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MINIMIZING PROBLEMS FOR THE HARDY-SOBOLEV TYPE INEQUALITY WITH THE SINGULARITY ON THE BOUNDARY

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Abstract. In this paper, we consider the existence of minimizers of the Hardy-Sobolev type variational problem. Recently, Ghoussoub and Robert [9, 10] proved that the Hardy-Sobolev best constant admits its minimizers provided the bounded smooth domain has the negative mean curvature at the origin on the boundary. We generalize their results by using the idea of Brézis and Nirenberg, and as a consequence, we shall prove the existence of positive solutions to the elliptic equation involving two different kinds of Hardy-Sobolev critical exponents.

1. Introduction. Let $n \geq 3$, $0 \leq s \leq 2$ and $2^* = 2^*(s) := 2(n-s)/(n-2)$. The Hardy-Sobolev inequality says that there is a constant $C > 0$ such that

$$\left(\int_{\boldsymbol{R}^n} |u|^{2^*}/|x|^s dx\right)^{2/2^*} \leq C \int_{\boldsymbol{R}^n} |\nabla u|^2 dx \quad \text{for } u \in H^1(\boldsymbol{R}^n)\,. \tag{1.1}$$

We note that for $s = 0$, the inequality (1.1) is reduced to the Sobolev inequality, and for $s = 2$, it is reduced to the Hardy inequality:

$$\int_{\boldsymbol{R}^n} |u|^2/|x|^2 dx \leq (2/(n-2))^2 \int_{\boldsymbol{R}^n} |\nabla u|^2 dx\,.$$

Naturally by (1.1), one is led to study the problem of the best constant. Let Ω be a domain in $\boldsymbol{R}^n$ such that $0 \in \overline{\Omega}$. Set

$$\mu_s(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2 dx\,;\ u \in H_0^1(\Omega) \text{ and } \int_\Omega |u|^{2^*}/|x|^s dx = 1\right\}. \tag{1.2}$$

The question whether $\mu_s(\Omega)$ can be attained in $H_0^1(\Omega)$ has been recently considered by many people. Among others, we refer to [8] and [9, 10, 11]. Furthermore, see [4, 5] and [13] for minimizers of the best constant of the Caffarelli-Kohn-Nirenberg inequality.

When $\Omega = \boldsymbol{R}^n$, the classical results state that (1.2) is attained by

$$u(x) = (\kappa + |x|^{2-s})^{-(n-2)/(2-s)}, \quad x \in \boldsymbol{R}^n \text{ and } \kappa > 0$$

if $0 \leq s < 2$ (see [12] and [15]). However, it is easy to see for $s = 2$ that $\mu_2(\boldsymbol{R}^n) = ((n-2)/2)^2$ is never attained. If $0 \in \Omega$, due to the scaling invariance, we have $\mu_s(\Omega) = \mu_s(\boldsymbol{R}^n)$. Thus $\mu_s(\Omega)$ is never attained if $\Omega \neq \boldsymbol{R}^n$.

However, if $0 \in \partial\Omega$, Ghoussoub and Robert [9, 10] proved the following theorem:

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THEOREM A. *Let $n \geq 3, 0 < s < 2$ and Ω a bounded smooth domain of $\boldsymbol{R}^n$ with $0 \in \partial\Omega$. Suppose the mean curvature $H(0)$ of $\partial\Omega$ at 0 is negative. Then $\mu_s(\Omega)$ is attained by some function in $H_0^1(\Omega)$.*

The proofs of Theorem A in [9, 10] are rather technical compared to [2]. In [2], Brézis and Nirenberg studied a minimizing problem

$$\mu_{0,2}^{\lambda}(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega |u|^2 dx\ ;\ u \in H_0^1(\Omega) \text{ and } \int_\Omega |u|^{2n/(n-2)} dx = 1\right\}$$

for $\lambda < 0$, and they showed the existence of minimizers by proving the inequality

$$\mu_{0,2}^{\lambda}(\Omega) < S_n\,,$$

where S_n is the best Sobolev constant defined by

$$S_n := \inf\left\{\int_\Omega |\nabla u|^2 dx\ ;\ u \in H_0^1(\Omega) \text{ and } \int_\Omega |u|^{2n/(n-2)} dx = 1\right\}. \tag{1.3}$$

Of course, the minimizing problem in [2] is simpler than the one considered in [9, 10], because the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact. However, we will see that the idea in [2] can be extended for the non-compact case, and by using it, the proof could be much easier. To exploit the idea in [2] further, we consider the problem

$$\mu_{s,p}^{\lambda}(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2 dx + \lambda\left(\int_\Omega |u|^p dx\right)^{2/p}\ ;\ u \in H_0^1(\Omega) \text{ and } \int_\Omega |u|^{2^*}/|x|^s dx = 1\right\},$$

where $\lambda \in \boldsymbol{R}, 0 < s < 2, 2 \leq p \leq 2n/(n-2)$. Throughout the paper, we always assume Ω is a bounded domain and $0 \in \partial\Omega$. Now we state our main results. First, for the case $\lambda > 0$, we shall prove the following theorem:

THEOREM 1.1. *Let $n \geq 3$ and $\partial\Omega$ be C^3 at 0. Suppose the mean curvature $H(0)$ of $\partial\Omega$ at 0 is negative. If $\lambda > 0$ and $2 \leq p < 2n/(n-1)$, then $\mu_{s,p}^{\lambda}(\Omega)$ is attained in $H_0^1(\Omega)$.*

Next, for the case $\lambda \leq 0$, we shall prove the following theorem:

THEOREM 1.2. *Let $n \geq 3$ and $\partial\Omega$ be C^3 at 0. Suppose the mean curvature $H(0)$ of $\partial\Omega$ at 0 is negative. Then the following hold*:

(i) *If $\lambda \leq 0$ and $2 \leq p < 2n/(n-2)$, then $\mu_{s,p}^{\lambda}(\Omega)$ is attained in $H_0^1(\Omega)$.*

(ii) *If $p = 2n/(n-2)$ and $-S_n < \lambda \leq 0$, then $\mu_{s,p}^{\lambda}(\Omega)$ is attained in $H_0^1(\Omega)$.*

The special cases $\lambda = 0$ in Theorem 1.2 and $p = 2$ in Theorems 1.1 and 1.2 were considered in Ghoussoub and Robert [9, 10]. If the mean curvature $H(0)$ is non-negative, then we have weaker results:

THEOREM 1.3. *Assume that $2 < p < 2n/(n-2), \lambda < 0, \partial\Omega$ is C^3 at 0 and $H(0) = 0$. Then $\mu_{s,p}^{\lambda}(\Omega)$ is attained.*

THEOREM 1.4. *Assume that $2n/(n-1) < p < 2n/(n-2), \lambda < 0$ and $\partial\Omega$ is C^2 at 0. Then $\mu_{s,p}^{\lambda}(\Omega)$ is attained.*

REMARK 1.5. If $0 \in \Omega$, we can prove that $\mu^{\lambda}_{s,p}(\Omega)$ is attained in $H^1_0(\Omega)$ provided that $\lambda < 0$ and

$$\begin{cases} 3 < p < 2n/(n-2) & \text{if } n = 3\,, \\ 2 < p < 2n/(n-2) & \text{if } n = 4\,, \\ 2 \le p < 2n/(n-2) & \text{if } n \ge 5\,. \end{cases}$$

The proof of the existence of minimizers is much easier than Theorem 1.4. It can be proved by the same method as in [2] without any modification.

To prove main theorems, we first show the inequalities:

$$\text{(1.4)} \qquad \begin{cases} \mu^{\lambda}_{s,p}(\Omega) < \mu_s(\boldsymbol{R}^n_+) & \text{for } 2 \le p < 2n/(n-2)\,, \\ \mu^{\lambda}_{s,*}(\Omega) < \mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+) & \text{for } p = 2n/(n-2)\,, \end{cases}$$

where $\mu^{\lambda}_{s,*}(\Omega) := \mu^{\lambda}_{s,p}(\Omega)$ with $p = 2n/(n-2)$, and then minimizers for $\mu^{\lambda}_{s,p}(\Omega)$, $2 \le p \le 2n/(n-2)$, can be obtained by using inequalities (1.4). This procedure is similar to [2, Lemma 1.1 and Lemma 1.2]. However, there is a major difference between our work and [2]. As it is well-known, the Sobolev best constant S_n of (1.3) is actually independent of Ω. This important fact was used in [2] implicitly. In our problem, $\mu_s(\Omega)$ does depend on the domain Ω. So, some blowing up argument is needed. See Sections 3 and 4.

The Euler-Lagrangian equation for a minimizer u of $\mu^{\lambda}_{s,*}(\Omega)$ is given by

$$\text{(1.5)} \qquad \begin{cases} \Delta u - \lambda \|u\|^{-4/(n-2)}_{L^{2n/(n-2)}(\Omega)} u^{(n+2)/(n-2)} + \mu^{\lambda}_{s,*}(\Omega)\, u^{2^*-1}/|x|^s = 0 \ \text{ in } \Omega\,, \\ u > 0 \ \text{ in } \Omega\,, \quad u = 0 \ \text{ on } \partial\Omega \quad \text{and} \quad \int_\Omega u^{2^*}/|x|^s dx = 1\,. \end{cases}$$

If Ω is star-shaped with respect to 0, then by the Pohozaev identity, the equation (1.5) does not admit any solution. A consequence of the scaling argument in Section 4 is the existence of entire solutions of

$$\text{(1.6)} \qquad \begin{cases} \Delta u + \mu\, u^{(n+2)/(n-2)} + u^{2^*-1}/|x|^s = 0 \quad \text{in } \boldsymbol{R}^n, \\ u > 0 \quad \text{in } \boldsymbol{R}^n \quad \text{and} \quad u \in H^1(\boldsymbol{R}^n) \end{cases}$$

and

$$\text{(1.7)} \qquad \begin{cases} \Delta u + \mu\, u^{(n+2)/(n-2)} + u^{2^*-1}/|x|^s = 0 \quad \text{in } \boldsymbol{R}^n_+\,, \\ u > 0 \quad \text{in } \boldsymbol{R}^n_+ \quad \text{and} \quad u = 0 \quad \text{on } \partial\boldsymbol{R}^n_+ \end{cases}$$

for some positive constant $\mu > 0$. Both equations (1.6) and (1.7) are involved with two different kinds of Hardy-Sobolev critical exponents. Therefore, it should be an interesting issue to study entire solutions of (1.6) and (1.7) or asymptotic behaviors of singular solutions as the case of the Sobolev exponent (see [3], [6, 7] and [14]). We will discuss this problem in a forthcoming paper.

2. Calculation of $\mu^{\lambda}_{s,p}(\Omega)$.

We first prove the following lemma:

LEMMA 2.1. *Let $n \ge 3$, $2 \le p < 2n/(n-2)$ and $\lambda \in \boldsymbol{R}$. Then we have*

$$\mu^{\lambda}_{s,p}(\Omega) \le \mu_s(\boldsymbol{R}^n_+)\,.$$

PROOF. For arbitrary $\alpha > 0$, there exists a function $u \in C_c^\infty(\boldsymbol{R}_+^n) \setminus \{0\}$ such that

$$\left(\int_{\boldsymbol{R}_+^n} |\nabla u|^2 dx\right)\left(\int_{\boldsymbol{R}_+^n} |u|^{2^*}/|x|^s dx\right)^{-2/2^*} \le \mu_s(\boldsymbol{R}_+^n) + \alpha.$$

Without loss of generality, we may assume that in a neighborhood of 0, $\partial\Omega$ can be represented by $x_n = \varphi(x')$, $x' = (x_1, \dots, x_{n-1})$, where $\varphi(0) = 0$ and $\nabla'\varphi(0) = 0$ with $\nabla' = (\partial_1, \dots, \partial_{n-1})$. Let $U \subset \boldsymbol{R}^n$ be a neighborhood of 0 such that $\Phi(U)$ is the open ball $B_{r_0}(0)$, where

$$\Phi(x) = (x', x_n - \varphi(x')) \quad \text{for } x \in \overline{\Omega} \cap \overline{U}.$$

We define

$$u_\varepsilon(x) := \varepsilon^{-(n-2)/2} u\,(\Phi(x)/\varepsilon) \quad \text{for } x \in \Omega \cap U,$$

and $B_{r_0}^+ := \{y \in \boldsymbol{R}_+^n\,;\, |y| < r_0\}$. Then with a change of the variable, we get

$$\int_\Omega |u_\varepsilon|^{2^*}/|x|^s dx = \int_{B_{r_0/\varepsilon}^+} |u|^{2^*}/|\Phi^{-1}(\varepsilon y)/\varepsilon|^s dy\,.$$

Note that $\left|\Phi^{-1}(\varepsilon y)/\varepsilon\right| \to |y|$ uniformly for $y \in \operatorname{supp} u$ as $\varepsilon \to 0$. Then letting $\varepsilon \to 0$ yields

$$\lim_{\varepsilon\to 0} \int_{B_{r_0/\varepsilon}^+} |u|^{2^*}/\left|\Phi^{-1}(\varepsilon y)/\varepsilon\right|^s dy = \int_{\boldsymbol{R}_+^n} |u|^{2^*}/|y|^s dy\,.$$

Next, by the direct calculation with a change of the variable, we have

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2 dx = &\int_{\boldsymbol{R}_+^n} |\nabla u|^2 dy \\ &+ \int_{\operatorname{supp} u} (\partial_n u(y))^2\, |(\nabla'\varphi)(\varepsilon y')|^2 dy \\ &- 2\int_{\operatorname{supp} u} \partial_n u(y)\, \nabla' u(y) \cdot (\nabla'\varphi)(\varepsilon y') dy\,.\end{aligned}$$

Since we have $\nabla'\varphi(0) = 0$, it holds

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 dx = \int_{\boldsymbol{R}_+^n} |\nabla u|^2 dy\,.$$

Furthermore, the subcritical case $p < 2n/(n-2)$ implies that

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^p dx = \lim_{\varepsilon\to 0} \varepsilon^{n-(n-2)p/2} \int_{\boldsymbol{R}_+^n} |u|^p dy = 0\,.$$

As a consequence, we have

$$\begin{aligned}\mu_{s,p}^\lambda(\Omega) &\le \left(\int_\Omega |\nabla u_\varepsilon|^2 dx + \lambda\left(\int_\Omega |u_\varepsilon|^p dx\right)^{2/p}\right)\left(\int_\Omega |u_\varepsilon|^{2^*}/|x|^s dx\right)^{-2/2^*} \\ &= \left(\int_{\boldsymbol{R}_+^n} |\nabla u|^2 dy\right)\left(\int_{\boldsymbol{R}_+^n} |u|^{2^*}/|y|^s dy\right)^{-2/2^*} + o(1) \\ &\le \mu_s(\boldsymbol{R}_+^n) + \alpha + o(1)\,.\end{aligned}$$

Letting $\varepsilon \to 0$ and $\alpha \to 0$ yield the desired estimate. □

To study a minimizing problem of $\mu^{\lambda}_{s,p}(\Omega)$, we start with the subcritical case in the following sense. For any small $\varepsilon > 0$, we set

$$\mu^{\lambda,\varepsilon}_{s,p}(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2 dx + \lambda\left(\int_\Omega |u|^p dx\right)^{2/p} ; u \in H^1_0(\Omega) \text{ and } \int_\Omega |u|^{2^*-\varepsilon}/|x|^s dx = 1\right\}.$$

It is easy to see that $\mu^{\lambda,\varepsilon}_{s,p}(\Omega)$ is attained by some positive function $u_\varepsilon \in H^1_0(\Omega)$, and u_ε satisfies

$$\begin{cases} \Delta u_\varepsilon - \lambda\, \|u_\varepsilon\|^{-(p-2)}_{L^p(\Omega)} u^{p-1}_\varepsilon + \mu^{\lambda,\varepsilon}_{s,p}(\Omega)\, u^{2^*-1-\varepsilon}_\varepsilon/|x|^s = 0 \quad \text{in } \Omega\,, \\ u_\varepsilon > 0 \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

We first consider the case $\lambda \le -\Lambda_p(\Omega)$ in the subcritical case $2 \le p < 2n/(n-2)$, where

$$\Lambda_p(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2 dx\,;\ u \in H^1_0(\Omega) \text{ and } \int_\Omega |u|^p dx = 1\right\}.$$

Note that $\mu^{\lambda}_{s,p}(\Omega) \le 0$ in this case.

THEOREM 2.2. *Suppose $\lambda \le -\Lambda_p(\Omega)$. Then $\mu^{\lambda}_{s,p}(\Omega)$ is attained.*

PROOF. If $\lambda = -\Lambda_p(\Omega)$, then obviously $\mu^{\lambda}_{s,p}(\Omega) = 0$ and is obtained. Hence, we may assume $\lambda < -\Lambda_p(\Omega)$. In this case, it follows $\mu^{\lambda,\varepsilon}_{s,p}(\Omega) < 0$.

Let u_ε be a minimizer for $\mu^{\lambda,\varepsilon}_{s,p}(\Omega)$. We first prove

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)} \le C\,, \quad \text{and} \quad |\mu^{\lambda,\varepsilon}_{s,p}(\Omega)| \le C\,. \tag{2.2}$$

Since u_ε is a minimizer for $\mu^{\lambda,\varepsilon}_{s,p}(\Omega)$, we have

$$\int_\Omega |\nabla u_\varepsilon|^2 dx + |\mu^{\lambda,\varepsilon}_{s,p}(\Omega)| = |\lambda|\left(\int_\Omega u^p_\varepsilon dx\right)^{2/p}, \quad \text{and} \quad \int_\Omega u^{2^*-\varepsilon}_\varepsilon/|x|^s dx = 1\,,$$

and $\int_\Omega u^2_\varepsilon dx$ can be bounded by $\int_\Omega u^{2^*-\varepsilon}_\varepsilon/|x|^s dx$ because of $2^* > 2$. Thus (2.2) holds in the case $p = 2$. If $p > 2$, from the interpolation inequality

$$\|u_\varepsilon\|_{L^p(\Omega)} \le C\|\nabla u_\varepsilon\|^{\theta}_{L^2(\Omega)}\|u_\varepsilon\|^{1-\theta}_{L^2(\Omega)} \le C\|\nabla u_\varepsilon\|^{\theta}_{L^2(\Omega)}$$

with some $0 < \theta < 1$, we obtain

$$\int_\Omega |\nabla u_\varepsilon|^2 dx + |\mu^{\lambda,\varepsilon}_{s,p}(\Omega)| = |\lambda|\left(\int_\Omega u^p_\varepsilon dx\right)^{2/p} \le C\left(\int_\Omega |\nabla u_\varepsilon|^2 dx\right)^{\theta},$$

which implies (2.2).

Next, we prove that u_ε is uniformly bounded. To see it, let w_ε be the solution of

$$\begin{cases} \Delta w_\varepsilon = \lambda\, \|u_\varepsilon\|^{-(p-2)}_{L^p(\Omega)} u^{p-1}_\varepsilon \quad \text{in } \Omega, \\ w_\varepsilon = 0 \quad \text{on } \partial\Omega\,. \end{cases}$$

Note that

$$\left(\int_\Omega u_\varepsilon^p dx\right)^{2/p} \geq \left|\mu_{s,p}^{\lambda,\varepsilon}(\Omega)\right|/|\lambda| \geq C > 0\,. \tag{2.3}$$

Thus by (2.2), (2.3) and the L^q-estimate, we have

$$\|w_\varepsilon\|_{W^{2,q}(\Omega)} \leq C|\lambda|\,\|u_\varepsilon\|_{L^p(\Omega)}^{-(p-2)}\left(\int_\Omega u_\varepsilon^{2n/(n-2)}dx\right)^{1/q} \leq C\,, \tag{2.4}$$

where $q := 2n/(p-1)(n-2)$. If $q > n/2$, by (2.4), we have $\|w_\varepsilon\|_{L^\infty(\Omega)} \leq C$. Thus the maximum principle shows that

$$0 < u_\varepsilon(x) \leq w_\varepsilon(x) \leq C \quad \text{for all } x \in \Omega\,. \tag{2.5}$$

In the case $q = n/2$, by (2.4), the Sobolev embedding and the maximum principle, we have

$$\|u_\varepsilon\|_{L^r(\Omega)} \leq \|w_\varepsilon\|_{L^r(\Omega)} \leq C$$

for all $1 < r < \infty$, and then the L^r-estimate shows that $\|w_\varepsilon\|_{W^{2,r}(\Omega)} \leq C$ for all $1 < r < \infty$, which implies (2.5). In the case $q < n/2$, the Sobolev embedding and the maximum principle yield that

$$\|u_\varepsilon\|_{L^{r_1}(\Omega)} \leq \|w_\varepsilon\|_{L^{r_1}(\Omega)} \leq C \quad \text{for } 1/r_1 = 1/q - 2/n\,,$$

and then the L^{q_1}-estimate shows that

$$\|w_\varepsilon\|_{W^{2,q_1}(\Omega)} \leq C\left(\int_\Omega u_\varepsilon^{r_1}dx\right)^{1/q_1} \leq C\,, \quad \text{where } q_1 = r_1/(p-1) > q\,.$$

If $q_1 > n/2$, we get (2.5). Otherwise, we continue the above procedure finite times so that we get $\|w_\varepsilon\|_{W^{2,q}(\Omega)} \leq C$ for some $q > n/2$, which is possible since $p < 2n/(n-2)$. Thus we have (2.5).

Hence, by letting $\varepsilon \to 0$, u_ε converges to some $u_0 \in H_0^1(\Omega)$ with $\int_\Omega u_0^{2^*}/|x|^s dx = 1$, and u_0 is a minimizer for $\mu_{s,p}^\lambda(\Omega)$. □

THEOREM 2.3. *Assume that* $2n/(n-1) < p < 2n/(n-2)$, $-\Lambda_p(\Omega) < \lambda < 0$ *and* $\partial\Omega$ *is* C^2 *at* 0, *then*

$$\mu_{s,p}^\lambda(\Omega) < \mu_s(\boldsymbol{R}_+^n)\,.$$

We prove the following theorems for the case the mean curvature $H(0)$ of $\partial\Omega$ at 0 is negative.

THEOREM 2.4. *Assume that* $\partial\Omega$ *is* C^3 *at* 0 *and the mean curvature* $H(0)$ *of* $\partial\Omega$ *at* 0 *is negative. If* $\lambda > 0$ *and* $2 \leq p < 2n/(n-1)$, *then* $\mu_{s,p}^\lambda(\Omega) < \mu_s(\boldsymbol{R}_+^n)$.

THEOREM 2.5. *Assume that* $\partial\Omega$ *is* C^3 *at* 0 *and the mean curvature* $H(0)$ *of* $\partial\Omega$ *at* 0 *is negative. Then the following hold*:

(i) *If* $\lambda \leq 0$ *and* $2 \leq p < 2n/(n-2)$, *then* $\mu_{s,p}^\lambda(\Omega) < \mu_s(\boldsymbol{R}_+^n)$.

(ii) *If* $-S_n < \lambda \leq 0$ *and* $p = 2n/(n-2)$, *then* $\mu_{s,p}^\lambda(\Omega) < \mu_{s,p}^\lambda(\boldsymbol{R}_+^n)$.

To prove Theorems 2.3 through 2.5, we need the following decay estimates of entire solutions:

LEMMA 2.6. *There exist entire positive solutions* $u \in H_0^1(\boldsymbol{R}_+^n)$ *of*

$$\begin{cases} \Delta u + \mu_s(\boldsymbol{R}_+^n)\, u^{2^*-1}/|x|^s = 0 & \text{in } \boldsymbol{R}_+^n\,, \\ u = 0 & \text{on } \partial \boldsymbol{R}_+^n\,, \end{cases} \tag{2.6}$$

and

$$\begin{cases} \Delta u - \lambda\, \|u\|_{L^{2n/(n-2)}(\boldsymbol{R}_+^n)}^{-4/(n-2)}\, u^{(n+2)/(n-2)} + \mu_{s,*}^{\lambda}(\boldsymbol{R}_+^n)\, u^{2^*-1}/|x|^s = 0 & \text{in } \boldsymbol{R}_+^n\,, \\ u = 0 & \text{on } \partial \boldsymbol{R}_+^n\,, \end{cases} \tag{2.7}$$

with $\int_{\boldsymbol{R}_+^n} u^{2^*}/|x|^s dx = 1$. *Furthermore, the following hold*:

(i)

$$\begin{cases} u \in C^2(\overline{\boldsymbol{R}_+^n}) & \text{if } s < (n+2)/n\,, \\ u \in C^{1,\beta}(\overline{\boldsymbol{R}_+^n}) & \text{for all } 0 < \beta < 1 \quad \text{if } s = (n+2)/n, \\ u \in C^{1,\beta}(\overline{\boldsymbol{R}_+^n}) & \text{for all } 0 < \beta < n(2-s)/(n-2) \quad \text{if } s > (n+2)/n\,. \end{cases}$$

(ii) *There is a constant* C *such that* $|u(x)| \le C(1+|x|)^{-(n-1)}$ *and* $|\nabla u(x)| \le C(1+|x|)^{-n}$.

(iii) $u(x', x_n)$ *is axially symmetric with respect to the* x_n*-axis, i.e.,* $u(x', x_n) = u(|x'|, x_n)$.

We first prove Theorems 2.3 through 2.5 with assuming Lemma 2.6.

PROOF OF THEOREMS 2.3 THROUGH 2.5. First, we treat the subcritical case $p < 2n/(n-2)$. Without loss of generality, we may assume that in a neighborhood of 0, $\partial\Omega$ can be represented by $x_n = \varphi(x')$, $x' = (x_1, \dots, x_{n-1})$, where $\varphi(0) = 0$, $\nabla'\varphi(0) = 0$, and the outer normal of $\partial\Omega$ at 0 is $-e_n$.

We take an entire solution u of (2.6) with $\int_{\boldsymbol{R}_+^n} u^{2^*}/|x|^s dx = 1$ given by Lemma 2.6. Let U and $\tilde{U}$ be neighborhoods of 0 such that $\Phi(U) = B_{r_0}(0)$ and $\Phi(\tilde{U}) = B_{r_0/2}(0)$, respectively. We define

$$v_\varepsilon(x) := \varepsilon^{-(n-2)/2} u\,(\Phi(x)/\varepsilon) \quad \text{for } x \in \Omega \cap U \quad \text{and} \quad \hat{v}_\varepsilon := \eta\, v_\varepsilon \quad \text{in } \Omega\,,$$

where $\eta \in C_c^\infty(U)$ is a positive cut-off function with $\eta \equiv 1$ in $\tilde{U}$, and

$$\Phi(x) = (x', x_n - \varphi(x')) \quad \text{for } x \in \overline{\Omega} \cap \overline{U}.$$

Then we have

$$\mu_{s,p}^{\lambda}(\Omega) \le \left(\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx + \lambda \left(\int_\Omega \hat{v}_\varepsilon^p dx \right)^{2/p} \right) \left(\int_\Omega \hat{v}_\varepsilon^{2^*}/|x|^s dx \right)^{-2/2^*}.$$

By the direct calculation with a change of the variable $\Phi(x)/\varepsilon = y$, we get

$$\begin{aligned} \int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx &= \int_{\Omega\cap U} \eta^2 |\nabla v_\varepsilon|^2 dx - \int_{\Omega \cap U} \eta(\Delta\eta) v_\varepsilon^2 dx \\ &= \int_{B_{r_0/\varepsilon}^+} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2\, |\nabla u(y)|^2\, dy \end{aligned}$$

$$
\begin{aligned}
&- 2 \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y)\, \nabla' u(y) \cdot (\nabla'\varphi)(\varepsilon y')dy \\
&+ \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 (\partial_n u(y))^2\, |(\nabla'\varphi)(\varepsilon y')|^2 dy \\
&- \varepsilon^2 \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)(\Delta\eta)\big(\Phi^{-1}(\varepsilon y)\big)u(y)^2 dy \\
=:\; & I_1 + I_2 + I_3 + I_4\,.
\end{aligned}
$$

We estimate each integral precisely. First, by the decay estimate of $|\nabla u|$ in Lemma 2.6, we have

$$
\begin{aligned}
I_1 &= \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2\, |\nabla u(y)|^2\, dy \\
&\le \int_{\{r_0/2 \le |\varepsilon y| < r_0\}_+} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2\, |\nabla u(y)|^2\, dy + \int_{\{|\varepsilon y| \ge r_0/2\}_+} |\nabla u(y)|^2\, dy + \int_{\boldsymbol{R}^n_+} |\nabla u(y)|^2\, dy \\
&\le C \int_{\{|\varepsilon y| \ge r_0/2\}} |y|^{-2n} dy + \int_{\boldsymbol{R}^n_+} |\nabla u(y)|^2\, dy \\
&= O(\varepsilon^n) + \mu_s(\boldsymbol{R}^n_+)\,,
\end{aligned}
$$

where $\{r_0/2 \le |\varepsilon y| < r_0\}_+ := \{y \in \boldsymbol{R}^n\,;\ r_0/2 \le |\varepsilon y| < r_0\} \cap \boldsymbol{R}^n_+$, and so on. Next, by using $|\nabla'\varphi(y')| = O(|y'|)$ and the decay estimate of $|\nabla u|$ in Lemma 2.6, we see

$$
\begin{aligned}
I_3 &= \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 (\partial_n u(y))^2\, |(\nabla'\varphi)(\varepsilon y')|^2 dy \\
&\le C\varepsilon^2 \int_{\boldsymbol{R}^n} (1 + |y|)^{-2n} |y|^2 dy = O(\varepsilon^2)\,.
\end{aligned}
$$

For I_4, since $(\Delta\eta)\big(\Phi^{-1}(\varepsilon y)\big) = 0$ for $y \in B^+_{r_0/(2\varepsilon)}$, by the decay estimate of u, we have

$$
\begin{aligned}
|I_4| &= \varepsilon^2 \int_{\{r_0/2 \le |\varepsilon y| < r_0\}_+} \eta\big(\Phi^{-1}(\varepsilon y)\big)\big|(\Delta\eta)\big(\Phi^{-1}(\varepsilon y)\big)\big| u(y)^2 dy \\
&\le C\varepsilon^2 \int_{\{r_0/2 \le |\varepsilon y| < r_0\}} |y|^{2(1-n)} dy = O(\varepsilon^n).
\end{aligned}
$$

The integral I_2 can be estimated as follows. Using integration by parts,

$$
\begin{aligned}
I_2 &= -(2/\varepsilon) \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y)\, \nabla' u(y) \cdot \nabla' \big[\varphi(\varepsilon y')\big]\, dy \\
&= (4/\varepsilon) \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big) \nabla'\big[\eta\big(\Phi^{-1}(\varepsilon y)\big)\big] \cdot \partial_n u(y)\, \nabla' u(y)\, \varphi(\varepsilon y')dy \\
&\quad + (2/\varepsilon) \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \nabla' \partial_n u(y) \cdot \nabla' u(y)\, \varphi(\varepsilon y')dy
\end{aligned}
$$

$$+(2/\varepsilon)\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y) \sum_{i=1}^{n-1} \partial_{ii} u(y)\, \varphi(\varepsilon y') dy$$
$$=: I_{2,1} + I_{2,2} + I_{2,3}\,.$$

Since

$$\nabla\big[\eta\big(\Phi^{-1}(\varepsilon y)\big)\big] = 0 \quad \text{for } y \in B^+_{r_0/(2\varepsilon)} \quad \text{and} \quad \big|\nabla\big[\eta\big(\Phi^{-1}(\varepsilon y)\big)\big]\big| = O(\varepsilon)\,,$$

we have

$$I_{2,1} = (4/\varepsilon)\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)\nabla'\big[\eta\big(\Phi^{-1}(\varepsilon y)\big)\big]\cdot \partial_n u(y)\, \nabla' u(y)\, \varphi(\varepsilon y') dy = O(\varepsilon^n)\,.$$

□

For $I_{2,2}$, using integration by parts and $\nabla' u \equiv 0$ on $\partial \boldsymbol{R}^n_+$, we have

$$\begin{aligned} I_{2,2} =&\ (2/\varepsilon)\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \nabla'\partial_n u(y)\cdot \nabla' u(y)\, \varphi(\varepsilon y') dy \\ =&\ \varepsilon^{-1}\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \partial_n\big[|\nabla' u(y)|^2\big]\varphi(\varepsilon y') dy \\ =&\ -\varepsilon^{-1}\int_{B^+_{r_0/\varepsilon}} \partial_n\big[\eta\big(\Phi^{-1}(\varepsilon y)\big)^2\big]|\nabla' u(y)|^2 \varphi(\varepsilon y') dy\,, \end{aligned}$$

and then

$$|I_{2,2}| \le C\varepsilon^2 \int_{\{r_0/2\le|\varepsilon y|<r_0\}_+} |\nabla u(y)|^2 |y|^2 dy \le C\varepsilon^2 \int_{\{r_0/2\le|\varepsilon y|<r_0\}} |y|^{-2n+2} dy = O(\varepsilon^n)\,.$$

For the integral $I_{2,3}$, by using the equation (2.6) and integration by parts, we have

$$\begin{aligned} I_{2,3} =&\ (2/\varepsilon)\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y)\, \varphi(\varepsilon y') \sum_{i=1}^{n-1} \partial_{ii} u(y)\, dy \\ =&\ -\big(2\mu_s(\boldsymbol{R}^n_+)/(2^*\varepsilon)\big)\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \varphi(\varepsilon y')\, (\partial_n\big[u(y)^{2^*}\big])/|y|^s dy \\ &-\varepsilon^{-1}\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \varphi(\varepsilon y')\, \partial_n\big[(\partial_n u(y))^2\big] dy \\ =&\ -\big(2s\mu_s(\boldsymbol{R}^n_+)/(2^*\varepsilon)\big)\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \varphi(\varepsilon y')\, u(y)^{2^*} y_n/|y|^{s+2} dy \\ &+\varepsilon^{-1}\int_{\{y\in\boldsymbol{R}^n\,;\,|\varepsilon y'|\le r_0,\ y_n=0\}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \varphi(\varepsilon y')\big(\partial_n u(y)\big)^2 dS_y + O(\varepsilon^n) \\ =&:\ J_1 + J_2 + O(\varepsilon^n)\,. \end{aligned}$$

If $\partial\Omega$ is C^3 at 0, φ can be expanded by

$$\varphi(y') = \sum_{j=1}^{n-1} \lambda_j y_j^2 + O(|y'|^3)\,, \tag{2.8}$$

where λ_j's are principal curvatures at $0 \in \partial\Omega$. Thus we see that

$$\begin{aligned} J_1 &= -\big(2s\mu_s(\boldsymbol{R}^n_+)/(2^*\varepsilon)\big) \int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \varphi(\varepsilon y')\, u(y)^{2^*} y_n/|y|^{s+2} dy \\ &= -\big(2s\mu_s(\boldsymbol{R}^n_+)/(2^*\varepsilon)\big) \int_{\{r_0/2 \le |\varepsilon y| < r_0\}_+} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \varphi(\varepsilon y')\, u(y)^{2^*} y_n/|y|^{s+2} dy \\ &\quad -\big(2s\mu_s(\boldsymbol{R}^n_+)/(2^*\varepsilon)\big) \int_{\{|\varepsilon y| < r_0/2\}_+} \varphi(\varepsilon y')\, u(y)^{2^*} y_n/|y|^{s+2} dy \\ &=: J_{1,1} + J_{1,2}\,, \end{aligned}$$

and

$$|J_{1,1}| \le C\varepsilon \int_{\{r_0/2 \le |\varepsilon y| < r_0\}} |y|^{-2^*(n-1)+1-s} dy = O(\varepsilon^{n(n-s)/(n-2)}).$$

Note that

$$\begin{cases} \varepsilon \int_{\{|\varepsilon y| \ge r_0/2\}_+} u(y)^{2^*} |y|^{1-s} dy = O(\varepsilon^{n(n-s)/(n-2)})\,, \\ \varepsilon^2 \int_{\{|\varepsilon y| < r_0/2\}_+} u(y)^{2^*} |y|^{2-s} dy = O(\varepsilon^2)\,, \end{cases} \tag{2.9}$$

where the latter integrand is integrable because $-2^*(n-1) + 2 - s + n$ is negative, i.e., $n^2 - (2+s)n + 4 > 0$. Thus by using (2.8) and (2.9), we get

$$\begin{aligned} J_{1,2} &= -\big(2s\varepsilon\mu_s(\boldsymbol{R}^n_+)/2^*\big) \sum_{i=1}^{n-1} \lambda_i \int_{\boldsymbol{R}^n_+} u(y)^{2^*} y_i^2 y_n/|y|^{s+2} dy + O(\varepsilon^2) \\ &= -\big(2s\varepsilon\mu_s(\boldsymbol{R}^n_+)/(2^*(n-1))\big) \int_{\boldsymbol{R}^n_+} u(y)^{2^*} |y'|^2 y_n/|y|^{s+2} dy \sum_{i=1}^{n-1} \lambda_i + O(\varepsilon^2) \\ &= -C_1 H(0)\varepsilon + O(\varepsilon^2)\,, \end{aligned}$$

where

$$\begin{aligned} H(0) &:= (n-1)^{-1} \sum_{i=1}^{n-1} \lambda_i \quad \text{and} \\ C_1 &:= \big(2s\mu_s(\boldsymbol{R}^n_+)/2^*\big) \int_{\boldsymbol{R}^n_+} u(y)^{2^*} |y'|^2 y_n/|y|^{s+2} dy\,. \end{aligned} \tag{2.10}$$

In the above estimate, we used the fact $u(y', y_n) = u(|y'|, y_n)$.

Next, we see that

$$\begin{aligned} J_2 &= \varepsilon^{-1} \int_{\{y \in \boldsymbol{R}^n\,;\, |\varepsilon y'| \le r_0,\, y_n = 0\}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \big(\partial_n u(y)\big)^2 \varphi(\varepsilon y') dS_y \\ &= \varepsilon^{-1} \int_{\{y \in \boldsymbol{R}^n\,;\, r_0/2 < |\varepsilon y'| \le r_0,\, y_n = 0\}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \big(\partial_n u(y)\big)^2 \varphi(\varepsilon y') dS_y \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^{-1}\int_{\{y\in \boldsymbol{R}^n\,;\,|\varepsilon y'|\le r_0/2,\,y_n=0\}}\big(\partial_n u(y)\big)^2\varphi(\varepsilon y')dS_y\\
&=:\,J_{2,1}+J_{2,2}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
|J_{2,1}| &\le (C/\varepsilon)\int_{\{y'\in \boldsymbol{R}^{n-1}\,;\,r_0/2<|\varepsilon y'|\le r_0\}}|(\partial_n u)(y',0)|^2|\varphi(\varepsilon y')|dy'\\
&\le C\varepsilon\int_{\{y'\in \boldsymbol{R}^{n-1}\,;\,r_0/2<|\varepsilon y'|\le r_0\}}|y'|^{-2n+2}dy' = O(\varepsilon^n)\,.
\end{aligned}
$$

For $J_{2,2}$, note that $|(\partial_n u)(y',0)|^2|y'|^3 = O(|y'|^{-2n+3})$ for large $|y'|$ and $2n-3>n-1$ for $n\ge 3$. Hence, it is integrable and

$$
(2.11)\qquad \begin{cases}\varepsilon\int_{\{y'\in \boldsymbol{R}^{n-1}\,;\,|\varepsilon y'|>r_0/2\}}|(\partial_n u)(y',0)|^2|y'|^2dy' = O(\varepsilon^n)\,,\\ \varepsilon^2\int_{\{y'\in \boldsymbol{R}^{n-1}\,;\,|\varepsilon y'|\le r_0/2\}}|(\partial_n u)(y',0)|^2|y'|^3dy' = O(\varepsilon^2)\,.\end{cases}
$$

Thus by using (2.8) and (2.11), we get

$$
\begin{aligned}
J_{2,2} &= \varepsilon\sum_{i=1}^{n-1}\lambda_i\int_{\boldsymbol{R}^{n-1}}\big((\partial_n u)(y',0)\big)^2 y_i^2dy' + O(\varepsilon^2)\\
&= \big(\varepsilon/(n-1)\big)\int_{\boldsymbol{R}^{n-1}}|(\nabla u)(y',0)|^2|y'|^2dy'\sum_{i=1}^{n-1}\lambda_i + O(\varepsilon^2)\\
&= C_2H(0)\varepsilon + O(\varepsilon^2),
\end{aligned}
$$

where

$$
(2.12)\qquad C_2 := \int_{\boldsymbol{R}^{n-1}}|(\nabla u)(y',0)|^2|y'|^2dy'\,.
$$

After all, we get

$$
(2.13)\qquad \int_\Omega|\nabla\hat{v}_\varepsilon|^2dx = \mu_s(\boldsymbol{R}^n_+) - C_1H(0)\varepsilon + C_2H(0)\varepsilon + O(\varepsilon^2)\,.
$$

Next, by changing the variable $\Phi(x)/\varepsilon = y$, we have

$$
(2.14)\qquad \left(\int_\Omega\hat{v}_\varepsilon^p dx\right)^{2/p} = \varepsilon^{2n/p-(n-2)}\left(\int_{\boldsymbol{R}^n_+}u^pdy\right)^{2/p} + O(\varepsilon^{np/2})\,.
$$

Finally, the integral $\int_\Omega\hat{v}_\varepsilon^{2^*}/|x|^sdx$ can be estimated as follows. By a change of the variable $\Phi(x)/\varepsilon = y$, we have

$$
(2.15)\qquad \int_\Omega\hat{v}_\varepsilon^{2^*}/|x|^sdx \ge \int_{\Omega\cap\tilde{U}}v_\varepsilon^{2^*}/|x|^sdx = \int_{B^+_{r_0/(2\varepsilon)}}u^{2^*}/\big|\Phi^{-1}(\varepsilon y)/\varepsilon\big|^sdy\,.
$$

Since $\Phi^{-1}(y) = (y', y_n + \varphi(y'))$, it holds that $|\Phi^{-1}(y)|^2 = |y|^2 + 2y_n\varphi(y') + \big(\varphi(y')\big)^2$, and then

$$\begin{aligned} \big|\Phi^{-1}(\varepsilon y)/\varepsilon\big|^{-s} &= |y|^{-s}\big(1 + 2y_n\varphi(\varepsilon y')/(\varepsilon|y|^2) + \big(\varphi(\varepsilon y')\big)^2/(\varepsilon^2|y|^2)\big)^{-s/2} \\ &= |y|^{-s}\big(1 - sy_n\varphi(\varepsilon y')/(\varepsilon|y|^2) - s\big(\varphi(\varepsilon y')\big)^2/(2\varepsilon^2|y|^2)\big) \\ &\quad + C|y|^{-s}\big(2y_n\varphi(\varepsilon y')/(\varepsilon|y|^2) + \big(\varphi(\varepsilon y')\big)^2/(\varepsilon^2|y|^2)\big)^2 . \end{aligned} \tag{2.16}$$

Thus from (2.15) and (2.16), we obtain

$$\begin{aligned} \int_\Omega \hat{v}_\varepsilon^{2^*}/|x|^s dx &\geq \int_{\boldsymbol{R}^n_+} u^{2^*}/|y|^s dy - (s/\varepsilon)\int_{B^+_{r_0/(2\varepsilon)}} u(y)^{2^*}\, y_n\varphi(\varepsilon y')/|y|^{2+s}dy + O(\varepsilon^2) \\ &= 1 - s\varepsilon\sum_{i=1}^{n-1}\lambda_i\int_{\boldsymbol{R}^n_+} u(y)^{2^*} y_i^2 y_n/|y|^{2+s}dy + O(\varepsilon^2) \\ &= 1 - \big(s\varepsilon/(n-1)\big)\int_{\boldsymbol{R}^n_+} u(y)^{2^*}|y'|^2 y_n/|y|^{2+s}dy\sum_{i=1}^{n-1}\lambda_i + O(\varepsilon^2) \\ &= 1 - \big(2^*C_1/(2\mu_s(\boldsymbol{R}^n_+))\big)H(0)\varepsilon + O(\varepsilon^2) , \end{aligned}$$

where C_1 is the same positive constant as in (2.10), and then

$$\left(\int_\Omega \hat{v}_\varepsilon^{2^*}/|x|^s dx\right)^{-2/2^*} \leq 1 + \big(C_1/\mu_s(\boldsymbol{R}^n_+)\big)H(0)\varepsilon + O(\varepsilon^2) . \tag{2.17}$$

Thus by (2.13), (2.14) and (2.17), we have

(2.18)

$$\begin{aligned} \mu^\lambda_{s,p}(\Omega) &\leq \left(\mu_s(\boldsymbol{R}^n_+) + \lambda\,\varepsilon^{2n/p-(n-2)}\left(\int_{\boldsymbol{R}^n_+} u^p dy\right)^{2/p} - C_1H(0)\varepsilon + C_2H(0)\varepsilon + O(\varepsilon^2)\right) \\ &\qquad \times\big(1 + \big(C_1/\mu_s(\boldsymbol{R}^n_+)\big)H(0)\varepsilon + O(\varepsilon^2)\big) \\ &= \mu_s(\boldsymbol{R}^n_+) + \lambda\,\varepsilon^{2n/p-(n-2)}\left(\int_{\boldsymbol{R}^n_+} u^p dy\right)^{2/p} \\ &\quad + \big(C_1\lambda H(0)/\mu_s(\boldsymbol{R}^n_+)\big)\left(\int_{\boldsymbol{R}^n_+} u^p dy\right)^{2/p}\varepsilon^{2n/p-(n-2)+1} + C_2H(0)\varepsilon + O(\varepsilon^2) . \end{aligned}$$

If $\lambda < 0$ and $2n/(n-1) < p < 2n/(n-2)$, i.e., $2n/p - (n-2) < 1$, we have

$$\mu^\lambda_{s,p}(\Omega) \leq \mu_s(\boldsymbol{R}^n_+) + \lambda\,\varepsilon^{2n/p-(n-2)}\left(\int_{\boldsymbol{R}^n_+} u^p dy\right)^{2/p} + O(\varepsilon) < \mu_s(\boldsymbol{R}^n_+) .$$

Thus Theorem 2.3 is proved.

Next, assume $H(0) < 0$, $\lambda > 0$ and $2 \leq p < 2n/(n-1)$, i.e., $2n/p - (n-2) > 1$. Then from (2.18), we obtain

$$\mu^\lambda_{s,p}(\Omega) \leq \mu_s(\boldsymbol{R}^n_+) + C_2H(0)\varepsilon + o(\varepsilon) < \mu_s(\boldsymbol{R}^n_+) .$$

Thus Theorem 2.4 is proved.

Furthermore, if $H(0) < 0$ and $\lambda \le 0$, for any $2 \le p < 2n/(n-2)$, then $2n/p - (n-2) + 1 > 1$. Hence, (2.18) yields that

$$\mu^{\lambda}_{s,p}(\Omega) \le \mu_s(\boldsymbol{R}^n_+) + C_2 H(0)\varepsilon + o(\varepsilon) < \mu_s(\boldsymbol{R}^n_+)\,.$$

Thus Theorem 2.5 (i) is proved.

If $p = 2n/(n-2)$, then u should be replaced by an entire solution of (2.7) given by Lemma 2.6. We can compute $\mu^{\lambda}_{s,*}(\Omega)$ in the same way as the subcritical case except for the estimate of the integral $I_{2,3}$. By using the equation (2.7), we have

$$\begin{aligned} I_{2,3} &= (2/\varepsilon)\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y)\,\varphi(\varepsilon y')\sum_{i=1}^{n-1}\partial_{ii}u(y)dy \\ &= \big(\lambda(n-2)/(\varepsilon\, n)\big)\|u\|^{-4/(n-2)}_{L^{2n/(n-2)}(\boldsymbol{R}^n_+)}\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2\varphi(\varepsilon y')\,\partial_n\big[u(y)^{2n/(n-2)}\big]dy \\ &\quad -\big(2\mu^{\lambda}_{s,n}(\boldsymbol{R}^n_+)/(\varepsilon\, 2^*)\big)\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2\varphi(\varepsilon y')\,(\partial_n\big[u(y)^{2^*}\big])/|y|^s dy \\ &\quad -\varepsilon^{-1}\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)^2\varphi(\varepsilon y')\,\partial_n\big[\big(\partial_n u(y)\big)^2\big]dy \\ &=: \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3\,, \end{aligned}$$

and

$$\begin{aligned} |\tilde{J}_1| &\le C\int_{B^+_{r_0/\varepsilon}} \eta\big(\Phi^{-1}(\varepsilon y)\big)|(\partial_n\eta)\big(\Phi^{-1}(\varepsilon y)\big)||\varphi(\varepsilon y')||u(y)|^{2n/(n-2)}dy \\ &\le C\varepsilon^2\int_{\{r_0/2\le|\varepsilon y|<r_0\}} |y|^{-2n(n-1)/(n-2)+2}dy = O(\varepsilon^{n^2/(n-2)})\,. \end{aligned}$$

Moreover, by using integration by parts along the e_n-direction, we get

$$\tilde{J}_2 + \tilde{J}_3 = -\tilde{C}_1 H(0)\varepsilon + C_2 H(0)\varepsilon + O(\varepsilon^2)\,,$$

where C_2 is the same positive constant as in (2.12) and

$$\tilde{C}_1 := (2s/2^*)\,\mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+)\int_{\boldsymbol{R}^n_+} u(y)^{2^*}|y'|^2 y_n/|y|^{2+s}dy\,.$$

Thus we have

$$\int_{\Omega} |\nabla\hat{v}_\varepsilon|^2 dx = \int_{\boldsymbol{R}^n_+} |\nabla u|^2 dy - \tilde{C}_1 H(0)\varepsilon + C_2 H(0)\varepsilon + O(\varepsilon^2)\,,$$

and

$$\Big(\int_{\Omega} \hat{v}_\varepsilon^{2n/(n-2)}dx\Big)^{(n-2)/n} = \Big(\int_{\boldsymbol{R}^n_+} u^{2n/(n-2)}dy\Big)^{(n-2)/n} + O(\varepsilon^{n^2/(n-2)})\,.$$

Since we have

$$\int_{\boldsymbol{R}^n_+} |\nabla u|^2 dy + \lambda\Big(\int_{\boldsymbol{R}^n_+} u^{2n/(n-2)}dy\Big)^{(n-2)/n} = \mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+),$$

(2.18) becomes

$$\mu^{\lambda}_{s,*}(\Omega) \le \big(\mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+) - \tilde{C}_1 H(0)\varepsilon + C_2 H(0)\varepsilon + o(\varepsilon)\big)\big(1 + \big(\tilde{C}_1/\mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+)\big)H(0)\varepsilon + o(\varepsilon)\big)$$
$$= \mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+) + C_2 H(0)\varepsilon + o(\varepsilon) < \mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+).$$

Thus Theorem 2.5 (ii) is also proved. □

If $H(0) = 0$, then we obtain the following weaker result:

THEOREM 2.7. *Assume that* $2 < p < 2n/(n-2)$, $\lambda < 0$, $\partial\Omega$ *is* C^3 *at* 0 *and the mean curvature* $H(0) = 0$, *then* $\mu^{\lambda}_{s,p}(\Omega) < \mu_s(\boldsymbol{R}^n_+)$.

PROOF. Since $H(0) = 0$, (2.18) becomes

$$\mu^{\lambda}_{s,p}(\Omega) \le \mu_s(\boldsymbol{R}^n_+) + \lambda\, \varepsilon^{2n/p-(n-2)}\left(\int_{\boldsymbol{R}^n_+} u^p dy\right)^{2/p} + O(\varepsilon^2) < \mu_s(\boldsymbol{R}^n_+)\,,$$

where $2n/p - (n-2) < 2$ if $p > 2$. Thus Theorem 2.7 is proved. □

Finally, we give a proof of Lemma 2.6.

PROOF OF LEMMA 2.6. First, for the existence of a positive solution of (2.6), see [1]. We shall show the existence of a positive solution of (2.7) in Section 4 (see Theorem 4.4 and Remark 4.5 (i)). Hence, we prove here the regularity of the positive solutions of (2.6) and (2.7). It is enough to consider the regularity theorem at $0 \in \partial\boldsymbol{R}^n_+$. By the Moser iteration method, u is locally bounded (see e.g., [2, Lemma 1.5]). Then we have $u \in C^{\alpha}(\overline{B^+_1})$ for $0 < \alpha < \min\{2-s, 1\}$, where $B^+_1 := B_1(0) \cap \boldsymbol{R}^n_+$. Set

$$\alpha_0 := \sup\left\{\alpha\,;\ \sup_{B^+_1}(|u(x)|/|x|^{\alpha}) < \infty,\ \ 0 < \alpha < 1\right\}.$$

Then for any $0 < \alpha < \alpha_0$, we have $|u(x)| \le C|x|^{\alpha}$ for $x \in \overline{B^+_1}$, and

$$|u(x)|^{2^*-1}/|x|^s \le C|x|^{(2^*-1)\alpha - s} \quad \text{for } x \in B^+_1\,. \tag{2.19}$$

We claim $\alpha_0 = 1$. Suppose $\alpha_0 < 1$. Then by (2.19), we have $(2^*-1)\alpha_0 - s < 0$. Otherwise, $u \in W^{2,q}(B^+_{1/2})$ for any $1 \le q < \infty$, and then $\alpha_0 = 1$. Therefore we get

$$|u|^{2^*-1}/|x|^s \in L^q(B^+_1) \quad \text{for all } 1 \le q < n/(s-(2^*-1)\alpha_0)\,.$$

By the Sobolev embedding, if $1/(s-(2^*-1)\alpha_0) \ge 1$, we have $u \in C^{\alpha}(\overline{B^+_{1/2}})$ for any $0 < \alpha < 1$, and then $\alpha_0 = 1$, which is a contradiction. Thus we may assume $1/(s-(2^*-1)\alpha_0) < 1$. Then $u \in C^{\alpha}(\overline{B^+_{1/2}})$ for all $0 < \alpha < 2-(s-(2^*-1)\alpha_0)$. The definition of α_0 yields $2-(s-(2^*-1)\alpha_0) \le \alpha_0$, but it implies $(2-s)+(2^*-2)\alpha_0 = \big(1+2\alpha_0/(n-2)\big)(2-s) \le 0$, which is impossible. Thus $\alpha_0 = 1$ is proved, i.e., for any $0 < \alpha < 1$,

$$|u(x)|^{2^*-1}/|x|^s \le C|x|^{(2^*-1)\alpha - s} \quad \text{for } x \in B^+_1\,.$$

Furthermore, if $2^*-1-s \ge 0$, i.e., $s \le (n+2)/n$, by taking α close to 1, we see that

$$|u|^{2^*-1}/|x|^s \in L^q(B^+_1) \quad \text{for all } 1 < q < \infty\,,$$

and then $u \in C^{1,\beta}(\overline{B^+_{1/2}})$ for all $0 < \beta < 1$. In particular, in the case $s < (n+2)/n$, there exists $q_0 > n$ such that

$$\|u\|_{W^{3,q_0}(B^+_{1/2})} \le C\big(1 + \|u^{4/(n-2)}\nabla u\|_{L^{q_0}(B^+_1)} + \big\|u^{2^*-2}\nabla u/|x|^s\big\|_{L^{q_0}(B^+_1)} + \big\|u^{2^*-1}/|x|^{s+1}\big\|_{L^{q_0}(B^+_1)}\big) < \infty .$$

Thus we get $u \in C^2(\overline{B^+_{1/2}})$. If $s > (n+2)/n$, by taking α close to 1, we have $u \in C^{1,\beta}(\overline{B^+_{1/2}})$ for all $0 < \beta < n(2-s)/(n-2)$.

To prove (ii), we apply the Kelvin transformation:

$$u^*(y) := |y|^{-(n-2)}u\big(y/|y|^2\big), \quad y \in \boldsymbol{R}^n_+ .$$

Thus u^* is in $H^1_0(\boldsymbol{R}^n_+)$ and satisfies the same equation as u. By (i), $u^* \in C^{1,\beta}(\overline{\boldsymbol{R}^n_+})$ for some $\beta > 0$. Then $|u^*(y)| \le C|y|$ for $y \in B^+_1$. By going back to u, we then have $|u(y)| \le C|y|^{-(n-1)}$ for $y \in \boldsymbol{R}^n_+$. By the gradient estimate, we have $|\nabla u(y)| \le C|y|^{-n}$ for $y \in \boldsymbol{R}^n_+$. Thus (ii) is proved.

The part (iii) can be proved by the well-known method of moving planes. Since it is a standard method, we skip the argument here. Hence, Lemma 2.6 is proved. □

3. Proof of main theorems. This section will be devoted to prove Theorem 1.1 through Theorem 1.4 excepting Theorem 1.2 (ii), which we will prove in the last section. If $\lambda \le -\Lambda_p(\Omega)$, then Theorem 2.2 implies the existence of minimizers. Thus we assume $-\Lambda_p(\Omega) < \lambda$. Let u_ε be a minimizer of $\mu^{\lambda,\varepsilon}_{s,p}(\Omega)$. It is easy to see that $\lim_{\varepsilon\to 0}\mu^{\lambda,\varepsilon}_{s,p}(\Omega) = \mu^{\lambda}_{s,p}(\Omega)$, and

$$\int_\Omega |\nabla u_\varepsilon|^2 dx + \lambda\left(\int_\Omega u_\varepsilon^p dx\right)^{2/p} = \mu^{\lambda,\varepsilon}_{s,p}(\Omega) \quad \text{and} \quad \int_\Omega u_\varepsilon^{2^*-\varepsilon}/|x|^s dx = 1 .$$

Then the Sobolev embedding yields that

$$(\Lambda_p(\Omega)+\lambda)\left(\int_\Omega u_\varepsilon^p dx\right)^{2/p} \le \int_\Omega |\nabla u_\varepsilon|^2 dx + \lambda\left(\int_\Omega u_\varepsilon^p dx\right)^{2/p} = \mu^{\lambda,\varepsilon}_{s,p}(\Omega) \le C ,$$

which implies

$$\int_\Omega u_\varepsilon^p dx \le C \quad \text{and} \quad \int_\Omega |\nabla u_\varepsilon|^2 dx \le C$$

because of the inequality $\Lambda_p(\Omega) + \lambda > 0$. Thus there exist $\{\varepsilon_j\}_{j\in\boldsymbol{N}} \subset (0, 2^*-2)$ with $\varepsilon_j \to 0$ as $j \to \infty$ and $u_0 \in H^1_0(\Omega)$ such that

$$\text{(3.1)} \qquad \begin{cases} u_j \rightharpoonup u_0 \quad \text{weakly in } H^1_0(\Omega) , \\ u_j/|x|^{s/2^*} \rightharpoonup u_0/|x|^{s/2^*} \quad \text{weakly in } L^{2^*}(\Omega) , \\ u_j \to u_0 \quad \text{strongly in } L^p(\Omega) , \\ u_j \to u_0 \quad \text{a.e. in } \Omega \end{cases}$$

as $j \to \infty$, where we put $u_j := u_{\varepsilon_j}$.

LEMMA 3.1. *If $u_0 \ne 0$, then u_0 is a minimizer for $\mu^{\lambda}_{s,p}(\Omega)$.*

PROOF. Since u_j is the minimizer for $\mu_{s,p}^{\lambda,\varepsilon_j}(\Omega)$, with u_0 as a test function, we have

$$\int_\Omega \nabla u_j \cdot \nabla u_0 dx + \lambda \|u_j\|_{L^p(\Omega)}^{-(p-2)} \int_\Omega u_j^{p-1} u_0 dx = \mu_{s,p}^{\lambda,\varepsilon_j}(\Omega) \int_\Omega u_j^{2^*-1-\varepsilon_j} u_0/|x|^s dx . \tag{3.2}$$

By using (3.1), we have

$$\int_\Omega \nabla u_j \cdot \nabla u_0 dx \to \int_\Omega |\nabla u_0|^2 dx \quad \text{and} \quad \int_\Omega u_j^{2^*-1} u_0/|x|^s dx \to \int_\Omega u_0^{2^*}/|x|^s dx .$$

Thus letting $j \to \infty$ in (3.2) yields that

$$\int_\Omega |\nabla u_0|^2 dx + \lambda \Big(\int_\Omega u_0^p dx \Big)^{2/p} = \mu_{s,p}^\lambda(\Omega) \int_\Omega u_0^{2^*}/|x|^s dx ,$$

and then we have

$$\begin{aligned} \mu_{s,p}^\lambda(\Omega) &\le \Big(\int_\Omega |\nabla u_0|^2 dx + \lambda \Big(\int_\Omega u_0^p\, dx \Big)^{2/p} \Big) \Big(\int_\Omega u_0^{2^*}/|x|^s dx \Big)^{-2/2^*} \\ &= \mu_{s,p}^\lambda(\Omega) \Big(\int_\Omega u_0^{2^*}/|x|^s dx \Big)^{(2^*-2)/2^*} , \end{aligned}$$

which implies

$$1 \le \int_\Omega u_0^{2^*}/|x|^s dx .$$

But since $u_j/|x|^{s/2^*} \rightharpoonup u_0/|x|^{s/2^*}$ weakly in $L^{2^*}(\Omega)$, we have

$$1 = \lim_{j\to\infty} \int_\Omega u_j^{2^*-\varepsilon_j}/|x|^s dx \ge \int_\Omega u_0^{2^*}/|x|^s dx .$$

Thus $\int_\Omega u_0^{2^*}/|x|^s dx = 1$ and u_0 is a minimizer for $\mu_{s,p}^\lambda(\Omega)$. □

THEOREM 3.2. *If $u_0 = 0$, then $\mu_{s,p}^\lambda(\Omega) = \mu_s(\boldsymbol{R}_+^n)$.*

PROOF. Let $x_j \in \Omega$ be a maximum point of u_j, that is, $0 < \max_\Omega u_j = u_j(x_j)$ holds. Then we see that up to a subsequence, $u_j(x_j) \to \infty$ as $j \to \infty$. Indeed, assume $\{u_j(x_j)\}_{j\in \boldsymbol{N}}$ is bounded. Then by the Lebesgue dominated convergence theorem, we have

$$\int_\Omega u_0^{2^*}/|x|^s dx = \lim_{j\to\infty} \int_\Omega u_j^{2^*-\varepsilon_j}/|x|^s dx = 1 ,$$

which implies $u_0 \neq 0$, and a contradiction occurs.

Next, we set $\kappa_j := u_j(x_j)^{-(p_j-2)/2}$, where $p_j := 2n/(n-2) - 2\varepsilon_j/(2-s)$. Then we shall show the following lemma:

LEMMA 3.3. *$|x_j| = O(\kappa_j)$ as $j \to \infty$.*

PROOF. Suppose that up to taking a subsequence,

$$\lim_{j\to\infty} |x_j|/\kappa_j = \infty .$$

By scaling, we set

$$v_j(y) := u_j(x_j + \beta_j y)/u_j(x_j) \quad \text{in } \Omega_j ,$$

where $\Omega_j := \{y \in \boldsymbol{R}^n\,;\ x_j + \beta_j y \in \Omega\}$ and

$$\beta_j := |x_j|^{s/2}\kappa_j^{(2-s)/2} = \big(|x_j|/\kappa_j\big)^{s/2}\kappa_j\,.$$

By (2.1), v_j satisfies

$$\Delta v_j - \lambda\beta_j^\delta \|v_j\|_{L^p(\Omega_j)}^{-(p-2)} v_j^{p-1} + \mu_{s,p}^{\lambda,\varepsilon_j}(\Omega)\beta_j^2 u_j(x_j)^{2^*-2-\varepsilon_j} v_j^{2^*-1-\varepsilon_j}/\big|x_j+\beta_j y\big|^s = 0 \quad \text{in}\ \ \Omega_j\,,$$

where $\delta := 2n/p - (n-2) > 0$. Note that

$$u_j(x_j)^{2^*-2-\varepsilon_j}\beta_j^2 |x_j|^{-s} = 1\,,$$

and since $s < 2$ and $\lim_{j\to\infty} |x_j|/\kappa_j = \infty$, we have $\beta_j/|x_j| = \big(|x_j|/\kappa_j\big)^{-(2-s)/2} \to 0$ as $j \to \infty$.

To prove the convergence of v_j, we have to obtain a lower bound of $\|v_j\|_{L^p(\Omega_j)}$. To see it, by applying the interior estimate and $v_j \le 1$ in Ω_j, we have

$$\|v_j\|_{W^{2,p/(p-2)}(B_{1/2}(0)\cap\Omega_j)} \le C\big(1 + \beta_j^\delta \|v_j\|_{L^p(\Omega_j)}^{-(p-2)} \|v_j^{p-1}\|_{L^{p/(p-2)}(B_1(0)\cap\Omega_j)}\big).$$

Since $p/(p-2) > n/2$ and

$$\|v_j\|_{L^p(\Omega_j)}^{-(p-2)} \|v_j^{p-1}\|_{L^{p/(p-2)}(B_1(0)\cap\Omega_j)} \le 1,$$

the Sobolev embedding implies $v_j \in C^\alpha(\overline{B_{1/2}(0)\cap\Omega_j})$ for some $\alpha > 0$. In particular,

$$v_j(x) \ge 1/2 \quad \text{for}\ \ x \in B_{r_0}(0)$$

with some small $r_0 > 0$ because $v_j(0) = 1$. Thus we have

$$\|v_j\|_{L^p(\Omega_j)} \ge C > 0, \quad \text{and then} \quad \beta_j^\delta \|v_j\|_{L^p(\Omega_j)}^{-(p-2)} \le C\beta_j^\delta \to 0 \quad \text{as}\ j \to \infty\,.$$

Therefore, v_j converges to some function v uniformly on any compact subset of $\boldsymbol{R}^n$. Moreover, v satisfies $v(0) = 1$ and

$$\Delta v + \mu_{s,p}^\lambda(\Omega)\, v^{2^*-1} = 0 \quad \text{in}\ \ \boldsymbol{R}^n \tag{3.3}$$

provided that Ω_j tends to $\boldsymbol{R}^n$, or

$$\begin{cases} \Delta v + \mu_{s,p}^\lambda(\Omega)\, v^{2^*-1} = 0 \quad \text{in}\ H\,, \\ v = 0 \quad \text{on}\ \partial H \end{cases} \tag{3.4}$$

provided that after a linear transformation, Ω_j tends to $H := \{y \in \boldsymbol{R}^n\,;\ y_n > -a\}$ for some $a > 0$. By a well-known result, both equations (3.3) and (3.4) have no positive solution. Thus we get a contradiction and Lemma 3.3 is proved. □

We go back to the proof of Theorem 3.2. As in the proof in Lemma 3.3, we set

$$v_j(y) := u_j(x_j + \kappa_j y)/u_j(x_j) \quad \text{in}\ \Omega_j\,,$$

where $\Omega_j := \{y \in \boldsymbol{R}^n\,;\ x_j + \kappa_j y \in \Omega\}$. Then v_j satisfies

$$\begin{cases} \Delta v_j - \lambda\,\kappa_j^\delta \|v_j\|_{L^p(\Omega_j)}^{-(p-2)} v_j^{p-1} + \mu_{s,p}^{\lambda,\varepsilon_j}(\Omega) v_j^{2^*-1-\varepsilon_j}/\big|x_j/\kappa_j + y\big|^s = 0 \quad \text{in}\ \ \Omega_j\,, \\ v_j(0) = 1 \quad \text{and}\ v_j = 0 \quad \text{on}\ \ \partial\Omega_j\,. \end{cases} \tag{3.5}$$

First, we assume up to taking a subsequence, $x_j/\kappa_j \to 0$ as $j \to \infty$. As in the proof in Lemma 3.3, we have to find a lower bound of $\|v_j\|_{L^p(\Omega_j)}^{p-2}$. Thus we may assume $p > 2$.

To obtain a lower bound of $\|v_j\|_{L^p(\Omega_j)}$, we write $v_j = \tilde{v}_j + w_j$, where

$$\begin{cases} \Delta w_j + \mu_{s,p}^{\lambda,\varepsilon_j}(\Omega) v_j^{2^*-1-\varepsilon_j} / \left|x_j/\kappa_j + y\right|^s = 0 & \text{in } B_1(0) \cap \Omega_j \,, \\ w_j = 0 \quad \text{on } \partial(B_1(0) \cap \Omega_j) \,, \end{cases}$$

and

$$\begin{cases} \Delta \tilde{v}_j - \lambda \kappa_j^\delta \|v_j\|_{L^p(\Omega_j)}^{-(p-2)} v_j^{p-1} = 0 & \text{in } B_1(0) \cap \Omega_j \,, \\ \tilde{v}_j = v_j \quad \text{on } \partial(B_1(0) \cap \Omega_j) \,. \end{cases}$$

By the interior estimate and $v_j \le 1$, we have for any $q < n/s$,

$$\|w_j\|_{W^{2,q}(B_{1/2}(0) \cap \Omega_j)} \le C \,.$$

By noting

$$\|v_j\|_{L^p(\Omega_j)}^{-(p-2)} \|v_j^{p-1}\|_{L^{p/(p-2)}(B_1(0) \cap \Omega_j)} \le 1 \,,$$

and again by the interior estimate, we have

$$\|\tilde{v}_j\|_{W^{2,p/(p-2)}(B_{1/2}(0) \cap \Omega_j))} \le C \,.$$

Since $2 < p < 2n/(n-2)$, i.e., $p/(p-2) > n/2$, there exists $q > n/2$ such that

$$\|v_j\|_{W^{2,q}(B_{1/2}(0) \cap \Omega_j))} \le C$$

for some constant C independent of j. Then the Sobolev embedding implies

$$\|v_j\|_{C^\alpha(\overline{B_{1/2}(0) \cap \Omega_j})} \le C$$

for some $\alpha > 0$. Thus $v_j(x) \ge 1/2$ for $x \in B_{r_0}(0) \cap \Omega$ with some small $r_0 > 0$ independent of j. Hence, $\|v_j\|_{L^p(\Omega_j)} \ge C > 0$ for some constant $C > 0$. Since $v_j \le 1$, it is standard to show that $v_j \to v$ uniformly in any compact set in $\overline{\boldsymbol{R}^n_+}$, and v satisfies

$$\begin{cases} \Delta v + \mu_{s,p}^{\lambda}(\Omega)\, v^{2^*-1}/|y|^s = 0 & \text{in } \boldsymbol{R}^n_+ \,, \\ v = 0 \quad \text{on } \partial \boldsymbol{R}^n_+ \,. \end{cases}$$

But $v(0) = 1$, which yields a contradiction to $v = 0$ on $\partial \boldsymbol{R}^n_+$, and we get $\liminf_{j\to\infty} |x_j|/\kappa_j > 0$.

Let $y_j := -x_j/\kappa_j$ and $\lim_{j\to\infty} y_j = -y_0$. After a linear transformation, we may assume $\Omega_j \to \boldsymbol{R}^n_+$ and $v_j \to v$ uniformly in any compact set in $\overline{\boldsymbol{R}^n_+}$. Then by (3.5), v satisfies

$$\begin{cases} \Delta v + \mu_{s,p}^{\lambda}(\Omega)\, v^{2^*-1}/|y|^s = 0 & \text{in } \boldsymbol{R}^n_+ \,, \\ v = 0 \quad \text{on } \partial \boldsymbol{R}^n_+ \,, \quad v(y_0) = \max_{\boldsymbol{R}^n_+} v = 1 \,, \end{cases}$$

and

$$\begin{aligned} \int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy &\le \lim_{j\to\infty} \int_{\Omega_j} v_j^{2^*-\varepsilon_j} / \left|y - y_j\right|^s dy \\ &= \lim_{j\to\infty} \kappa_j^{(n-2)\varepsilon_j/(2^*-2-\varepsilon_j)} \int_\Omega u_j^{2^*-\varepsilon_j}/|x|^s dx \le 1 \,. \end{aligned} \tag{3.6}$$

Thus we have

$$\int_{\boldsymbol{R}^n_+} |\nabla v|^2 dy = \mu^{\lambda}_{s,p}(\Omega) \int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy \,. \tag{3.7}$$

Then from (3.6), (3.7) and Lemma 2.1, we obtain

$$\begin{aligned}\mu_s(\boldsymbol{R}^n_+) &\le \left(\int_{\boldsymbol{R}^n_+} |\nabla v|^2 dy \right) \left(\int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy \right)^{-2/2^*} \\ &= \mu^{\lambda}_{s,p}(\Omega) \left(\int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy \right)^{(2^*-2)/2^*} \le \mu_s(\boldsymbol{R}^n_+).\end{aligned}$$

As a consequence, it follows that $\mu^{\lambda}_{s,p}(\Omega) = \mu_s(\boldsymbol{R}^n_+)$ and $\int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy = 1$. Thus Theorem 3.2 is proved. □

In view of Theorem 3.2, Theorems 1.1, 1.3, 1.4 and Theorem 1.2 (i) follow immediately from Theorems 2.4, 2.7, 2.3 and Theorem 2.5 (i), respectively.

4. The critical case $p = 2n/(n-2)$**.** In this section, we want to discuss a minimizing problem of $\mu^{\lambda}_{s,*}(\Omega)$:

$$\mu^{\lambda}_{s,*}(\Omega) := \mu^{\lambda}_{s,p}(\Omega) \quad \text{with} \;\; p = 2n/(n-2)\,.$$

First, we have the following result:

LEMMA 4.1. *If* $\lambda < -S_n$, *then* $\mu^{\lambda}_{s,*}(\Omega) = -\infty$.

PROOF. Let u be the positive solution of

$$\begin{cases} \Delta u + S_n u^{(n+2)/(n-2)} = 0 & \text{in } \boldsymbol{R}^n\,, \\ u(x) = u(|x|) \quad \text{and} & \int_{\boldsymbol{R}^n} u^{2n/(n-2)} dx = 1\,, \end{cases}$$

$x_0 \in \Omega$ and $0 \le \eta \in C^{\infty}_c(B_{r_0}(x_0))$, where $\eta \equiv 1$ in $B_{r_0/2}(x_0)$ and $r_0 < \operatorname{dist}(x_0, \partial\Omega)$. Set $\varphi_\varepsilon(x) := \eta(x)\, u\big((x-x_0)/\varepsilon\big)$ for $x \in \Omega$. Then we have

$$\mu^{\lambda}_{s,*}(\Omega) \le \left(\int_\Omega |\nabla \varphi_\varepsilon|^2 dx + \lambda \left(\int_\Omega \varphi_\varepsilon^{2n/(n-2)} dx \right)^{(n-2)/n} \right) \left(\int_\Omega \varphi_\varepsilon^{2^*}/|x|^s dx \right)^{-2/2^*} .$$

Thus by using the decay property of u, we see that

$$\int_\Omega |\nabla \varphi_\varepsilon|^2 dx \le \varepsilon^{n-2} S_n + O(\varepsilon^{2(n-2)}) \quad \text{and} \quad \left(\int_\Omega \varphi_\varepsilon^{2n/(n-2)} dx \right)^{(n-2)/n} \ge \varepsilon^{n-2} + O(\varepsilon^{2n})\,.$$

Moreover, we get

$$\int_\Omega \varphi_\varepsilon^{2^*}/|x|^s dx = O(\varepsilon^n) \quad \text{if } \; n \ge 4\,,$$

and in the case $n = 3$, we have

$$\int_\Omega \varphi_\varepsilon^{2^*}/|x|^s dx = \begin{cases} O(\varepsilon^3) & \text{if } \; 0 < s < 3/2\,, \\ O(\varepsilon^3 |\log \varepsilon|) & \text{if } \; s = 3/2\,, \\ O(\varepsilon^{6-2s}) & \text{if } \; 3/2 < s < 2\,. \end{cases}$$

As a consequence, we get

$$\mu^{\lambda}_{s,*}(\Omega) \le \lim_{\varepsilon\to 0} \big((S_n+\lambda)+O(\varepsilon^{n-2})\big)/(C_1\varepsilon^{\alpha}) = -\infty$$

with some positive constants C_1 and α. □

LEMMA 4.2. *For $\lambda > -S_n$, the inequality $\mu^{\lambda}_{s,*}(\Omega) \le \mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+)$ holds.*

The proof of Lemma 4.2 is similar to that of Lemma 2.1. So, the detailed argument is omitted here. To study $\mu^{\lambda}_{s,*}(\Omega)$, we consider $\mu^{\lambda,\varepsilon}_{s,*}(\Omega)$, where

$$\mu^{\lambda,\varepsilon}_{s,*}(\Omega) := \mu^{\lambda,\varepsilon}_{s,2n/(n-2)-2\varepsilon/(2-s)}(\Omega) \quad \text{for } 0<\varepsilon\ll 1\,.$$

Then we have $\lim_{\varepsilon\to 0}\mu^{\lambda,\varepsilon}_{s,*}(\Omega) = \mu^{\lambda}_{s,*}(\Omega)$ if $-S_n < \lambda \le 0$. Let u_ε be a minimizer for $\mu^{\lambda,\varepsilon}_{s,*}(\Omega)$ with $\int_\Omega u_\varepsilon^{2^*-\varepsilon}/|x|^s dx = 1$. It is easy to see that $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$ is bounded because $\lambda > -S_n$. Thus by passing to a subsequence $\varepsilon_j \to 0$, there exists $u_0 \in H^1_0(\Omega)$ such that

$$\text{(4.1)} \qquad \begin{cases} u_j \rightharpoonup u_0 & \text{weakly in } H^1_0(\Omega)\cap L^{2n/(n-2)}(\Omega)\,, \\ u_j/|x|^{s/2^*} \rightharpoonup u_0/|x|^{s/2^*} & \text{weakly in } L^{2^*}(\Omega) \end{cases}$$

as $j\to\infty$, where we put $u_j := u_{\varepsilon_j}$.

Next, we prove the following lemma in the same way as before except small modification.

LEMMA 4.3. *$\mu^{\lambda}_{s,*}(\Omega)$ is attained if $u_0 \ne 0$.*

PROOF. Recall that u_j satisfies

$$\Delta u_j - \lambda\,\|u_j\|^{-(p_j-2)}_{L^{p_j}(\Omega)}\, u_j^{p_j-1} + \mu^{\lambda,\varepsilon_j}_{s,*}(\Omega)\, u_j^{2^*-1-\varepsilon_j}/|x|^s = 0 \quad \text{in } \Omega\,,$$

where $p_j := 2n/(n-2) - 2\varepsilon_j/(2-s)$. By taking u_0 as a test function, we have

$$\int_\Omega \nabla u_j\cdot\nabla u_0\,dx + \lambda\,\|u_j\|^{-(p_j-2)}_{L^{p_j}(\Omega)}\int_\Omega u_j^{p_j-1}u_0\,dx = \mu^{\lambda,\varepsilon_j}_{s,*}(\Omega)\int_\Omega u_j^{2^*-1-\varepsilon_j}u_0/|x|^s dx\,.$$

By using (4.1) and letting $j\to\infty$, we get

$$\int_\Omega |\nabla u_0|^2 dx + \lambda\, C_0^{-(2n/(n-2)-2)}\int_\Omega u_0^{2n/(n-2)}dx = \mu^{\lambda}_{s,*}(\Omega)\int_\Omega u_0^{2^*}/|x|^s dx\,,$$

where

$$\text{(4.2)} \qquad C_0 := \lim_{j\to\infty}\|u_j\|_{L^{p_j}(\Omega)} \ge \|u_0\|_{L^{2n/(n-2)}(\Omega)}\,.$$

Thus by using (4.2) and $\lambda\le 0$, we see that

$$\int_\Omega |\nabla u_0|^2 dx + \lambda\left(\int_\Omega u_0^{2n/(n-2)}dx\right)^{(n-2)/n} \le \mu^{\lambda}_{s,*}(\Omega)\int_\Omega u_0^{2^*}/|x|^s dx\,,$$

and then

$$\mu^{\lambda}_{s,*}(\Omega) \le \left(\int_\Omega |\nabla u_0|^2 dx + \lambda\left(\int_\Omega u_0^{2n/(n-2)}dx\right)^{(n-2)/n}\right)\left(\int_\Omega u_0^{2^*}/|x|^s dx\right)^{-2/2^*}$$

$$\leq \mu^{\lambda}_{s,*}(\Omega)\Big(\int_{\Omega} u_0^{2^*}/|x|^s dx\Big)^{(2^*-2)/2^*},$$

which implies $1 \leq \int_{\Omega} u_0^{2^*}/|x|^s dx$. On the other hand, by (4.1), we have

$$1 = \lim_{j\to\infty}\int_{\Omega} u_j^{2^*-\varepsilon_j}/|x|^s dx \geq \int_{\Omega} u_0^{2^*}/|x|^s dx\,.$$

Hence, we get $\int_{\Omega} u_0^{2^*}/|x|^s dx = 1$, which implies u_0 is a minimizer for $\mu^{\lambda}_{s,*}(\Omega)$. □

If $u_0 = 0$, then we have the following theorem:

THEOREM 4.4. *Assume* $-S_n < \lambda \leq 0$. *Then* $\mu^{\lambda}_{s,*}(\Omega) = \mu^{\lambda}_{s,*}(\boldsymbol{R}^n_+)$ *and the equation* (2.7) *admits an entire positive solution.*

REMARK 4.5. (i) By the Pohozaev identity, it can be shown that the equation

$$\begin{cases} \Delta u - \lambda\,\|u\|^{-4/(n-2)}_{L^{2n/(n-2)}(\Omega)} u^{(n+2)/(n-2)} + \mu^{\lambda}_{s,*}(\Omega)\,u^{2^*-1}/|x|^s = 0 \quad \text{in } \Omega\,, \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \quad \text{with} \quad \int_{\Omega} u^{2^*}/|x|^s dx = 1 \end{cases}$$

admits no solutions provided that Ω is a star-shaped domain with respect to 0. Thus for such a domain Ω, u_0 must be 0, that is, u_ε must blow up as $\varepsilon \to 0$. Hence, Theorem 4.4 yields the existence of a positive solution of the equation (2.7). This entire solution was used to prove Theorem 2.5 (ii) in Section 2.

(ii) Obviously, by Theorem 2.5 (ii), Lemma 4.3 and Theorem 4.4 again, we can prove Theorem 1.2 (ii).

PROOF OF THEOREM 4.4. We first show the lower bound of $\|u_j\|_{L^{p_j}(\Omega)}$. We use the following interpolation inequality

$$\Big(\int_{\Omega} u_j^{2^*-\varepsilon_j}/|x|^s dx\Big)^{1/(2^*-\varepsilon_j)} \leq C\|\nabla u_j\|^{a_j}_{L^2(\Omega)}\|u_j\|^{1-a_j}_{L^{p_j}(\Omega)} \tag{4.3}$$

for some $a_j \in (0,1)$ such that $a_j \to s(n-2)/(2(n-s)) \in (0,1)$ as $j \to \infty$. The inequality (4.3) can be obtained as follows. First, by the interpolation inequality, we have

$$\Big(\int_{\Omega} u_j^{2^*-\varepsilon_j}/|x|^s dx\Big)^{1/(2^*-\varepsilon_j)} \leq C\Big(\int_{\Omega} u_j^2/|x|^2 dx\Big)^{a_j/2}\Big(\int_{\Omega} u_j^{p_j} dx\Big)^{(1-a_j)/p_j},$$

and then by the Hardy inequality, we have

$$\int_{\Omega} u_j^2/|x|^2 dx \leq C\int_{\Omega} |\nabla u_j|^2 dx\,.$$

Combining the above two inequalities, we get (4.3). Hence, from (4.3), we obtain

$$1 = \Big(\int_{\Omega} u_j^{2^*-\varepsilon_j}/|x|^s dx\Big)^{1/(2^*-\varepsilon_j)} \leq C\|\nabla u_j\|^{a_j}_{L^2(\Omega)}\|u_j\|^{1-a_j}_{L^{p_j}(\Omega)}\,.$$

Thus a lower bound of $\|u_j\|_{L^{p_j}(\Omega)}$ follows from the upper bound of $\|\nabla u_j\|_{L^2(\Omega)}$.

We divide the proof of Theorem 4.4 into several steps. Since the minimizers $u_j \rightharpoonup 0$ in $H_0^1(\Omega)$, u_j blows up at some point in $\overline{\Omega}$. Let $x_j \in \Omega$ be a maximum point of u_j. Then we have

$$u_j(x_j) = \max_{\Omega} u_j \to \infty \quad \text{as } j \to \infty .$$

STEP 1. We claim that 0 is the only blow up point. Since $\|u_j\|_{L^{p_j}(\Omega)}$ is bounded from above and below, we may assume

$$\lim_{j\to\infty} \|u_j\|_{L^{p_j}(\Omega)}^{p_j-2} = C_1 > 0 .$$

Suppose that u_j blows up at some point $\tilde{x}_0 \neq 0$. The function $1/|x|^s$ is smooth in $\overline{\Omega} \setminus \{0\}$. If u_j blows up at $\tilde{x}_0 \neq 0$, then by some well-known works (see e.g., [6, 7]), v_j converges to some v on any compact subset of $\boldsymbol{R}^n$, where

$$v_j(y) := \lambda_j^{-1} u_j(\tilde{x}_j + \lambda_j^{-2/(n-2)} y) \quad \text{in } \Omega_j ,$$

and

$$\begin{cases} \Omega_j := \{y \in \boldsymbol{R}^n \,;\ \tilde{x}_j + \lambda_j^{-2/(n-2)} y \in \Omega\} , \\ \lambda_j := u_j(\tilde{x}_j) := \max\limits_{B_{r_0}(\tilde{x}_0)} u_j \to \infty \end{cases}$$

with some small fixed constant r_0. Then v satisfies $v(0) = 1$ and

$$\Delta v - (\lambda/C_1)\, v^{(n+2)/(n-2)} = 0 \quad \text{in } \boldsymbol{R}^n \quad \text{with } C_1 \geq \|v\|_{L^{2n/(n-2)}(\boldsymbol{R}^n)}^{4/(n-2)} .$$

Thus we have

$$\begin{aligned} -\lambda &= \left(C_1 \int_{\boldsymbol{R}^n} |\nabla v|^2 dy\right)\left(\int_{\boldsymbol{R}^n} v^{2n/(n-2)} dy\right)^{-1} \\ &= C_1\, \|\nabla v\|_{L^2(\boldsymbol{R}^n)}^2 \|v\|_{L^{2n/(n-2)}(\boldsymbol{R}^n)}^{-2} \|v\|_{L^{2n/(n-2)}(\boldsymbol{R}^n)}^{-4/(n-2)} \\ &\geq C_1\, S_n\, \|v\|_{L^{2n/(n-2)}(\boldsymbol{R}^n)}^{-4/(n-2)} \geq S_n , \end{aligned}$$

which contradicts $\lambda > -S_n$, and Step 1 is proved.

STEP 2. Set $\kappa_j := u_j(x_j)^{-(p_j-2)/2}$. We claim that $|x_j| = O(\kappa_j)$. Suppose $\lim_{j\to\infty} |x_j|/\kappa_j = \infty$, and set

$$v_j(y) := u_j(x_j + \kappa_j y)/u_j(x_j) \quad \text{in } \Omega_j ,$$

where

$$\Omega_j := \{y \in \boldsymbol{R}^n \,;\ x_j + \kappa_j y \in \Omega\} .$$

Then v_j satisfies

$$\Delta v_j - \lambda\, \|u_j\|_{L^{p_j}(\Omega)}^{-(p_j-2)} v_j^{p_j-1} + \mu_{s,*}^{\lambda,\varepsilon_j}(\Omega)\, \kappa_j^2\, u_j(x_j)^{2^*-2-\varepsilon_j} v_j^{2^*-1-\varepsilon_j}/|x_j + \kappa_j y|^s = 0 \quad \text{in } \Omega_j .$$

By noting

$$\kappa_j^2\, u_j(x_j)^{2^*-2-\varepsilon_j} |x_j|^{-s} = \big(\kappa_j/|x_j|\big)^s \to 0 \quad \text{as } j \to \infty ,$$

and $\|u_j\|_{L^{p_j}(\Omega)}$ is bounded from below, we know that v_j converges to some function v uniformly on any compact set, and v satisfies $v(0) = 1$ and

$$\Delta v - (\lambda/C_1)\, v^{(n+2)/(n-2)} = 0 \quad \text{in } \boldsymbol{R}^n, \quad \text{or} \tag{4.4}$$

$$\begin{cases} \Delta v - (\lambda/C_1)\, v^{(n+2)/(n-2)} = 0 & \text{in some half space } H\,, \\ v = 0 \quad \text{on } \partial H\,, \end{cases} \tag{4.5}$$

where $C_1 := \lim_{j\to\infty} \|u_j\|^{p_j-2}_{L^{p_j}(\Omega_j)}$. Since (4.5) has no positive solution, v satisfies (4.4). By scaling, we have

$$\|u_j\|^{p_j-2}_{L^{p_j}(\Omega)} = \big(u_j(x_j)^{p_j}\kappa_j^n\big)^{(p_j-2)/p_j} \|v_j\|^{p_j-2}_{L^{p_j}(\Omega_j)}\,, \tag{4.6}$$

where

$$u_j(x_j)^{p_j}\kappa_j^n = u_j(x_j)^{p_j - n(p_j-2)/2} = u_j(x_j)^{(n-2)\varepsilon_j/(2-s)} \geq 1\,.$$

Thus by passing to the limit in (4.6), we have

$$C_1 \geq \|v\|^{4/(n-2)}_{L^{2n/(n-2)}(\boldsymbol{R}^n)}\,,$$

and then

$$\begin{aligned} S_n\bigg(\int_{\boldsymbol{R}^n} v^{2n/(n-2)}dy\bigg)^{(n-2)/n} &\leq \int_{\boldsymbol{R}^n} |\nabla v|^2 dy \\ &= -(\lambda/C_1)\int_{\boldsymbol{R}^n} v^{2n/(n-2)}dy \leq -\lambda\bigg(\int_{\boldsymbol{R}^n} v^{2n/(n-2)}dy\bigg)^{(n-2)/n}, \end{aligned}$$

which contradicts $\lambda > -S_n$. Thus Step 2 is proved.

STEP 3. We prove that $\lim_{j\to\infty} |x_j|/\kappa_j > 0$. We note that

$$\kappa_j^2\, u_j(x_j)^{2^*-2-\varepsilon_j}\kappa_j^{-s} = 1\,.$$

Hence, v_j satisfies

$$\Delta v_j - \lambda\, \|u_j\|^{-(p_j-2)}_{L^{p_j}(\Omega)} v_j^{p_j-1} + \mu^{\lambda,\varepsilon_j}_{s,*}(\Omega)\, v_j^{2^*-1-\varepsilon_j}/\big|x_j/\kappa_j + y\big|^s = 0 \quad \text{in } \Omega_j\,.$$

Suppose $\lim_{j\to\infty} |x_j|/\kappa_j = 0$. By passing to a subsequence (still denoted by v_j), v_j converges to some v smoothly on any compact subset of $\overline{\boldsymbol{R}^n_+}$, and v satisfies

$$\begin{cases} \Delta v - (\lambda/C_1)\, v^{(n+2)/(n-2)} + \mu^{\lambda}_{s,*}(\Omega)\, v^{2^*-1}/|y|^s = 0 & \text{in } \boldsymbol{R}^n_+\,, \\ v = 0 \quad \text{on } \partial\boldsymbol{R}^n_+\,. \end{cases}$$

This is impossible since $v(0) = 1$. This proves Step 3.

To complete the proof of Theorem 4.4, we note that after a linear transformation and a translation, v_j converges to some v uniformly on any compact subset of $\overline{\boldsymbol{R}^n_+}$, and v satisfies

$$\begin{cases} \Delta v - (\lambda/C_1)\, v^{(n+2)/(n-2)} + \mu^{\lambda}_{s,*}(\Omega)\, v^{2^*-1}/|y|^s = 0 \quad \text{in } \boldsymbol{R}^n_+\,, \\ v = 0 \quad \text{on } \partial\boldsymbol{R}^n_+ \quad \text{and} \quad v(y_0) = \max_{\boldsymbol{R}^n_+} v = 1 \quad \text{for some } y_0 \in \boldsymbol{R}^n_+\,, \end{cases}$$

where

$$C_1 \geq \left(\int_{\boldsymbol{R}^n_+} v^{2n/(n-2)} dy \right)^{2/n} .$$

Thus we have

$$\begin{aligned} \mu^\lambda_{s,*}(\boldsymbol{R}^n_+) &\leq \left(\int_{\boldsymbol{R}^n_+} |\nabla v|^2 dy + \lambda \left(\int_{\boldsymbol{R}^n_+} v^{2n/(n-2)} dy \right)^{(n-2)/n} \right) \left(\int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy \right)^{-2/2^*} \\ &\leq \left(\int_{\boldsymbol{R}^n_+} |\nabla v|^2 dy + (\lambda/C_1) \int_{\boldsymbol{R}^n_+} v^{2n/(n-2)} dy \right) \left(\int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy \right)^{-2/2^*} \qquad (4.7) \\ &= \mu^\lambda_{s,*}(\Omega) \left(\int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy \right)^{(2^*-2)/2^*} \leq \mu^\lambda_{s,*}(\boldsymbol{R}^n_+) , \end{aligned}$$

where the condition $\lambda \leq 0$ is necessary in the second inequality of (4.7). Thus we have $\mu^\lambda_{s,*}(\Omega) = \mu^\lambda_{s,*}(\boldsymbol{R}^n_+)$ and $\int_{\boldsymbol{R}^n_+} v^{2^*}/|y|^s dy = 1$. In addition, from the second equality in (4.7), we obtain

$$C_1 = \left(\int_{\boldsymbol{R}^n_+} v^{2n/(n-2)} dy \right)^{2/n} .$$

□

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