SMALL NOISE ASYMPTOTIC EXPANSIONS FOR STOCHASTIC PDE'S, I. THE CASE OF A DISSIPATIVE POLYNOMIALLY BOUNDED NONLINEARITY

Sergio Albeverio, Luca Di Persio and Elisa Mastrogiacomo

Abstract. We study a reaction-diffusion evolution equation perturbed by a Gaussian noise. Here the leading operator is the infinitesimal generator of a C_0-semigroup of strictly negative type, the nonlinear term has at most polynomial growth and is such that the whole system is dissipative.

The corresponding Itô stochastic equation describes a process on a Hilbert space with dissipative nonlinear, non globally Lipschitz drift and a Gaussian noise.

Under smoothness assumptions on the nonlinearity, asymptotics to all orders in a small parameter in front of the noise are given, with uniform estimates on the remainders. Applications to nonlinear SPDEs with a linear term in the drift given by a Laplacian in a bounded domain are included. As a particular example we consider the small noise asymptotic expansions for the stochastic FitzHugh-Nagumo equations of neurobiology around deterministic solutions.

1. Introduction. In many problems of natural sciences and engineerings, modeling of dynamical systems by nonlinear deterministic partial differential equations (PDEs) is heavily used. This is for example the case for the equations of classical hydrodynamics and more generally classical field theory, as well as for the equations used in the description of certain neurodynamical processes (see, e.g., [4] and [15], respectively). Due to the uncertainty concerning stochastic influences on the systems (for example by additive *random forcing*) an addition of stochastic terms in the equations describing such systems is appropriate. This generates the necessity to study stochastic partial differential equations.

The problem of the study of a deterministic evolution equation of first order in time and finite dimensional state spaces perturbed by an additive Gaussian noise and the associated small noise expansions has been discussed by several authors. Roughly speaking, the work concerning this problem discusses either individual solutions or expectations of functionals of the solution process. For the first category let us mention [41, 37], [24] for example. Work concerning the second category uses methods which go back to Donsker's school (see for example [35, 8, 11, 3] and references therein). To the latter category belongs also the Laplace method for infinite dimensional integrals (see for example [8, 11, 5, 32, 36, 22, 23]) and work related to *semiclassical expansions for Wiener type integrals* (see for example [8]). These expansions go beyond the large deviations estimates which are on the other hand valid without

smoothness assumption on the drift (see e.g. [17, 18, 20] and [19, 20]). Astonishing enough corresponding work for the case of evolving systems with infinite dimensional state space, i.e., for SPDEs, is much more sparse, see however for example, [14, 10, 34]. R. Marcus studied in [31, 30] problems of this type in the case of globally Lipschitz nonlinear terms.

Our present paper extends the latter work in the direction of dropping the global Lipschitz condition and allowing for nonlinearities of at most polynomial growth and of dissipative character, like the ones occurring in the case of the stochastic FitzHugh-Nagumo equation studied in stochastic neurodynamics (see [12] and references therein). More precisely our paper considers a system whose deterministic part corresponds to a nonlinear PDEs of the semilinear type with an (unbounded) linear term and a nonlinearity which is smooth and at most polynomially growing at infinity. This deterministic PDEs is perturbed additively by a space-time noise term of the Gaussian type, with a small positive coefficient ε in front of it. Mathematically the problem can be looked upon as described by a stochastic differential equation for an infinite dimensional stochastic process with a small parameter in front of the noise term, given by an infinite dimensional Wiener process. Since we allow for polynomial growth of the nonlinear part of the drift, in order to assure existence and uniqueness of mild solutions (in the sense of [18]) we assume that the total drift term is dissipative. In turn this is assured by assumptions on the linear drift term and by one sided dissipative type conditions on the nonlinear term.

The study of such equations was from the very beginning influenced by motivations from areas like quantum field theory (such as stochastic quantization equation (see [9, 6, 16, 26, 33]) and the references therein, the Ginzburg-Landau equation of classical statistical mechanics (the equation describing growth of surfaces in solid state physics), biology (for example in the study of stochastic neurodynamics [38, 39, 40, 12]) and economics (for example interest rate models [7, 21]).

In many problems it is interesting to know how the solutions of the perturbed problem depend on the small parameter ε describing the random forcing. For example, in connection with FitzHugh-Nagumo models of neurodynamics this has been discussed heuristically by Tuckwell (see [38, 39, 40], and also [28] for motivations in connection with the description of phenomena in the study of epilepsy and [1] for connections with problems of synchronization in neuronal systems). The exploitation of the dissipativity permits to compensate for the lack of a global Lipschitz condition on the nonlinear drift.

Our aim is to provide asymptotic expansions of the solution to all orders in the perturbation parameter ε, with explicit expressions both for the expansion coefficients and the remainder. The technique used is general and also covers the case of deterministic *forcing terms*. An application to models of stochastic FitzHugh-Nagumo dynamics on networks, used for the description of systems of biological neurons, will be given in a subsequent paper [2].

In a further paper we shall extend our method to study other SPDEs having other types of nonlinear polynomially growing drifts.

2. Outline of the paper.

Let us consider the following deterministic problem

$$(2.1) \quad \begin{cases} d\phi(t) = [A\phi(t) + F(\phi(t))]dt \,, & t \in [0, +\infty) \\ \phi(0) = u^0 \,, & u^0 \in H \,, \end{cases}$$

where A is a linear operator on a separable Hilbert space H which generates a C_0-semigroup of strict negative type. The term F is a *smooth* nonlinear, quasi-m-dissipative mapping from the domain $D(F) \subset H$ (dense in H) with values in H; this means that there exists $\omega \in \mathbf{R}$ such that $(F - \omega I)$ is m-dissipative in the sense of [18, p. 73], with (at most) polynomial growth at infinity (and satisfying some further assumptions which will be specified in Hypothesis 3.1 below) while $D(F)$ is the domain of F, assumed to be dense in H. Existence and uniqueness of solutions for equation (2.1) is discussed in Proposition 3.7 below.

Our aim is to study a stochastic (white noise) perturbation of (2.1) and to write its (unique) solution as an expansion in powers of a parameter $\varepsilon > 0$, which controls the strength of the noise, as ε goes to zero. More precisely, we are concerned with the following stochastic Cauchy problem on the Hilbert space H

$$(2.2) \quad \begin{cases} du(t) = [Au(t) + F(u(t))]dt + \varepsilon\sqrt{Q}dW(t) \,, & t \in [0, +\infty) \\ u(0) = u^0 \,, & u^0 \in K \,, \end{cases}$$

where A and F are as described above, W is a cylindrical Wiener process on H, Q is a positive trace class linear operator from H to H and $\varepsilon > 0$ is the parameter which determines the magnitude of the stochastic perturbation. The initial datum u^0 takes value into a continuously embedded Banach space K of H. A unique solution of the problem (2.2) can be shown to exist exploiting as in [12] results on stochastic differential equations (contained, e.g., in [17, 18]). Our purpose is to show that the solution of the equation (2.2), which will be denoted by $u = u(t), t \in [0, +\infty)$, can be written as

$$u(t) = \phi(t) + \varepsilon u_1(t) + \cdots + \varepsilon^n u_n(t) + R_n(t, \varepsilon) \,,$$

where n depends on the differentiability order of F. Further, $R_n(t, \varepsilon)$ is a suitable process which goes to 0 with order ε^n as ε goes to 0. The function $\phi(t)$ solves the associated deterministic problem (2.1), $u_1(t)$ is the stochastic process which solves the following linear stochastic (non-autonomous) equation

$$(2.3) \quad \begin{cases} du_1(t) = [Au_1(t) + \nabla F(\phi(t))[u_1(t)]]dt + \sqrt{Q}dW(t) \,, t \in [0, +\infty) \\ u_1(0) = 0 \,, \end{cases}$$

while for each $k = 2, \ldots, n$, $u_k(t)$ solves the following non-homogeneous linear differential equation with stochastic coefficients

$$(2.4) \quad \begin{cases} du_k(t) = [Au_k(t) + \nabla F(\phi(t))[u_k(t)]]dt + \Phi_k(t)dt \,, \\ u_k(0) = 0 \,. \end{cases}$$

$\Phi_k(t)$ is a stochastic process which depends on $u_1(t), \ldots, u_{k-1}(t)$ and the Fréchet derivatives of F up to order k, see Section 4 for details.

The paper is organized as follows. In Section 3 we recall standard results for the solution of equations of types (2.1), (2.2), (2.3) and (2.4). Section 4 is devoted to the study of some

properties of the nonlinear term F, in particular the n-th remainder of its Taylor expansion. Section 5 is concerned with the proof of the main result on the asymptotic expansion in powers of ε of the solution of the stochastic equation (2.2), with explicit coefficients and remainders, and estimates thereof. We conclude with some remarks on applications of the results, in particular concerning the stochastic FitzHugh-Nagumo equation.

3. Assumptions and basic estimates. Before recalling some known results on problems of the types (2.1), (2.2), (2.3) and (2.4), we begin by presenting our notation and assumptions. We are concerned with a real separable Hilbert space, with the inner product $\langle \cdot, \cdot \rangle$; on H there are given a linear operator $A : D(A) \subset H \to H$, a nonlinear operator $F : D(F) \subset H \to H$ with dense domain in H and a bounded linear operator Q. Moreover, we are given a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \boldsymbol{P})$ which satisfies the usual conditions, i.e., the probability space is complete, $\mathcal{F}$ contains all $\boldsymbol{P}$-null subsets of sets in $\mathcal{F}$ and the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right continuous. Further, for any trace-class linear operator Q, we will denote by $\mathrm{Tr}\, Q$ its trace; if f is any mapping on H which is Fréchet differentiable up to order n, $n \in \boldsymbol{N}$, we will denote by $f^{(i)}$, $i = 1, \ldots, n$ its i-th Fréchet derivative and by $D(f^{(i)})$ its domain (for a short survey on Fréchet differentiable mappings we refer to Section 4). For any $j \in \boldsymbol{N}$, $L(H^j; H)$ denotes the space of j-linear bounded mappings from H^j into H while the space of linear bounded mappings from H into $L(H^j; H)$ is denoted by $L^j(H)$. We denote by $|\cdot|_H$ the norm on H, by $\|\cdot\|_{L^j(H)}$ the norm of any j-linear operator on H and by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm of any linear operator on H. Finally, we will denote by $\mathcal{L}^p(\Omega; C([0, T]; H))$ the space of continuous and adapted processes taking values in H such that the following norm is finite

$$\|u\| = \boldsymbol{E}\left(\sup_{t \in [0, T]} |u(t)|_H^p \right)^{1/p} < \infty.$$

HYPOTHESIS 3.1.

1.   The operator $A : D(A) \subset H \to H$ generates an analytic semigroup $(e^{tA})_{t \geq 0}$, on H of strict negative type such that

$$\|e^{tA}\|_{L(H)} \leq e^{-\omega t}, \quad t \geq 0$$

with ω a strictly positive, real constant.
Let assume that there exists a Banach space K, densely and continuously embedded (as a Borel subset) into H, endowed with the norm $|\cdot|_K$. Moreover, if A_K denotes the part of A in K, that is

$$D(A_K) := \{x \in D(A) \cap K; Ax \in K\}, \quad A_K x = Ax,$$

then A_K generates an analytic semigroup (of negative type) e^{tA_K}, $t \geq 0$ on K.

2.   The mapping $F : D(F) \subset H \to H$ is continuous, nonlinear, Fréchet differentiable up to order n for some positive integer n and quasi-m-dissipative, i.e., there exist $\eta > 0$ such

that

$$\langle F(u) - F(v) - \eta(u - v), u - v \rangle < 0, \quad for \ all \ u, v \in D(F).$$

3. If $F_K^{(j)}$, $j = 1, \ldots, n$ denotes the part of $F^{(j)}$ in K, that is

$$D(F_K^{(j)}) := \{ x \in D(F^{(j)}) \cap K; F_K^{(j)}(x) \in K \}, \quad F_K^{(j)}(x) = F^{(j)}(x),$$

then the following estimates hold

(a)   there exist a positive real number γ and a natural positive number m such that

$$|F_K(u)|_K \leq \gamma(1 + |u|_K^m), \quad u \in K,$$

(b)   for some $n \in N$ and any $u \in D(F_K^{(i)})$, $i = 1, \ldots, n$, there exist positive real constants γ_i, $i = 1, \ldots, n$ such that

$$\|F_K^{(i)}(u)\|_{L^j(K)} \leq \gamma_i(1 + |u|_K^{m-i}), \quad \text{with } m \text{ as in (3a)}, u \in K.$$

4.   The constants ω, η satisfy the inequality $\omega - \eta > 0$; this implies that the term $A + F$ is m-dissipative in the sense of [17], [18, p. 73].

5.   The term W is an H-cylindrical Wiener process (for example in the sense of [17, 18]).

6. Q is a positive linear bounded operator on H of trace class, that is $\text{Tr } Q < \infty$.

EXAMPLE 3.2.   Let us give an example of a mapping F satisfying the above hypothesis (in view of the application to stochastic neuronal models). Let $H = L^2(\Lambda)$ with $\Lambda \subset R^n$, bounded and open; set $K := C(\bar{\Lambda})$ (with $\bar{\Lambda}$ the closure of Λ in R^n) and let F be a multinomial of degree $m \in N$, i.e., a mapping of the form $F(u) = g_m(u)$, where $g_m(u)$, $u \in H$, is a polynomial of degree m, that is, $g_m(u) = a_0 + a_1 u + \cdots + a_m u^m$, with $a_i \in R$, $i = 0, \ldots, m$. Then it is easy to prove that $D(F) = L^{2m}(\Lambda) \subseteq L^2(\Lambda)$, $m > 0$, $D(F) = L^{2m}(\Lambda) = H$, $m = 0$ and (by using the Hölder inequality) $D(F^{(i)}) = L^{2i}(\Lambda)$. Moreover, it turns out that, for any $u \in D(F)$, $F^{(i)}(u)$ can be identified with the element $g_m^{(i)}(u)$ (both in $D(F)$ and K). Consequently,

$$\begin{aligned} |F(u)|_K &= \sup_{\xi \in \bar{\Lambda}} |F(u(\xi))| \\ &= \sup_{\xi \in \bar{\Lambda}} |g_m(u(\xi))| \\ &\leq C_m(1 + \sup_{\xi \in \bar{\Lambda}} |u(\xi)|^m) \\ &= C_m(1 + |u|_K^m) \end{aligned}$$

and, similarly,

$$\begin{aligned} |\nabla^{(j)} F(u)|_{L^j(K)} &\leq C_m(1 + \sup_{\xi \in \bar{\Lambda}} |u(\xi)|^{m-j}) \\ &= C_m(1 + |u|_K^{m-j}), \quad j = 0, 1, \ldots, m. \end{aligned}$$

Hence F satisfies Hypothesis 3.1 (2), (3). Further, in the case $g_3(u) = -u(u-1)(u-\xi)$, $0 < \xi < 1$ the corresponding mapping F coincides with the nonlinear term of the first equation in the FitzHugh-Nagumo system (see Example 5.4 below).

We recall the notion of mild solution for the deterministic and stochastic problems (2.1), (2.2); next we recall the definition of stochastic convolution and we list some of its properties

DEFINITION 3.3. Let $u^0 \in K$; we say that the function $\phi : [0, \infty) \to H$ is a mild solution of equation (2.1) if it is continuous (in t), with values in H and it satisfies

$$(3.1) \qquad \phi(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(\phi(s))ds, \quad t \in [0, +\infty),$$

with the integral existing in the sense of Bochner integrals on Hilbert spaces.

DEFINITION 3.4. Let $u^0 \in K$. A predictable H-valued process $u := (u(t))_{t \geq 0}$ is called a mild solution to the Cauchy problem (2.2) with initial condition $u^0 \in D(F)$ if for arbitrary $t \geq 0$ we have

$$u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(u(s))ds + \varepsilon \int_0^t e^{(t-s)A}\sqrt{Q}dW(s), \quad \boldsymbol{P}\text{-a.s.}$$

Moreover $W_A(t) := \int_0^t e^{(t-s)A}\sqrt{Q}dW(s)$ is called a stochastic convolution and under our hypothesis it is a well defined mean square continuous $\mathcal{F}_t$-adapted Gaussian process with values in H (see e.g., [17, Theorem 5.2, p. 119]).

The first integral on the right-hand side is defined pathwise in the Bochner sense, $\boldsymbol{P}$-almost surely.

For further use, in the following we introduce some additional condition on the stochastic convolution:

HYPOTHESIS 3.5. The stochastic convolution $W_A(t)$, $t \geq 0$ introduced in Definition 3.4, admits a K-valued version such that, for any $T > 0$ and $m \in \boldsymbol{N}$, it satisfies the following estimate

$$(3.2) \qquad\qquad \boldsymbol{E}\left(\sup_{t \in [0,T]} |W_A(t)|_K^{2m} \right) \leq C_T$$

for some positive constant C_T (possibly depending on T).

EXAMPLE 3.6. Let us give an example for the setting (H, L, A, Q) where W_A is well-defined and Hypothesis 3.5 is satisfied. This example is related to the application to the stochastic FitzHugh-Nagumo model which we discuss in Example 5.4. Let H, K be as in Example 3.2. Let $A = \Delta$ be the Laplacian in $L^2(\Lambda)$ with Neumann boundary conditions. Let Q be a bounded trace class operator commuting with A. By [17, Proposition 5.15] $W_A(t) \in D((-A)^\gamma)$, $\gamma \in (0, 1/2)$; in particular $W_A(t) \in K$, W_A being in addition a Gaussian process. This implies the bound in Hypothesis 3.5.

We shall use Hypothesis 3.5 below concerning the solution of the stochastic equation (2.2). First, let us have a look at the deterministic equation.

PROPOSITION 3.7.   *Under Hypothesis* 3.1 *there exists a unique mild solution* $\phi = \phi(t), t \in [0, \infty)$ *of the deterministic problem* (2.1) *such that*

(3.3)
$$|\phi(t)|_H \leq e^{-2(\omega - \eta)t} |u^0|_H, \quad t \geq 0.$$

PROOF.   The proof of the existence and the uniqueness can be found, among the others, in [17, Theorem 7.13, p. 203], while the estimate (3.3) is a direct consequence of the application of Gronwall's lemma to the following inequality

$$\frac{d}{dt}|\phi(t)|_H^2 = 2\langle A\phi(t), \phi(t)\rangle dt + 2\langle F(\phi(t)), \phi(t)\rangle$$
$$\leq -2(\omega - \eta)|\phi(t)|_H^2. \qquad \square$$

REMARK 3.8.   It can be shown that, under Hypothesis 3.1 there exists a K-continuous version of the unique solution of equation (3.1) such that, for any $T > 0$, $p \geq 1$

$$\sup_{t \in [0,T]} |\phi(t)|_K^p < \infty$$

(see [18, Section 5.5.2, Proposition 5.5.6]). Hence, in the following, by ϕ we will understand this K-valued version of the solution of (2.1).

PROPOSITION 3.9.   *Assume that A and F satisfy Hypothesis* 3.1. *Assume that A and Q satisfy Hypothesis* 3.5. *Then for any* $u^0 \in D(F)$ *and* $T > 0$, *there exists a unique mild solution* $u = (u(t))_{0 \leq t \leq T}$ *of the equation* (2.2) (*cf. Definition* 3.4) *which belongs to the space* $\mathcal{L}^p(\Omega; C([0, T]; H))$, *i.e., such that*

(3.4)
$$E\left(\sup_{t \in [0,T]} |u(t)|_H^p \right) < +\infty,$$

for any $p \in [2, \infty)$.

PROOF.   For the existence and the uniqueness of the solution see for instance, [17, Theorem 7.13, p. 203]. Hence we only have to prove the estimate (3.4). Let $z(t) := u(t) - W_A(t)$; then it is not difficult to show that $z(t)$ is the unique solution of the following deterministic equation:

$$\begin{cases} z'(t) = Az(t) + F(z(t) + W_A(t)) \\ z(0) = u^0 \end{cases}$$

with $z'(t) := \frac{d}{dt}z(t)$.

With no loss of generality (because of inclusion results for L^p-spaces with respect to bounded measures) we can assume that $p = 2a$, $a \in N$. Now combining condition (1) with

(2) in Hypothesis 3.1 and recalling Newton's binomial formula we have

$$
\begin{aligned}
\frac{d}{dt}|z(t)|_H^{2a} &= 2a\langle z'(t), z(t)\rangle |z(t)|_H^{2a-2} \\
&= 2a\langle Az(t) + F(z(t) + W_A(t)), z(t)\rangle |z(t)|_H^{2a-2} \\
&\leq -2a\omega |z(t)|_H^{2a} + 2a\langle F(z(t) + W_A(t)), z(t)\rangle |z(t)|_H^{2a-2} \\
&\leq -2a(\omega - \eta)|z(t)|_H^{2a} + 2a|F(W_A(t))|_H |z(t)|_H^{2a-1} \\
&\leq -2a(\omega - \eta)|z(t)|_H^{2a} + 2a\frac{C_a}{\xi}|F(W_A(t))|_H^{2a} + C_a 2a\xi|z(t)|_H^{2a},
\end{aligned}
$$

(3.5)

for some constant $C_a > 0$ and a sufficiently small $\xi > 0$ such that $-2a(\omega - \eta) + 2a\xi C_a < 0$. Applying the previous inequality and Gronwall's lemma we get:

$$
|z(t)|_H^{2a} \leq e^{(-2a(\omega-\eta)+\xi C_a 2a)t}|u^0|_H^{2a} + \frac{2aC_a}{\xi}\int_0^t e^{-2a(\omega-\eta)(t-s)}|F(W_A(s))|_H^{2a}ds.
$$

Then there exists a positive constant C such that:

$$
\begin{aligned}
|u(t)|_H^{2a} \leq C\Bigg(&e^{(-2a(\omega-\eta)+\xi C_a 2a)t}|u^0|_H^{2a} \\
&+ 2a\int_0^t e^{-2a(\omega-\eta)(t-s)}|F(W_A(s))|_H^{2a}ds + |W_A(t)|_H^{2a}\Bigg).
\end{aligned}
$$

(3.6)

Since by condition $(3a)$ in Hypothesis 3.1, the restriction of F to K has (at most) polynomial growth at infinity in the K-norm and, by the assumption on $W_A(t)$ made in Hypothesis 3.5, W_A takes value in K, for any $a \in N$ we have:

$$
|F(W_A(t))|_H^{2a} \leq C_{a,m}(1 + |W_A(t)|_K^m)^{2a} \leq C_{a,m}(1 + |W_A(t)|_K^{2am}),
$$

for some positive constant $C_{a,m}$ depending on m and a. Moreover, we observe that, again by Hypothesis 3.5, it holds that

$$
E\left(\sup_{t\in[0,T]}|W_A(t)|_K^{2am}\right) \leq C'_{a,m,T},
$$

where $C'_{a,m,T}$ is again a positive constant depending on m, a and T; hence

$$
\begin{aligned}
&E\left[\sup_{t\in[0,T]}\int_0^t e^{-2a(\omega-\eta)(t-s)}|F(W_A(s))|_H^{2a}ds\right] \\
&\quad \leq \tilde{C}E\left[\sup_{t\in[0,T]}\int_0^t e^{-2a(\omega-\eta)(t-s)}(1 + |W_A(t)|_K^{2am})ds\right] \\
&\quad \leq \tilde{C}E\left[\sup_{t\in[0,T]}\int_0^t e^{-2a(\omega-\eta)(t-s)}ds + C'_{a,m}\int_0^t e^{-2a(\omega-\eta)}ds\right] \leq \bar{C},
\end{aligned}
$$

(3.7)

for some positive constants $\widetilde{C}, \bar{C}$ depending on a, m and T. Consequently, putting together inequalities (3.6), (3.7), we obtain

$$E\left(\sup_{t\in[0,T]}|u(t)|_H^{2a}\right) \leq C|u^0|_H^{2a} + \bar{\bar{C}},$$

for some positive constant $\bar{\bar{C}}$, so that the proposition follows.   □

4. Properties of the nonlinear term F and Taylor expansions. In this section we study the nonlinear term F in order to write its Taylor expansion around the solution $\phi(t)$ of (3.1) with respect to an increment given in terms of powers of ε. In order to do that we recall some basic properties of Fréchet differentiable functions.

Let U and V be two real Banach spaces. For a mapping $F : U \to V$ the Gâteaux differential at $u \in U$ in the direction $h \in U$ is defined as

$$\nabla F(u)[h] = \lim_{s\to 0} \frac{F(u+sh) - F(u)}{s},$$

whenever the limit exists in the topology of V (see for example [29, p. 12]).

We notice that if $\nabla F(u)[h]$ exists in a neighborhood of $u_0 \in U$ and is continuous in u at u_0 and also continuous in h at $h = 0$, then $\nabla F(u)[h]$ is linear in h (see for instance [29, Problem 1.6.1, p. 15]). If in addition $h \mapsto \nabla F(u)[h]$ is bounded from U to V, then $\nabla F \equiv F'(u)$ is called Gâteaux derivative of F at u. If $\nabla F(u_0)[h]$ has this property for all $u_0 \in U_0 \subseteq U$ and all $h \in U$ we shall say that F belongs to the space $\mathcal{G}^1(U_0; V)$. If F is continuous from U to V and $F \in \mathcal{G}^1(U_0; V)$ and one has $F(u+h) = F(u) + \nabla F(u)[h] + R(u, h)$, for any $u \in U_0$ with

$$(4.1) \qquad\qquad \lim_{|h|_U\to 0} \frac{|R(u, h)|_V}{|h|_U} = 0,$$

with $|\cdot|_V$ and $|\cdot|_U$ denoting respectively the norm in V and U, then the map $h \to \nabla F(u)[h]$ is a bounded linear operator from U_0 to V, and $\nabla F(u)[h]$ is, by definition, the unique Fréchet differential of F at $u \in U_0$ with increment $h \in U$. The function $R(u, h)$ is called the remainder of this Fréchet differential, while the operator sending h into $\nabla F(u)[h]$ is then called the Fréchet derivative of F at u and is usually denoted by $F'(u)$ (see for instance [29, pp. 15–16, Problem 1.6.2 and Lemma 1.6.3]). We have then $\nabla F(u)[h] = F'(u) \cdot h$, with the symbol $\cdot$ denoting the action of the linear bounded operator $F'(u)$ on h.

The mapping $F'(u)$ is also called the gradient of F at u (see for example [29, p. 15]) and it coincides with the Gâteaux derivative of F at u. We shall denote by $\mathcal{F}^{(1)}(U_0, V)$ the subset of $\mathcal{G}^1(U_0, V)$ such that the Fréchet derivative exists at any point of U_0. Similarly we introduce the Fréchet derivative $F''(u)$ of F' at $u \in U$. This is a bounded linear map from a subset $D(F')$ of U into $L(U, V)$ ($L(U, V)$ being the space of bounded linear operators from U to V). One has thus $F'' \in L(U, L(U, V))$. If we choose $h, k \in U$ then $F''(u)\cdot k \in L(U, V)$ and $(F''(u) \cdot k) \cdot h \in V$. The latter is also written $F''(u)\ h\ k$ or $F''(u)[h, k]$. The mapping $F''(u)[h, k]$ is bilinear in h, k, for any given $u \in D(F'')$ and it can be identified with the Gâteaux differential $\nabla^{(2)} F(u)[h, k]$ of $\nabla F(u)[h]$ in the direction k, the latter looked upon

as a map from U to $L(U, V)$. Similarly one defines the j-th Fréchet derivative $F^{(j)}(u)$ and the j-th Gâteaux differential $\nabla F^{(j)}(u)[h_1, \ldots, h_j]$. The function $F^{(j)}(u)$ acts j-linearly on $h_1, \ldots, h_j$ with $h_i \in U$ for any $i = 1, \ldots, j$. Let U_0 be an open subset of U and consider the space $\mathcal{F}^{(j)}(U_0, V)$ of maps F from U to V such that $F^{(j)}(u)$ exists at all $u \in U_0$ and is uniformly continuous on U_0. The following Taylor formula holds for any $u, h \in U$ for which $F(h)$ and $F(u + h)$ are well defined (i.e., h and $u + h$ are elements of $D(F)$), and $j = 1, \ldots, n + 1$ with $u \in \bigcap_{j=1}^{n+1} \mathcal{F}^{(j)}(U_0, V)$

$$F(u + h) = F(u) + \nabla F(u)[h] + \frac{1}{2} \nabla^{(2)} F(u)[h, h] + \cdots$$

(4.2)

$$+ \frac{1}{n!} \nabla^{(n)} F(u) \underbrace{[h, \ldots, h]}_{n\text{-terms}} + R^{(n)}(u; h),$$

where $|R^{(n)}(u; h)|_U \leq C_{u,n} \cdot |h|_U^n$ for some constant $C_{u,n}$ depending only on u and n (see for example [27, Theorem X.1.2]).

Now let us consider the case $U = H$, with H being the same Hilbert space appearing in problem (2.1). Let F be as in Hypothesis 3.1 and set $U_0 = D(F)$. Let us define for $0 < \varepsilon \leq 1$ the function $h(t), t \geq 0$

$$h(t) = \sum_{k=1}^{n} \varepsilon^k u_k(t) + r^{(n)}(t; \varepsilon),$$

where the functions $u_k(t), k = 1, \ldots, n$ and $r^{(n)}(t; \varepsilon)$ are p-mean integrable continuous stochastic processes with values in H, defined on the whole interval $[0, T]$ for $p \in [2, \infty)$. Moreover we suppose $r^{(n)}(\cdot; \varepsilon) = \mathbf{o}(\varepsilon^n)$, i.e.,

$$\lim_{\varepsilon \to 0} E\left[\sup_{t \in [0, T]} \frac{|r^{(n)}(t; \varepsilon)|^p}{\varepsilon^n} \right] = 0, \quad \text{for any } T > 0.$$

Let ϕ be a p-mean integrable continuous stochastic process with values in the Banach space K. Then using the above Taylor formula we have

$$F(\phi(t) + h(t)) = F(\phi(t)) + \nabla F(\phi(t))[h(t)] + \frac{1}{2} \nabla^{(2)} F[h(t), h(t)] + \cdots$$

(4.3)

$$+ \frac{1}{n!} \nabla^{(n)} F(u) \underbrace{[h(t), \ldots, h(t)]}_{n\text{-terms}} + R^{(n)}(\phi(t); h(t)),$$

and, recalling that for any $j = 1, \ldots, n, \nabla^{(j)} F(\phi(t))$ is multilinear, we have

$$\frac{1}{j!} \nabla^{(j)} F(\phi(t)) \underbrace{[h(t), \ldots, h(t)]}_{j\text{-terms}}$$

(4.4)

$$= \frac{1}{j!} \sum_{k_1 + \cdots + k_j = j}^{nj} \varepsilon^{k_1 + \cdots + k_j} \nabla^{(j)} F(\phi(t))[u_{k_1}(t), \ldots, u_{k_j}(t)] + \mathbf{o}_j(\varepsilon^{nj})$$

where $\mathbf{o}_j(\varepsilon^{nj})$ is the contribution to the right member of the above equality coming from the term $r^{(n)}(t; \varepsilon)$ and satisfies the estimate

$$\lim_{\varepsilon \to 0} E\left[\sup_{t \in [0,T]} \frac{|\mathbf{o}_j(\varepsilon^{nj})|^p}{\varepsilon^{nj}} \right] = 0, \quad \text{for any } T > 0.$$

We notice that any derivative appearing in the member on the right-hand side of (4.4) is multiplied by the parameter ε raised to a power between j and nj.

Taking into account the above equality we can rewrite (4.3) as

$$
\begin{aligned}
F(\phi(t) + h(t)) &= F(\phi(t)) + \sum_{k=1}^{n} \varepsilon^k \nabla F(\phi(t))[u_k(t)] \\
&\quad + \sum_{j_1+j_2=2}^{n} \frac{\varepsilon^{j_1+j_2}}{2!} \nabla^{(2)} F(\phi(t))[u_{j_1}(t), u_{j_2}(t)] + \cdots \\
&\quad + \sum_{j_1+\cdots+j_k=k}^{n} \frac{\varepsilon^{j_1+\cdots+j_k}}{k!} \nabla^{(k)} F(\phi(t))[u_{j_1}(t), \ldots, u_{j_k}(t)] + \cdots \\
&\quad + \frac{\varepsilon^n}{n!} \nabla^{(n)} F(\phi(t))[u_1(t), \ldots, u_1(t)] + R_1^{(n)}(\phi(t); h(t), \varepsilon),
\end{aligned}
$$
(4.5)

where the quantity $R_1^{(n)}(\phi(t); h(t), \varepsilon)$ is given in terms of the derivatives of F with the parameter ε raised to powers greater than n, in terms of the n-th remainder $R^{(n)}(\phi(t); h(t))$ in the Taylor expansion of the map F (as stated in equation (4.2)) and in terms of the remainders $\mathbf{o}_j(\varepsilon^{nj})$, $j = 2, \ldots, n$ introduced in (4.4). Namely, we have:

$$
\begin{aligned}
R_1^{(n)}(\phi(t); h(t), \varepsilon) &= \sum_{j=2}^{n} \sum_{i_1+\cdots+i_j=n+1}^{nj} \varepsilon^{i_1+\cdots+i_j} \frac{1}{j!} \nabla^{(j)} F(\phi(t))[u_{i_1}(t), \ldots, u_{i_j}(t)] \\
&\quad + \sum_{j=2}^{n} \mathbf{o}_j(\varepsilon^{nj}) + R^{(n)}(\phi(t); h(t)),
\end{aligned}
$$
(4.6)

$R^{(n)}(\phi(t); h(t))$ being as in (4.2) (with u replaced by ϕ). In this way equation (4.5) can be rearranged as

$$
\begin{aligned}
F(\phi(t) + h(t)) &= F(\phi(t)) + \sum_{j=2}^{n} \varepsilon^j \left(\sum_{i_1+\cdots+i_j=j} \frac{1}{j!} \nabla^{(j)} F(\phi(t))[u_{i_1}(t), \ldots, u_{i_j}(t)] \right) \\
&\quad + R_1^{(n)}(\phi(t); h(t), \varepsilon).
\end{aligned}
$$
(4.7)

LEMMA 4.1. *Let $R_1^{(n)}$ be as in formula (4.6). Then for all $p \in [2, \infty)$ and $T > 0$ there exists a constant $C > 0$, depending on $|\phi|_K, |u_1|, \ldots, |u_n|_H, \nabla^{(1)} F, \ldots, \nabla^{(n)} F, p, n$, such that*:

$$E\left[\sup_{t \in [0,T]} |R_1^{(n)}(\phi(t); h(t), \varepsilon)|_H^p \right] \leq C \varepsilon^{p(n+1)}$$

for all $0 < \varepsilon \leq 1$.

PROOF.   First of all we notice that

$$\sum_{j=2}^{n} \mathbf{o}_j(\varepsilon^{nj}) = \mathbf{O}(\varepsilon^{2n}),$$

meaning that

(4.8) $$\left| \sum_{j=2}^{n} \mathbf{o}(\varepsilon^{nj}) \right| \leq C_n \varepsilon^{2n}, \quad \varepsilon \to 0,$$

for some constant $C_n > 0$. Now since:

$$R_1^{(n)}(\phi(t); h(t), \varepsilon) = \sum_{j=2}^{n} \sum_{i_1 + \cdots + i_j = n+1}^{nj} \varepsilon^{i_1 + \cdots + i_j} \frac{1}{j!} \nabla^{(j)} F(\phi(t)) [u_{i_1}(t), \ldots, u_{i_j}(t)]$$

$$+ \sum_{j=2}^{n} \mathbf{o}_j(\varepsilon^{nj}) + R^{(n)}(\phi(t); h(t)),$$

using the estimate given in condition (3.b) in Hypothesis 3.1 and (4.8), for $\varepsilon \in (0, 1]$ we have

$$|R_1^{(n)}(\phi(t); h(t), \varepsilon)|_H^p$$

$$\leq C_{n,p}^1 \varepsilon^{(n+1)p} \left[\left(\max_{j=1,\ldots,n} \|\nabla^{(j)} F(\phi(t))\|_{L^j(K)} \right)^p \left(\sum_{i=1}^{n} |u_i(t)|_H^p \right) \right]$$

$$+ (\mathbf{O}(\varepsilon^{2n}))^p + C_{n,p}^2 |R^{(n)}(\phi(t); h(t))|_H^p$$

(4.9) $$\leq C_{n,p}^{(1)} \varepsilon^{(n+1)p} \max_{j=1,\ldots,n} \left[\gamma_j^p (1 + |\phi(t)|_K^{m-j})^p \right] \left(\sum_{i=1}^{n} |u_i(t)|_H^p \right)$$

$$+ C_n \varepsilon^{2np} + C_{n,p}^{(2)} |R^{(n)}(\phi(t); h(t))|_H^p$$

$$\leq \tilde{C}_n \varepsilon^{(n+1)p} + C_{n,p}^{(2)} |R^{(n)}(\phi(t); h(t))|_H^p,$$

where $C_{n,p}^1, C_{n,p}^{(1)}, C_{n,p}^{(2)}$ are constants depending only on n, p and the constant C_n in (4.8) while $\tilde{C}_n$ is a suitable positive constant depending on $p, n, \max_{j=1,\ldots,n}[\gamma_j^p(1 + |\phi(t)|_K^{m-j})^p]$ (γ_i being the constants appearing in Hypothesis 3.1, condition (3)) and $|u_i(t)|_H^p, i = 1, \ldots, n$. We notice that the above inequality follows by recalling that the deterministic function $\phi(t)$ is bounded in the H-norm (see Proposition 3.7).

Now by the bound on $R^{(n)}$ in the equation (4.2) we have that

$$|R^{(n)}(\phi(t); h(t))|_H^p \leq \hat{C}_n |h(t)|_H^{(n+1)p}$$

with $\hat{C}_n$ depending on $\phi(t)$ and n but independent of $h(t)$. Since $h(t) = \sum_{k=1}^{n} \varepsilon^k u_k(t) + r^{(n)}(t; \varepsilon)$ with $|r^{(n)}(t; \varepsilon)| \leq C_n \varepsilon^{n+1}$ for some $\tilde{C}_n$, then:

(4.10) $$|R^{(n)}(\phi(t); h(t))|_H^p \leq \varepsilon^{(n+1)p} \hat{C}_{n,p}(|u_1(t)|_H, \ldots, |u_n(t)|_H)$$

with $\hat{C}_{n,p} = \hat{C}_{n,p}(|u_1(t)|_H, \ldots, |u_n(t)|_H)$ independent of ε.

Hence by (4.9) and (4.10) we have that

$$E\left[\sup_{t \in [0,T]} |R_1^{(n)}(\phi(t); h(t), \varepsilon)|_H^p\right] \leq C_n' \varepsilon^{n+1},$$

where $C_n' := C_n'(p, \nabla^{(1)}F, \ldots, \nabla^{(n)}F, |\phi|_H, \ldots, |u_n|_H)$ is independent of ε. This gives the lemma, with $C = C_n'$. □

As we said before, we want to expand the solution of the equation (2.2) around $\phi(t)$, that is we want to write $u(t)$ as:

$$(4.11) \qquad u(t) = \phi(t) + \varepsilon u_1(t) + \cdots + \varepsilon^n u_n(t) + R_n(t, \varepsilon),$$

(with the term $R_n(t, \varepsilon) = \mathbf{O}(\varepsilon^{n+1})$), for any $t \geq 0$), where the processes $(u_i(t))_{t \geq 0}, i = 1, \ldots, n$ can be found by using the Taylor expansion of F around $\phi(t)$ and *matching terms* in the equation (2.2) for u. Given predictable H-valued stochastic processes $w(t), v_1(t), \ldots, v_n(t)$ let us use the notation:

$$(4.12) \quad \Phi_k(w(t))[v_1(t), \ldots, v_k(t)] := \sum_{j=2}^{k} \sum_{i_1 + \cdots + i_j = k} \nabla^{(j)}F(w(t))[v_{i_1}(t), \ldots, v_{i_j}(t)],$$

with $i_1, \ldots, i_j$, running from 0 to k and the given restriction $i_1 + \cdots + i_n = k$. With the above notation the processes $u_1(t), \ldots, u_n(t)$ occurring in (4.11) satisfy the following equations:

$$\begin{cases} du_1(t) = [Au_1(t) + \nabla F(\phi(t))[u_1(t)]]dt + \sqrt{Q}dW(t), \\ u_1(0) = 0, \end{cases}$$

and

$$(4.13) \qquad \begin{cases} du_k(t) = [Au_k(t) + \nabla F(\phi(t))[u_k(t)]]dt + \Phi_k(t)dt, \\ u_k(0) = 0, \end{cases}$$

with

$$(4.14) \qquad \Phi_k(t) := \Phi_k(\phi(t))[u_1(t), \ldots, u_{k-1}(t)]:, \quad k \in N, \quad n \geq k \geq 2.$$

Notice that while $u_1(t)$ is the solution of a linear stochastic differential equation (with time dependent drift operator $A + \nabla F(\phi(t))$), the processes $u_2, \ldots, u_n$ are solutions of non-homogenous differential equations with random coefficients whose meaning is given below.

DEFINITION 4.2.   Let $2 \leq k \leq n$. Then a predictable H-valued stochastic process $u_k = u_k(t), t \geq 0$ is a solution of the problem (2.4) (i.e., (4.13)) if almost surely it satisfies the following integral equation

$$u_k(t) = \int_0^t e^{(t-s)A} \nabla F(\phi(s))[u_k(s)]ds + \int_0^t \Phi_k(s)ds, \quad t \geq 0, \ 2 \leq k \leq n,$$

with ϕ as in Proposition 3.7 and Φ_k as in (4.12) and (4.14).

In the following result we estimate the norm of Φ_k in H by means of the norms of the Frechet derivatives of F and the norms of $v_j(t)$, $j = 1, \ldots, k-1$, where $v_j(t)$ are H-valued stochastic processes.

LEMMA 4.3.   *Let us fix $2 \le k \le n$; let $w(t)$ and $v_1(t), \ldots, v_{k-1}(t)$ be respectively a K-valued process and H-valued stochastic processes. Then $\Phi_k(w(t))\big[v_1(t), \ldots, v_{k-1}(t)\big]$ as in (4.12) satisfies the following inequality*

$$|\Phi_k(w(t))[v_1(t), \ldots, v_{k-1}(t)]|_H \le C(1 + |w(t)|_K^{m-2})k^2(k + |v_1(t)|_H^{k-1} + \cdots + |v_{k-1}(t)|_H^{k-1}),$$

where C is some positive constants depending on k and the constant γ_j, $j = 2, \ldots, k$ introduced in Hypothesis 3.1.

PROOF.   We have

$$
\begin{aligned}
&|\Phi_k(w(t))[v_1(t), \ldots, v_{k-1}(t)]|_H \\
(4.15) \qquad &= \left| \sum_{j=2}^{k} \sum_{i_1+\cdots+i_j=k} \frac{\nabla^{(j)} F(w(t))[v_{i_1}(t), \ldots, v_{i_j}(t)]}{j!} \right|_H \\
&\le \sum_{j=2}^{k} \sum_{i_1+\cdots+i_j=k} \left| \frac{\nabla^{(j)} F(w(t))[v_{i_1}(t), \ldots, v_{i_j}(t)]}{j!} \right|_H
\end{aligned}
$$

and using the assumption (3) in Hypothesis 3.1, we get

$$
\begin{aligned}
|\Phi_k(t)|_H &\le \sum_{j=2}^{k} \sum_{i_1+\cdots+i_j=k} \frac{1}{j!} \|\nabla F^{(j)}(w(t))\|_{L^j(H)} \prod_{l=1}^{j} |v_{i_l}(t)|_H \\
&\le \sum_{j=2}^{k} \frac{1}{j!} \gamma_j (1 + |w(t)|_K)^{m-j} \sum_{i_1+\cdots+i_j=k} \sum_{l=1}^{j} |v_{i_l}(t)|_H^j \\
(4.16) \qquad &\le \sum_{j=2}^{k} \frac{1}{j!} \gamma_j (1 + |w(t)|_K)^{m-j} \sum_{i_1+\cdots+i_j=k} \left(j + \sum_{l=1}^{k-1} |v_l(t)|_H^{k-1} \right) \\
&\le \sum_{j=2}^{k} \frac{1}{j!} \gamma_j (1 + |w(t)|_K)^{m-j} k^2 \left(k + \sum_{l=1}^{k-1} |v_l(t)|_H^{k-1} \right) \\
&\le C(1 + |w(t)|_K^{m-2}) k^2 \left(k + \sum_{l=1}^{k-1} |v_l(t)|_H^{k-1} \right),
\end{aligned}
$$

for some positive constant C, from which the assertion in Lemma 4.3 follows.   □

REMARK 4.4.   Notice that by Lemma 4.3, if $v_1, \ldots, v_{k-1}$ are p-mean ($p \in [2, \infty)$), integrable continuous stochastic processes then the same holds for Φ_k.

5. Main results.

PROPOSITION 5.1. *Under Hypothesis* 3.1 *the following stochastic differential equation*:

$$
(5.1) \quad \begin{cases} du_1(t) = [Au_1(t) + \nabla F(\phi(t))[u_1(t)]]dt + \sqrt{Q}dW(t), & t \in [0, +\infty), \\ u_1(0) = 0, \end{cases}
$$

has, with ϕ as in Proposition 3.7, *a unique mild solution satisfying, for any $p \geq 2$, the following estimate*:

$$
(5.2) \quad E\left[\sup_{t \in [0,T]} |u_1(t)|_H^p\right] < +\infty, \quad \text{for any } T > 0.
$$

PROOF. First we show the uniqueness. Let us suppose that $w_1(t)$ and $w_2(t)$ are two solutions of (5.1). Then by Itô's formula we have:

$$
d|w_1(t) - w_2(t)|_H^2 = \langle A(w_1(t) - w_2(t)), w_1(t) - w_2(t)\rangle \, dt \\ + \langle \nabla F(\phi(t))[w_1(t) - w_2(t)], w_1(t) - w_2(t)\rangle \, dt,
$$

so that, by the dissipativity condition on A and the estimate on ∇F in Hypothesis 3.1, (3), we have

$$
d|w_1(t) - w_2(t)|_H^2 \leq -\omega|w_1(t) - w_2(t)|_H^2 + \gamma_1(1 + |\phi|_K^{m-1})|w_1(t) - w_2(t)|_H^2.
$$

Now uniqueness follows by applying Gronwall's lemma.

As far as the existence is concerned, we proceed by a fixed point argument. We introduce the mapping Γ from $\mathcal{L}^p(\Omega; C([0, T]; H))$ into itself defined by

$$
\Gamma(w(t)) := \int_0^t e^{(t-s)A}\nabla F(\phi(s)))[w(s)]ds + W_A(t).
$$

We are going to prove that there exists $\tilde{T} > 0$ such that Γ is a contraction on $\mathcal{L}^p(\Omega; C([0, \tilde{T}]; H))$. In fact, for any $v, w \in \mathcal{L}^p(\Omega; C([0, \tilde{T}]; H))$ we have, for any $0 \leq t \leq \tilde{T}$:

$$
\|\Gamma(v(t)) - \Gamma(w(t))\|^p = E\left[\sup_{t \in [0,\tilde{T}]} \left|\int_0^t e^{(t-s)A}\nabla F(\phi(s)))[v(s) - w(s)]ds\right|_H^p\right]
$$

$$
\leq E\left[\sup_{t \in [0,\tilde{T}]} \int_0^t \|e^{(t-s)A}\|_{L(H)}^p \left|\nabla F(\phi(s))[v(s) - w(s)]\right|_H^p ds\right]
$$

$$
\leq E\left[\sup_{s \in [0,\tilde{T}]} |\nabla F(\phi(s))[v(s) - w(s)]|_H^p\right] \int_0^{\tilde{T}} \|e^{(\tilde{T}-s')A}\|_{L(H)}^p ds'
$$

$$
\leq E\left[\sup_{s \in [0,\tilde{T}]} |v(s) - w(s)|_H^p\right] \gamma_1^p \left(1 + |\phi(s)|_K^{m-1}\right)^p \frac{1}{\omega p}\left(1 - e^{-\omega p\tilde{T}}\right)
$$

$$
\leq \gamma_1^p (1 + |u^0|_K^{m-1})^p \|v - w\|^p \frac{1}{\omega p}\left(1 - e^{-\omega p\tilde{T}}\right),
$$

where we used condition (3) in Hypothesis 3.1 for the third inequality and Proposition 3.7 for the last inequality. Then if $\tilde{T}$ is sufficiently small (depending on ω, p, γ_1, ϕ), we see that Γ is a contraction on $\mathcal{L}^p(\Omega; C([0, \tilde{T}]; H))$.

By considering the map Γ on intervals $[0, \tilde{T}]$, $[\tilde{T}, 2\tilde{T}]$, ..., $[(N - 1)\tilde{T}, T]$, $\tilde{T} \equiv T/N$, $N \in \mathbf{N}$, we have that Γ is a contraction on $\mathcal{L}^p(\Omega; C([0, T]; H))$ and hence we have the existence and the uniqueness of the solution for the equation (5.1) in the space $\mathcal{L}^p(\Omega; C([0, T]; H))$ for any $p \in [2, \infty)$.

Let us now consider the estimate (5.2). By condition (3) in Hypothesis 3.1 we have for all points in the probability space and $p = 2a$ with $a \in \mathbf{N}$:

(5.3)
$$
\begin{aligned}
\frac{d}{dt}|u_1(t)|_H^{2a} &= 2a\langle Au_1(t), u_1(t)\rangle|u_1(t)|_H^{2a-2} + 2a\langle \nabla F(\phi(t))[u_1(t)], u_1(t)\rangle|u_1(t)|_H^{2a-2} \\
&\quad + 2a\langle W_A(t), u_1(t)\rangle|u_1(t)|_H^{2a-2} \\
&\leq -2a\omega|u_1(t)|_H^{2a} + 2a\gamma(1 + |u^0|_K^{m-1})|u_1(t)|_H^{2a} + 2a\langle W_A(t), u_1(t)\rangle|u_1(t)|_H^{2a-1} \\
&\leq -2a\tilde{\omega}|u_1(t)|_H^{2a} + C_a|W_A(t)|_H^{2a},
\end{aligned}
$$

where $\tilde{\omega} := \omega - \gamma(1 + |u^0|_H)$. By Hypothesis 3.5 we have that:

$$
\mathbf{E}\left[\sup_{t \in [0,T]} |W_A(t)|_H^{2a}\right] \leq C_a', \quad T > 0
$$

(first with K replacing H, but then with H, due to the assumption on H, K). C_a' is some positive constant. Integrating on $[0, T]$ both sides in (5.3), taking the expectation of both members in the inequality and applying Gronwall's lemma to (5.3) we obtain:

$$
\mathbf{E}\left[\sup_{t \in [0,T]} |u_1(t)|_H^{2a}\right] \leq C_{a,T}' e^{-2a\tilde{\omega}T} < C_{a,T},
$$

where $C_{a,T}$ is a positive constant and (5.2) follows. $\square$

THEOREM 5.2. *Let us fix $2 \leq k \leq n$, assume that Hypothesis 3.1 holds, and let u_1 be the solution of the problem (2.3). Suppose moreover that u_j is the unique mild solution of the following Abstract Cauchy Problem* (ACP):

(ACP$_j$) $\qquad \begin{cases} du_j(t) = [Au_j(t) + \nabla F(\phi(t))[u_j(t)]]dt + \Phi_j(t)dt, \\ u_j(0) = 0 \end{cases}$

for $j = 2, \ldots, k - 1$ satisfying:

(5.4) $\qquad \mathbf{E}\left[\sup_{t \in [0,T]} |u_j(t)|_H^p\right] < +\infty, \quad T > 0, \ for \ any \ p \in [2, \infty);$

then there exists a unique mild solution $u_k(t)$ of the following non-homogeneous linear differential equation with stochastic coefficients (*in the sense of Definition 4.2*) :

(ACP$_k$) $\qquad \begin{cases} du_k(t) = [Au_k(t) + \nabla F(\phi(t))[u_k(t)]]dt + \Phi_k(t)dt, \quad t \in [0, +\infty), \\ u_k(0) = 0 \end{cases}$

and it satisfies the following estimate, for any $T > 0$:

$$(5.5) \qquad E\left[\sup_{t \in [0,T]} |u_k(t)|_H^p\right] < +\infty.$$

PROOF. We proceed by a fixed point argument, where the contraction is given by

$$\Gamma(y(t)) := \int_0^t e^{(t-s)A} \nabla F(\phi(t))[y(t)]ds + \int_0^t e^{(t-s)A} \Phi_k(s)ds$$

on $\mathcal{L}^p(\Omega; C([0,T];H))$. In fact, arguing as in Proposition 5.1, we see that for $\tilde{T} \in [0,T]$ sufficiently small, Γ is a contraction on $\mathcal{L}^p(\Omega; C([0,\tilde{T}];H))$, $p \in [2,\infty)$, so that the existence and the uniqueness of the solution for (ACP$_k$) follows.

Let us consider the estimate (5.5). By the condition (4) in Hypothesis 3.1 we have, for $p = 2a$ with $a \in N$ (and all points in the probability space) :

$$(5.6)$$

$$\begin{aligned}
\frac{d}{dt}|u_k(t)|_H^{2a} &= 2a\langle Au_k(t), u_k(t)\rangle |u_k(t)|_H^{2a-2} + 2a\langle \nabla F(\phi(t))[u_k(t)], u_k(t)\rangle |u_k(t)|_H^{2a-2} \\
&\quad + 2a\langle \Phi_k(t), u_k(t)\rangle |u_k(t)|_H^{2a-2} \\
&\leq -2a\omega|u_k(t)|_H^{2a} + 2a\gamma(1 + |u^0|_K)|u_k(t)|_H^{2a} + 2a|\Phi_k(t)|_H |u_k(t)|_H^{2a-1} \\
&\leq -2a\tilde{\omega}|u_k(t)|_H^{2a} + C_a|\Phi_k(t)|_H^{2a},
\end{aligned}$$

where $\tilde{\omega} := \omega - \gamma(1 + |u^0|_K)$ as in the proof of Proposition (5.1). By the assumption (5.4) made on $u_j(t)$, $j = 1, \ldots, k-1$ and Lemma 4.3 we have that:

$$E\left[\sup_{t \in [0,T]} |\Phi_k(t)|_H^{2a}\right] \leq C_a', \quad T > 0,$$

so that taking the expectation of inequality (5.6) and applying Gronwall's lemma (similarly as in the proof of Proposition 5.1) we obtain:

$$E\left[\sup_{t \in [0,T]} |u_k(t)|_H^{2a}\right] \leq C_a' e^{-2a\tilde{\omega}T} < C_a,$$

where C_a is a positive constant, and the theorem follows.   □

We are now able to state the main result of this section:

THEOREM 5.3. *Under Hypothesis* 3.1 *the mild solution* $u(t)$ *of* (2.2) (*in the sense of Definition* 3.4) *can be expanded in powers of* $\varepsilon > 0$ *in the following form*

$$u(t) = \phi(t) + \varepsilon u_1(t) + \cdots + \varepsilon^n u_n(t) + R_n(t,\varepsilon), \quad n \in N,$$

where u_1 is the solution of

$$\begin{cases} du_1(t) = [Au_1(t) + \nabla F(\phi(t))[u_1(t)]]dt + \sqrt{Q}dW(t), \\ u_1(0) = 0, \end{cases}$$

while u_k, $k = 2, \ldots, n$ is the solution of

$$(\text{ACP}_k) \qquad \begin{cases} du_k(t) = [Au_k(t) + \nabla F(\phi(t))[u_k(t)]]dt + \Phi_k(t)dt\,, \\ u_k(0) = 0\,. \end{cases}$$

The remainder $R_n(t, \varepsilon)$ is defined by

$$R_n(t, \varepsilon) := u(t) - \phi(t) - \sum_{k=1}^{n} \varepsilon^k u_k(t)$$

$$(5.7) \qquad = \int_0^t e^{(t-s)A}\left(F(u(s)) - F(\phi(s)) - \sum_{k=1}^{n} \varepsilon^k \nabla F(\phi(s))[u_k(s)] \right.$$

$$\left. - \sum_{k=2}^{n} \varepsilon^k \Phi_k(s) \right)ds\,,$$

and verifies the following inequality

$$E\left[\sup_{t\in[0,T]} |R_n(t,\varepsilon)|_H^p \right] \le C_p \varepsilon^{n+1}\,,$$

with a constant $C_p > 0$.

PROOF.   Let us define $R_n(t, \varepsilon)$, $n \in N$, as stated in the theorem. Since by construction

- $\phi(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(\phi(s))ds$ (cf. Definition 3.3);
- $u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(u(s))ds + \varepsilon W_A(t)$ (cf. Definition 3.4);
- $u_1(t) = \int_0^t e^{(t-s)A}\nabla F(\phi(s))[u_1(s)]ds + W_A(t)$ (cf. Proposition 5.1 and Definition 3.4);
- $u_k(t) = \int_0^t e^{(t-s)A}\nabla F(\phi(s))[u_k(s)]ds + \int_0^t e^{(t-s)A}\Phi_k(s)ds$ for $k = 2, \ldots, n$, with $\Phi_k(s) := \Phi_k(\phi(s))[u_1(s), \ldots, u_{k-1}(s)]$ defined in (4.14) (cf. Theorem 5.2 and Definition 3.4);

we have

$$R_n(t, \varepsilon) = \int_0^t e^{(t-s)A}\left(F(u(s)) - F(\phi(s)) - \sum_{k=1}^{n} \varepsilon^k \nabla F(\phi(s))[u_k(s)] - \sum_{k=2}^{n} \varepsilon^k \Phi_k(s) \right)ds\,.$$

Recalling that $R_1^{(n)}(\phi(s); h(s), \varepsilon) = F(u(s)) - F(\phi(s)) - \sum_{k=1}^{n} \varepsilon^k \nabla F(\phi(s))[u_k(s)] - \sum_{k=2}^{n} \varepsilon^k \Phi_k(s)$ we get:

$$E\left[\sup_{t\in[0,T]} |R_n(t,\varepsilon)|_H^p \right] \le E\left[\sup_{t\in[0,T]} \left| \int_0^t e^{(t-s)A} R_1^{(n)}(\phi(s); h(s), \varepsilon)ds \right|_H^p \right]$$

$$(5.8) \qquad \le E\left[\sup_{t\in[0,T]} \int_0^t \|e^{(t-s)A}\|_{L(H)}^p |R_1^{(n)}(\phi(s); h(s), \varepsilon)|_H^p ds \right]$$

$$\le E\left[\sup_{t\in[0,T]} |R_1^{(n)}(\phi(t); h(t), \varepsilon)|_H^p \int_0^t e^{-\omega(t-s)p}ds \right]$$

$$\le C_{n,p} \varepsilon^{p(n+1)},$$

for some positive constant $C_{n,p}$ (depending on n, p, but not on ε), where in the second and third inequality we have used the contraction property of the semigroup generated by A. Now recalling Lemma 4.1 the inequality in Theorem 5.3 follows.   □

EXAMPLE 5.4.   Our results apply in particular to stochastic PDEs describing the FitzHugh-Nagumo equation with a Gaussian noise perturbation (as those studied, for example, in [38, 39, 40] and [12]).

The reference equation is given by (see [12, equation (1.1)])

$$
(5.9) \quad
\begin{cases}
\partial_t v(t, x) = \partial_x(c(x)\partial_x v(t, x)) - p(x)v(t, x) - w(t, x) + f(v(t, x)) + \varepsilon\dot{\beta}_1(t, x), \\
\partial_t w(t, x) = \gamma v(t, x) - \alpha w(t, x) + \varepsilon\dot{\beta}_2(t, x), \\
\partial_x v(t, 0) = \partial_x v(t, 1) = 0, \\
v(0, x) = v_0(x), \quad w(0, x) = w_0(x),
\end{cases}
$$

with the parameter $\varepsilon > 0$ in front of the noise, where u, w are real valued random variables, α, γ are strictly positive phenomenological constants and c, p are strictly positive smooth functions on $[0, 1]$. Moreover, the initial values v_0, w_0 are in $C([0, 1])$. The nonlinear term is of the form $f(v) = -v(v - 1)(v - \xi)$, where $\xi \in (0, 1)$. Finally β_1, β_2 are independent Q_i-Brownian motions with values in $L^2(0, 1)$, with Q_i positive trace class commuting operators, commuting also with A_0, A_0 being defined below. The above equation can be rewritten in the form of an infinite dimensional stochastic evolution equation on the space

$$
(5.10) \quad H := L^2(0, 1) \times L^2(0, 1)
$$

by introducing the following operators:

$$
A_0 := \partial_x c(x)\partial_x,
$$

$$
D(A_0) := \{u \in H^2(0, 1); v_x(0) = v_x(1)\},
$$

and

$$
A = \begin{pmatrix} A_0 - p & -I \\ \gamma I & \alpha I \end{pmatrix},
$$

with domain $D(A) := D(A_0) \times L^2(0, 1)$, and

$$
F\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -v(v - 1)(v - \xi) \\ 0 \end{pmatrix}, \quad \text{with } D(F) := L^6(0, 1) \times L^2(0, 1).
$$

Further, we introduce the Banach space $K := C[0, 1] \times L^2(0, 1)$, endowed with the norm $|\cdot|_K := |\cdot|_\infty + |\cdot|_2$ and consider $u^0 \in K$. In this way, the equation (5.9) can be rewritten as

$$
\begin{cases}
du(t) = Au(t) + F(u(t))dt + \sqrt{Q}dW(t), \\
u(0) = u^0 := (v^0, w^0) \in K,
\end{cases}
$$

with A and F satisfying Hypothesis 3.1 when $\xi^2 - \xi + 1 \leq 3\min_{x\in[0,1]} p(x)$. In fact, the properties of the two operators A and F can be determined starting from the problems considered

in [12] and S. Cerrai [13]. In particular from [13, Section 2.2] the estimates on the nonlinear term F and its derivatives can be easily deduced. Moreover we claim that the stochastic convolution

$$W_A(t) := \int_0^t e^{(t-s)A} \mathrm{d}W(s),$$

(where e^{tA}, $t \geq 0$ denotes the semigroup generated by A) is well-defined and admits a continuous version with values into the space K. This fact can be proved by an application of [17, Theorem 5.16] and its proof, taking into account that the domain of fractional powers of A are contained in K (cf. Appendix A - in particular Example A.5.2 - in [17]) and moreover we are assuming Tr $Q < \infty$.

Then by Theorem 5.3 we get an asymptotic expansion in powers of $\varepsilon > 0$ of the solution, in terms of solutions of the corresponding deterministic FitzHugh-Nagumo equation and the solution of a system of (explicit) linear (non homogeneous) stochastic equations. The expansion holds for all orders in $\varepsilon > 0$. The remainders are estimated according to Theorem 5.3. We can use these results to carry through a discussion similar to the one made by Tuckwell [37, 40] in the case where $Q = (Q_1, Q_2)$ with Q_i the identity. Tuckwell, in particular, has made heuristic expansions up to second order in ε for the mean and the variance of the solution process $u = (u(t))_{t \geq 0}$ (see [37, 40]), proving in particular that one has enhancement (respectively reduction) of the mean according to whether the expansion is around which stable point of the stationary deterministic equation.

In a future work [2] we shall apply these results to the case of networks of FitzHugh-Nagumo neurons. Moreover in the second part of the present work we shall study asymptotic expansions for the case where the dissipativity condition is replaced by other conditions on the non Lipschitz drift term.

REFERENCES

[1]   S. ALBEVERIO AND C. CEBULLA, Synchronizability of stochastic network ensembles in a model of interacting dynamical units, Physica A Stat. Mech. Appl 386 (2007), 503–512.

[2]   S. ALBEVERIO, L. DI PERSIO AND E. MASTROGIACOMO, Invariant measures for stochastic differential equations on networks, in preparation.

[3]   S. ALBEVERIO, V. FATALOV AND V. I. PITERBARG, Asymptotic behavior of the sample mean of a function of the Wiener process and the Macdonald function, J. Math. Sci. Univ. Tokyo 16 (2009), 55–93.

[4] S. ALBEVERIO, F. FLANDOLI AND Y. G. SINAI, SPDE in hydrodynamic: recent progress and prospects, Lectures given at the C.I.M.E. Summer School held in Cetraro, August 29–September 3, 2005, Edited by G. Da Prato and M. Röckner, Lecture Notes in Mathematics, vol. 1942, Springer-Verlag, Berlin, 2008.

[5] S. ALBEVERIO AND S. LIANG, Asymptotic expansions for the Laplace approximations of sums of Banach space-valued random variables, Ann. Probab. 33 (2005), 300–336.

[6] S. ALBEVERIO, Wiener and Feynman-path integrals and their applications, Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994), pp. 153–194. Proc. Sympos. Appl. Math. 52, Amer. Math. Soc., Providence, RI, 1997.

[7] S. ALBEVERIO, E. LYTVYNOV AND A. MAHNIG, A model of the term structure of interest rates based on Lévy fields, Stochastic Process. Appl. 114 (2004), 251–263

[8] S. ALBEVERIO AND S. MAZZUCCHI, The trace formula for the heat semigroup with polynomial potential, Proc. Seminar Stochastic Analysis, Random Fields and Applications VI, Ascona 2008, (Edited by R. Delang, M. Dozzi, F. Russo.), pp. 3–22, Birkhäuser, Basel, 2011.

[9] S. ALBEVERIO AND M. RÖCKNER, Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms, Probab. Theory Related Fields 89 (1991), 347–386.

[10] S. ALBEVERIO, H. RÖCKLE AND V. STEBLOVSKAYA, Asymptotic expansions for Ornstein-Uhlenbeck semigroups perturbed by potentials over Banach spaces, Stochastics Stochastics Rep. 69 (2000), 195–238.

[11] S. ALBEVERIO AND V. STEBLOVSKAYA, Asymptotics of infinite-dimensional integrals with respect to smooth measures. I, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 2 (1999), 529–556.

[12] S. BONACCORSI AND E. MASTROGIACOMO, Analysis of the stochastic FitzHugh-Nagumo system, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 11 (2008), 427–446.

[13] S. CERRAI, Differentiability of Markov semigroups for stochastic reaction-diffusion equations and applications to control, Stochastic Process. Appl. 83 (1999), 15–37.

[14] S. CERRAI AND M. FREIDLIN, Smoluchowski-Kramers approximation for a general class of SPDEs, J. Evol. Equ. 6 (2006), 657–689.

[15] S. CARDANOBILE AND D. MUGNOLO, Analysis of a FitzHugh-Nagumo-Rall model of a neuronal network, Math. Methods Appl. Sci. 30 (2007), 2281–2308.

[16] G. DA PRATO AND L. TUBARO, Self-adjointness of some infinite-dimensional elliptic operators and application to stochastic quantization, Probab. Theory Related Fields 118 (2000), 131–145.

[17] G. DA PRATO AND J. ZABCZYK, Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, Cambridge, 1992.

[18] G. DA PRATO AND J. ZABCZYK, Ergodicity for infinite-dimensional systems, London Mathematical Society Lecture Note Series 229, Cambridge University Press, Cambridge, 1996.

[19] J.-D. DEUSCHEL AND D. W. STROOCK, Large deviations, Pure and Applied Mathematics 137, Academic Press Inc., Boston, MA, 1989.

[20] A. DEMBO AND O. ZEITOUNI, Large deviations techniques and applications, Applications of Mathematics 38, Second edition, Springer-Verlag, New York, 1998.

[21] B. FORSTER, E. LÜTKEBOHMERT AND J. TEICHMANN, Absolutely continuous laws of jump-diffusions in finite and infinite dimensions with applications to mathematical finance, SIAM J. Math. Anal. 40 (2008/09), 2132–2153.

[22] Y. INAHAMA AND H. KAWABI, Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths, J. Funct. Anal. 243 (2007), 270–322.

[23] Y. INAHAMA AND H. KAWABI, On the Laplace-type asymptotics and the stochastic Taylor expansion for Itô functionals of Brownian rough paths, Proceedings of RIMS Workshop on Stochastic Analysis and Applications, 139–152, RIMS Kôkyûroku Bessatsu, B6, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

[24] N. IKEDA AND S. WATANABE, Stochastic differential equations and diffusion processes, North-Holland Mathematical Library 24, Second edition, North-Holland Publishing Co., Amsterdam, 1989.

[25] G. JONA-LASINIO AND P. K. MITTER, Large deviation estimates in the stochastic quantization of ϕ_2^4, Comm.

Math. Phys. 130 (1990), 111–121.

[26]   G. JONA-LASINIO AND P. K. MITTER, On the stochastic quantization of field theory, Comm. Math. Phys. 101 (1985), 409–436.

[27]   A. N. KOLMOGOROV AND S. V. FOMIN, Elements of the theory of functions and functional analysis. Vol. 2: Measure. The Lebesgue integral. Hilbert space, Translated from the first (1960) Russian ed. by H. Kamel and H. Komm, Graylock Press, Albany, N.Y., 1961.

[28]   K. LEHNERTZ, J. ARNHOLD, P. GRASSBERGER AND C. E. ELGER, Chaos in Brain ?, World Scientific, Singapore, 2000.

[29]   G. E. LADAS AND V. LAKSHMIKANTHAM, Differential equations in abstract spaces, Mathematics in Science and Engineering 85, Academic Press, New York, 1972.

[30]   R. MARCUS, Parabolic Itô equations, Trans. Amer. Math. Soc. 198 (1974), 177–190.

[31]   R. MARCUS, Parabolic Itô equations with monotone nonlinearities, J. Funct. Anal. 29 (1978), 275–286.

[32]   P. MALLIAVIN AND S. TANIGUCHI, Analytic functions, Cauchy formula, and stationary phase on a real abstract Wiener space, J. Funct. Anal. 143 (1997), 470–528.

[33]   G. PARISI, Statistical field theory, Frontiers in Physics 66, With a foreword by David Pines, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, MA, 1988.

[34]   C. ROVIRA AND S. TINDEL, Sharp Laplace asymptotics for a parabolic SPDE, Stochastics Stochastics Rep. 69 (2000), 11–30.

[35]   M. SCHILDER, Some asymptotic formulas for Wiener integrals, Trans. Amer. Math. Soc. 125 (1966), 63–85.

[36]   B. SIMON, Functional integration and quantum physics, Second edition, AMS Chelsea Publishing, Providence, RI, 2005.

[37]   H. C. TUCKWELL, Analytical and simulation results for the stochastic spatial FitzHugh-Nagumo model neuron, Neural Comput. 20 (2008), 3003–3033.

[38]   H. C. TUCKWELL, Introduction to theoretical neurobiology. Vol. 1, Linear cable theory and dendritic structure, Cambridge Studies in Mathematical Biology 8, Cambridge University Press, Cambridge, 1988.

[39]   H. C. TUCKWELL, Introduction to theoretical neurobiology. Vol. 2, Nonlinear and stochastic theories, Cambridge Studies in Mathematical Biology 8, Cambridge University Press, Cambridge, 1988.

[40]   H. C. TUCKWELL, Random perturbations of the reduced FitzHugh-Nagumo equation, Phys. Scripta 46 (1992), 481–484.

[41]   S. WATANABE, Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels, Ann. Probab. 15 (1987), 1–39.

INSTITUTE FOR APPLIED MATHEMATICS
UNIVERSITY OF BONN
HCM; SFB611, BIBOS, IZKS
D-53115 BONN
GERMANY

E-mail address: albeverio@uni-bonn.de

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TRENTO
VIA SOMMARIVE 14
38050 POVO, TRENTO
ITALY

E-mail address: dipersio@science.unitn.it

POLITECNICO DI MILANO
DIPARTIMENTO DI MATEMATICA
F. BRIOSCHI, PIAZZA LEONARDO DA VINCI
32, 20133 MILANO
ITALY

E-mail address: elisa.mastrogiacomo@polimi.it