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ON A CLASS OF FOLIATED NON-KÄHLERIAN COMPACT COMPLEX SURFACES

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Abstract. Motivated by recent results on non-Kählerian compact complex surfaces with small second Betti number, we classify those on which a holomorphic foliation (with singularities) exists.

Introduction. According to Kodaira, a compact connected complex surface S belongs to the class VII_0 if it is minimal and its first Betti number $b_1(S)$ is equal to 1. It is still an open and fundamental problem to get a classification of these surfaces, which are not Kählerian and hence rather elusive. Let us shortly recall some advances about this problem [Nak], [DOT], [Tel], in order to place and to motivate our result. Because of the important rôle of the *second* Betti number in the following discussion, it is convenient to denote by VII_0^n, $n \in \boldsymbol{N}$, the class of VII_0 surfaces S with $b_2(S) = n$, and to set $\mathrm{VII}_0^+ = \bigsqcup_{n>0} \mathrm{VII}_0^n$.

Surfaces of class VII_0^0 have been completely classified in a series of works by Kodaira, Inoue, Bogomolov, Li-Yau-Zheng and Teleman. Let us henceforth concentrate our attention to surfaces of class VII_0^+.

Around 1977, Kato [Kat] discovered a large collection of VII_0^+ surfaces, nowadays called *Kato surfaces* (a.k.a. surfaces with a global spherical shell). They are, in some sense, generalizations of the classical Hopf surfaces (which belong to class VII_0^0), and a significant number of papers has been dedicated to them, so that Kato surfaces may be today considered as "well known" surfaces. No other examples of VII_0^+ surfaces have been discovered so far, and indeed some authors courageously conjecture that every VII_0^+ surface should be a Kato surface. An important result in that direction has been proved by Nakamura [Na1], [Na2], in some particular cases, and then Dloussky-Oeljeklaus-Toma [DOT], in the general case: if S is a surface of class VII_0^n ($n > 0$) and contains n rational curves, then S is a Kato surface (the converse also being true, by construction).

That result motivates the search for rational curves on VII_0^+ surfaces. In recent years, Teleman developed a general strategy for finding those rational curves, using methods of gauge theory [Tel]. Up to now, his strategy has been successfull for small values of the second Betti number: it is proved in [Te1] and [Te2] that every surface of class VII_0^1 or VII_0^2 contains at least one rational curve. We shall give in Section 1 more details on this spectacular result.

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As a consequence, we get a complete classification of VII_0^1 surfaces: any such surface is a Kato surface. However, the same conclusion cannot be drawn for VII_0^2 surfaces: we need, for that purpose, to find a *second* rational curve, besides the one founded by Teleman.

Since the works of Kodaira and Inoue on the subject, holomorphic foliations (possibly with singularities) have played a distinguished rôle. For instance, the contribution of Inoue to the classification of VII_0^0 surfaces consisted precisely in getting such a classification under the additional hypothesis that the surface admits a foliation [Ino]. Remark that this is not an anodyne hypothesis, due to the nonalgebraic nature of these surfaces, and only later Bogomolov [Bog], Li-Yau-Zheng [LYZ] and Teleman [Te0] showed that any VII_0^0 surface does admit a foliation. Our contribution consists in completing the classification of VII_0^2 surfaces when a foliation exists.

THEOREM 0.1. *Let S be a surface of class* VII_0^2 *and suppose that S admits a foliation. Then S is a Kato surface.*

It is here worth observing that, conversely, every Kato surface admits a foliation [D-O]. The presence of foliations on Kato surfaces is the guiding principle of [DOT].

Looking at the above discussion, the strategy of the proof of Theorem 0.1 should be clear: we shall use the foliation on S to get the second rational curve, which actually will be a leaf of the foliation. This is done in a sequence of steps which involve several different techniques and ideas from the theory of foliations: residue formulae, leafwise uniformisation, invariant measures, complete flows, etc. In fact, we think that it may be precisely this mixture of arguments the interesting part of the paper, rather than its eventual consequences on the classification of surfaces.

A natural question is about the extension of Theorem 0.1 to surfaces of class VII_0^n, $n \geq 3$. Of course, the first obstruction is that we need Teleman's results on rational curves, which for the moment are not yet proven for "large" second Betti numbers. But, even assuming those results, there are very big difficulties in extending our arguments to VII_0^n surfaces when $n \geq 3$: at several places (but especially at the beginning) the condition $b_2(S) = 2$ seems insurmountable. The point is that the gap between Teleman's program, which would lead to possibly *only one* rational curve, and Nakamura-Dloussky-Oeljeklaus-Toma theorem, which needs $b_2(S)$ rational curves, becomes obviously larger when $b_2(S)$ increases.

In spite of these difficulties, we hope that this paper may be read as a further step toward the general problem of classifying foliated non-Kählerian surfaces, which we begun to study in [Br5]. It fits perfectly into the general philosophy described at the end of the introduction of that paper, and the results of that paper will be an essential piece of the proof of Theorem 0.1.

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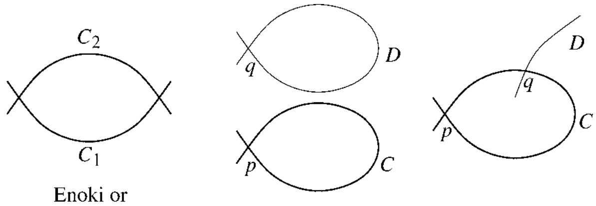
1. Some preliminary results. In this section we firstly recall some results on VII_0^2 surfaces that we shall need, and then we begin the study of foliations on those surfaces.

1.1. Teleman's cycle. Let S be a VII_0^2 surface. According to [Te2], S contains a cycle of rational curves. Because a VII_0^n surface can contain at most n rational curves [Na1, (3.5)], there are two possibilities for such a cycle: either it is composed by two smooth rational curves C_1 and C_2, or by only one rational curve C with a point p of transverse selfintersection (a node). In the former case, by results of Enoki and Nakamura [Na1, (10.3)] S is a Kato surface, and more precisely either an odd Inoue-Hirzebruch surface (a.k.a. a half-Inoue surface) or an Enoki surface (a.k.a. an exceptional compactification of an affine line bundle). For our purposes, we shall therefore consider only the latter case, in which Teleman's cycle is a single rational curve C.

Our objective is to find a second rational curve D on S. There will be two possibilities: either D will be another rational curve with a node q, disjoint from C, or D will be a smooth rational curve intersecting transversely C at some point q. In the former case, by [Na1, (8.1)] S is a Kato surface, more precisely an even Inoue-Hirzebruch surface (a.k.a. a hyperbolic Inoue surface). In the latter case, by [DOT] S is a Kato surface of so-called intermediate type.



By another result of Nakamura [Na2, (1.5)], the surface S can be deformed to a blown-up primary Hopf surface, in such a way that moreover C is deformed to a smooth elliptic curve. In particular, S is diffeomorphic to the connected sum of $\boldsymbol{S}^3 \times \boldsymbol{S}^1$ and two copies of $\overline{\boldsymbol{CP}^2}$, and its fundamental group is infinite cyclic. Denote by

$$\pi : \widetilde{S} \longrightarrow S$$

the universal covering, and set

$$\widetilde{C} = \pi^{-1}(C)\,.$$

The natural map from $\pi_1(C) \simeq \boldsymbol{Z}$ to $\pi_1(S) \simeq \boldsymbol{Z}$ is an isomorphism [Na1, (9.2)], hence $\widetilde{C}$ is an infinite chain of smooth rational curves in $\widetilde{S}$. We fix a generator

$$\varphi : \widetilde{S} \longrightarrow \widetilde{S}$$

of the group of covering transformations, and we denote by $\{C_j\}_{j\in\boldsymbol{Z}}$ the rational curves composing $\widetilde{C}$, labeled in such a way that

$$\varphi(C_j) = C_{j+1}\,.$$

Following [Na2, (1.7)], we may fix on S two line bundles $L_1, L_2 \in \mathrm{Pic}(S)$ such that

(1) $L_1 \cdot L_1 = L_2 \cdot L_2 = -1,\ L_1 \cdot L_2 = 0,$

(2) $K_S = L_1 + L_2$,

(3) $\mathcal{O}_S(C) = -L_1$

(here we use the additive notation for tensor products of line bundles). In particular, we see that

$$C \cdot C = -1$$

and therefore $C_j \cdot C_j = -3$ for every $j \in \boldsymbol{Z}$.

Observe also that

$$\pi_1(\widetilde{S} \setminus \widetilde{C}) = 0\,.$$

Indeed, after a Nakamura deformation [Na2, (1.5)] the cycle C becomes a smooth elliptic curve of selfintersection -1 in a blown-up Hopf surface, and necessarily such an elliptic curve intersects, at a single point, a rational curve contained in the exceptional divisor of the blow-up. Therefore, in the surface S we can find a smooth (non-holomorphic) sphere Σ which intersects C at exactly one point, so that $\pi^{-1}(\Sigma)$ is a countable collection of spheres $\{\Sigma_j\}$ in $\widetilde{S}$ with $\Sigma_j \cdot C_k = \delta_{jk}$. Because $\pi_1(\widetilde{S} \setminus \widetilde{C})$ is generated by small cycles turning around the curves C_j, and these cycles can be chosen inside the spheres Σ_j, the claim above follows.

We shall denote by $\approx$ the numerical equivalence between line bundles, i.e., the equality between their Chern classes. Thus, every line bundle on S is numerically equivalent to $a_1L_1 + a_2L_2$ for suitable $a_1, a_2 \in \boldsymbol{Z}$. Remark that from $H^1(S, \mathcal{O}) \simeq H^1(S, \boldsymbol{C})$ it follows that a line bundle on S is numerically trivial if and only if it is *flat*, i.e., it can be defined by a locally constant cocycle.

1.2. Singularities of the foliation. Suppose now that on the surface S we have a foliation $\mathcal{F}$. We refer to [Br2] for the basic results that we shall use, in particular for the residue formulae of Baum-Bott and Camacho-Sad.

Let $N_{\mathcal{F}}$ be the normal bundle of $\mathcal{F}$, and write

$$N_{\mathcal{F}} \approx a_1L_1 + a_2L_2, \quad a_1, a_2 \in \boldsymbol{Z}\,.$$

If $\mathrm{Det}(\mathcal{F})$ is the total number of singularities of $\mathcal{F}$, counted with multiplicity, then

$$\mathrm{Det}(\mathcal{F}) = c_2(S) + K_S \cdot N_{\mathcal{F}} + N_{\mathcal{F}} \cdot N_{\mathcal{F}} = 2 - [a_1(a_1+1) + a_2(a_2+1)]$$

[Br2, p. 21] and if $\mathrm{Tr}(\mathcal{F})$ is the sum of Baum-Bott residues of those singularities, then

$$\mathrm{Tr}(\mathcal{F}) = N_{\mathcal{F}} \cdot N_{\mathcal{F}} = -(a_1^2 + a_2^2)$$

[Br2, p. 34]. The singular set $\mathrm{Sing}(\mathcal{F})$ cannot be empty, otherwise $\mathrm{Det}(\mathcal{F}) = \mathrm{Tr}(\mathcal{F}) = 0$ and this contradicts the above two formulae. Hence $\mathrm{Det}(\mathcal{F}) \geq 1$, and the first formula immediately implies

$$a_1, a_2 \in \{-1, 0\}$$

and so

$$\mathrm{Det}(\mathcal{F}) = 2\,.$$

Remark that for the moment this does not exclude that $\mathcal{F}$ has only one singular point, of multiplicity 2.

LEMMA 1.1. *The curve C is $\mathcal{F}$-invariant.*

PROOF. If not, then we compute the degree of $N_{\mathcal{F}}$ on C using the formula in [Br2, p. 23]:

$$N_{\mathcal{F}} \cdot C = \chi(C) + \mathrm{tang}(\mathcal{F}, C)\,.$$

The arithmetic Euler characteristic $\chi(C)$ is here equal to 0, and $\mathrm{tang}(\mathcal{F}, C) > 0$ because the node p gives a strictly positive contribution $\mathrm{tang}(\mathcal{F}, C, p)$. Hence $N_{\mathcal{F}} \cdot C > 0$. On the other side,

$$N_{\mathcal{F}} \cdot C = -(a_1 L_1 + a_2 L_2) \cdot L_1 = a_1 \in \{-1, 0\}\,,$$

giving a contradiction. □

In particular, the node $p \in C$ is a singularity of the foliation, and the two local branches of C through p are separatrices [Br2, p. 9] for that singularity.

LEMMA 1.2. *The singularity of $\mathcal{F}$ at p has multiplicity* 1.

PROOF. If not, then $\mathrm{Sing}(\mathcal{F}) = \{p\}$ and p has multiplicity 2. Let (z, w) be local coordinates centered at p such that $\{zw = 0\}$ is an equation for C. The foliation is generated by a vector field of the form

$$A(z, w) z \frac{\partial}{\partial z} + B(z, w) w \frac{\partial}{\partial w}\,.$$

The only way to get multiplicity 2 is, up to a permutation, $A(0, 0) \neq 0$ and $B(0, 0) = 0$, $\frac{\partial B}{\partial w}(0, 0) \neq 0$. This means that p is a saddle-node singularity, with strong separatrix $\{w = 0\}$ and weak separatrix $\{z = 0\}$. In suitable formal coordinates, still denoted by (z, w), the foliation is generated by a vector field of the form [Br2, p. 11]

$$(1 + \nu w) z \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w}\,, \qquad \nu \in \boldsymbol{C}\,.$$

In the following we shall use this formal normal form to compute some residues, and it is an easy matter to check the computation even when the formal conjugacy is not convergent (e.g. use Dulac's normal form instead of the formal one).

Concerning Camacho-Sad residues [Br2, p. 39] we get

$$\mathrm{CS}(\mathcal{F}, \{w = 0\}, p) = 0\,,$$

$$\mathrm{CS}(\mathcal{F}, \{z = 0\}, p) = \nu\,,$$

$$\mathrm{CS}(\mathcal{F}, \{zw = 0\}, p) = \nu + 2\,,$$

and concerning Baum-Bott residue [Br2, p. 34] we get

$$\mathrm{BB}(\mathcal{F}, p) = \mathrm{Res}_0 \left\{ \frac{[(1 + \nu w) + 2w]^2}{z w^2 (1 + \nu w)} dz \wedge dw \right\} = 2(\nu + 2)\,.$$

From Camacho-Sad formula $C \cdot C = \mathrm{CS}(\mathcal{F}, C, p)$ and $C \cdot C = -1$ we get $\nu = -3$, and consequently $\mathrm{BB}(\mathcal{F}, p) = -2$. From Baum-Bott formula we then get $N_{\mathcal{F}} \cdot N_{\mathcal{F}} = -2$, which in turn implies $a_1 = a_2 = -1$. Therefore,

$$N_{\mathcal{F}} \cdot C = -(-L_1 - L_2) \cdot L_1 = -1\,.$$

On the other side, this degree can be computed also by the formula in [Br2, p. 25]:

$$N_{\mathcal{F}} \cdot C = C \cdot C + \mathrm{Z}(\mathcal{F}, C)\,.$$

In our case, $\mathrm{Z}(\mathcal{F}, C) = \mathrm{Z}(\mathcal{F}, C, p) = 1$ [Br2, p. 38] and $C \cdot C = -1$, whence $N_{\mathcal{F}} \cdot C = 0$, which is a contradiction. □

By this lemma (and $\mathrm{Det}(\mathcal{F}) = 2$) we have

$$\mathrm{Sing}(\mathcal{F}) = \{p, q\}\,,$$

where both p and q are nondegenerate singularities.

We shall distinguish from now on two possibilities:

(A) q belongs to C.

(B) q does not belong to C.

Case (A) will lead to an intermediate Kato surface, and Case (B) will lead to an even Inoue-Hirzebruch surface.

2. Numerical class of the normal bundle. We have already seen that $N_{\mathcal{F}} \approx a_1 L_1 + a_2 L_2$, with a_1 and a_2 in $\{-1, 0\}$. Here we shall see a more precise statement. Some of our arguments can be found also in [DOT, §2], but we repeat them for sake of completeness.

2.1. Case (A). First of all we observe that

$$N_{\mathcal{F}} \cdot C = C \cdot C + \mathrm{Z}(\mathcal{F}, C, p) + \mathrm{Z}(\mathcal{F}, C, q) = -1 + 0 + 1 = 0$$

due to the nondegeneracy of p and q. Therefore, from $\mathcal{O}_S(C) = -L_1$ we infer $a_1 = 0$, that is

$$N_{\mathcal{F}} \approx a_2 L_2\,, \quad a_2 \in \{-1, 0\}\,.$$

LEMMA 2.1. *In Case* (A) *we have* $N_{\mathcal{F}} \approx -L_2$.

PROOF. By contradiction, suppose that $N_{\mathcal{F}}$ is numerically trivial, i.e., flat. Thus, for a suitable open covering $\{U_j\}$ of S, the foliation is generated by holomorphic 1-forms $\omega_j \in \Omega^1(U_j)$ with $\omega_j = g_{jk}\omega_k$ on $U_j \cap U_k$, where $g_{jk} \in \boldsymbol{C}^*$ and $\{g_{jk}\}$ is a cocycle defining $N_{\mathcal{F}} = F$. By differentiating ($d\omega_j = g_{jk} d\omega_k$) we get a section of $K_S \otimes F$. This section must be identically zero: otherwise K_S would be numerically equivalent to a non-negative divisor $\sum m_j D_j$, $m_j \geq 0$, the minimality of S would give $K_S \cdot D_j \geq 0$ and hence $K_S \cdot K_S \geq 0$, contradicting $K_S \cdot K_S = -2$. Therefore, each 1-form ω_j is closed, and even exact if we assume (as we can) that each U_j is contractible:

$$\omega_j = df_j\,, \quad f_j \in \mathcal{O}(U_j)\,.$$

The functions f_j are submersions outside p and q, which are Morse-type critical points due to their nondegeneracy. On $U_j \cap U_k$ we have

$$f_j = g_{jk} f_k + c_{jk}$$

with $c_{jk} \in \boldsymbol{C}$. The foliation is a so-called *transversely affine* foliation.

When we pass to the universal covering $\widetilde{S}$, the lifted foliation $\widetilde{\mathcal{F}}$ can be defined by (the differential of) a global holomorphic function $f \in \mathcal{O}(\widetilde{S})$, with Morse-type singularities at

$\mathrm{Sing}(\widetilde{\mathcal{F}})$ (developing map). We may choose such a function so that $f|_{\widetilde{C}} \equiv 0$. The covering transformation $\varphi : \widetilde{S} \to \widetilde{S}$ acts on f in an affine way, and taking into account that $\widetilde{C}$ is φ-invariant we get

$$f \circ \varphi = \lambda \cdot f$$

for some $\lambda \in \boldsymbol{C}^*$.

Consider now the curve $L = \{f = 0\} \subset \widetilde{S}$. Because $\varphi(L) = L$, its projection to S is a compact curve $L_0 \subset S$. One irreducible component of L_0 is the curve C, but there is (at least) another irreducible component D, cutting C at q. Using Camacho-Sad formula it is immediate to see that D is a smooth rational curve of selfintersection -1, but this contradicts the minimality of S. □

REMARK 2.1. In the non-minimal case, a situation like the one described in the previous proof exists. Indeed, take S_0 of class VII_0^1 and of Enoki type. There is on S_0 a foliation $\mathcal{F}_0$, tangent to the unique rational curve C_0, which is a nodal rational curve with zero self-intersection. The normal bundle of $\mathcal{F}_0$ is numerically trivial. If we blow-up a point on C_0, different from the node, we get a foliation $\mathcal{F}$ whose normal bundle is still numerically trivial, and whose structure is like the one in the previous proof, the only difference being that the ambient surface S is not minimal.

2.2. Case (B). In this case, where $q \notin C$, we have

$$N_{\mathcal{F}} \cdot C = C \cdot C + \mathrm{Z}(\mathcal{F}, C, p) = -1\,,$$

from which we infer $a_1 = -1$, that is

$$N_{\mathcal{F}} \approx -L_1 + a_2 L_2\,, \quad a_2 \in \{-1, 0\}\,.$$

LEMMA 2.2. *In Case* (B) *we have* $N_{\mathcal{F}} \approx -L_1 - L_2$.

PROOF. By contradiction, suppose that $N_{\mathcal{F}} \approx -L_1$, i.e., that $N_{\mathcal{F}} \otimes \mathcal{O}(-C) = F$ is flat. This means that the foliation is generated by logarithmic 1-forms $\omega_j \in \Omega^1(\log C)(U_j)$ with $\omega_j = g_{jk}\omega_k$ on $U_j \cap U_k$, $g_{jk} \in \boldsymbol{C}^*$ (the logarithmic property comes from the fact that C is $\mathcal{F}$-invariant). The cocycle $\{g_{jk}\}$ defines F. By taking differentials we now get a section of $K_S \otimes \mathcal{O}_S(C) \otimes F$, which, similarly to Case (A), must be identically zero: the canonical bundle K_S cannot be numerically equivalent to a divisor $\sum m_j D_j - C$, $m_j \geq 0$, because we would get $K_S \cdot K_S \geq -K_S \cdot C = -1$. Hence, every ω_j is a closed logarithmic 1-form.

When we pass to $\widetilde{S}$, the foliation $\widetilde{\mathcal{F}}$ can be therefore defined by a global closed logarithmic 1-form $\omega \in \Omega^1(\log \widetilde{C})(\widetilde{S})$, with

$$\varphi^*(\omega) = \lambda \cdot \omega$$

for a suitable $\lambda \in \boldsymbol{C}^*$.

However, we know that $\widetilde{S} \setminus \widetilde{C}$ is simply connected (Subsection 1.1), hence the periods of $\omega|_{\widetilde{S}\setminus\widetilde{C}}$ must vanish and consequently ω has no poles at all along $\widetilde{C}$. This is a contradiction with the construction of ω. □

REMARK 2.2. At the end, in Case (B) we shall find a second nodal rational curve $D \subset S$, and the lifted foliation will be defined by a closed logarithmic 1-form with poles on $\widetilde{C} \cup \widetilde{D}$, and with periods additively generated by $\{\lambda^j\}$, with λ a quadratic irrational (see [D-O]). But here one finds also that

$$H_1(\widetilde{S} \setminus (\widetilde{C} \cup \widetilde{D}), \boldsymbol{R}) = \boldsymbol{R}^2 ,$$

and so everything is coherent (the additive group generated by $\{\lambda^j\}$ has rank 2). It is worth observing that, whereas in Lemma 2.1 we used only the minimality of the surface, in Lemma 2.2 we use a more subtle topological argument. This announces that the proof in Case (B) will be more difficult than in Case (A).

2.3. The canonical bundle. From now on our attention will be shifted from the normal bundle $N_{\mathcal{F}}$ to the canonical bundle $K_{\mathcal{F}}$. It is therefore convenient to restate Lemmata 2.1 and 2.2 in terms of $K_{\mathcal{F}}$, which is immediate from the adjunction formula $K_S = N^*_{\mathcal{F}} \otimes K_{\mathcal{F}}$.

PROPOSITION 2.1. *In Case* (A) *the line bundle* $K_{\mathcal{F}} \otimes \mathcal{O}_S(C)$ *is numerically trivial. In Case* (B) *the line bundle* $K_{\mathcal{F}}$ *is numerically trivial.*

3. Uniformisation. In this section we pause the proof of Theorem 0.1, and we work in a much more general setting. Our aim is to prove a uniformisation result analogous to the one in [Br3], but in a possibly non-Kählerian case, and following a program initiated in [Br5].

3.1. Setting and statement. We consider an arbitrary compact connected complex surface S equipped with a foliation $\mathcal{F}$. We shall assume that $\mathcal{F}$ is *uniformisable* in the sense of [Br5, Def. 2.1], which is an innocuous assumption since nonuniformisable foliations are completely classified in [Br5]. Hence, given a local transversal $T \subset S^\circ = S \setminus \mathrm{Sing}(\mathcal{F})$ (say, isomorphic to the disc), we can construct the *covering tube*

$$U_T \xrightarrow{P_T} T$$

by glueing together the universal coverings of the leaves cutting T. We shall also assume that $\mathcal{F}$ is *relatively minimal* in the sense of [Br5, Rem. 2.1], i.e., that there does not exist a smooth rational curve of negative selfintersection over which the canonical bundle $K_{\mathcal{F}}$ has negative degree. This is also an innocuous assumption (at least, if we allow S to have some quotient singularities). In that case, the natural map

$$\Pi_T : U_T \to S$$

sending fibers to leaves is a holomorphic (and not merely meromorphic) immersion. Moreover, we do not need to worry about the definition of "leaves" of $\mathcal{F}$, since they are just the usual leaves outside $\mathrm{Sing}(\mathcal{F})$: there are no vanishing ends.

THEOREM 3.1. *Suppose that there exists a holomorphic vector field v on U_T, which is tangent to the fibers of P_T and nowhere vanishing. Then U_T is a Stein surface.*

Remark that, conversely, if U_T is Stein then such a vertical nowhere vanishing vector field v certainly exists, since the relative tangent bundle of the fibration is topologically and hence holomorphically trivial.

By the classical results of Nishino and Yamaguchi (see [Br3] and references therein), the previous theorem has the following corollaries:

(1) If at least one fiber of P_T is hyperbolic, then the fiberwise Poincaré metric on U_T has a subharmonic variation; in particular, the parabolic fibers fill a complete pluripolar subset.

(2) If every fiber of P_T is parabolic, then $U_T \simeq T \times \boldsymbol{C}$.

By functoriality of the Poincaré metric we get the following corollary from (1):

COROLLARY 3.1. *Suppose that the hypothesis of Theorem* 3.1 *is satisfied for every covering tube associated to $\mathcal{F}$. If at least one leaf of $\mathcal{F}$ is hyperbolic, then the leafwise Poincaré metric induces a (singular) hermitian metric on $K_{\mathcal{F}}$ whose curvature is a closed positive current. In that case, parabolic leaves and singularities fill a complete pluripolar subset of S.*

Of course, it is sufficient to verify the hypothesis of Theorem 3.1 only on a finite number of tubes, whose union covers S°.

Remark that Theorem 3.1 does not exclude the algebraic (or Kähler) case already studied in [Br3]. However, in that paper there is no hypothesis on the holomorphic triviality of the relative tangent bundle.

3.2. Developing map. Let us firstly recall a one dimensional construction. Suppose that we have a connected and simply connected complex curve R equipped with a nowhere vanishing holomorphic vector field v. Pick a point $p \in R$. Then we have a canonically defined immersion $\mathcal{D} : R \to \boldsymbol{C}$ as follows. We take on R the holomorphic 1-form β dual to v (i.e., $\beta(v) \equiv 1$), which is closed and even exact because R is simply connected. Then $\mathcal{D}$ is just the primitive of β, normalized by $\mathcal{D}(p) = 0$. It is an immersion because β is nowhere vanishing. Incidentally, the Riemann domain $R \to \boldsymbol{C}$ can be also understood as the maximal domain over which the (uncomplete) flow of v, with initial condition at p, is defined.

Turning to Theorem 3.1, the vertical vector field v on U_T is exploited in the following way. For each $t \in T$, by the previous construction we get an immersion $\mathcal{D}_t : P_T^{-1}(t) \to \boldsymbol{C}$; here the base point on $P_T^{-1}(t)$ is just $p_T(t)$, where $p_T : T \to U_T$ is the canonical section of U_T. By gluing together these maps we then get a holomorphic immersion

$$\mathcal{D} : U_T \to T \times \boldsymbol{C}$$

sending $P_T^{-1}(t)$ to $\{t\} \times \boldsymbol{C}$. In particular, U_T is a Riemann domain over $T \times \boldsymbol{C} \simeq \boldsymbol{D} \times \boldsymbol{C}$. Also this Riemann domain can be understood as the maximal domain over which the flow of v, with initial condition on T, is defined.

The fact that U_T is a Riemann domain over $\boldsymbol{C}^2$ gives a drastic simplification in the proof of Theorem 3.1, with respect to [Br3]: in order to prove that some manifold is Stein, it is much simpler when the manifold is presented as a Riemann domain over the Euclidean space, or over another Stein manifold. In some sense, we are closer to [Ily] than to [Br3], and indeed the proof that we shall give is somewhat reminiscent of Il'yashenko's ideas.

More concretely, by Oka's Theorem in order to prove that U_T is Stein it is sufficient to prove that it is Hartogs-convex, i.e., that every embedding of a Hartogs figure

$$\mathsf{H}_\varepsilon = \{(z, w) \in \boldsymbol{D}^2 \,;\ |w| < \varepsilon \text{ or } |z| > 1 - \varepsilon\,\}$$

into U_T extends to an embedding of its completion $\boldsymbol{D}^2$ into U_T. This is the path that we will follow.

3.3. Hartogs-convexity. Take a holomorphic embedding

$$f : \mathsf{H}_\varepsilon \to U_T\,.$$

The composite map $\mathcal{D} \circ f : \mathsf{H}_\varepsilon \to T \times \boldsymbol{C}$ is an immersion which, of course, can be holomorphically extended to $\boldsymbol{D}^2$. This extension, denoted by $\widehat{\mathcal{D} \circ f}$, is still an immersion. Let $\mathcal{G}$ be the foliation on $\boldsymbol{D}^2$ obtained by pulling-back the vertical foliation on $T \times \boldsymbol{C}$, and note that $\mathcal{G}$ is a *nonsingular* foliation.

Consider the immersion

$$\Pi_T \circ f : \mathsf{H}_\varepsilon \to S\,.$$

According to [Iva, Cor.1], this map extends meromorphically to $\boldsymbol{D}^2 \setminus \Gamma$, where $\Gamma \subset \boldsymbol{D}^2$ is a discrete subset of essential singularities. Outside Γ and the indeterminacy points $\Sigma \subset \boldsymbol{D}^2 \setminus \Gamma$, this extension $\widehat{\Pi_T \circ f}$ is still an immersion, which sends leaves of $\mathcal{G}$ to leaves of $\mathcal{F}$.

LEMMA 3.1. *We have* $\Sigma = \Gamma = \emptyset$.

PROOF. Because $\mathcal{F}$ is relatively minimal, and because $\mathcal{G}$ is nonsingular, the map $\widehat{\Pi_T \circ f}$ cannot have indeterminacy points: indeed, by [Br3, Lemma 1] an indeterminacy point would produce a rational curve in S over which the canonical bundle $K_{\mathcal{F}}$ would have negative degree (see also [Br5, Remark 2.1]). Or, in other words, an indeterminacy point would produce a vanishing end of some leaf of $\mathcal{F}$, which cannot exist by relative minimality.

Because $\mathcal{F}$ is uniformisable, the set of essential singularities is empty too. Indeed, a point $q \in \Gamma$ would produce a vanishing cycle [Br5, Definition 2.2]: take the image by $\widehat{\Pi_T \circ f}$ of a small cycle around q in the leaf of $\mathcal{G}$ through q. □

By this lemma, we have now at our disposal a holomorphic immersion

$$\widehat{\Pi_T \circ f} : \boldsymbol{D}^2 \to S\,.$$

Note that this map avoids Sing$(\mathcal{F})$, since $\mathcal{G} = (\widehat{\Pi_T \circ f})^*(\mathcal{F})$ is nonsingular. We shall use the continuity method to lift such a map to U_T, i.e., to find

$$\widehat{f} : \boldsymbol{D}^2 \to U_T$$

such that $\widehat{\Pi_T \circ f} = \Pi_T \circ \widehat{f}$ and $\widehat{f}|_{\mathsf{H}_\varepsilon} = f$. This gives the Hartogs-convexity.

As a preliminary fact, observe that, up to a small perturbation of our initial f (which does not affect Hartogs-convexity), we may assume that no horizontal disc of $\boldsymbol{D}^2$ is a leaf of $\mathcal{G}$. Thus, for every $w \in \boldsymbol{D}$, the disc $D_w = \boldsymbol{D} \times \{w\} \subset \boldsymbol{D}^2$ is generically transverse to $\mathcal{G}$, but may have also some tangency points.

Suppose that the desired lifting $\widehat{f}$ has been already constructed on

$$\mathsf{H}_{\varepsilon,r} = \{(z,w) \in \boldsymbol{D}^2\,;\ |w| < r \text{ or } |z| > 1-\varepsilon\,\}$$

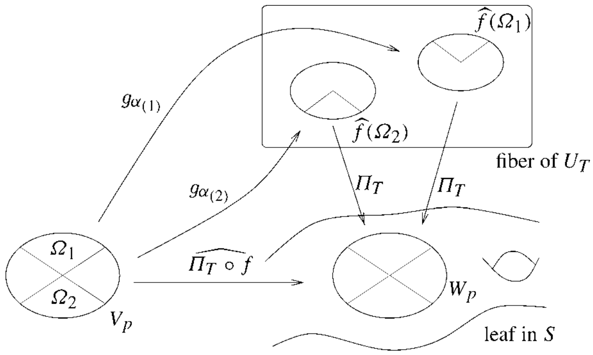
for some $r < 1$, and take a point $p = (z_0, w_0)$ with $|w_0| = r$ and $|z_0| \le 1-\varepsilon$. Let V_p be a small disc centered at p and contained in the leaf of $\mathcal{G}$ through p. By the map $\widehat{\Pi_T \circ f}$, V_p is sent to a disc W_p inside some leaf of $\mathcal{F}$, which can be lifted to the universal covering of the leaf, i.e., to some fiber of U_T. Of course, such a lifting is usually not unique, and we have to select the "good" one. Denote by $\{g_\alpha\}_{\alpha \in A}$ the (countable) set of maps $g_\alpha : V_p \to U_T$ such that

$$\Pi_T \circ g_\alpha = \widehat{\Pi_T \circ f}|_{V_p}\,.$$

Observe now that $V_p \cap \mathsf{H}_{\varepsilon,r}$ is a *nonempty* open subset of V_p, which however may have several connected components (when p is a tangency point of $\mathcal{G}$ with D_{w_0}). On $V_p \cap \mathsf{H}_{\varepsilon,r}$ the map $\widehat{f}$ is defined, and satisfies $\widehat{\Pi_T \circ f} = \Pi_T \circ \widehat{f}$, and so for every connected component Ω_j of $V_p \cap \mathsf{H}_{\varepsilon,r}$ there exists one and only one g_α which extends $\widehat{f}|_{\Omega_j}$ to the full V_p.

LEMMA 3.2. *α does not depend on j.*

PROOF. This is a manifestation of the fact that U_T is Hausdorff [Br5]. Note that if $q = (z_0, w_0')$, $|w_0'| < r$, is close to p and $V_q \subset \mathcal{G}$ is close to V_p, then the intersection $V_q \cap \mathsf{H}_{\varepsilon,r}$ becomes *connected*. Hence, by the argument above, we may find $g : V_q \to U_T$ which extends $\widehat{f}|_{V_q \cap \mathsf{H}_{\varepsilon,r}}$. By taking a limit as q tends to p we then get the conclusion. □



an impossible situation

Therefore, we obtain a well defined extension of $\widehat{f}$ to the disc V_p. By repeating this procedure for every p on $\{|w| = r\}$ we obtain the extension of $\widehat{f}$ to $\mathsf{H}_{\varepsilon,r'}$ for some $r' > r$. Hence, if R is the supremum of those $r < 1$ such that $\widehat{f}$ extends to $\mathsf{H}_{\varepsilon,r}$, we find $R = 1$. This means precisely that the sought extension to $\boldsymbol{D}^2$ has been found.

4. Case (A): conclusion. We now return to the proof of Theorem 0.1, and to the setting and notation of Sections 1 and 2. With the help of Proposition 2.1 and Theorem 3.1, we can complete the proof of Theorem 0.1 in the Case (A).

First of all, we may suppose that $\mathcal{F}$ is uniformisable. Indeed, nonuniformisable foliations are classified in [Br5], and it follows from that result that the only minimal compact complex surfaces admitting nonuniformisable foliations are Hopf and Kato surfaces, in which case the conclusion of Theorem 0.1 holds (and, more precisely, S is an intermediate Kato surface).

We may also suppose that $\mathcal{F}$ is relatively minimal. Indeed, in the opposite case we have on S a rational curve D with $K_{\mathcal{F}} \cdot D < 0$. This curve cannot coincide with C, for $K_{\mathcal{F}} \cdot C = 1$. In fact, it is easy to see that D must be a smooth rational curve, invariant by $\mathcal{F}$, and cutting C at the point q. By [DOT] we get, again, that S is an intermediate Kato surface.

Remark that on any intermediate Kato surface there exists an unique foliation [D-O], which is *not* uniformisable *nor* relatively minimal [Br5, p. 737]. Therefore it is not a surprise that in the following we shall get a contradiction.

By Proposition 2.1, the foliation $\widetilde{\mathcal{F}}$ on the universal covering $\widetilde{S}$ is defined by a holomorphic vector field v which vanishes on $\widetilde{C}$, and only there. Because any covering tube $U_T \overset{\Pi_T}{\to} S$ can be lifted to $\widetilde{S}$, we can pull-back v to U_T, getting a vector field v' tangent to the fibers. If $T \cap C = \emptyset$, then v' is nowhere vanishing. If $T \cap C \neq \emptyset$, say $T \cap C = \{t_0\}$ to fix ideas, then we can multiply v' by the function $P_T^*(1/(t - t_0))$, and again we get a vertical nowhere vanishing vector field. In conclusion, the hypothesis of Theorem 3.1 is satisfied, for every covering tube.

Observe now that $\mathcal{F}$ has at least one hyperbolic leaf: the one contained in C, which is isomorphic to the projective line minus three points. We stress that here we are using the relative minimality (see again [Br5, p. 737]). Therefore, from Corollary 3.1 we deduce that $K_{\mathcal{F}}$ is *pseudoeffective*.

Following [Tom, Remark 8] and [Lam], recall now that the pseudoeffectivity of a line bundle on a class VII_0 surface implies that the line bundle is numerically equivalent to an *effective* one. In our case, we therefore get that $K_{\mathcal{F}}$ is numerically equivalent to $\mathcal{O}_S(\alpha C)$ for some $\alpha \geq 0$ (we assume that C is the only curve on S, otherwise S is already a Kato surface). On the other side, $K_{\mathcal{F}}$ is numerically equivalent to $\mathcal{O}_S(-C)$. This is a contradiction, because $\mathcal{O}_S(C)$ is *not* numerically trivial.

5. Case (B): first integral. From now on we shall consider only Case (B). The proof, in this case, becomes more elaborate. A first step consists in the construction of a plurisubharmonic first integral for $\widetilde{\mathcal{F}}$ on $\widetilde{S}$.

As in Case (A), we may assume that $\mathcal{F}$ is uniformisable and relatively minimal. Note, however, that contrary to Case (A) the model that we are looking for (one of the two foliations on an Inoue-Hirzebruch surface [D-O]) *is* uniformisable and relatively minimal [Br5, p. 737]. This explains, perhaps, why Case (B) is more difficult than Case (A).

By the same argument already used in Case (A), Proposition 2.1 implies that we are in the domain of application of Theorem 3.1 and Corollary 3.1.

5.1. Existence of parabolic leaves. The nodal curve C contains a parabolic leaf of $\mathcal{F}$, isomorphic to $\boldsymbol{C}^*$. Besides that, we have:

PROPOSITION 5.1. *There exists at least one parabolic leaf in $S \setminus C$.*

PROOF. By contradiction, suppose that all the leaves outside C are hyperbolic. The standard Brody-type argument [Br3, Proposition 2] shows that the leafwise Poincaré metric is then continuous, as a hermitian metric on $K_{\mathcal{F}}$ with poles on $C \cup \{q\}$.

By Corollary 3.1, the curvature of this metric is a closed positive current Ω. This current is exact, because $K_{\mathcal{F}}$ is numerically trivial, and therefore by [Tom, Remark 8] and [Lam] the algebraic component of its Siu decomposition is trivial.

A stronger regularity holds: Ω is absolutely continuous (i.e., its coefficients are absolutely continuous measures w.r.t. the Lebesgue measure). This can be seen by checking that the proof of such a statement given in [Br4] is valid on any compact complex surface, and not only on projective ones (there are just some local computations and Stokes theorem). Therefore the pointwise wedge product $\Omega \wedge \Omega$ is well-defined, and it is an absolutely continuous positive measure on S whose total mass is not greater than $[\Omega]^2$, $[\cdot]$ denoting the cohomology class. See [Dem, Corollary 7.6]. But this cohomology class is trivial, hence the total mass of $\Omega \wedge \Omega$ must be zero and therefore

$$\Omega \wedge \Omega \equiv 0 .$$

We are now in the position of applying [Br3, Proposition 6], which produces a *holomorphic* Monge-Ampère foliation $\mathcal{G}$ in the Kernel of Ω, transverse to $\mathcal{F}$ along hyperbolic leaves. In particular, the point q is an isolated tangency point between $\mathcal{F}$ and $\mathcal{G}$, which is an evident absurdity because the tangency set between two foliations is always of pure dimension 1. This contradiction completes the proof. □

5.2. Invariant measure and first integral. Let $L \subset S \setminus C$ be a parabolic leaf of $\mathcal{F}$. By the usual Ahlfors-type procedure [Br1], we can associate to L a (nontrivial!) closed positive current T invariant by the foliation (also called an invariant transverse measure). Note that, even if in [Br1] the ambient manifold is supposed Kählerian, the arguments given in [Br1, p. 197–198] for the construction of the closed positive current do not use the closedness of the Kähler form, and they hold on any compact Hermitian manifold. We shall assume that C is the only compact curve on S (otherwise, S is already a Kato surface), so that the Siu decomposition of T is

$$T = T_0 + \nu\delta_C , \quad \nu \geq 0 ,$$

where T_0 is an exact positive current, by [Tom, Remark 8] and [Lam]. Of course, T_0 is still invariant by the foliation.

Note that T_0 is nontrivial too. Indeed, the curve C can be analytically collapsed to one point, giving a surface S' with a singular point p'. The parabolic curve L projects to S' to a parabolic curve L' outside p', and the direct image of T on S' is a closed positive current T' associated to L' (and hence nontrivial). This current T' cannot be supported only on $\{p'\}$, whence T cannot be supported only on C.

Denote by $\widetilde{T}_0$ the lifting of T_0 to $\widetilde{S}$. By [Tom, Proposition 4], $\widetilde{T}_0$ is dd^c-exact. More precisely, there exists a plurisubharmonic function F on $\widetilde{S}$ such that

$$\widetilde{T}_0 = dd^c F\ , \quad F \circ \varphi = F + c$$

for some real constant c ($\neq 0$ because $T_0 \not\equiv 0$). Take care to the fact that F could be highly irregular. The fact that T_0 is $\mathcal{F}$-invariant reflects into the fact that F is harmonic (or $-\infty$) along the leaves of $\widetilde{\mathcal{F}}$.

PROPOSITION 5.2. *The function F is constant along the leaves of $\widetilde{\mathcal{F}}$.*

PROOF. Take a point in $\widetilde{S}$ outside $\mathrm{Sing}(\widetilde{\mathcal{F}})$ and let (z_1, z_2) be local coordinates such that the foliation is expressed by $dz_1 = 0$. The current $\widetilde{T}_0$ is locally written as $\sum_{j,k} A_{j,k} i dz_j \wedge d\bar{z}_k$, where $A_{j,k}$ are (complex) measures. The $\widetilde{\mathcal{F}}$-invariance means that $A_{j,k} \equiv 0$ for $(j,k) \neq (1,1)$, and in turn this means that the (distributional) partial derivatives $F_{z_j,\bar{z}_k}$ are equal to zero when $(j,k) \neq (1,1)$. It follows that F_{z_2} is a *holomorphic* function.

In a more intrinsic way, the $(1,0)$-differential ∂F, restricted to the leaves, is a (a priori irregular) section of $K_{\widetilde{\mathcal{F}}}$. The previous local computation shows that this section is actually a holomorphic one (by Hartogs, it is sufficient to check the holomorphicity outside the singular points). Such a section descends to the quotient S, because ∂F is φ-invariant. We therefore obtain a holomorphic section β of $K_{\mathcal{F}}$.

Because $K_{\mathcal{F}}$ is numerically trivial, and moreover S contains no divisor homologous to zero, we have only two possibilities: either β is identically zero, or β is nowhere vanishing, in which case $K_{\mathcal{F}}$ is holomorphically trivial. But we shall see in the next section that the holomorphic class of $K_{\mathcal{F}}$ can be easily computed, and it is not the trivial one. Hence $\beta \equiv 0$. By construction, this precisely means that F is constant along the leaves. □

REMARK 5.1. On an Inoue-Hirzebruch surface we have two foliations, $\mathcal{F}_h$ and $\mathcal{F}_p$, which are transverse to each other outside the rational curves [D-O]. The leaves of $\mathcal{F}_h$ outside the rational curves are all hyperbolic, whereas the ones of $\mathcal{F}_p$ are all parabolic. The foliation $\widetilde{\mathcal{F}}_p$ has a plurisubharmonic first integral like our F above (something like $-\log(-G)$, where G is the Green function [D-O], [Tom]), but *not* the foliation $\widetilde{\mathcal{F}}_h$. In our arguments above, we have "lost" the foliation $\mathcal{F}_h$ when we wrote that the nonalgebraic part T_0 of the Siu decomposition of T is nontrivial: the only parabolic leaf of $\mathcal{F}_h$ outside C is contained in the second rational curve D, and that leaf can generate only the current δ_D.

6. Case (B): completeness. Here we shall use the plurisubharmonic first integral F to prove that $\widetilde{\mathcal{F}}$ is generated by a complete vector field.

6.1. Some qualitative remarks. Let us firstly discuss some properties of the foliation $\widetilde{\mathcal{F}}$.

By Proposition 2.1, $\widetilde{\mathcal{F}}$ is generated by a holomorphic vector field v which vanishes only at $\mathrm{Sing}(\widetilde{\mathcal{F}})$ and which satisfies

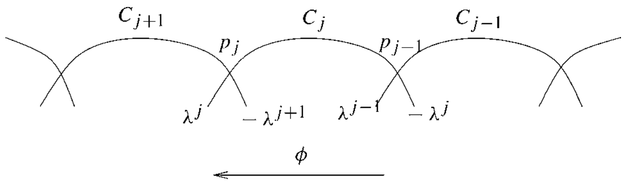
$$\varphi^*(v) = \lambda \cdot v$$

for a suitable $\lambda \in \boldsymbol{C}^*$ (representing the holomorphic class of $K_{\mathcal{F}}$). Let us compute it.

We label the singularities of $\widetilde{\mathcal{F}}$ as

$$\{p_j, q_j\}_{j\in\mathbf{Z}}$$

where $p_j = C_j \cap C_{j+1}$, so that $\varphi(p_j) = p_{j+1}$, and $q_j \in \pi^{-1}(q)$ are chosen so that $\varphi(q_j) = q_{j+1}$. The vector field $v|_{C_j}$ has two zeroes, p_j and p_{j-1}, and the two corresponding eigenvalues of the linear part of $v|_{C_j}$ are necessarily one the opposite of the other. We may normalize v in such a way that $v|_{C_0}$ has eigenvalues $+1$ at p_0 and -1 at p_{-1}. From $\varphi^*(v) = \lambda \cdot v$ and $\varphi(C_j) = C_{j+1}$ we then deduce that $v|_{C_j}$ has eigenvalues λ^j at p_j and $-\lambda^j$ at p_{j-1}. Hence, the two eigenvalues of v at p_j are λ^j and $-\lambda^{j+1}$, so that $-\lambda$ is the quotient of them.



This quotient can be computed via the Camacho-Sad formula on C. Indeed, $C \cdot C$ is equal to $\mathrm{CS}(\mathcal{F}, C, p)$, which in turn is equal to $2 - \lambda - 1/\lambda$, and so we get that λ is a solution of

$$\lambda + \frac{1}{\lambda} - 3 = 0\,.$$

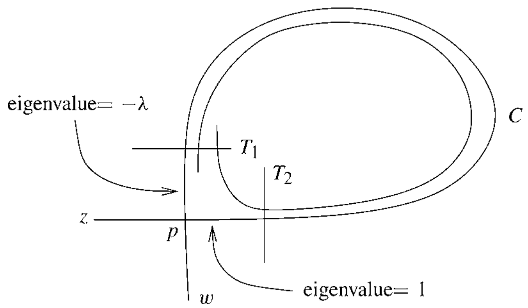
There are two solutions, one smaller and one bigger than 1, but up to changing φ with φ^{-1} we may assume

$$\lambda = \frac{3+\sqrt{5}}{2} > 1\,.$$

Remark that λ is a quadratic irrational, hence a vector field generating the foliation around p can be linearized, by Siegel's theorem. This permits to understand the structure of $\mathcal{F}$ around C, as we now explain.

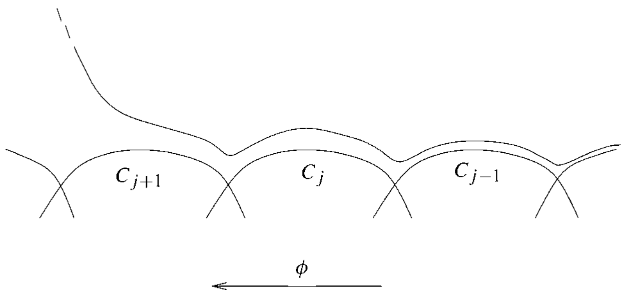
Choose local coordinates (z, w) centered at p such that $\mathcal{F}$ is generated by $z\frac{\partial}{\partial z} - \lambda w \frac{\partial}{\partial w}$. We have there the real first integral $H = |z|^\lambda \cdot |w|$. However, such a first integral does not extend to a neighbourhood of C, due to a monodromy along a path $\gamma \subset C$ generating $\pi_1(C) \simeq \mathbf{Z}$.

More precisely, on a neighbourhood U of C we have a real codimension one foliation $\mathcal{H}$, singular along C, whose leaves are the closures of the leaves of $\mathcal{F}$. Around p, such a foliation is given by the level sets of H. Take the transversals $T_1 = \{w = 1,\ |z| < 1\}$ and $T_2 = \{z = 1,\ |w| < 1\}$ (we may assume that the domain of the local chart contains the closed unit bidisc). Then the level set $\{H = c\}$ intersects T_1 along the circle of radius $c^{1/\lambda}$ and T_2 along the circle of radius c. In other words, the Dulac-type holonomy of $\mathcal{H}$ from T_1 to T_2 is $r \mapsto r^\lambda$, r being the radius. Note that this holonomy is highly contracting, since $\lambda > 1$.



The holonomy of $\mathcal{H}$ along the generator $\gamma \subset C$ is obtained by composing the previous Dulac-type holonomy from T_1 to T_2 with the holonomy along a path in $C \setminus \{p\}$ from T_2 to T_1. This second factor is a diffeomorphism at 0 (in the radial coordinate), and so its composition with $r \mapsto r^\lambda$ is still a highly contracting map. In conclusion, the structure of $\mathcal{H}$ is the one drawn in the picture above: the leaves of $\mathcal{H}$ tend to C when "travelling in the anticlockwise sense".

Now, in our pictures of the universal covering $\widetilde{S}$, this anticlockwise sense corresponds to "travelling to the right". We thus see that, when we go to the right, the leaves of $\widetilde{\mathcal{H}}$ (which is a foliation defined on $\widetilde{U} = \pi^{-1}(U)$) stay close to $\widetilde{C}$, or even tend to $\widetilde{C}$ if we use on $\widetilde{U}$ a φ-invariant metric.



More formally, and returning to the foliation $\widetilde{\mathcal{F}}$, whose leaves on $\widetilde{U}$ are still dense in the ones of $\widetilde{\mathcal{H}}$, here is the statement that we shall actually use. We denote by P a compact fundamental domain for the action of φ on $\widetilde{S}$.

LEMMA 6.1. *For every $x \in \widetilde{S} \setminus \widetilde{C}$ sufficiently close to $\widetilde{C}$ there exists $n_x \in \mathbf{Z}$ such that, if L_x is the leaf of $\widetilde{\mathcal{F}}$ through x, then*

$$L_x \cap \varphi^{-n}(\mathsf{P}) \neq \emptyset \quad \text{for every} \;\; n \geq n_x \,.$$

Let us now turn to the plurisubharmonic first integral F, which has not been used so far. Recall that $F \circ \varphi = F + c$, where c is a nonzero real constant.

LEMMA 6.2. *The function F tends to $-\infty$ to the left*:

$$\max_{\varphi^n(\mathsf{P})} F \to -\infty \quad \text{as } n \to +\infty .$$

In particular, if $L \subset \widetilde{S} \setminus \widetilde{C}$ is a leaf of $\widetilde{\mathcal{F}}$ over which F is not $-\infty$, then L cannot accumulate to the left: *there exists $n_L \in \boldsymbol{Z}$ such that*

$$L \cap \varphi^n(\mathsf{P}) = \emptyset \quad \text{for every } n \geq n_L .$$

PROOF. We have $\max_{\varphi^n(\mathsf{P})} F = \max_{\mathsf{P}} F + nc$ and so F tends to $-\infty$ when travelling either to the left or to the right, depending on the sign of c. However, the fact that there is an open set of leaves which stay close to $\widetilde{C}$ when travelling to the right (Lemma 6.1), and the fact that F is constant on these leaves, excludes $F \to -\infty$ to the right. □

Remark that the polar set $\{F = -\infty\}$ is not empty: by the maximum principle, F must be constant on the chain of compact curves $\widetilde{C}$, and $F \circ \varphi = F + c$, $c \neq 0$, implies

$$F|_{\widetilde{C}} = -\infty .$$

In principle, however, $\{F = -\infty\}$ could be much larger than $\widetilde{C}$, and could be a non-analytic subset of $\widetilde{S}$. Remark also that, by construction, F has vanishing Lelong number except, possibly, at the singular points of the foliation (in fact, by intersection theory [Dem], even at those points the Lelong number must be zero); hence the eventual analyticity of $\{F = -\infty\}$ cannot be detected by Siu's theorem. A posteriori, the polar set of F will be composed by $\widetilde{C}$ and a second chain of rational curves $\widetilde{D}$.

6.2. Completeness of the flow. We can now prove that our vector field v on $\widetilde{S}$ is complete. The argument is somewhat reminiscent of [DOT, §3], but we shall need also our uniformisation result Theorem 3.1 (basically, to get rid of the poles of F).

PROPOSITION 6.1. *The flow of v is complete.*

PROOF. Over the compact fundamental domain $\mathsf{P} \subset \widetilde{S}$ we can find $\varepsilon > 0$ such that the local flow

$$\psi : \boldsymbol{D}(\varepsilon) \times \mathsf{P} \to \widetilde{S}$$

is well defined. From $\varphi^*(v) = \lambda \cdot v$ it follows that also on $\boldsymbol{D}(\lambda^n \varepsilon) \times \varphi^{-n}(\mathsf{P})$ the local flow is well defined, for every $n \in \boldsymbol{Z}$. Since $\lambda > 1$, we get a well defined local flow on

$$\boldsymbol{D}(\varepsilon) \times \bigcup_{n \geq 0} \varphi^{-n}(\mathsf{P}) .$$

Take now a leaf L of $\widetilde{\mathcal{F}}$ not contained in $\{F = -\infty\}$. Up to a translation by φ, we may assume, by Lemma 6.2, that L is contained in $\bigcup_{n \geq 0} \varphi^{-n}(\mathsf{P})$. Hence the local flow of $v|_L$ is defined on the uniformly thick domain $\boldsymbol{D}(\varepsilon) \times L$, and from this fact the completeness of $v|_L$ immediately follows.

In particular, any leaf outside the polar set of F is parabolic. Obviously, the set $\{F \neq -\infty\}$ is not pluripolar, and so by Theorem 3.1 we deduce that *every* leaf is parabolic, even those contained in the poles of F.

Take now a covering tube U_T. Again by Theorem 3.1, we have $U_T \simeq T \times \boldsymbol{C}$. The vector field v lifts to U_T to a vector field of the form

$$h(z, w)\frac{\partial}{\partial w},$$

where h is holomorphic and nowhere vanishing, $z \in T$, $w \in \boldsymbol{C}$. There is a large set of fibers (with pluripolar complement) over which this vector field is complete, and this means that on those fibers the function h is constant. Of course, this in turn implies that h is constant on *every* fiber, i.e., h depends only on the variable z and so the lifted vector field on U_T is complete. This holds for every covering tube, whence the completeness of v. □

7. Case (B): conclusion. Having proved the completeness of v, we can now conclude the proof of Theorem 0.1 also in Case (B).

The equivariance property $\varphi^*(v) = \lambda \cdot v$ with $\lambda \in \boldsymbol{R}$ allows to define, for every $\vartheta \in [0, 2\pi)$, a real one dimensional foliation $\mathcal{L}_\vartheta$ on S, tangent to $\mathcal{F}$ and singular only at $\mathrm{Sing}(\mathcal{F})$: just take the *unparametrized* trajectories on $\widetilde{S}$ given by the flow of v restricted to the real time $\boldsymbol{R} \cdot \mathrm{e}^{\mathrm{i}\vartheta} \subset \boldsymbol{C}$, and project them to S. Actually, this can be done even without the completeness of v, for the local flow is already sufficient.

Let us look at the singularity $q \in \mathrm{Sing}(\mathcal{F})$ outside C. The quotient of the eigenvalues of that singularity can be computed via Baum-Bott formula: we already know that $N_\mathcal{F} \cdot N_\mathcal{F} = -2$ (Lemma 2.2), and moreover $\mathrm{BB}(\mathcal{F}, p) = (1-\lambda)^2/(-\lambda) = -1$, hence $\mathrm{BB}(\mathcal{F}, q) = -1$ and the quotient of eigenvalues at q is the same as at p, namely $-\lambda$.

In particular, the vector field v is linearizable at any q_j, and we can choose $\vartheta \in [0, 2\pi)$ such that $\mathcal{L} = \mathcal{L}_\vartheta$ coincides, around q and in suitable coordinates, with the real trajectories of the vector field

$$\mathrm{i}\left(z\frac{\partial}{\partial z} - \lambda w\frac{\partial}{\partial w}\right).$$

Remark that every round ball centered at q is completely invariant by $\mathcal{L}$. The leaves of $\mathcal{L}$ in the two separatrices are circles, whereas the ones outside the separatrices are real lines, each one dense in a torus $\{|z| = c_1, |w| = c_2\}$.

Let now L be the leaf of $\mathcal{F}$ containing the separatrix $\{w = 0\}$. The completeness of v implies the following two capital properties:

(1) L is isomorphic to $\boldsymbol{C}^*$.

(2) $\mathcal{L}|_L$ is a foliation by circles.

Consider now the second end of L, i.e., the one which does not correspond to $\{w = 0\}$. The structure of $\mathcal{L}$ around q implies that, if this second end accumulates to q, then necessarily it corresponds to the second separatrix $\{z = 0\}$. In that case, the closure of L is a rational curve with a node at q, and by [Na1, (8.1)] the surface S is an even Inoue-Hirzebruch surface.

Suppose now that the second end of L does not accumulate to q, and let us reach a contradiction.

Since L intersects a neighbourhood of q only along $\{w = 0\}$, the same L cannot accumulate to itself (i.e., L is properly embedded in $S \setminus \lim L$, where $\lim L = \overline{L} \setminus L$). Recall now the discussion in Subsection 6.1 about the structure of $\mathcal{F}$ around C: it follows from that discussion that every leaf of $\mathcal{F}$ passing close to C certainly accumulates to itself, being dense in a real hypersurface. Hence, our L does not accumulate to C.

The second end of L being of parabolic type (and transcendental), we may associate to it a closed positive current T' invariant by $\mathcal{F}$ as in Subsection 5.2. By the previous considerations, we have

$$\mathrm{Supp}(T') \cap C = \emptyset\,.$$

However, as in Subsection 5.2 this current T' has a plurisubharmonic potential F' on $\widetilde{S}$, and we already observed that necessarily

$$F'|_{\widetilde{C}} = -\infty\,.$$

This is in evident contradiction with $\mathrm{Supp}(T') \cap C = \emptyset$, which implies that F' is pluriharmonic, and hence finite, on a neighbourhood of $\widetilde{C}$. This completes the proof.

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