varphi w)|\,, \\ g(B(X_1, X_1), \varphi w) = 0\,, & g(B(X_1, X_1), \varphi X_1) \geq 2g(B(w, w), \varphi X_1)\,, \end{cases}$$

where w is a unit vector in T_pM orthogonal to X_1. It is easy to see that X_1 is an eigenvector of $A_1 = A_{\varphi X_1}$ with corresponding eigenvalue λ_1. Then, since A_1 is symmetric, we consider X_2 and X_3 to be unit eigenvectors of A_1 orthogonal to each other and to X_1. Further, we distinguish two cases.

If $\lambda_2 \neq \lambda_3$, we can choose X_2 and X_3 such that

$$\begin{cases} g(B(X_2, X_2), \varphi X_2) \geq 0\,, & g(B(X_3, X_3), \varphi X_3) \geq 0\,, \\ g(B(X_2, X_2), \varphi X_2) \geq g(B(X_3, X_3), \varphi X_3)\,. \end{cases}$$

If $\lambda_2 = \lambda_3$, we consider $f_{1,p}$ the restriction of f_p to $\{w \in U_pM;\ g(w, X_1) = 0\}$, and we have two subcases:

(1) The function $f_{1,p}$ is identically zero. In this case, we have

$$\begin{cases} g(B(X_2, X_2), \varphi X_2) = 0\,, & g(B(X_2, X_2), \varphi X_3) = 0\,, \\ g(B(X_2, X_3), \varphi X_3) = 0\,, & g(B(X_3, X_3), \varphi X_3) = 0\,. \end{cases}$$

(2) The function $f_{1,p}$ does not vanish identically. Then we choose X_2 such that $f_{1,p}(X_2)$ is an absolute maximum. We have that

$$\begin{cases} g(B(X_2, X_2), \varphi X_2) > 0\,, & g(B(X_2, X_2), \varphi X_2) \geq g(B(X_3, X_3), \varphi X_3) \geq 0\,, \\ g(B(X_2, X_2), \varphi X_3) = 0\,, & g(B(X_2, X_2), \varphi X_2) \geq 2g(B(X_3, X_3), \varphi X_2)\,. \end{cases}$$

Now, with respect to the orthonormal basis $\{X_1, X_2, X_3\}$, the shape operators A_1, $A_2 = A_{\varphi X_2}$ and $A_3 = A_{\varphi X_3}$, at p, can be written as

$$A_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \lambda_2 & 0 \\ \lambda_2 & \alpha & \beta \\ 0 & \beta & \mu \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & \lambda_3 \\ 0 & \beta & \mu \\ \lambda_3 & \mu & \delta \end{pmatrix}. \tag{4.1}$$

We also have $A_0 = A_\xi = 0$. With these notations we have

$$\lambda_1 > 0\,, \quad \lambda_1 \geq |\alpha|\,, \quad \lambda_1 \geq |\delta|\,, \quad \lambda_1 \geq 2\lambda_2\,, \quad \lambda_1 \geq 2\lambda_3\,. \tag{4.2}$$

For $\lambda_2 \neq \lambda_3$ we get

$$\alpha \geq 0\,, \quad \delta \geq 0 \quad \text{and} \quad \alpha \geq \delta\,, \tag{4.3}$$

and for $\lambda_2 = \lambda_3$ we obtain that

$$\alpha = \beta = \mu = \delta = 0 \tag{4.4}$$

or

$$\alpha > 0\,, \quad \delta \geq 0\,, \quad \alpha \geq \delta\,, \quad \beta = 0 \quad \text{and} \quad \alpha \geq 2\mu\,. \tag{4.5}$$

We can extend X_1 on a neighbourhood V_p of p such that $X_1(q)$ is a maximal point of $f_q : U_q M \to \boldsymbol{R}$ for any point q of V_p.

If the eigenvalues of A_1 have constant multiplicities, then the above basis $\{X_1, X_2, X_3\}$, defined at p, can be smoothly extended and we can work on the open dense subset of M defined by this property.

Using this basis, in [6], the authors proved that, when M is an integral $\mathcal{C}$-parallel submanifold, the functions λ_i, $i \in \{1, 2, 3\}$, and α, β, μ, δ are constant on V_p. Then, they classified all 3-dimensional integral $\mathcal{C}$-parallel submanifolds in a 7-dimensional Sasakian space form.

According to that classification, if $c + 3 > 0$ then M is a non-minimal integral $\mathcal{C}$-parallel submanifold if and only if either:

Case I. M is flat, it is locally a product of three curves which are helices of osculating orders $r \leq 4$, and $\lambda_1 = (\lambda^2 - (c+3)/4)/\lambda$, $\lambda_2 = \lambda_3 = \lambda = \text{constant} \neq 0$, $\alpha = \text{constant}$, $\beta = 0$, $\mu = \text{constant}$, $\delta = \text{constant}$, such that $-\sqrt{c+3}/2 < \lambda < 0$, $0 < \alpha \leq \lambda_1$, $\alpha > 2\mu$, $\alpha \geq \delta \geq 0$, $(c+3)/4 + \lambda^2 + \alpha\mu - \mu^2 = 0$ and $((3\lambda^2 - (c+3)/4)/\lambda)^2 + (\alpha+\mu)^2 + \delta^2 > 0$, or

Case II. M is locally isometric to a product $\gamma \times \bar{M}^2$, where γ is a curve and $\bar{M}^2$ is a $\mathcal{C}$-parallel surface, and either

(1) $\lambda_1 = 2\lambda_2 = -\lambda_3 = \sqrt{c+3}/(2\sqrt{2})$, $\alpha = \mu = \delta = 0$, $\beta = \pm\sqrt{3(c+3)}/(4\sqrt{2})$. In this case γ is a helix in N with curvatures $\kappa_1 = 1/\sqrt{2}$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere of radius $\rho = \sqrt{8/(3(c+3))}$, or

(2) $\lambda_1 = (\lambda^2 - (c+3)/4)/\lambda$, $\lambda_2 = \lambda_3 = \lambda = \text{constant}$, $\alpha = \beta = \mu = \delta = 0$, such that $-\sqrt{c+3}/2 < \lambda < 0$ and $\lambda^2 \neq (c+3)/12$. In this case, γ is a helix in N with curvatures $\kappa_1 = \lambda_1$ and $\kappa_2 = 1$, and $\bar{M}^2$ is the 2-dimensional Euclidean sphere of radius $\rho = 1/\sqrt{(c+3)/4 + \lambda^2}$.

Now, identifying the shape operators A_i with the corresponding matrices, from Proposition 3.7, we get the following proposition.

PROPOSITION 4.1. *A non-minimal integral $\mathcal{C}$-parallel submanifold M^3 of a Sasakian space form $N^7(c)$, $c > -1/3$, is proper-biharmonic if and only if*

$$\left(\sum_{i=1}^{3} A_i^2\right)\begin{pmatrix} \operatorname{trace} A_1 \\ \operatorname{trace} A_2 \\ \operatorname{trace} A_3 \end{pmatrix} = \frac{3c+1}{2}\begin{pmatrix} \operatorname{trace} A_1 \\ \operatorname{trace} A_2 \\ \operatorname{trace} A_3 \end{pmatrix}, \tag{4.6}$$

where matrices A_i are given by (4.1).

Now, we can state the theorem.

THEOREM 4.2. *A* 3*-dimensional integral $\mathcal{C}$-parallel submanifold M^3 of a Sasakian space form $N^7(c)$ is proper-biharmonic if and only if either*:

(1) *$c > -1/3$ and M^3 is flat and it is locally a product of three curves*:
- *a helix with curvatures $\kappa_1 = (\lambda^2 - (c+3)/4)/\lambda$ and $\kappa_2 = 1$,*
- *a helix of order 4 with curvatures $\kappa_1 = \sqrt{\lambda^2 + \alpha^2}$, $\kappa_2 = (\alpha/\kappa_1)\sqrt{\lambda^2+1}$ and $\kappa_3 = -(\lambda/\kappa_1)\sqrt{\lambda^2+1}$,*
- *a helix of order 4 with curvatures $\kappa_1 = \sqrt{\lambda^2+\mu^2+\delta^2}$, $\kappa_2 = (\delta/\kappa_1)\sqrt{\lambda^2+\mu^2+1}$ and $\kappa_3 = (\kappa_2/\delta)\sqrt{\lambda^2+\mu^2}$, if $\delta \neq 0$, or a circle with curvature $\kappa_1 = \sqrt{\lambda^2+\mu^2}$, if $\delta = 0$,*

where $\lambda, \alpha, \mu, \delta$ are constants given by

$$
\begin{cases}
\left(3\lambda^2 - \dfrac{c+3}{4}\right)\left(3\lambda^4 - 2(c+1)\lambda^2 + \dfrac{(c+3)^2}{16}\right) + \lambda^4((\alpha+\mu)^2 + \delta^2) = 0\,, \\
(\alpha+\mu)\left(5\lambda^2 + \alpha^2 + \mu^2 - \dfrac{7c+5}{4}\right) + \mu\delta^2 = 0\,, \\
\delta\left(5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - \dfrac{7c+5}{4}\right) = 0\,, \\
\dfrac{c+3}{4} + \lambda^2 + \alpha\mu - \mu^2 = 0
\end{cases} \tag{4.7}
$$

such that $-\sqrt{c+3}/2 < \lambda < 0$, $0 < \alpha \leq (\lambda^2 - (c+3)/4)/\lambda$, $\alpha \geq \delta \geq 0$, $\alpha > 2\mu$ and $\lambda^2 \neq (c+3)/12$;

or

(2) *M^3 is locally isometric to a product $\gamma \times \bar{M}^2$ between a curve and a $\mathcal{C}$-parallel surface of N, and either*

(a) *$c = 5/9$, γ is a helix in $N^7(5/9)$ with curvatures $\kappa_1 = 1/\sqrt{2}$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere with radius $\sqrt{3}/2$, or*

(b) *$c \in [(-7+8\sqrt{3})/13, +\infty) \setminus \{1\}$, γ is a helix in $N^7(c)$ with curvatures $\kappa_1 = (\lambda^2 - (c+3)/4)/\lambda$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere with radius $2/\sqrt{4\lambda^2+c+3}$, where*

$$
\lambda < 0 \quad \text{and} \quad \lambda^2 = \begin{cases}
\dfrac{4c+4 \pm \sqrt{13c^2+14c-11}}{12} & \text{if} \quad c < 1\,, \\
\dfrac{4c+4 - \sqrt{13c^2+14c-11}}{12} & \text{if} \quad c > 1\,.
\end{cases} \tag{4.8}
$$

PROOF. Let M^3 be a proper-biharmonic integral $\mathcal{C}$-parallel submanifold of a Sasakian space form $N^7(c)$. From Proposition 4.1 we see that $c > -1/3$.

Next, we easily get that the equation (4.6) is equivalent to the system

$$
(4.9)\quad \begin{cases}
\Big(\sum_{i=1}^{3}\lambda_i\Big)\Big(\sum_{i=1}^{3}\lambda_i^2-\frac{3c+1}{2}\Big)+(\alpha+\mu)(\alpha\lambda_2+\mu\lambda_3)\\
\qquad+(\beta+\delta)(\beta\lambda_2+\delta\lambda_3)=0\,,\\
\Big(\sum_{i=1}^{3}\lambda_i\Big)(\alpha\lambda_2+\mu\lambda_3)+(\alpha+\mu)\Big(2\lambda_2^2+\alpha^2+3\beta^2+\mu^2+\beta\delta-\frac{3c+1}{2}\Big)\\
\qquad+\mu(\beta+\delta)^2=0\,,\\
\Big(\sum_{i=1}^{3}\lambda_i\Big)(\beta\lambda_2+\delta\lambda_3)+\beta(\alpha+\mu)^2\\
\qquad+(\beta+\delta)\Big(2\lambda_3^2+\delta^2+3\mu^2+\beta^2+\alpha\mu-\frac{3c+1}{2}\Big)=0\,.
\end{cases}
$$

In the following, we shall split the study of this system, as M^3 is given by Case I or Case II of the classification.

Case I. The system (4.9) is equivalent to the system given by the first three equations of (4.7). Now, M is not minimal if and only if at least one of the components of the mean curvature vector field H does not vanish and, from the first equation of (4.7), it follows that λ^2 must be different from $(c+3)/12$. Thus, again using [6] for the expressions of the curvatures of the three curves, we obtain the first case of the theorem.

Case II. (1) In this case, the second equation of (4.9) is identically satisfied and the other two are equivalent to $c = 5/9$. Thus, from the classification of the integral $\mathcal{C}$-parallel submanifolds, we get the first part of the second case of the theorem.

(2) The second and the third equation of (4.9) are satisfied, in this case, and the first equation is equivalent to

$$3\lambda^4-2(c+1)\lambda^2+\frac{(c+3)^2}{16}=0\,.$$

This equation has solutions if and only if

$$c\in\Big(-\infty,\frac{-7-8\sqrt{3}}{13}\Big]\cup\Big[\frac{-7+8\sqrt{3}}{13},+\infty\Big)\,,$$

and these solutions are given by

$$\lambda^2=\frac{4c+4\pm\sqrt{13c^2+14c-11}}{12}\,.$$

Since $c > -1/3$ it follows that $c\in[(-7+8\sqrt{3})/13,+\infty)$. Moreover, if $c = 1$, from the above relation, it follows that λ^2 must be equal to 1 or 1/3, which is a contradiction, and therefore $c\in[(-7+8\sqrt{3})/13,+\infty)\setminus\{1\}$. Further, it is easy to check that $\lambda^2 = (4c+4+\sqrt{13c^2+14c-11})/12 < (c+3)/4$ if and only if $c\in[(-7+8\sqrt{3})/13,1)$ and $\lambda^2 = (4c+4-\sqrt{13c^2+14c-11})/12 < (c+3)/4$ if and only if $c\in[(-7+8\sqrt{3})/13,+\infty)\setminus\{1\}$. □

5. Proper-biharmonic submanifolds in the 7-sphere. In this section we shall work with the standard model for simply connected Sasakian space forms $N^7(c)$ with $c+3>0$, which is the sphere $\boldsymbol{S}^7$ endowed with its canonical Sasakian structure or with the deformed Sasakian structure introduced by Tanno.

In [6] the authors obtained the explicit equation of the 3-dimensional integral $\mathcal{C}$-parallel flat submanifolds in $\boldsymbol{S}^7(1)$, whilst in [22] we gave the explicit equation of such submanifolds in $\boldsymbol{S}^7(c)$, $c+3>0$.

Using these results and Theorem 4.2 we easily get the following theorem.

THEOREM 5.1. *A 3-dimensional integral $\mathcal{C}$-parallel submanifold M^3 of $\boldsymbol{S}^7(c)$, $c = 4/a-3>-3$, is proper-biharmonic if and only if either*:

(1) *$c>-1/3$ and M^3 is flat, it is locally a product of three curves and its position vector in $\boldsymbol{C}^4$ is*

$$\begin{aligned} x(u,v,w) &= \frac{\lambda}{\sqrt{\lambda^2+1/a}}\exp\left(\mathrm{i}\left(\frac{1}{a\lambda}u\right)\right)\mathcal{E}_1 \\ &+\frac{1}{\sqrt{a(\mu-\alpha)(2\mu-\alpha)}}\exp(-\mathrm{i}(\lambda u-(\mu-\alpha)v))\mathcal{E}_2 \\ &+\frac{1}{\sqrt{a\rho_1(\rho_1+\rho_2)}}\exp(-\mathrm{i}(\lambda u+\mu v+\rho_1 w))\mathcal{E}_3 \\ &+\frac{1}{\sqrt{a\rho_2(\rho_1+\rho_2)}}\exp(-\mathrm{i}(\lambda u+\mu v-\rho_2 w))\mathcal{E}_4\,, \end{aligned}$$

where $\rho_{1,2}=(\sqrt{4\mu(2\mu-\alpha)+\delta^2}\pm\delta)/2$ and $\lambda,\alpha,\mu,\delta$ are real constants given by (4.7) *such that $-1/\sqrt{a}<\lambda<0$, $0<\alpha\le(\lambda^2-1/a)/\lambda$, $\alpha\ge\delta\ge 0$, $\alpha>2\mu$, $\lambda^2\ne 1/(3a)$ and $\{\mathcal{E}_i\}_{i=1}^4$ is an orthonormal basis of $\boldsymbol{C}^4$ with respect to the usual Hermitian inner product*;

or

(2) *M^3 is locally isometric to a product $\gamma\times\bar{M}^2$ between a curve and a $\mathcal{C}$-parallel surface of N, and either*

(a) *$c=5/9$, γ is a helix in $\boldsymbol{S}^7(5/9)$ with curvatures $\kappa_1=1/\sqrt{2}$ and $\kappa_2=1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere with radius $\sqrt{3}/2$, or*

(b) *$c\in[(-7+8\sqrt{3})/13,+\infty)\setminus\{1\}$, γ is a helix in $\boldsymbol{S}^7(c)$ with curvatures $\kappa_1=(\lambda^2-(c+3)/4)/\lambda$ and $\kappa_2=1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere with radius $2/\sqrt{4\lambda^2+c+3}$, where*

$$\lambda<0 \quad \textit{and} \quad \lambda^2=\begin{cases} \dfrac{4c+4\pm\sqrt{13c^2+14c-11}}{12} & \textit{if} \quad c<1\,,\\ \dfrac{4c+4-\sqrt{13c^2+14c-11}}{12} & \textit{if} \quad c>1\,. \end{cases}$$

Now, applying this theorem in the case of the 7-sphere endowed with its canonical Sasakian structure we get the following corollary, which also shows that, for $c = 1$, the system (4.7) can be completely solved.

COROLLARY 5.2. *A 3-dimensional integral $\mathcal{C}$-parallel submanifold M^3 of $\boldsymbol{S}^7(1)$ is proper-biharmonic if and only if it is flat, it is locally a product of three curves and its position vector in $\boldsymbol{C}^4$ is*

$$x(u, v, w) = -\frac{1}{\sqrt{6}} \exp(-\mathrm{i}\sqrt{5}u)\mathcal{E}_1 + \frac{1}{\sqrt{6}} \exp\left(\mathrm{i}\left(\frac{1}{\sqrt{5}}u - \frac{4\sqrt{3}}{\sqrt{10}}v\right)\right)\mathcal{E}_2$$
$$+\frac{1}{\sqrt{6}} \exp\left(\mathrm{i}\left(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v - \frac{3\sqrt{2}}{2}w\right)\right)\mathcal{E}_3$$
$$+\frac{1}{\sqrt{2}} \exp\left(\mathrm{i}\left(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v + \frac{\sqrt{2}}{2}w\right)\right)\mathcal{E}_4\,,$$

where $\{\mathcal{E}_i\}_{i=1}^4$ is an orthonormal basis of $\boldsymbol{C}^4$ with respect to the usual Hermitian inner product. Moreover, the x_u-curve is a helix with curvatures $\kappa_1 = 4\sqrt{5}/5$ and $\kappa_2 = 1$, the x_v-curve is a helix of order 4 with curvatures $\kappa_1 = \sqrt{29}/\sqrt{10}$, $\kappa_2 = 9\sqrt{2}/\sqrt{145}$ and $\kappa_3 = 2\sqrt{3}/\sqrt{145}$ and the x_w-curve is a helix of order 4 with curvatures $\kappa_1 = \sqrt{5}/\sqrt{2}$, $\kappa_2 = 2\sqrt{3}/\sqrt{10}$ and $\kappa_3 = \sqrt{3}/\sqrt{10}$.

PROOF. Since $c = 1$ the system (4.7) becomes

$$\begin{cases} (3\lambda^2 - 1)^2(\lambda^2 - 1) + \lambda^4((\alpha + \mu)^2 + \delta^2) = 0\,, \\ (\alpha + \mu)(5\lambda^2 + \alpha^2 + \mu^2 - 3) + \mu\delta^2 = 0\,, \\ \delta(5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - 3) = 0\,, \\ \lambda^2 + \alpha\mu - \mu^2 + 1 = 0 \end{cases} \tag{5.1}$$

with the supplementary conditions

$$-1 < \lambda < 0\,, \quad 0 < \alpha \le \frac{\lambda^2 - 1}{\lambda}\,, \quad \alpha \ge \delta \ge 0\,, \quad \alpha > 2\mu \quad \text{and} \quad \lambda^2 \ne \frac{1}{3}\,. \tag{5.2}$$

We note that, since $\alpha > 2\mu$, from the fourth equation of (5.1) it results that $\mu < 0$.

The third equation of system (5.1) suggests that, in order to solve this system, we need to split our study in two cases as δ is equal to 0 or not.

Case 1: $\delta = 0$. In this case the third equation holds whatever the values of λ, α and μ are, and so does the condition $\alpha \ge \delta$. We also note that $\alpha \ne -\mu$, since otherwise, from the first equation, it results $\lambda^2 = 1$ or $\lambda^2 = 1/3$, which are both contradictions.

In the following, we shall look for α of the form $\alpha = \omega\mu$, where $\omega \in (-\infty, 0) \setminus \{-1\}$, since $\alpha > 0$, $\mu < 0$ and $\alpha \ne -\mu$. From the second and the fourth equations of the system we have $\lambda^2 = -(\omega^2 + 3\omega - 2)/((\omega - 2)(\omega - 3))$, $\mu^2 = 8/((\omega - 2)(\omega - 3))$ and then $\alpha^2 = 8\omega^2/((\omega - 2)(\omega - 3))$. Replacing in the first equation, after a straightforward computation, it can be written as

$$\frac{8(\omega + 1)^3(1 - 3\omega)}{(\omega - 3)^3(\omega - 2)} = 0$$

and its solutions are -1 and $1/3$. But $\omega \in (-\infty, 0) \setminus \{-1\}$ and therefore we conclude that there are no solutions of the system that verify all conditions (5.2) when $\delta = 0$.

Case 2: $\delta > 0$. In this case the third equation of (5.1) becomes

$$5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - 3 = 0\,.$$

Now, since $\alpha > 0$ and $\mu < 0$, we can take again $\alpha = \omega\mu$, with $\omega \in (-\infty, 0)$, and then, from the last three equations of the system, we easily get $\lambda^2 = -(\omega^2 + 5\omega + 2)/((\omega - 1)(\omega - 2))$, $\alpha^2 = 8\omega^3/((\omega - 1)^2(\omega - 2))$, $\mu^2 = 8\omega/((\omega - 1)^2(\omega - 2))$ and $\delta^2 = 8(\omega + 1)^2/(\omega - 1)^2$. Next, from the first equation of (5.1), after a straightforward computation, one obtains

$$\frac{16(\omega + 1)^3(\omega + 3)}{(\omega - 2)(\omega - 1)^3} = 0\,,$$

whose solutions are -3 and -1. If $\omega = -1$ it follows that $\lambda^2 = 1/3$, which is a contradiction, and therefore we obtain that $\omega = -3$. Hence

$$\lambda^2 = \frac{1}{5}\,, \quad \alpha^2 = \frac{27}{10}\,, \quad \mu^2 = \frac{3}{10} \quad \text{and} \quad \delta^2 = 2\,.$$

As $\lambda < 0$, $\alpha > 0$, $\mu < 0$ and $\delta > 0$ it results that $\lambda = -1/\sqrt{5}$, $\alpha = 3\sqrt{3}/\sqrt{10}$, $\mu = -\sqrt{3}/\sqrt{10}$ and $\delta = \sqrt{2}$. It can be easily seen that also the conditions (5.2) are verified by these values, and then, by the meaning of the first statement of Theorem 5.1, we come to the conclusion. □

REMARK 5.3. A proper-biharmonic compact submanifold M of $\boldsymbol{S}^n$ of constant mean curvature $|H| \in (0, 1)$ is of 2-type and mass-symmetric (see [8, 9]). In our case, the Riemannian immersion x can be written as $x = x_1 + x_2$, where

$$\begin{aligned} x_1(u, v, w) &= \frac{1}{\sqrt{2}} \exp\left(\mathrm{i}\left(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v + \frac{\sqrt{2}}{2}w\right)\right)\mathcal{E}_4\,, \\ x_2(u, v, w) &= -\frac{1}{\sqrt{6}} \exp(-\mathrm{i}\sqrt{5}u)\mathcal{E}_1 + \frac{1}{\sqrt{6}} \exp\left(\mathrm{i}\left(\frac{1}{\sqrt{5}}u - \frac{4\sqrt{3}}{\sqrt{10}}v\right)\right)\mathcal{E}_2 \\ &\quad + \frac{1}{\sqrt{6}} \exp\left(\mathrm{i}\left(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v - \frac{3\sqrt{2}}{2}w\right)\right)\mathcal{E}_3\,, \end{aligned}$$

and $\Delta x_1 = 3(1 - |H|)x_1 = x_1$, $\Delta x_2 = 3(1 + |H|)x_2 = 5x_2$, $|H| = 2/3$. Now, Corollary 5.2 could also be proved by using the main result in [5] and [11, Proposition 4.1].

REMARK 5.4. By a straightforward computation we can deduce that the map x factorizes to a map from the torus $\mathcal{T}^3 = \boldsymbol{R}^3/\Lambda$ into $\boldsymbol{R}^8$, where Λ is the lattice generated by the vectors $a_1 = (6\pi/\sqrt{5}, \sqrt{3}\pi/\sqrt{10}, \pi/\sqrt{2})$, $a_2 = (0, -3\sqrt{5}\pi/\sqrt{6}, -\pi/\sqrt{2})$ and $a_3 = (0, 0, -4\pi/\sqrt{2})$, and the quotient map is a Riemannian immersion.

By the meaning of Theorem 2.6 we know that the cylinder over x, given by

$$y(t, u, v, w) = \phi_t(x(u, v, w))\,,$$

is a proper-biharmonic map into $\boldsymbol{S}^7(1)$. Moreover, we have the following proposition.

PROPOSITION 5.5. *The cylinder over x determines a proper-biharmonic Riemannian embedding from the torus $\mathcal{T}^4 = \boldsymbol{R}^4/\Lambda$ into $\boldsymbol{S}^7$, where the lattice Λ is generated by* $a_1 = (2\pi/\sqrt{6}, 0, 0, 0)$, $a_2 = (0, 2\pi/\sqrt{6}, 0, 0)$, $a_3 = (0, 0, 2\pi/\sqrt{6}, 0)$ *and* $a_4 = (0, 0, 0, 2\pi/\sqrt{2})$. *The image of this embedding is the Riemannian product between a Euclidean circle of radius* $1/\sqrt{2}$ *and three other Euclidean circles, each of radius* $1/\sqrt{6}$.

PROOF. As the flow of the characteristic vector field ξ is given by $\phi_t(z) = \exp(-\mathrm{i}t)z$ we get

$$\begin{aligned} y(t,u,v,w) = &-\frac{1}{\sqrt{6}}\exp(-\mathrm{i}(t+\sqrt{5}u))\mathcal{E}_1 + \frac{1}{\sqrt{6}}\exp\Big(\mathrm{i}\Big(-t+\frac{1}{\sqrt{5}}u-\frac{4\sqrt{3}}{\sqrt{10}}v\Big)\Big)\mathcal{E}_2 \\ &+\frac{1}{\sqrt{6}}\exp\Big(\mathrm{i}\Big(-t+\frac{1}{\sqrt{5}}u+\frac{\sqrt{3}}{\sqrt{10}}v-\frac{3\sqrt{2}}{2}w\Big)\Big)\mathcal{E}_3 \\ &+\frac{1}{\sqrt{2}}\exp\Big(\mathrm{i}\Big(-t+\frac{1}{\sqrt{5}}u+\frac{\sqrt{3}}{\sqrt{10}}v+\frac{\sqrt{2}}{2}w\Big)\Big)\mathcal{E}_4\,, \end{aligned}$$

where $\{\mathcal{E}_i\}_{i=1}^4$ is an orthonormal basis of $\boldsymbol{C}^4$ with respect to the usual Hermitian inner product.

Now, we consider the following two orthogonal transformations of $\boldsymbol{R}^4$:

$$\begin{cases} \dfrac{1}{\sqrt{2}}t+\dfrac{1}{\sqrt{10}}u+\dfrac{\sqrt{3}}{2\sqrt{5}}v+\dfrac{1}{2}w = t', \\ \dfrac{2}{\sqrt{5}}u-\dfrac{\sqrt{6}}{4\sqrt{5}}v-\dfrac{\sqrt{2}}{4}w = u', \\ \dfrac{\sqrt{5}}{2\sqrt{2}}v-\dfrac{\sqrt{3}}{2\sqrt{2}}w = v', \\ \dfrac{1}{\sqrt{2}}t-\dfrac{1}{\sqrt{10}}u-\dfrac{\sqrt{3}}{2\sqrt{5}}v-\dfrac{1}{2}w = w'\,, \end{cases}$$

and

$$\begin{cases} \dfrac{\sqrt{2}}{\sqrt{6}}t'+\dfrac{2}{\sqrt{6}}u' = \tilde{t}\,, \\ -\dfrac{\sqrt{2}}{\sqrt{6}}t'+\dfrac{1}{\sqrt{6}}u'-\dfrac{\sqrt{3}}{\sqrt{6}}v' = \tilde{u}\,, \\ -\dfrac{\sqrt{2}}{\sqrt{6}}t'+\dfrac{1}{\sqrt{6}}u'+\dfrac{\sqrt{3}}{\sqrt{6}}v' = \tilde{v}\,, \\ w' = \tilde{w}\,. \end{cases}$$

Then we obtain

$$\begin{aligned} \tilde{y}(\tilde{t},\tilde{u},\tilde{v},\tilde{w}) = &-\frac{1}{\sqrt{6}}\exp(-\mathrm{i}(\sqrt{6}\tilde{t}))\mathcal{E}_1+\frac{1}{\sqrt{6}}\exp(\mathrm{i}(\sqrt{6}\tilde{u}))\mathcal{E}_2+\frac{1}{\sqrt{6}}\exp(\mathrm{i}(\sqrt{6}\tilde{v}))\mathcal{E}_3 \\ &+\frac{1}{\sqrt{2}}\exp(\mathrm{i}(\sqrt{2}\tilde{w}))\mathcal{E}_4\,, \end{aligned}$$

which ends the proof. □

REMARK 5.6. We see that y can be written as $y = y_1 + y_2$, where $y_1(t, u, v, w) = \exp(-\mathrm{i}t)x_1$, $y_2(t, u, v, w) = \exp(-\mathrm{i}t)x_2$, and $\Delta y_1 = 2y_1$, $\Delta y_2 = 6y_2$, the mean curvature of y being equal to $1/2$.

REMARK 5.7. It is known that the parallel flat $(n + 1)$-dimensional compact anti-invariant submanifolds in $\mathbf{S}^{2n+1}(1)$ are Riemannian products of circles of radii r_i, $i = 1, \dots, n+1$, where $\sum_{i=1}^{n+1} r_i^2 = 1$ (see [32]). The biharmonicity of such submanifolds was solved in [33].

6. Proper-biharmonic parallel Lagrangian submanifolds of $\mathbf{C}P^3$. We consider the Hopf fibration $\pi : \mathbf{S}^{2n+1}(1) \to \mathbf{C}P^n(4)$, and $\overline{M}$ a Lagrangian submanifold of $\mathbf{C}P^n$. Then $\tilde{M} = \pi^{-1}(\overline{M})$ is an $(n + 1)$-dimensional anti-invariant submanifold of $\mathbf{S}^{2n+1}$ invariant under the flow-action of the characteristic vector field ξ_0 and, locally, $\tilde{M}$ is isometric to $\mathbf{S}^1 \times M^n$. The submanifold $\overline{M}$ is a parallel Lagrangian submanifold if and only if M is an integral $\mathcal{C}$-parallel submanifold (see [27]), and it was proved in [18] that a parallel Lagrangian submanifold $\overline{M}$ is biharmonic if and only if M is (-4)-biharmonic.

We recall here that a map $\psi : (M, g) \to (N, h)$ is (-4)-*biharmonic* if it is a critical point of the (-4)-bienergy $E_2(\psi) - 4E(\psi)$, i.e., ψ verifies $\tau_2(\psi) + 4\tau(\psi) = 0$. Also, a real submanifold $\overline{M}$ of $\mathbf{C}P^n$ is called *Lagrangian* if it has dimension n and the complex structure $\overline{J}$ of $\mathbf{C}P^n$ maps the tangent space to $\overline{M}$ onto the normal one.

Thus, in order to determine all proper-biharmonic parallel Lagrangian submanifolds of $\mathbf{C}P^3$, we shall determine the (-4)-biharmonic integral $\mathcal{C}$-parallel submanifolds of $\mathbf{S}^7(1)$.

Just as in the case of Theorem 3.1 we obtain the following theorem.

THEOREM 6.1. *The integral submanifold* $\mathbf{i} : M^3 \to \mathbf{S}^7(1)$ *is* (-4)-*biharmonic if and only if*

$$\begin{cases} \Delta^{\perp} H + \operatorname{trace} B(\cdot, A_H \cdot) - 7H = 0 \\ 4 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + 3 \operatorname{grad}(|H|^2) = 0 \,. \end{cases}$$

Therefore it follows the next proposition.

PROPOSITION 6.2. *A non-minimal integral $\mathcal{C}$-parallel submanifold M^3 of $\mathbf{S}^7(1)$ is (-4)-biharmonic if and only if*

$$\operatorname{trace} B(\cdot, A_H \cdot) = 6H \,. \tag{6.1}$$

Now, we can state the theorem.

THEOREM 6.3. *A 3-dimensional integral $\mathcal{C}$-parallel submanifold M^3 of $\mathbf{S}^7(1)$ is (-4)-biharmonic if and only if either*:

(1) *M^3 is flat and it is locally a product of three curves*:
- *a helix with curvatures* $\kappa_1 = (\lambda^2 - 1)/\lambda$ *and* $\kappa_2 = 1$,
- *a helix of order 4 with curvatures* $\kappa_1 = \sqrt{\lambda^2 + \alpha^2}$, $\kappa_2 = (\alpha/\kappa_1)\sqrt{\lambda^2 + 1}$ *and* $\kappa_3 = -(\lambda/\kappa_1)\sqrt{\lambda^2 + 1}$,

- *a helix of order* 4 *with curvatures* $\kappa_1 = \sqrt{\lambda^2+\mu^2+\delta^2}$, $\kappa_2 = (\delta/\kappa_1)\sqrt{\lambda^2+\mu^2+1}$ *and* $\kappa_3 = (\kappa_2/\delta)\sqrt{\lambda^2+\mu^2}$, *if* $\delta \neq 0$, *or a circle with curvature* $\kappa_1 = \sqrt{\lambda^2+\mu^2}$, *if* $\delta = 0$,

where $\lambda, \alpha, \mu, \delta$ *are constants given by*

$$(6.2)\qquad \begin{cases} (3\lambda^2-1)(3\lambda^4-8\lambda^2+1)+\lambda^4((\alpha+\mu)^2+\delta^2)=0\,, \\ (\alpha+\mu)(5\lambda^2+\alpha^2+\mu^2-7)+\mu\delta^2=0\,, \\ \delta(5\lambda^2+\delta^2+3\mu^2+\alpha\mu-7)=0\,, \\ 1+\lambda^2+\alpha\mu-\mu^2=0 \end{cases}$$

such that $-1<\lambda<0$, $0<\alpha\leq(\lambda^2-1)/\lambda$, $\alpha\geq\delta\geq 0$, $\alpha>2\mu$ *and* $\lambda^2\neq 1/3$;

or

(2) M^3 *is locally isometric to a product* $\gamma\times\bar{M}^2$ *between a helix with curvatures* $\kappa_1 = (\sqrt{13}-1)/\sqrt{12-3\sqrt{13}}$ *and* $\kappa_2 = 1$, *and a* $\mathcal{C}$*-parallel surface of* $\boldsymbol{S}^7(1)$ *which is locally isometric to the* 2*-dimensional Euclidean sphere with radius* $\sqrt{3/(7-\sqrt{13})}$.

PROOF. It is easy to see that the equation (6.1) is equivalent to the system

$$(6.3)\qquad \begin{cases} \Big(\sum_{i=1}^{3}\lambda_i\Big)\Big(\sum_{i=1}^{3}\lambda_i^2-6\Big)+(\alpha+\mu)(\alpha\lambda_2+\mu\lambda_3)+(\beta+\delta)(\beta\lambda_2+\delta\lambda_3)=0\,, \\ \Big(\sum_{i=1}^{3}\lambda_i\Big)(\alpha\lambda_2+\mu\lambda_3)+(\alpha+\mu)(2\lambda_2^2+\alpha^2+3\beta^2+\mu^2+\beta\delta-6) \\ \quad +\mu(\beta+\delta)^2=0\,, \\ \Big(\sum_{i=1}^{3}\lambda_i\Big)(\beta\lambda_2+\delta\lambda_3)+\beta(\alpha+\mu)^2+(\beta+\delta)(2\lambda_3^2+\delta^2+3\mu^2+\beta^2+\alpha\mu-6) \\ \quad =0\,. \end{cases}$$

In the same way as for the study of biharmonicity, we shall split the study of this system, as M^3 is given by Case I or Case II of the classification.

Case I. The system (6.3) is equivalent to the system given by the first three equations of (6.2) and, just like in the proof of Theorem 4.2, we conclude the result.

Case II. (1) It is easy to verify that this case cannot occur in this setting.

(2) The second and the third equation of system (6.3) are satisfied and the first equation is equivalent to $3\lambda^4-8\lambda^2+1=0$, whose solutions are $\lambda^2=(4\pm\sqrt{13})/3$. Since $\lambda^2<1$ it follows that $\lambda^2=(4-\sqrt{13})/3$ and this, together with the classification of the integral $\mathcal{C}$-submanifolds, leads to the conclusion. □

Using the explicit equation of the 3-dimensional integral $\mathcal{C}$-parallel flat submanifolds in $\boldsymbol{S}^7(1)$ (see [6]), we obtain the following corollary.

COROLLARY 6.4. *Any 3-dimensional flat (-4)-biharmonic integral $\mathcal{C}$-parallel submanifold M^3 of $\boldsymbol{S}^7(1)$ is given locally by*

$$\begin{aligned} x(u, v, w) = {} & \frac{\lambda}{\sqrt{\lambda^2+1}} \exp\Big(\mathrm{i}\Big(\frac{1}{\lambda}u\Big)\Big)\mathcal{E}_1 + \frac{1}{\sqrt{(\mu-\alpha)(2\mu-\alpha)}} \exp(-\mathrm{i}(\lambda u - (\mu-\alpha)v))\mathcal{E}_2 \\ & + \frac{1}{\sqrt{\rho_1(\rho_1+\rho_2)}} \exp(-\mathrm{i}(\lambda u + \mu v + \rho_1 w))\mathcal{E}_3 \\ & + \frac{1}{\sqrt{\rho_2(\rho_1+\rho_2)}} \exp(-\mathrm{i}(\lambda u + \mu v - \rho_2 w))\mathcal{E}_4 , \end{aligned}$$

where $\rho_{1,2} = (\sqrt{4\mu(2\mu-\alpha)+\delta^2} \pm \delta)/2$, $-1 < \lambda < 0$, $0 < \alpha \le (\lambda^2-1)/\lambda$, $\alpha \ge \delta \ge 0$, $\alpha > 2\mu$, $\lambda^2 \ne 1/3$, the tuple $(\lambda, \alpha, \mu, \delta)$ being one of the following

$$\Bigg(-\sqrt{\frac{4-\sqrt{13}}{3}},\ \sqrt{\frac{7-\sqrt{13}}{6}},\ -\sqrt{\frac{7-\sqrt{13}}{6}},\ 0\Bigg),$$

$$\Bigg(-\sqrt{\frac{1}{5+2\sqrt{3}}},\ \sqrt{\frac{45+21\sqrt{3}}{13}},\ -\sqrt{\frac{6}{21+11\sqrt{3}}},\ 0\Bigg),$$

or

$$\Bigg(-\sqrt{\frac{1}{6+\sqrt{13}}},\ \sqrt{\frac{523+139\sqrt{13}}{138}},\ -\sqrt{\frac{79-17\sqrt{13}}{138}},\ \sqrt{\frac{14+2\sqrt{13}}{3}}\Bigg),$$

and $\{\mathcal{E}_i\}_{i=1}^4$ is an orthonormal basis of $\boldsymbol{C}^4$ with respect to the usual Hermitian inner product.

PROOF. In order to solve the system (6.2), we first note that, since $\alpha > 2\mu$, from the fourth equation it results $\mu < 0$.

The third equation suggests that we need to split our study in two cases as δ is equal to 0 or not.

Case 1: $\delta = 0$. In this case the third equation holds whatever the values of λ, α and μ are, and so does the condition $\alpha \ge \delta$.

If $\alpha = -\mu$ we easily obtain that the solution of the system is

$$\lambda = -\sqrt{\frac{4-\sqrt{13}}{3}}, \quad \alpha = \sqrt{\frac{7-\sqrt{13}}{6}}, \quad \mu = -\sqrt{\frac{7-\sqrt{13}}{6}}.$$

In the following, we shall look for α of the form $\alpha = \omega\mu$, where $\omega \in (-\infty, 0) \setminus \{-1\}$, since $\alpha > 0$ and $\mu < 0$. From the second and the fourth equations of the system we have $\lambda^2 = -(\omega^2+7\omega-6)/((\omega-2)(\omega-3))$, $\mu^2 = 12/((\omega-2)(\omega-3))$ and then $\alpha^2 = 12\omega^2/((\omega-2)(\omega-3))$. Replacing in the first equation, after a straightforward computation, it can be written as

$$3\omega^6 + 16\omega^5 - 58\omega^4 - 140\omega^3 + 531\omega^2 - 444\omega + 108 = 0,$$

which is equivalent to

$$(\omega-2)^2(3\omega^4 + 28\omega^3 + 42\omega^2 - 84\omega + 27) = 0,$$

whose solutions are 2, $-3 \pm 2\sqrt{3}$ and $(-5 \pm 2\sqrt{13})/3$. From these solutions the only one to verify the supplementary conditions is $\omega = -3 - 2\sqrt{3}$, for which we have

$$\lambda = -\sqrt{\frac{1}{5+2\sqrt{3}}}\,, \quad \alpha = \sqrt{\frac{45+21\sqrt{3}}{13}}\,, \quad \mu = -\sqrt{\frac{6}{21+11\sqrt{3}}}\,.$$

Case 2: $\delta > 0$. In this case the third equation of (6.2) becomes

$$5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - 7 = 0\,.$$

Now, again taking $\alpha = \omega\mu$, this time with $\omega \in (-\infty, 0)$, from the last three equations of the system, we easily get

$$\lambda^2 = -\frac{\omega^2 + 9\omega + 2}{(\omega-1)(\omega-2)}\,, \quad \alpha^2 = \frac{12\omega^3}{(\omega-1)^2(\omega-2)}\,,$$
$$\mu^2 = \frac{12\omega}{(\omega-1)^2(\omega-2)}\,, \quad \delta^2 = \frac{12(\omega+1)^2}{(\omega-1)^2}\,.$$

Replacing in the first equation of the system we obtain the solutions $-2 \pm \sqrt{3}$ and $-4 \pm \sqrt{13}$, from which only $\omega = -4 - \sqrt{13}$ verifies the supplementary conditions. Therefore, we obtain

$$\lambda = -\sqrt{\frac{1}{6+\sqrt{13}}}\,, \quad \alpha = \sqrt{\frac{523+139\sqrt{13}}{138}}\,,$$
$$\mu = -\sqrt{\frac{79-17\sqrt{13}}{138}}\,, \quad \delta = \sqrt{\frac{14+2\sqrt{13}}{3}}\,,$$

and we are done. □

REMARK 6.5. By some straightforward computations we can check that the images of the cylinders over the above x are, respectively: the Riemannian product of a circle of radius $\sqrt{(5-\sqrt{13})/12}$ and three circles, each of radius $\sqrt{(7+\sqrt{13})/36}$; the Riemannian product of two circles each of radius $\sqrt{(3+\sqrt{3})/12}$ and two circles each of radius $\sqrt{(3-\sqrt{3})/12}$; the Riemannian product of a circle of radius $\sqrt{(5+\sqrt{13})/12}$ and three circles each of radius $\sqrt{(7-\sqrt{13})/36}$.

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