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COLLAPSING THREE-MANIFOLDS WITH A LOWER CURVATURE BOUND

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Abstract. We survey works on collapsing Riemannian manifolds with a lower bound of sectional curvature, focusing on the three-dimensional case. We also explain the basics of Seifert manifolds and Alexandrov spaces quickly and a key idea of our proof of the volume collapsing theorem.

Introduction. In 2002 and 2003, G. Perelman published three preprints [24, 25, 26], in which he gave the proofs of the Poincaré conjecture and Thurston's geometrization conjecture. In his proofs, he used a claim on collapsing Riemannian manifolds, say the *Volume Collapsing Theorem* (see Theorem 1.8). However, he did not publish the proof of that. It is a coincidence that the author and Takao Yamaguchi together had proved the Volume Collapsing Theorem at the same time, for which we published the first version of a preprint in the same year 2003 and the final version had been published as [29]. After our paper, Bessires-Besson-Boileau-Maillot-Porti [1] gave another proof of it in a different approach. Morgan-Tian [19] and J. Cao-Ge [4] presented a detailed proof based on our proof. Kleiner-Lott [17] gave a proof similar to ours, but with some original ideas. Our proof completely depends on our previous paper [28] published in 2000, in which we considered the case where the diameter is bounded above. A key idea of our proof is 'the critical point-rescaling argument', which is already contained in [28]. Further, the prior works [9, 30] of Fukaya and Yamaguchi were necessary to achieve all of our results. In this article, we first survey works on collapsing Riemannian manifolds with a lower bound of sectional curvature, focusing on the three-dimensional case. We also explain the basics of Seifert manifolds and Alexandrov spaces quickly, and the key idea of the proof of the Volume Collapsing Theorem, i.e., the critical point-rescaling argument.

In the case where the sectional curvature is not only bounded from below but also from above, we obtain a more complete information on the topology of manifolds (cf. [5]), which we do not mention in this article. We emphasize that the study (the technique and the idea of proofs) of collapsing manifolds in the case of only a lower curvature bound and that in the case of a bound of the absolute values of curvatures are completely different from each other. The difference comes from the difference between geometries of (almost) nonnegatively curved manifolds and (almost) flat manifolds in a sense. Flat manifolds are completely characterized

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by the Deck transformation on the universal covering space $\boldsymbol{R}^n$, but nonnegatively curved manifolds are much more difficult to characterize. We have yet to know the complete structure of nonnegatively curved manifolds. We refer to Fukaya's article [8] for a more comprehensive survey on Riemannian geometry.

1. Survey of collapsing Riemannian manifolds. Throughout this article, all manifolds are assumed to be connected. For given constants $n \in \boldsymbol{N}$ and $D > 0$, let $\mathcal{M}(n, D)$ denote the set of the isometry classes of n-dimensional closed Riemannian manifolds of sectional curvature $K_M \geq -1$ and diameter $\operatorname{diam}(M) \leq D$. We have the following theorem.

THEOREM 1.1 (Finiteness Theorem; Grove-Petersen [12] (see also Grove-Petersen-Wu [13], Perelman [22])). *For any number $v > 0$, the set of $M \in \mathcal{M}(n, D)$ with $\operatorname{vol}(M) \geq v$ contains at most finitely many homeomorphism types, where $\operatorname{vol}(M)$ indicates the volume of M.*

On the other hand, we have infinitely many topological types of manifolds in $\mathcal{M}(n, D)$ with small volume. One of our motivations to study collapsing Riemannian manifolds is to classify the topology of such manifolds. For that purpose, taking a sequence of manifolds $M_i \in \mathcal{M}(n, D)$, $i = 1, 2, \dots$, with volume converging to zero, we investigate the topology of M_i for large i. The Gromov compactness theorem (cf. [11]) implies that, by replacing $\{M_i\}$ with a subsequence, the sequence $\{M_i\}$ converges to a compact metric space X with respect to the Gromov-Hausdorff convergence (see Definition 3.1). The limit X becomes an Alexandrov space whose dimension is an integer less than n (see Theorem 3.11). In this stuation, we say that M_i *collapses to* X. We consider the two following problems.

- What sort of a space is the limit Alexandrov space X?
- What is the relation between X and M_i for large i?

Solving these two problems, we are able to determine the topology of M_i for large i. If we remove the condition that the volume of M_i converges to zero, then the dimension of the limit X is less than or equal to n. If it is equal to n, then M_i does not collapse to X, (but converges to X) and M_i for sufficiently large i is homeomorphic to X, in which case X becomes an n-dimensional topological manifold (see Perelman's Stability Theorem 4.6 in §4).

In the situation where M_i collapses to X, we have the following important theorem.

THEOREM 1.2 (Yamaguchi's Fibration Theorem [30]). *If a sequence $\{M_i\}$ in $\mathcal{M}(n, D)$ converges to a complete Riemannian manifold N with respect to the Gromov-Hausdorff convergence, then, for every sufficiently large i, we have the following*:

(1) *M_i is diffeomorphic to a fiber bundle over N.*

(2) *The first Betti number, say b_1, of a fiber is less than or equal to $n - \dim N$.*

(3) *There exists a finite covering space of a fiber that is diffeomorphic to a fiber bundle over a torus T^{b_1} of dimension b_1.*

The Hopf fibration $S^1 \to S^{2n+1} \to \boldsymbol{C}P^n$ gives an example of the theorem, where S^k denotes a k-dimensional sphere and $\boldsymbol{C}P^n$ the n-dimensional complex projective space. There is a sequence of Riemannian metrics $g_1, g_2, \dots$ of positive sectional curvature on S^{2n+1} such that (S^{2n+1}, g_i) collapses to $\boldsymbol{C}P^n$.

In general, a limit of a sequence $M_i \in \mathcal{M}(n, D)$ has singularity. However, if we have no singularity in the limit Alexandrov space, then the theorem says that, for sufficiently large i, M_i becomes a fiber bundle over the limit. We remark that an Alexandrov space without singularity becomes a complete C^0 Riemannian manifold [21] and Yamaguchi's Fibration Theorem is true even in the case where N is a complete C^0 Riemannian manifold.

If the limit space consists of a single point, we have the following.

THEOREM 1.3 (Fukaya-Yamaguchi [9]). *For any natural number $n \geq 2$, there exists a constant $\delta_n > 0$ such that any manifold $M \in \mathcal{M}(n, \delta_n)$ satisfies the following* (1) *and* (2).

(1) *The fundamental group $\pi_1(M)$ of M has a nilpotent subgroup of finite index.*

(2) *If $n = 3$, then M has a finite covering space homeomorphic to either $S^1 \times S^2$, T^3, a nil-manifold, or a homotopy sphere.*

REMARK 1.4. (1) By the Poincaré conjecture solved by Perelman, a homotopy sphere in the above theorem can be replaced with a sphere.

(2) If two three-dimensional manifolds are homeomorphic to each other, then they are diffeomorphic to each other.

(3) The theorem can be considered to be an extension of Gromov's almost flat theorem [10] in the sense of π_1.

(4) A fiber of M_i in Yamaguchi's Fibration Theorem 1.2 is not necessarily totally geodesic, but is almost totally geodesic in some sense, so that the fiber has curvature bounded below in a generalized sense and satisfies (1) and (2) of Theorem 1.3.

In the case where the limit Alexandrov space has singularity, we have the following theorem for three-dimensional M_i. We explain Seifert manifolds in §2 below. Denote a two-dimensional disk by D^2.

THEOREM 1.5 (Shioya-Yamaguchi [28]). *Assume that a sequence of three-dimensional orientable Riemannian manifolds $M_i \in \mathcal{M}(3, D)$ converges to a two-dimensional Alexandrov space X.* (*It is known that such an X is a two-dimensional topological manifold possibly with boundary ∂X.*) *For every sufficiently large i, we have the following*:

(1) *If X has no boundary, then M_i is homeomorphic to a Seifert manifold over X.*

(2) *If X has non-empty boundary, then M_i is homeomorphic to the gluing of a Seifert manifold over X and $\partial X \times D^2$ along their boundaries such that the fiber of M_i over each point $p \in \partial X$ is glued with $\{p\} \times \partial D^2$.*

(3) *In both cases of* (1) *and* (2), *the orbit type, say (μ, ν), of the fiber over each point $p \in X \setminus \partial X$ satisfies $\mu \leq 2\pi / L(\Sigma_p)$, where $L(\Sigma_p)$ is the length of the space of directions Σ_p at p of X.*

See §3 for the definition of the space of directions Σ_p.

We next consider the case where a sequence of orientable $M_i \in \mathcal{M}(3, D)$ converges to a one-dimensional Alexandrov space X. In this case, the limit X is isometric to either a circle or a line segment (see Proposition 3.12). If X is a circle, then Yamaguchi's Fibration Theorem implies that M_i for large i is a fiber bundle over S^1 and its fiber is either S^2 or T^2.

Let us describe a result in the case where X is a line segment. In the following, P^3 denotes a three-dimensional real projective space and $M\#N$ the connected sum of two manifolds M and N. Let $\text{Mö}\tilde{\times}S^1$ be the twisted S^1-bundle over the Möbius band Mö. (Although the usual product space $\text{Mö} \times S^1$ is not orientable, yet there exists an orientable S^1-bundle over the Möbius band Mö, which is called the *twisted* S^1*-bundle over Mö*.)

THEOREM 1.6 (Shioya-Yamaguchi [28]). *Assume that a sequence of three-dimensional orientable Riemannian manifolds* $M_i \in \mathcal{M}(3, D)$ *converges to a line segment. Then, for every sufficiently large* i, M_i *is homeomorphic to either* S^3, P^3, $P^3\#P^3$, *or* $U \cup V$, *where* U *and* V *are either* $S^1 \times D^2$ *or* $\text{Mö}\tilde{\times}S^1$ *and* $U \cup V$ *is the gluing of them along their boundaries.*

We have a four-dimensional version of the above theorems.

THEOREM 1.7 (Yamaguchi [31]). *If a sequence of four-dimensional orientable Riemannian manifolds* $M_i \in \mathcal{M}(4, D)$ *converges to an Alexandrov space* X, *then* M_i *has a fibration structure over* X *for every sufficiently large* i.

We here omit the definition of a fibration structure.

We hope to extend it to the case of general dimension, which seems to be a difficult problem.

The following is a corollary to Theorems 1.3 (2), 1.5 and 1.6: There is a number $v(D) > 0$ depending on a real number $D > 0$ such that if an orientable manifold $M \in \mathcal{M}(3, D)$ has volume $\mathrm{vol}(M) < v(D)$, then either M is a graph manifold or the fundamental group of M is finite. A *graph manifold* is defined to be a three-dimensional topological manifold that is a gluing of finitely many Seifert manifolds with boundary tori, where the gluing is obtained along boundary tori.

In fact, $v(D)$ can be taken to be independent of D, i.e., we have the following theorem.

THEOREM 1.8 (Volume Collapsing Theorem; Perelman [25], Shioya-Yamaguchi [29]). *There exists a universal constant* $v_0 > 0$ *such that if a three-dimensional orientable closed Riemannian manifold* M *has sectional curvature* $K_M \geq -1$ *and volume* $\mathrm{vol}(M) < v_0$, *then we have one of the following* (1) *and* (2).

(1) M is a graph manifold.

(2) The fundamental group of M is finite.

In [25], Perelman claimed (a version of) this theorem and used it in his proof of the geometrization conjecture. However, he did not publish his proof of Theorem 1.8. Our proof in [29] is based on those of Theorems 1.5 and 1.6 proved in [28].

REMARK 1.9. (1) Perelman stated the theorem under a weaker assumption, called 'local lower curvature bound'. However, our proof works without any essential modification under local lower curvature bound.

(2) By the geometrization conjecture, any closed manifold satisfying (2) in the theorem has a Riemannian metric of positive constant curvature and becomes a Seifert manifold. Therefore, (1) in the theorem always holds.

We have the converse to the above theorem in the following sense.

PROPOSITION 1.10 (Cheeger-Gromov [7]). *For any closed graph manifold* M, *there exists a sequence* $\{g_i\}_{i=1,2,\dots}$ *of Riemannian metrics on* M *such that* $|K_{g_i}| \leq 1$ *for all* i *and* $\mathrm{vol}(M, g_i) \to 0$ *as* $i \to \infty$.

Combining Theorem 1.8 with Proposition 1.10 we have the following.

COROLLARY 1.11. *Let* M *be a closed orientable three-dimensional manifold. Then,* M *is a graph manifold if and only if* M *has a Riemannian metric* g *such that* $K_g \geq -1$ *and* $\mathrm{vol}(M, g) < v_0$, *where* v_0 *is as in Theorem* 1.8.

2. Seifert manifolds.

In this section, we define the notions of a Seifert manifold, a Riemann-Seifert manifold, and a graph manifold. A Riemann-Seifert manifold gives a typical example of a collapsing manifold.

Let μ and ν be two coprime integers with $1 \leq \nu < \mu/2$, or $(\mu, \nu) = (1, 0)$. We take a Riemannian metric g_D on the unit disk $D^2 := \{(x, y) \in \boldsymbol{R}^2; x^2 + y^2 \leq 1\}$ that is invariant under the positive rotation of angle $2\pi/\mu$ centered at the origin of $\boldsymbol{R}^2$. Consider the Riemannian manifold $D := (D^2, g_D)$ and the product Riemannian manifold $D \times \boldsymbol{R}$. For $k \in \boldsymbol{Z}$ and $(x, t) \in D \times \boldsymbol{R}$, let $k \cdot x$ be the positively rotated point from x with angle $2\pi k\nu/\mu$ and $k \cdot (x, t) := (k \cdot x, t + kl)$, where l is a positive constant. This defines isometric $\boldsymbol{Z}$-actions on D and on $D \times \boldsymbol{R}$. We denote by $S_l^{\mu,\nu}$ the quotient of $D \times \boldsymbol{R}$ by the $\boldsymbol{Z}$-action. $S_l^{\mu,\nu}$ is a three-dimensional Riemannian manifold diffeomorphic to a solid torus, $D^2 \times S^1$. The product $D \times \boldsymbol{R}$ is the Riemannian universal covering space of $S_l^{\mu,\nu}$ and $D \times [0, l]$ is a fundamental domain. $S_l^{\mu,\nu}$ is also obtained from $D \times [0, l]$ by gluing $D \times \{0\}$ and $D \times \{l\}$ via $(x, 0) \sim (1 \cdot x, l)$ (see Figure 1). We denote by C the quotient of D by the $\boldsymbol{Z}$-action, which is a two-dimensional cone. If $\mu = 1$, then $C = D$ and $S_l^{\mu,\nu} = D \times (\boldsymbol{R}/l\boldsymbol{Z})$.

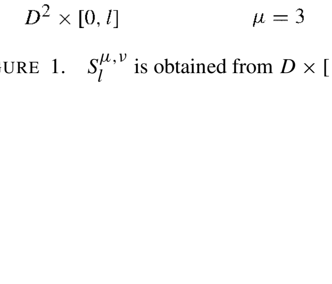

FIGURE 1. $S_l^{\mu,\nu}$ is obtained from $D \times [0, l]$.

For a point $x \in D$, we call $\{x\} \times \boldsymbol{R}$ a *vertical line* in $D \times \boldsymbol{R}$. A vertical line $\{x\} \times \boldsymbol{R}$ is mapped by the covering map to a simple closed curve in $S_l^{\mu,\nu}$, which we call a *fiber* of $S_l^{\mu,\nu}$. The fiber is said to be *singular* if $\mu \neq 1$ and if x is the origin o of D. The fiber is *regular* if it is not singular. If x is the origin o of D, then the fiber is called the *central fiber* of $S_l^{\mu,\nu}$. The image of a regular fiber of $S_l^{\mu,\nu}$ in the fundamental domain $D \times [\,0, l\,]$ consists of μ vertical line segments, so that a regular fiber has length μl, while a singular fiber has length μ. For each fiber of $S_l^{\mu,\nu}$ we find a point in C via the projection $D \to C$. This defines a natural projection from $S_l^{\mu,\nu}$ to C, which we denote by $\pi : S_l^{\mu,\nu} \to C$. Each regular fiber of $S_l^{\mu,\nu}$ can be moved continuously onto the central fiber as a μ-fold covering. The pair (μ, ν) is called the *orbit type* of the central fiber of $S_l^{\mu,\nu}$. As $l \to 0$, $S_l^{\mu,\nu}$ collapses to C.

A Seifert manifold is a manifold equipped with the structure of $S_l^{\mu,\nu}$ locally. Precisely, a three-dimensional topological manifold S is called a *Seifert manifold* if S is a union of simple closed curves, called *fibers*, and if for each fiber F there exist two integers μ and ν as above and a neighborhood U of F such that U is a union of fibers and is homeomorphic to $S_l^{\mu,\nu}$ by a homeomorphism that maps fibers to fibers. Note that the homeomorphism type of the fiber structure of $S_l^{\mu,\nu}$ is determined only by the orbit type (μ, ν). We call such a family of fibers of a Seifert manifold a *Seifert structure*. If we identify all points in each fiber to each other, then we have the quotient topological space, which we call the *orbit surface* of the Seifert manifold. Let X be the orbit surface of a Seifert manifold S. Then the projection $S \to X$ coincides with the projection $\pi : S_l^{\mu,\nu} \to C$ locally. Since C is homeomorphic to the disk D^2, the orbit surface is a two-dimensional topological manifold without boundary. We call a point in the orbit surface corresponding to a singular fiber a *singular point*. The set of singular points in the orbit surface is discrete.

A Seifert manifold S with a Riemannian metric is called a *Riemann-Seifert manifold* if for each fiber F of S there exist a Riemannian manifold $S_l^{\mu,\nu}$ as above and an isometry between a neighborhood of F and $S_l^{\mu,\nu}$ that maps fibers to fibers. The orbit surface of a Riemann-Seifert manifold is locally a cone C whose vertex angle is equal to $2\pi/\mu$, where μ depends on a fiber. Only in the case of $\mu \neq 1$, the vertex is a singular point. If we remove all singular points from the orbit surface, then it has a natural Riemannian metric. The orbit surface is, in fact, a Riemannian orbifold. The lengthes of regular fibers of a Riemann-Seifert manifold are all equal to each other. We have a one-parameter family of Riemann-Seifert metrics on a Riemann-Seifert manifold, parameterized by the length of regular fibers. As the length goes to zero, the Riemann-Seifert manifold collapses to its orbit surface. This is a typical example of Theorem 1.5.

Let S be a Seifert manifold, X the orbit surface, and $\pi : S \to X$ the projection. We take a subset $X' \subset X$ that is a two-dimensional topological manifold with boundary, and assume that the boundary contains no singular point. The three-dimensional manifold $S' := \pi^{-1}(X')$ has a fibration structure induced from that of S. We call such an S' with the fibration structure a *Seifert manifold with boundary*. Any compact connected component of the boundary of a Seifert manifold with boundary is homeomorphic to a torus. A closed three-dimensional manifold is called a *graph manifold* if it is a gluing of finitely many compact Seifert manifolds with boundary along their boundary tori.

3. Alexandrov spaces. This section is devoted to a quick introduction to Alexandrov spaces.

DEFINITION 3.1 (Gromov-Hausdorff convergence; [11]). We say that a sequence of metric spaces X_i, $i = 1, 2, \ldots$, *converges to a metric space* X *with respect to the Gromov-Hausdorff convergence* if there exist a sequence of (not necessarily continuous) maps $\varphi_i : X_i \to X$ and a sequence of positive real numbers $\varepsilon_i \to 0$ such that

(1) for any two points $x, y \in X_i$ we have

$$|d(\varphi_i(x), \varphi_i(y)) - d(x, y)| < \varepsilon_i ,$$

where d is the distance function,

(2) the ε_i-neighborhood of the image $\varphi_i(X_i)$ coincides with the whole space X.

The map φ_i is called an *approximation map.* For a sequence of points $p_i \in X_i$ and a point $p \in X$, we say that p_i *converges to* p if $\varphi_i(p_i) \to p$ as $i \to \infty$.

EXAMPLE 3.2. For a Riemann-Seifert manifold, as the length of regular fibers goes to zero, the manifold converges to the orbit surface with respect to the Gromov-Hausdorff convergence. The natural projection is an approximation map.

DEFINITION 3.3 (Pointed Gromov-Hausdorff convergence; [11]). We say that a sequence of pointed metric spaces (X_i, p_i), $i = 1, 2, \ldots$, *converges to a pointed metric space* (X, p) *with respect to the Gromov-Hausdorff convergence* if there exist three sequences of positive numbers $\varepsilon_i \to 0$, $r_i, r_i' \to +\infty$ with $|r_i - r_i'| \to 0$ and a sequence of maps $\varphi_i : B(p_i, r_i) \to B(p, r_i')$, where $B(p, r)$ indicates the open metric ball centered at p and of radius r, such that

(1) for any two points $x, y \in B(p_i, r_i)$,

$$|d(\varphi_i(x), \varphi_i(y)) - d(x, y)| < \varepsilon_i,$$

(2) the ε_i-neighborhood of the image $\varphi_i(B(p_i, r_i))$ contains $B(p, r_i')$,

(3) $\varphi_i(p_i) = p$.

EXAMPLE 3.4. Take a point p_r in an n-dimensional round sphere $S^n(r)$ of radius r in $\boldsymbol{R}^{n+1}$. As $r \to +\infty$, $(S^n(r), p_r)$ converges to $(\boldsymbol{R}^n, o)$ with respect to the Gromov-Hausdorff convergence.

DEFINITION 3.5 (Geodesic space). A complete metric space X is called a *geodesic space* if for any two points $x, y \in X$ there exists a length-minimizing curve, say xy, joining x to y whose length is equal to the distance $d(x, y)$ between x and y. We call a length-minimizing curve a *minimal geodesic.*

A complete Riemannian manifold is a geodesic space. Note that, for given two points x and y in a geodesic space, a minimal geodesic xy joining them is not necessarily unique.

DEFINITION 3.6 (Triangle). Let p, q, r be three distinct points in a geodesic space and pq, qr, rp minimal geodesics joining them. The set of p, q, r, pq, qr, rp is called a triangle and denoted by $\triangle pqr$. We call p, q, r the *vertices* of $\triangle pqr$ and pq, qr, rp the *edges* of $\triangle pqr$ respectively.

DEFINITION 3.7 (Comparison triangle). Let $\triangle pqr$ be a triangle in a geodesic space and $\triangle \tilde{p}\tilde{q}\tilde{r}$ a triangle in a complete simply connected space form of constant curvature κ, where κ is a real number. We call $\triangle \tilde{p}\tilde{q}\tilde{r}$ a *κ-comparison triangle* of $\triangle pqr$ if it satisfies that $d(p,q) = d(\tilde{p},\tilde{q})$, $d(q,r) = d(\tilde{q},\tilde{r})$ and $d(r,p) = d(\tilde{r},\tilde{p})$. We denote by $\tilde{\angle} pqr$ the angle at $\tilde{q}$ of a κ-comparison triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ of $\triangle pqr$.

Consider the following three conditions:

(i) $\kappa \le 0$.

(ii) $\kappa > 0$, $d(p,q) + d(q,r) + d(r,p) \le 2\pi/\sqrt{\kappa}$ and $\max\{d(p,q), d(q,r), d(r,p)\} < \pi/\sqrt{\kappa}$.

(iii) $\kappa > 0$, $d(p,q) + d(q,r) + d(r,p) \le 2\pi/\sqrt{\kappa}$ and $\max\{d(p,q), d(q,r), d(r,p)\} = \pi/\sqrt{\kappa}$.

A κ-comparison triangle exists and is unique if (i) or (ii) is satisfied. In the case of (iii), a κ-comparison triangle exists, but is not unique.

DEFINITION 3.8 (Alexandrov space). Let κ be a real number. A complete geodesic space X is called an *Alexandrov space of curvature* $\ge \kappa$ if, for any triangle $\triangle pqr$ in X and for any point s on the edge qr, there exists a κ-comparison triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ of $\triangle pqr$ such that

$$d(p,s) \ge d(\tilde{p},\tilde{s}),$$

where $\tilde{s}$ is the point on $\tilde{q}\tilde{r}$ with $d(q,s) = d(\tilde{q},\tilde{s})$.

Taking three points $p, q, r \in X$ and two minimal geodesics qp and qr in an Alexandrov space of curvature $\ge \kappa$, we can define the *angle* $\angle pqr$ between them, where we omit the precise definition. The angle satisfies the following.

PROPOSITION 3.9 (Toponogov comparison). *For any triangle $\triangle pqr$ in an Alexandrov space of curvature $\ge \kappa$ and for a κ-comparison triangle, we have*

$$\angle pqr \ge \tilde{\angle} pqr .$$

EXAMPLE 3.10. (1) Compact Riemannian orbifolds are Alexandrov spaces. In particular, the orbit surface of a closed Riemann-Seifert manifold is an Alexandrov space.

(2) The boundary of a convex body in a Euclidean space is an Alexandrov space of nonnegative curvature.

In general we have the following theorem.

THEOREM 3.11 (Burago-Gromov-Perelman [2]). *We take any $n \in \boldsymbol{N}$ and $\kappa \in \boldsymbol{R}$ and fix them. Then, any Gromov-Hausdorff limit of a sequence of n-dimensional complete Riemannian manifolds of sectional curvature $K_M \ge \kappa$ is an Alexandrov space of curvature $\ge \kappa$*

and dimension $\leq n$. *For any Alexandrov space*, *its Hausdorff dimension and topological* (*covering*) *dimension coincide with each other.*

PROPOSITION 3.12. *Any one-dimensional Alexandrov space is isometric to a one-dimensional complete Riemannian manifold possibly with boundary.*

We have a concept of the *tangent cone*, say K_p, at a point p in an Alexandrov space. Instead of stating the definition of the tangent cones, we give some examples.

EXAMPLE 3.13. (1) For a Riemannian manifold, the tangent cone is nothing but the tangent space at a point. The metric on the tangent space is just the Riemannian metric, so that the tangent cone at any point in a Riemannian manifold is isometric to a Euclidean space.

(2) As we stated in Example 3.10 (2), the boundary of a convex body in a Euclidean space is an Alexandrov space. The tangent cone at a boundary point p of a convex body coincides with the union of rays emanating from p tangent to the boundary of the convex body.

(3) Another example of the tangent cone comes from a Riemannian orbifold. At any point p in a Riemannian orbifold (e.g. the orbit surface of a Riemann-Seifert manifold), there is a neighborhood of p that is isometric to the quotient of a Riemannian manifold D diffeomorphic to a disk by an isometric finite group action fixing a point $\tilde{p} \in D$, where $\tilde{p}$ is mapped to p by the projection and the group is a subgroup of an orthogonal group. This group action on D induces a linear isometric action on the tangent space $T_{\tilde{p}}D$ at $\tilde{p}$. The tangent cone at p of the Riemannian orbifold is obtained as the quotient of $T_{\tilde{p}}D$ by this group action.

In general, the tangent cone has a Euclidean cone structure as explained in the following.

DEFINITION 3.14 (Euclidean cone). Let Y be a metric space. The *cone over* Y is defined to be

$$K(Y) := Y \times [0, +\infty)/Y \times \{0\}.$$

We define a metric on $K(Y)$ by

$$d((x, s), (y, t)) := \sqrt{s^2 + t^2 - 2st \cos \min\{d(x, y), \pi\}}, \quad x, y \in Y,\ s, t \in [0, +\infty).$$

The space $K(Y)$ equipped with this metric is called the *Euclidean cone over* Y. The point in $K(Y)$ corresponding to $Y \times \{0\}$ is called the *vertex of* $K(Y)$. We sometimes identify Y with the subset $Y \times \{1\}$ of $K(Y)$.

For example, if Y is a circle of length less than 2π equipped with the metric defined by arc-lengths, then the Euclidean cone $K(Y)$ over Y is a cone in a standard sense.

Let X be an n-dimensional Alexandrov space. In general, the tangent cone K_p at a point p in X is isometric to the Euclidean cone over a compact $(n-1)$-dimensional Alexandrov space, say Σ_p, of curvature ≥ 1. We call Σ_p the *space of directions at* p. Denote by o_p the vertex of K_p. Then Σ_p is identified with the set of unit elements in K_p, where 'unit' means that the distance from the vertex o_p is equal to one. For example, the space of directions Σ_p

at a point p in a Riemannian orbifold is isometric to the quotient of the unit tangent sphere at $\tilde{p}$ by the group action on the tanget space at $\tilde{p}$, where $\tilde{p}$ is as in Example 3.13 (3).

The tangent cone is obtained as the expansion-scaling limit of X.

THEOREM 3.15 (Burago-Gromov-Perelman [2]). *Let p be a point in an Alexandrov space X. As $r \to +\infty$, (rX, p) converges to the tangent cone (K_p, o_p) at p in the sense of Gromov-Hausdorff, where rX denotes the space X with the metric multiplied by r-times.*

We call an element of K_p a *tangent vector* at p. As well as Riemannian manifolds, for a given geodesic $\gamma : [0, l] \to X$ from a point p in X, the tangent vector $\dot{\gamma}(0)$ is defined as a point in K_p in a suitable way. The length of $\dot{\gamma}(0)$, i.e., the distance between o_p and $\dot{\gamma}(0)$, coincides with the speed of γ. If the speed of γ is unit, then we can consider the tangent vector $\dot{\gamma}(0)$ as an element of Σ_p. From now on, we assume that all geodesics are of unit speed unless otherwise stated. For two minimal geodesics pq and pr from p, if u, v are the tangent vectors of pq, pr at p respectively, then the distance, say $\angle(u, v)$, between u and v in Σ_p is equal to the angle $\angle qpr$ between qp and qr.

If the tangent cone K_p at p is not isometric to the Euclidean space $\boldsymbol{R}^n$, $n = \dim X$, we call the point p a *singular point*. For the orbit surface of a Riemann-Seifert manifold, this definition of a singular point is equivalent to that defined in §2. On one hand, the set of singular points in a two-dimensional Riemannian orbifold (e.g., the orbit surface) is discrete. On the other hand, the set of singular points of the boundary of a convex body in $\boldsymbol{R}^3$ is not necessarily discrete and is much more complex in general. We have an example of a convex body in $\boldsymbol{R}^3$ such that the set of singular points in the boundary of the body is dense in the boundary. This phenomenon makes the study of Alexandrov spaces difficult. The following theorem says that the set of singular points in an Alexandrov space is small in view of measure theory.

THEOREM 3.16 (Burago-Gromov-Perelman [2], Otsu-Shioya [21]). *The Hausdorff dimension of the set of singular points in an n-dimensional Alexandrov space is at most $n - 1$.*

4. Critical point theory for distance functions. In this section, we explain the critical point theory for distance functions, which is a powerful tool to investigate the topology of Riemannian manifolds and Alexandrov spaces.

Let M be a Riemannian manifold and p a point in M.

DEFINITION 4.1 (Critical point; [14]). A point $q \in M$ with $p \neq q$ is a *critical point* of the distance function $d(p, \cdot)$ from p if for any tangent vector $v \in T_qM \setminus \{o\}$ there exists a minimal geodesic pq joining p and q such that the angle at q between v and pq is not greater than $\pi/2$. We regard that p is a critical point of $d(p, \cdot)$. A point in M is called a *regular point* of $d(p, \cdot)$ if it is not a critical point of $d(p, \cdot)$.

It is easy to prove that the set of critical points in M is a closed set. The following theorem is important for the study of the topology of Riemannian manifolds.

THEOREM 4.2 (Grove-Shiohama [14]). *For any point $p \in M$, there exists a smooth vector field V on the set of regular points of $d(p, \cdot)$ such that V is non-zero everywhere and $d(p, \cdot)$ is strictly monotone increasing along any integral curve of V.*

COROLLARY 4.3. *Take any open metric ball $B(p, r)$ in M. If $B(p, r)$ contains no critical point of $d(p, \cdot)$, then $B(p, r)$ is diffeomorphic to an open disk domain.*

By using the theorem, the following famous theorem is obtained.

THEOREM 4.4 (Diameter Sphere Theorem; Grove-Shiohama [14]). *Let M be a closed Riemannian manifold. If the sectional curvature of M satisfies $K_M \geq 1$ and if the diameter satisfies $\mathrm{diam}(M) > \pi/2$, then M is homeomorphic to a sphere.*

Perelman generalized the method of Grove-Shiohama [14] and proved the following two theorems.

THEOREM 4.5 (Stratification Theorem; Perelman [22, 23]). *Let X be an n-dimensional Alexandrov space. Then, X has a stratification structure, i.e., there exists a sequence of subsets*

$$X = X^n \supset X^{n-1} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset$$

such that each $X^k \setminus X^{k-1}$ is either empty or a k-dimensional topological manifold.

THEOREM 4.6 (Stability Theorem; Perelman [22]). *Let X and X_i, $i = 1, 2, \ldots$, be n-dimensional compact Alexandrov spaces of curvature ≥ -1, where $n \in \boldsymbol{N}$ is a fixed natural number. If X_i converges to X with respect to the Gromov-Hausdorff convergence, then X_i is homeomorphic to X for every sufficiently large i.*

The proof of Stability Theorem leads us to the following.

COROLLARY 4.7. *Any two-dimensional Alexandrov space is a two-dimensional topological manifold possibly with boundary.*

5. Structure of noncompact spaces of nonnegative curvature. As we explain in the next section, §6, to determine the topology of collapsing manifolds, it is useful to investigate an expansion-scaling limit of the manifolds. The expansion-scaling limit is a noncompact Alexandrov space of nonnegative curvature, for which we discuss the structure in this section.

First of all, we see the following well-known theorem. A manifold is said to be *open* if it is noncompact and if it has no boundary.

THEOREM 5.1 (Soul Theorem; Cheeger-Gromoll [6]). *Let M be a complete open Riemannian manifold of nonnegative sectional curvature. Then, there exists a closed totally convex submanifold S of M such that the normal bundle over S is diffeomorphic to M. In particular, if S consists of a single point, then M is diffeomorphic to $\boldsymbol{R}^n$, where $n := \dim M$.*

A subset S of M is said to be *totally convex* if any geodesic joining two points in S is contained in S. Any totally convex set in M is a totally geodesic submanifold of M.

The submanifold S as in the above theorem is called a *soul* of M.

EXAMPLE 5.2. We take a complete and nonnegatively curved Riemannian metric g on $\boldsymbol{R}^2$ that is invariant under the $2\pi/\mu$-rotation around the origin of $\boldsymbol{R}^2$ for a natural number μ. Let D be the Riemannian manifold $(\boldsymbol{R}^2, g)$ and $S_l^{\mu,\nu}$ be as obtained in §2 for D. Then, $S_l^{\mu,\nu}$ is a complete open Riemannian manifold of nonnegative sectional curvature. If $\mu \neq 1$, the singular fiber in the center of $S_l^{\mu,\nu}$ is a unique soul of $S_l^{\mu,\nu}$.

As a generalization of Theorem 5.1, we have the following. Let X be a noncompact Alexandrov space of nonnegative curvature. An Alexandrov space is said to be *closed* if it is compact and if it has no boundary, where we omit the definition of the boundary (see [2]).

THEOREM 5.3 (Perelman [22]). *There exists a compact totally convex subset S of X such that S is a closed Alexandrov space of nonnegative curvature with respect to the induced metric, and is homotopy equivalent to X.*

We call the set S in the theorem a *soul* of X. In the same way as for a Riemannian manifold, we define the *normal bundle over S* to be the set of tangent vectors at points in S orthogonal to S.

THEOREM 5.4 (Shioya-Yamaguchi [28]). *If X is a three-dimensional topological manifold, then a soul S of X is a closed topological manifold of dimension at most two and X is homeomorphic to the normal bundle over S.*

We consider a contraction-scaling limit of X. Let p be a point in X and let $\varepsilon > 0$. It is known that, as $\varepsilon \to 0$, the scaled space $(\varepsilon X, p)$ converges to a noncompact Alexandrov space of nonnegative curvature in the sense of Gromov-Hausdorff and the limit is independent of the point p. We call the limit the *limit cone* of X and denote it by X_∞. The limit cone X_∞ is a Euclidean cone over a compact Alexandrov space, say $X(\infty)$. We call $X(\infty)$ the *ideal boundary* of X and an element of $X(\infty)$ a *point at infinity*. For a given sequence of points x_i, $i = 1, 2, \dots$, in X with $R_i := d(p, x_i) \to +\infty$, we consider each x_i as a point in $(R_i^{-1} X, p)$. Then the distance between x_i and p is equal to one. Therefore, a limit of x_i as $i \to \infty$ is identified with a point at infinity, say $\xi \in X(\infty)$, where we recall that $X(\infty)$ is identified with the unit sphere at the vertex of the limit cone X_∞. Such a convergence $x_i \to \xi$ as $i \to \infty$ induces a topology on the disjoint union $\bar{X} := X \sqcup X(\infty)$. $\bar{X}$ is a compactification of X.

A curve $\gamma : [0, +\infty) \to X$ is called a *ray* if we have $d(\gamma(s), \gamma(t)) = |s - t|$ for any $s, t \geq 0$.

LEMMA 5.5. *For any point $p \in X$ and any point at infinity $\xi \in X(\infty)$, there exists a ray from p to ξ, i.e., a ray $\gamma_{p\xi} : [0, +\infty) \to X$ with $\gamma_{p\xi}(0) = p$ such that $\gamma_{p\xi}(t)$ converges to ξ as $t \to +\infty$ in the topology of the compactification $\bar{X}$.*

As an extension to the Toponogov comparison (Proposition 3.9), we have the following.

PROPOSITION 5.6. *For any $p \in X$ and any $\xi, \eta \in X(\infty)$, we have*

$$\angle(\dot{\gamma}_{p\xi}(0), \dot{\gamma}_{p\eta}(0)) \geq d(\xi, \eta) .$$

In other words, the map $X(\infty) \ni \xi \mapsto \dot{\gamma}_{p\xi}(0) \in \Sigma_p$ expands the distances.

Moreover, if we take p as a point in a soul S of X, then we are able to take $\gamma_{p\xi}$ as a ray orthogonal to S. As a result we have the following theorem.

THEOREM 5.7 (Dimension Formula; Shioya-Yamaguchi [27, 28]). *For a soul S of X we have*

$$\dim S + \dim X_\infty \le \dim X .$$

It is an interesting problem to consider what happens if $\dim S + \dim X_\infty = \dim X$. For a complete open Riemannian manifold M of nonnegative sectional curvature with a soul S, if $\dim S + \dim M_\infty = \dim M$ holds, then there exists a finite covering space $\hat{M}$ of M that is isometric to the Riemannian product $N \times \hat{S}$, where $\hat{S}$ is a soul of $\hat{M}$ and N is a complete Riemannian manifold diffeomorphic to a Euclidean space (see [27]).

6. A key idea of the proof of Theorem 1.5: critical point-rescaling argument. We explain a key idea of the proof of Theorem 1.5 (1). The proofs of the other Theorems 1.5 (2), 1.6 and 1.8 are based on the same idea.

Let a sequence of three-dimensional orientable manifolds $M_i \in \mathcal{M}(3, D)$, $i = 1, 2, \dots,$ converge to a two-dimensional Alexandrov space X without boundary in the sense of Gromov-Hausdorff. For simplicity, we assume that X has at most finitely many singular points. Let $\Sigma = \{p^1, p^2, \dots, p^m\}$ be the set of singular points in X. For each singular point $p^k \in \Sigma$, we take a sequence of points $p_i^k \in M_i$, $i = 1, 2, \dots,$ converging to p^k as $i \to \infty$, and put $\Sigma_i := \{p_i^1, p_i^2, \dots, p_i^m\}$. We take any $\varepsilon > 0$ and fix it. Yamaguchi's Fibration Theorem implies that, for every sufficiently large i, there is a fiber bundle

$$S^1 \to M_i \setminus B(\Sigma_i, \varepsilon) \to X \setminus B(\Sigma, \varepsilon) .$$

It therefore suffices to determine the topology around p_i^k for each k. Fixing a number k, we put $p_i := p_i^k$ and $p := p^k$. To prove that M_i is a Seifert manifold, it is essential to prove that the closed ball $\bar{B}(p_i, \varepsilon)$ is homeomorphic to the solid torus $S^1 \times D^2$. We shall prove it in the following.

Taking a small number $\varepsilon > 0$, since $\partial B(p, \varepsilon) \simeq S^1$, we have

$$\partial B(p_i, \varepsilon) \simeq T^2$$

for every sufficiently large i. Since (rX, p) converges to K_p as $r \to \infty$, we may assume that $B(p, \varepsilon)$ itself is a cone. Therefore, for i large enough, $(B(p_i, \varepsilon), p_i)$ is close to a Euclidean cone in the Gromov-Hausdorff sense. If $\bar{B}(p_i, \varepsilon)$ contains no critical points of $d(p_i, \cdot)$, then we have $\bar{B}(p_i, \varepsilon) \simeq D^3$, which contradicts $\partial B(p_i, \varepsilon) \simeq T^2$. So there is at least one critical point of $d(p_i, \cdot)$ in $\bar{B}(p_i, \varepsilon)$. Let q_i be one of the critical points of $d(p_i, \cdot)$ in $\bar{B}(p_i, \varepsilon)$ furthest from p_i. We are able to prove that

$$d(p_i, q_i) \to 0 \quad \text{as } i \to \infty .$$

Let $\delta_i := d(p_i, q_i)$. By replacing $\{(\delta_i^{-1} M_i, p_i)\}_i$ with a subsequence, $(\delta_i^{-1} M_i, p_i)$ converges to a pointed Alexandrov space $(Y, \hat{p})$. We have $\dim Y \le 3$. For the moment we assume that

$$\dim Y = 3 .$$

(We discuss the case $\dim Y \le 2$ later.)

Using the critical point theory, we are able to prove that $B(\hat{p}, R) \simeq Y$ for every sufficiently large $R > 0$. By Perelman's Stability Theorem 4.6, $\delta_i^{-1} B(p_i, \varepsilon)$ for large i is homeomorphic to Y, i.e., we have

$$B(p_i, \varepsilon) \simeq Y .$$

Note that we here use Perelman's Stability Theorem for metric balls of Alexandrov spaces and Perelman's proof works for such metric balls to obtain the topological stability. Let S be a soul of Y. We investigate the topology of Y using Theorem 5.4.

If $\dim S = 0$, then S consists of a single point and $Y \simeq \boldsymbol{R}^3$. Therefore we have $\bar{B}(p_i, \varepsilon) \simeq D^3$, which contradicts $\partial B(p_i, \varepsilon) \simeq T^2$.

If $\dim S = 1$, then Y is an $\boldsymbol{R}^2$-bundle over $S \simeq S^1$, i.e., $Y \simeq S^1 \times \boldsymbol{R}^2$. Therefore we have $\bar{B}(p_i, \varepsilon) \simeq S^1 \times D^2$, which is what we want.

If $\dim S = 2$, then, by Dimension Formula (Theorem 5.7), we have

$$\dim Y_\infty \le \dim Y - \dim S = 1 .$$

Since Y_∞ is a Euclidean cone, we have $\dim Y_\infty = 1$ and Y_∞ is isometric to either $\boldsymbol{R}$ or $[\,0, +\infty\,)$. Roughly speaking, Y is very thin globally. On the other hand, Y is obtained by the magnification limit of the space close to the Euclidean cone K_p, which means that Y is globally more spreading than K_p, because the curvature is bounded below. This leads to a contradiction. Rigorously speaking, we are able to construct a map from Σ_p to $Y(\infty)$ that does not contract the distances, for which we omit the proof. We have

$$\dim Y(\infty) \ge \dim \Sigma_p = 1 , \quad \text{i.e.,} \quad \dim Y_\infty \ge 2 . \tag{6.1}$$

This is a contradiction and we have $\dim S = 1$ and $\bar{B}(p_i, \varepsilon) \simeq S^1 \times D^2$.

The rest of the proof is to show $\dim Y = 3$, but this is not true in general. In fact, we consider the case where the cone angle of $B(p, \varepsilon)$ is less than π and the convergence rate of $p_i \to p$ is much slower than the diameters of fibers in M_i, where one of our purposes is to find fibers, but we here assume that there are fibers. See Figure 2. Then, for the furthest critical

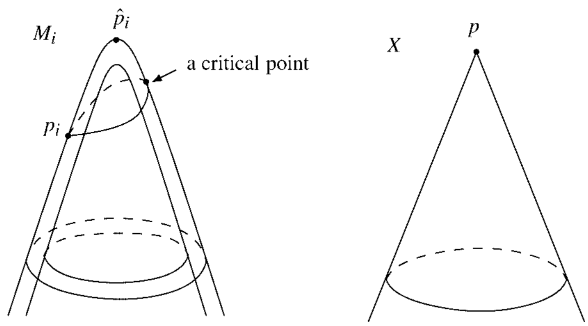

FIGURE 2. $\dim Y = 3$ does not hold in general.

point q_i of $d(p_i, \cdot)$ in $B(p_i, \varepsilon)$, the distance $\delta_i = d(p_i, q_i)$ is much greater than the diameters of fibers, so that $(\delta_i^{-1} M_i, p_i)$ collapses and we cannot expect to obtain the homeomorphism $B(p_i, \varepsilon) \simeq Y$ as stated before.

To avoid this problem, we try to replace the point p_i with a point at a pointed head of $B(p_i, \varepsilon)$. To find such a point, we define a function as

$$f_i(x) := \int_{\partial B(p_i,\varepsilon)} d(x, y)\, d\mu_i(y)\,, \quad x \in M_i\,,$$

where μ_i is the surface measure on $\partial B(p_i, \varepsilon)$. (The measure μ_i is a little different from the original proof in [28]). Since $B(p_i, \varepsilon)$ is close to a cone, f_i takes a local maximum near p_i, for which we omit the proof. Letting $\hat{p}_i$ be a local maximum point of f_i near p_i, we have $d(p_i, \hat{p}_i) \to 0$. Using the critical point theory, we prove that $B(p_i, \varepsilon) \simeq B(\hat{p}_i, \varepsilon)$, so that the above discussion also works if we replace p_i with $\hat{p}_i$. Let q_i be as above, i.e., a critical point in $\bar{B}(\hat{p}_i, \varepsilon)$ of $d(\hat{p}_i, \cdot)$ furthest from $\hat{p}_i$. We set $\delta_i := d(\hat{p}_i, q_i)$. The following lemma completes the proof.

LEMMA 6.1 (Key lemma). *If $(Y, \hat{p})$ is a limit of $(\delta_i^{-1} M_i, \hat{p}_i)$ in the sense of Gromov-Hausdorff, then we have*

$$\dim Y = 3\,.$$

PROOF. By replacing $\{q_i\}$ with a subsequence, q_i converges to a point in Y, say q. We fix a minimal geodesic segment joining $\hat{p}$ and q. Let $u \in K_{\hat{p}}$ be the unit vector at $\hat{p}$ tangent to the minimal geodesic segment. We prove the following.

CLAIM 6.2. *For any ray γ from $\hat{p}$, we have*

$$\angle(\dot{\gamma}(0), u) = \pi/2\,.$$

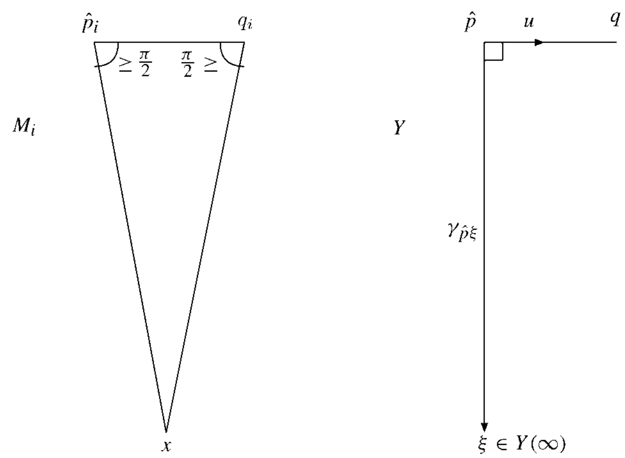

FIGURE 3. Proof of Key Lemma.

Idea of the proof of the claim. We assume, for simplicity, that a minimal geodesic joining given two points is always unique. Take any point $x \in \partial B(p_i, \varepsilon)$. Since q_i is a critical point of $d(\hat{p}_i, \cdot)$, we have $\angle \hat{p}_i q_i x \le \pi/2$ (see Figure 3). By Toponogov comparison (Proposition 3.9), the total sum of the angles of a triangle is almost greater than π, so that, recalling $d(\hat{p}_i, q_i) \to 0$, we have

$$\liminf_{i\to\infty} \angle q_i \hat{p}_i x \ge \pi/2 . \tag{6.2}$$

On the other hand, since $\hat{p}_i$ is a local maximum point of f_i, the directional derivative of f_i with the direction of $\hat{p}_i q_i$ is nonnegative. By the first variation formula,

$$-\int_{\partial B(p_i,\varepsilon)} \cos \angle q_i \hat{p}_i x \; d\mu_i(x) \ge 0 .$$

This together with (6.2) shows

$$\lim_{i\to\infty} \angle q_i \hat{p}_i x = \pi/2 .$$

By replacing $\hat{p}_i, q_i, x$ with $\hat{p}, q, \xi$ for $\xi \in Y(\infty)$, a discussion similar to the above leads to the claim, but we omit the details. □

For any point at infinity $\xi \in Y(\infty)$, we take a ray $\gamma_{\hat{p}\xi}$ from $\hat{p}$ with $\gamma_{\hat{p}\xi}(\infty) = \xi$ (see Figure 3). Proposition 5.6 says that the map $Y(\infty) \ni \xi \mapsto \dot{\gamma}_{\hat{p}\xi}(0) \in \Sigma_{\hat{p}}$ does not contract the distances. The claim implies that

$$\dot{\gamma}_{\hat{p}\xi}(0) \in A := \{v \in \Sigma_{\hat{p}} \,;\, \angle(u, v) = \pi/2\} .$$

Comparing the Hausdorff dimensions we have $\dim Y(\infty) \le \dim A$. By (6.1) we have $\dim Y(\infty) \ge 1$ and $\dim A = \dim \Sigma_{\hat{p}} - 1 = \dim Y - 2$, which leads to $\dim Y \ge 3$. This completes the proof. □

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