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PARABOLIC HARNACK INEQUALITY ON METRIC SPACES WITH A GENERALIZED VOLUME PROPERTY

GIACOMO DE LEVA

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Abstract. We study the parabolic Harnack inequality on metric measure spaces with the more general volume growth property than the volume doubling property. As applications we extend some Liouville theorems and heat kernel estimates for Riemannian manifolds to Alexandrov spaces satisfying a volume comparison condition of Bishop-Gromov type.

1. Introduction. In [KMS01] Sobolev and parabolic Harnack inequalities are proved for an n-dimensional Alexandrov space X with curvature bounded below with the n-dimensional Hausdorff measure. The results are all obtained on a relatively compact open set, since on such a set the volume doubling property and a weak Poincaré inequality hold. A global result can be obtained under the assumption that X has a (global) lower bound for the curvature. In this paper, using a general setting that includes spaces satisfying a volume comparison condition of Bishop-Gromov type as defined in [KS10], we prove a parabolic Harnack inequality. We recall that Moser proved the elliptic and parabolic Harnack inequality in [M64] and [M71], essentially under the assumption of the Poincaré, the Sobolev inequality, and the volume doubling property. In [J86] it was proved that the Poincaré inequality holds if the weak one holds. Later Moser's arguments were extended in more general settings, in particular for Riemannian manifolds (see [G91], [S92], [S02]) and for metric spaces carrying a Dirichlet form ([St96]).

NOTATION 1.1. Let (X, d, μ) be a locally compact separable metric measure space, that is (X, d) is a locally compact separable metric space and $\mu : \mathcal{B}(X) \to [0, +\infty]$ is a locally finite measure on the Borel σ-algebra $\mathcal{B}(X)$ of (X, d) with full support. We shall denote by $B(x, r)$ and $\bar{B}(x, r)$, $x \in X$ and $r > 0$, respectively, the open and closed ball centered at x and of radius r: $B(x, r) := \{y \in X; d(x, y) < r\}$ and $\bar{B}(x, r) := \{y \in X; d(x, y) \le r\}$. Let Ω be an open subset of X and $(\mathcal{E}, \mathcal{F})$ a regular, strongly local symmetric Dirichlet form on the real Hilbert space $L^2(\Omega, \mu)$ with the core $C_0^{\mathrm{Lip}}(\Omega)$, the set of the real valued Lipschitz functions with compact support. Let Γ be the energy measure associated to $\mathcal{E}$. Let Ω' be an open subset of Ω. We denote by $\mathcal{F}_{\mathrm{loc}}(\Omega')$ the set of all measurable functions $f : \Omega \to \boldsymbol{R}$ such that for every compact subset $K \subset \Omega'$, there exists a function $g_K \in \mathcal{F}$ such that $f = g_K$ on K (μ-a.e.). We set $\Gamma(f, f)(K) := \Gamma(g_K, g_K)(K)$. Notice that $\Gamma(f, f)(K)$ does not depend on the choice of g_K. Since the Borel σ-algebra $\mathcal{B}(\Omega')$ of Ω' is generated by

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the compact subsets of Ω', we can define in this way a measure $\Gamma(f, f)$ on $\mathcal{B}(\Omega')$ for any $f \in \mathcal{F}_{\mathrm{loc}}(\Omega')$.

ASSUMPTION 1.2. Let (X, d, μ) be a locally compact separable metric measure space, Ω an open set of X, and $(\mathcal{E}, \mathcal{F})$ a regular, strongly local symmetric Dirichlet form on the real Hilbert space $L^2(\Omega, \mu)$ with the core $C_0^{\mathrm{Lip}}(\Omega)$ such that, for any $x \in \Omega$, $\Gamma(d_x) \leq \mu$, where Γ is the energy measure associated to $\mathcal{E}$ and d_x is the function defined by $d_x(y) := d(x, y)$, $y \in \Omega$. Assume that the closure of any open ball $B(x, r) \subset \Omega$ is the closed ball $\bar{B}(x, r)$. Moreover, the closed balls $\bar{B}(x, r)$ are compact. For any open ball $B(x, r) \subset \Omega$ and $y \in B(x, r)$, there is a geodetic segment joining x and y, that is a continuous map $\gamma : [a, b] \to X$ such that $\gamma(a) = x$, $\gamma(b) = y$ and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all t_1 and t_2 in $[a, b]$. There exist a non-decreasing function $\Theta : (0, +\infty) \to [1, +\infty)$ and $\nu > 2$ such that for any $x \in \Omega$ and $0 < r_1 \leq r_2$, with $B(x, r_2) \subset \Omega$, we have

$$\mu(B(x, r_2)) \leq \left(\frac{r_2}{r_1}\right)^{\nu} \Theta(r_2)\mu(B(x, r_1))\,. \tag{1}$$

We denote by Δ the generator of $\mathcal{E}$. Moreover, assume that there exist a constant $k > 1$ and a non-decreasing function $\Upsilon : (0, +\infty) \to [1, +\infty)$ such that for any open ball $B(x, r)$ with $B(x, kr) \subset \Omega$ and $f \in \mathcal{F}$,

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \leq \Upsilon(r) r^2 \int_{B(x,kr)} d\Gamma(f)\,, \tag{2}$$

where

$$f_{B(x,r)} := \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f\, d\mu\,.$$

REMARK 1.3. The assumption $\nu > 2$ is not restrictive. If (1) holds for some $0 < \nu \leq 2$, then we may assume that the same inequality holds for $\nu := \nu_0$, where ν_0 is any fixed constant strictly bigger than 2.

We refer to [St95] for the definition of local subsolution (resp. local supersolution) on $I \times G$ of the equation

$$\Delta u = \frac{\partial u}{\partial t} \tag{3}$$

where I is any open interval of $\boldsymbol{R}$ and G any open subset of Ω. A local solution on $I \times G$ of Equation (3) is a local subsolution and a local supersolution on $I \times G$ of Equation (3).

THEOREM 1.4. *Let (X, d, μ), $\nu > 2$, $\Omega := X$, $\Theta : (0, +\infty) \to [1, +\infty)$, $(\mathcal{E}, \mathcal{F})$, $k > 1$, and $\Upsilon : (0, +\infty) \to [1, +\infty)$ be satisfying Assumption 1.2. Let $\tau > 0$ and $0 < \varepsilon < \eta < \delta \leq 1$ and $\xi \in (0, 1)$ such that*

$$\xi\eta - \varepsilon < (1 - \eta)(1 - \xi)\,. \tag{4}$$

Then, there exist two constants $H_1(\nu, k, \tau, \delta, \eta, \varepsilon, \xi)$, depending only on ν, k, τ, δ, η, ε and ξ, and $H_2(\nu, k)$, depending only on ν and k, as well as constants $C_2(k)$ and $C_4(k)$, depending *only on k, such that for any open ball $B(x,r)$, any $s \in \boldsymbol{R}$, and any nonnegative local solution u of Equation* (3) *in $(s-\tau r^2, s) \times B(x,r)$, we have*

$$\operatorname*{ess\,sup}_{Q_-} u \le H_1(\nu,k,\tau,\delta,\eta,\varepsilon,\xi)\,(\Theta(C_2 r)\Upsilon(C_4 r))^{H_2(\nu,k)} \operatorname*{ess\,inf}_{Q_+} u\,,$$

with

$$Q_- := \left(s-\delta\tau r^2, s-\eta\tau r^2\right) \times B(x,\xi r) \quad \text{and} \quad Q_+ := \left(s-\varepsilon\tau r^2, s\right) \times B(x,\xi r)\,.$$

The constants $C_2(k)$ and $C_4(k)$ are equal to those in Proposition 2.2.

The above theorem holds also if Assumption 1.2 is satisfied for some open set $\Omega \subset X$ strictly included in X (see Theorem 2.14).

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2. Mean values and parabolic Harnack inequalities. In our setting it is easy to extend all the proofs in [CW71] and [S02, Section 5.3] to prove Poincaré-type inequalities. We omit the proofs. It is convenient to give the following definition.

DEFINITION 2.1. Let (X,d,μ), $\nu > 2$, $\Omega \subset X$, $\Theta : (0,+\infty) \to [1,+\infty)$, $(\mathcal{E},\mathcal{F})$, $k > 1$, and $\Upsilon : (0,+\infty) \to [1,+\infty)$ be satisfying Assumption 1.2. Let $s_0 := \max\{s \in (0,1); k < (1-s^2)/(12s+20s^2)\}$. An open ball is said to be admissible if $B(x, 5\alpha r) \subset \Omega$, where $\alpha(k) := (17s_0^2 + 10s_0 + 1)/(2s_0^2 + 2s_0)$. If the stronger volume property

$$\mu(B(x,r_2)\cap\Omega) \le \left(\frac{r_2}{r_1}\right)^{\nu} \Theta(r_2)\mu(B(x,r_1)\cap\Omega) \tag{5}$$

holds for any $x \in \Omega$ and $0 < r_1 \le r_2$, then every open ball $B(x,r) \subset \Omega$ is said to be admissible.

PROPOSITION 2.2. *Let (X,d,μ), $\nu > 2$, $\Omega \subset X$, $\Theta : (0,+\infty) \to [1,+\infty)$, $(\mathcal{E},\mathcal{F})$, $k > 1$, and $\Upsilon : (0,+\infty) \to [1,+\infty)$ be satisfying Assumption* 1.2. *Let $s_0 := \max\{s \in (0,1); k < (1-s^2)/(12s+20s^2)\}$. Then, there exist constants $C_1(\nu,k)$, depending only on ν and k, $C_2(k)$, $C_3(k)$, and $C_4(k)$, depending only on k, such that for any admissible open ball $B(x,r)$ and $f \in \mathcal{F}$, we have*

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \le C_1(\nu,k)\Theta(C_2(k)r)^{C_3(k)}\Upsilon(C_4(k)r)r^2 \int_{B(x,r)} d\Gamma(f)\,. \tag{6}$$

The constants $C_2(k)$ and $C_4(k)$ may be chosen in such a way that $C_2(k) \le 5\alpha$ and $C_4(k) \le c_0$, where $c_0 := (3+5s_0)/(1-s_0)$.

COROLLARY 2.3. *Let (X,d,μ), $\nu > 2$, $\Omega \subset X$, $\Theta : (0,+\infty) \to [1,+\infty)$, $(\mathcal{E},\mathcal{F})$, $k > 1$, and $\Upsilon : (0,+\infty) \to [1,+\infty)$ be satisfying Assumption* 1.2. *Then, there exist constants $C_5(\nu,k)$, depending only on ν and k, $C_2(k)$, $C_6(k)$, and $C_4(k)$, depending only on k, such that for any admissible open ball $B(x,r)$ and $f \in \mathcal{F}$, we have*

$$\int_{B(x,r)} |f - f_{\Phi_x}|^2 \Phi_x \, d\mu \tag{7}$$

$$\leq C_5(\nu,k,\beta)\Theta(C_2(k)r)^{C_6(k)}\Upsilon(C_4(k)r)r^2\int_{B(x,r)}\Phi_x d\Gamma(f)\,,$$

where $\Phi_x(y) := ((1-d(x,y)/r)\vee 0)^2$, $y\in X$, *and*

$$f_{\Phi_x} := \frac{1}{\int_{B(x,r)}\Phi_x\,d\mu}\int_{B(x,r)} f\Phi_x d\mu\,.$$

The constants $C_2(k)$ *and* $C_4(k)$ *are those in Proposition* 2.2.

Extending similarly the proofs of [S02, Theorem 3.2.2] and [BM95, Proposition 1], we easily prove the following Sobolev inequality.

PROPOSITION 2.4. *Let* (X,d,μ), $\nu>2$, $\Omega\subset X$, $\Theta:(0,+\infty)\to[1,+\infty)$, $(\mathcal{E},\mathcal{F})$, $k>1$, *and* $\Upsilon:(0,+\infty)\to[1,+\infty)$ *be satisfying Assumption* 1.2. *Then, there exist constants* $S_1(\nu,k)$, *depending only on* ν *and* k, $C_2(k)$, $S_3(k)$, *and* $C_4(k)$, *depending only on* k, *such that for any admissible open ball* $B:=B(x,r)$ *with* $B(x,3r/2)\subset\Omega$ *and* $f\in\mathcal{F}$, *we have*

$$\left(\int_B |f|^{2\nu/(\nu-2)}d\mu\right)^{(\nu-2)/\nu}\leq\frac{S_1(\nu,k)\Theta(C_2(k)r)^{S_3(k)}\Upsilon(C_4(k)r)}{\mu(B)^{2/\nu}}\left(r^2\int_B\Gamma(f)+\int_B|f|^2d\mu\right).$$

The constants $C_2(k)$ *and* $C_4(k)$ *are those in Proposition* 2.2.

NOTATION 2.5. For every $x\in X$ and $r>0$, with $B(x,r)\subset\subset\Omega$, the function $d_{x,r}(y):=(r-d(x,y))\vee 0$ belongs to the set $\mathcal{F}\cap C_0(\Omega)$ and $\Gamma(d_{x,r},d_{x,r})\leq\mu$. We also denote by $\bar\mu$ the measure $dt\otimes\mu$ on $\boldsymbol{R}\times\Omega$

We can easily extend the proof of [St95, Lemma 2.2] and get the following L^p-estimates.

LEMMA 2.6. *Let* (X,d,μ), $\nu>2$, $\Omega\subset X$, $\Theta:(0,+\infty)\to[1,+\infty)$, $(\mathcal{E},\mathcal{F})$, $k>1$, *and* $\Upsilon:(0,+\infty)\to[1,+\infty)$ *be satisfying Assumption* 1.2. *Define, for any admissible open ball* $B:=B(x,r)$, *with* $B(x,3r/2)\subset\Omega$, *any* $\varepsilon\in(0,1]$, $s\in\boldsymbol{R}$, *and* $\tau\in(0,+\infty)$, $Q^-(\varepsilon):=(s-\varepsilon\tau r^2,s)\times B(x,\varepsilon r)$ *and* $Q^+(\varepsilon):=(s,s+\varepsilon\tau r^2)\times B(x,\varepsilon r)$. *Then, defining*

$$C:=64\frac{S_1\Theta(C_2r)^{S_3+1}\Upsilon(C_4r)r^2}{\mu(B)^{2/\nu}}8^{2/\nu}\left(1+\frac{1}{|1-p^{-1}|}\right)^{2+2/\nu}\times\left(\frac{1}{\tau\wedge 1}\right)^{1+2/\nu}\left(\frac{1}{(\delta-\delta')r}\right)^{2+4/\nu},$$

where $S_1:=S_1(\nu,k)$, $C_2:=C_2(k)$, $S_3:=S_3(k)$, *and* $C_4:=C_4(k)$ *are the constants given by Proposition* 2.4, *for any* $s\in\boldsymbol{R}$, $\tau\in(0,+\infty)$, *and all* $0<\delta'<\delta\leq 1$, *we have*

$$\int_{Q^-(\delta')}u^{p(1+2/\nu)}d\bar\mu\leq C\left(\int_{Q^-(\delta)}u^p d\bar\mu\right)^{1+2/\nu},$$

whenever $p>1$ *and* u *is a nonnegative local subsolution of Equation* (3) *on* $Q^-(1)$;

$$\int_{Q^+(\delta')}u^{p(1+2/\nu)}d\bar\mu\leq C\left(\int_{Q^+(\delta)}u^p d\bar\mu\right)^{1+2/\nu},$$

whenever $0 < p < 1$ *and u is a nonnegative local supersolution of Equation* (3) *on* $Q^+(1)$;

$$\int_{Q^-(\delta')} u^{p(1+2/\nu)} d\bar{\mu} \le C \left(\int_{Q^-(\delta)} u^p d\bar{\mu} \right)^{1+2/\nu},$$

whenever $p < 0$ *and u is a nonnegative local supersolution of Equation* (3) *on* $Q^-(1)$.

We are now ready to carry out the Moser iteration. The iterative arguments used in the next two propositions are the same as those used in [M71].

PROPOSITION 2.7. *Let* (X, d, μ), $\nu > 2$, $\Omega \subset X$, $\Theta : (0, +\infty) \to [1, +\infty)$, $(\mathcal{E}, \mathcal{F})$, $k > 1$, *and* $\Upsilon : (0, +\infty) \to [1, +\infty)$ *be satisfying Assumption* 1.2. *Define, for any admissible open ball* $B := B(x, r)$, *with* $B(x, 3r/2) \subset \Omega$, *any* $\varepsilon \in (0, 1]$, $s \in \boldsymbol{R}$, *and* $\tau \in (0, +\infty)$, $Q^-(\varepsilon) := (s - \varepsilon\tau r^2, s) \times B(x, \varepsilon r)$. *Then, there exist two constants* $B(\tau, p, \nu)$,

$$B(\tau, p, \nu) := 64^{\nu/2} 4^{\sum_{n\in N} n(1+2/\nu)^{-n+1}} 8 \left(1 + \frac{1}{|1 - p^{-1}|}\right)^{1+\nu} \left(\frac{1}{\tau \wedge 1}\right)^{1+\nu/2},$$

and $A(p, \nu)$,

$$A(p, \nu) := 4^{2+\nu} \left(\frac{4}{3}\right)^{p(1+\nu/2) \sum_{n\in N} (n-1)(1-p/2)^{n-1}},$$

such that, for any $s \in \boldsymbol{R}$, $\tau \in (0, +\infty)$, *and all* $0 < \delta' < \delta \le 1$, *we have*

$$\|u^p\|_{L^\infty(Q^-(\delta'),\bar{\mu})} \le B(\tau, p, \nu) \frac{\left(S_1 \Theta(C_2 r)^{S_3+1} \Upsilon(C_4 r)\right)^{\nu/2}}{(\delta - \delta')^{2+\nu} r^2 \mu(B)} \int_{Q^-(\delta)} u^p d\bar{\mu},$$

whenever $1 < p < +\infty$ *and u is a nonnegative local subsolution of Equation* (3) *on* $Q^-(1)$;

$$\|u^p\|_{L^\infty(Q^-(\delta'),\bar{\mu})} \le A(p, \nu) B(\tau, 2, \nu) \frac{\left(S_1 \Theta(C_2 r)^{S_3+1} \Upsilon(C_4 r)\right)^{\nu/2}}{(\delta - \delta')^{2+\nu} r^2 \mu(B)} \int_{Q^-(\delta)} u^p d\bar{\mu},$$

whenever $0 < p \le 1$ *and u is a nonnegative local subsolution of Equation* (3) *on* $Q^-(1)$;

$$\|u^p\|_{L^\infty(Q^-(\delta'),\bar{\mu})} \le B(\tau, p, \nu) \frac{\left(S_1 \Theta(C_2 r)^{S_3+1} \Upsilon(C_4 r)\right)^{\nu/2}}{(\delta - \delta')^{2+\nu} r^2 \mu(B)} \int_{Q^-(\delta)} u^p d\bar{\mu},$$

whenever $p < 0$ *and u is a nonnegative local supersolution of Equation* (3) *on* $Q^-(1)$, *where* $S_1 := S_1(\nu, k)$, $C_2 := C_2(k)$, $S_3 := S_3(k)$, *and* $C_4 := C_4(k)$ *are the constants given by Proposition* 2.4.

PROOF. Let u be a nonnegative local subsolution and $p > 1$ or a nonnegative local supersolution and $p < 0$. Define $\omega_n := (\delta - \delta')2^{-n}$, $d_1 := \delta$, $\delta_{n+1} := \delta_n - \omega_n$. Notice that $\delta_n \downarrow \delta'$ for $n \to +\infty$. Applying Lemma 2.6 (where $p(1 + 2/\nu)^{n-1}$ plays the role of the parameter p of the statement), we get

$$\int_{Q^-(\delta_{n+1})} u^{p(1+2/\nu)^n} d\bar{\mu} \le 4^{n(1+2/\nu)} A \left(\int_{Q^-(\delta_n)} u^{p(1+2/\nu)^{n-1}} d\bar{\mu} \right)^{1+2/\nu},$$

where

$$A := 64\frac{S_1\Theta(C_2r)^{S_3+1}\Upsilon(C_4r)r^2}{\mu(B)^{2/\nu}}8^{2/\nu}\left(1+\frac{1}{|1-p^{-1}|}\right)^{2+2/\nu}$$
$$\times\left(\frac{1}{\tau\wedge 1}\right)^{1+2/\nu}\left(\frac{1}{(\delta-\delta')r}\right)^{2+4/\nu}.$$

Iterating the above inequality, we obtain

$$\int_{Q^-(\delta_{n+1})} u^{p(1+2/\nu)^n}d\bar{\mu} \le 4^{\sum_{i=1}^n (n+1-i)(1+2/\nu)^i} A^{\sum_{i=1}^n (1+2/\nu)^{i-1}}\left(\int_{Q^-(\delta)} u^p d\bar{\mu}\right)^{(1+2/\nu)^n}.$$

We obtain the desired inequality by raising to the power of $(1+2/\nu)^{-n}$ and letting n tend to infinity. Finally, let u be a nonnegative local subsolution and $0 < p \le 1$. Fix $0 < \delta' < \delta \le 1$. We have already proved that

$$\|u^2\|_{L^\infty(Q^-(\delta_0'))} \le A(\delta_0-\delta_0')^{-2-\nu}\int_{Q^-(\delta_0)} u^2 d\bar{\mu}\,,$$

for any $0 < \delta_0' < \delta_0 \le 1$, where A is given by

$$A := B(\tau, 2, \nu)\frac{\big(S_1\Theta(C_2r)^{S_3+1}\Upsilon(C_4r)\big)^{\nu/2}}{r^2\mu(B)}.$$

Hence,

$$\|u\|_{L^\infty(Q^-(\delta_0'))} \le A^{1/2}(\delta_0-\delta_0')^{-1-\nu/2}\|u\|_{L^2(Q^-(\delta_2))}.$$

It follows

$$\|u\|_{L^\infty(Q^-(\delta_0'))} \le A^{1/2}(\delta_0-\delta_0')^{-1-\nu/2}\|u\|^{p/2}_{L^p(Q^-(\delta))}\|u\|^{1-p/2}_{L^\infty(Q^-(\delta_0))},$$

for any $0 < \delta_0' < \delta_0 \le \delta$. Define $\delta_1 := \delta'$ and $\delta_{n+1} := \delta_n + (\delta-\delta_n)/4$, $n \in \boldsymbol{N}$. Thus, $\delta_{n+1}-\delta_n = (1/4)(3/4)^{n-1}(\delta-\delta')$. Applying the above inequality, we obtain for any $n \in \boldsymbol{N}$

$$\|u\|_{L^\infty(Q^-(\delta_n))} \le \tilde{A}\left(\frac{4}{3}\right)^{(n-1)(1+\nu/2)}\|u\|^{1-p/2}_{L^\infty(Q^-(\delta_{n+1}))},$$

where $\tilde{A}$ is defined by

$$\tilde{A} := A^{1/2}\left(\frac{4}{\delta-\delta'}\right)^{1+\nu/2}\|u\|^{p/2}_{L^p(Q^-(\delta))}.$$

Hence, for any $n \in \boldsymbol{N}$, we have

$$\|u\|_{L^\infty(Q^-(\delta'))} \le \tilde{A}^{\sum_{i=1}^n (1-p/2)^{i-1}}\left(\frac{4}{3}\right)^{(1+\nu/2)\sum_{i=1}^n (i-1)(1-p/2)^{i-1}}\|u\|^{(1-p/2)^n}_{L^\infty(Q^-(\delta_{n+1}))}.$$

Letting n tend to infinity and raising to the power of p yields

$$\|u^p\|_{L^\infty(Q^-(\delta'))} \le \left(\frac{4}{3}\right)^{p(1+\nu/2)\sum_{n\in\boldsymbol{N}}(n-1)(1-p/2)^{n-1}} A\left(\frac{4}{\delta-\delta'}\right)^{2+\nu}\|u\|^p_{L^p(Q^-(\delta))}.$$

$\square$

REMARK 2.8. With the same notation of Proposition 2.7, we can estimate the constant $B(\tau, p, \nu)$ by a constant that does not depend on p. Indeed, we have $B(\tau, p, \nu) \le B(\tau, \nu)$, where

$$B(\tau, \nu) := 64^{\nu/2} 4^{\sum_{n\in N} n(1+2/\nu)^{-n+1}} 2^{4+\nu} \left(\frac{1}{\tau \wedge 1}\right)^{1+\nu/2} .$$

PROPOSITION 2.9. *Let (X, d, μ), $\nu > 2$, $\Omega \subset X$, $\Theta : (0, +\infty) \to [1, +\infty)$, $(\mathcal{E}, \mathcal{F})$, $k > 1$, and $\Upsilon : (0, +\infty) \to [1, +\infty)$ be satisfying Assumption 1.2. Define, for any admissible open ball $B := B(x, r)$, with $B(x, 3r/2) \subset \Omega$, any $\varepsilon \in (0, 1]$, $s \in \boldsymbol{R}$, and $\tau \in (0, +\infty)$, $Q^+(\varepsilon) := (s, s + \varepsilon\tau r^2) \times B(x, \varepsilon r)$. Moreover, set*

$$C(\tau, p_0, \nu) := 64^{\nu/2} 4^{\nu/2 \sum_{n\in N} n(1+\nu/2)^{-(n-2)}} 8 \left(\frac{1 + 2/\nu}{1 + 2/\nu - p_0}\right)^{1+\nu} \left(\frac{1}{\tau \wedge 1}\right)^{1+\nu/2} \tau .$$

Then, for any $s \in \boldsymbol{R}$, $\tau \in (0, +\infty)$, $0 < p_0 < 1 + 2/\nu$, and all $0 < \delta' < \delta \le 1$, we have

$$\|u\|_{L^{p_0}(Q^+(\delta'))} \le \left(\frac{C(\tau, p_0, \nu)^{4\nu} \left(S_1 \Theta(C_2 r)^{S_3+1} \Upsilon(C_4 r)\right)^{2\nu^2}}{(\delta - \delta')^{4\nu(2+\nu)} r^2 \mu(B)}\right)^{1/p - 1/p_0} \|u\|_{L^p(Q^+(\delta))} ,$$

whenever $0 < p < p_0(1 + 2/\nu)^{-1}$ and u is a nonnegative local supersolution of Equation (3) *on $Q^+(1)$, where $S_1 := S_1(\nu, k)$, $C_2 := C_2(k)$, $S_3 := S_3(k)$, and $C_4 := C_4(k)$ are the constants given by Proposition* 2.4.

PROOF. Define $p_m := p_0(1 + 2/\nu)^{-m}$, $m \in \boldsymbol{N}$, $\omega_n := (\delta - \delta')2^{-n}$, $d_1 := \delta$, and $\delta_{n+1} := \delta_n - \omega_n$. Since, for any $j \in \{1, \dots, m\}$, $p_m(1+2/\nu)^{j-1} \le p_0(1+2/\nu)^{-1}$, applying Lemma 2.6 (with $p := p_m(1 + 2/\nu)^{j-1}$), we have, for any $n \in \boldsymbol{N}$ and $j \in \{1, \dots, m\}$,

$$\int_{Q^+(\delta_{n+1})} u^{p_m(1+2/\nu)^j} d\bar{\mu} \le 4^{n(1+2\nu)} A \left(\int_{Q^+(\delta_n)} u^{p_m(1+2/\nu)^{j-1}} d\bar{\mu}\right)^{1+2/\nu} ,$$

where A is given by

$$A := 64 \frac{S_1 \Theta(C_2 r)^{S_3+1} \Upsilon(C_4 r)}{(\delta - \delta')^{2+4/\nu} \mu(B)^{2/\nu} r^{4/\nu}} 8^{2/\nu} \left(\frac{1 + 2/\nu}{1 + 2/\nu - p_0}\right)^{2+2/\nu} \left(\frac{1}{\tau \wedge 1}\right)^{1+2/\nu} .$$

Iterating the above inequality, we obtain

$$\begin{aligned}
&\int_{Q^+(\delta_{n+1})} u^{p_0} d\bar{\mu} \\
&\quad \le 4^{\sum_{i=1}^n (n+1-i)(1+2/\nu)^i} A^{\sum_{i=1}^n (1+2/\nu)^{i-1}} \left(\int_{Q^+(\delta)} u^{p_n} d\bar{\mu}\right)^{(1+2/\nu)^n} \\
&\quad \le 4^{\sum_{m\in N} m(1+\nu/2)^{-(m-2)} (\nu/2)(p_0/p_n - 1)} A^{(\nu/2)(p_0/p_n - 1)} \left(\int_{Q^+(\delta)} u^{p_n} d\bar{\mu}\right)^{(1+2/\nu)^n} ,
\end{aligned}$$

where the last inequality follows from

$$\sum_{i=1}^n (n + 1 - i) \left(1 + \frac{\nu}{2}\right)^i \le \left(1 + \frac{\nu}{2}\right)^n \sum_{m\in N} n \left(1 + \frac{\nu}{2}\right)^{-(n-1)}$$

$$\leq \frac{\nu}{2}\left(\left(1+\frac{\nu}{2}\right)^{n}-1\right)\sum_{m\in \boldsymbol{N}} n\left(1+\frac{\nu}{2}\right)^{-(n-2)} \quad \text{and}$$

$$\sum_{i=1}^{n}\left(1+\frac{\nu}{2}\right)^{i-1} = \frac{(1+2/\nu)^n-1}{(1+2/\nu)-1}.$$

Hence, for any $n \in \boldsymbol{N}$,

$$\left(\int_{Q^+(\delta')} u^{p_0} d\bar{\mu}\right)^{1/p_0} \leq \left(4^{\nu/2\sum_{m\in \boldsymbol{N}} m(1+\nu/2)^{-(m-2)}} A^{\nu/2}\right)^{1/p_n-1/p_0}\left(\int_{Q^+(\delta)} u^{p_n} d\bar{\mu}\right)^{1/p_n},$$

that is

$$\left(\int_{Q^+(\delta')} u^{p_0} d\bar{\mu}\right)^{1/p_0}$$
$$\leq \left(\frac{C(\tau, p_0, \nu)\left(S_1\Theta(C_2 r)^{S_3+1}\Upsilon(C_4 r)\right)^{\nu/2}}{(\delta-\delta')^{2+\nu} r^2 \mu(B)}\right)^{1/p_n-1/p_0}\left(\int_{Q^+(\delta)} u^{p_n} d\bar{\mu}\right)^{1/p_n}.$$

Now, let $p \in (0, p_0/(1+2/\nu))$. Let $k_\nu \in \boldsymbol{N}$ be the smallest integer m such that $(2/\nu)(1+2/\nu)^m \geq 1$, that is, $m \geq \log(\nu/2)/\log(1+2/\nu)$. Notice that $1/p_{k+2}-1/p_{k+1} \geq 1$ for all $k \geq k_\nu$. Denote by k_p the integer such that $p_{k_p+1} \leq p < p_{k_p}$. Set $n := \max\{k_\nu, k_p\}+2$. Observe that $1/p_n - 1/p_0 \leq (1+2/\nu)^{k_\nu+2}(1/p-1/p_0)$. Moreover, since we need to make explicit the dependence from τ, r and $\mu(B)$, we set

$$\tilde{A} := 64^{\nu/2} 4^{\nu/2\sum_{n\in \boldsymbol{N}} n(1+\nu/2)^{-(n-2)}} 8\left(\frac{1+2/\nu}{1+2/\nu-p_0}\right)^{1+\nu} \frac{\left(S_1\Theta(C_2 r)^{S_3+1}\Upsilon(C_4 r)\right)^{\nu/2}}{(\delta-\delta')^{2+\nu}}.$$

Hence, rewriting the above inequality and applying Jensen's inequality, we get

$$\left(\int_{Q^+(\delta')} u^{p_0} d\bar{\mu}\right)^{1/p_0} \leq \left(\frac{1}{\tau\wedge 1}\right)^{(1+\nu/2)(1/p_n-1/p_0)}\left(\frac{\tilde{A}}{r^2\mu(B)}\right)^{1/p_n-1/p_0}\bar{\mu}(Q^+(\delta))^{1/p_n}$$
$$\times\left(\frac{1}{\bar{\mu}(Q^+(\delta))}\int_{Q^+(\delta)} u^{p_n} d\bar{\mu}\right)^{1/p_n}$$
$$\leq \left(\frac{1}{\tau\wedge 1}\right)^{(1+\nu/2)(1/p_n-1/p_0)}\left(\frac{\tilde{A}}{r^2\mu(B)}\right)^{1/p_n-1/p_0}$$
$$\times(\tau r^2\mu(B))^{1/p_n-1/p}\left(\int_{Q^+(\delta)} u^{p} d\bar{\mu}\right)^{1/p}$$
$$\leq \left(\frac{1}{\tau\wedge 1}\right)^{(1+\nu/2)(1+2/\nu)^{k_\nu+2}(1/p-1/p_0)}\tilde{A}^{(1+2/\nu)^{k_\nu+2}(1/p-1/p_0)}$$
$$\times\, \tau^{(1+2/\nu)^{k_\nu+2}(1/p-1/p_0)}\left(\frac{1}{r^2\mu(B)}\right)^{1/p-1/p_0}\left(\int_{Q^+(\delta)} u^{p} d\bar{\mu}\right)^{1/p}.$$

□

The following lemma can be proved as [M64, Theorem 4] (see also [S02, Lemma 5.4.1]) for the case that u is such that $u_t \in L^\infty(B)$ and $u_t^{-1} \in L^\infty(B)$ for all $t \in (s-\tau r^2, s)$. The general case follows by an approximation argument considering the functions $(n \wedge u) \vee (1/n)$. We omit the details.

LEMMA 2.10. *Let* (X, d, μ), $\nu > 2$, $\Omega \subset X$, $\Theta : (0, +\infty) \to [1, +\infty)$, $(\mathcal{E}, \mathcal{F})$, $k > 1$, *and* $\Upsilon : (0, +\infty) \to [1, +\infty)$ *be satisfying Assumption* 1.2. *Then, for any admissible open ball* $B := B(x, r)$, *with* $B(x, 3r/2) \subset \Omega$, *any* $\tau > 0$, $\delta \in (0, 1)$, $\varepsilon \in (0, 1)$, $s \in \boldsymbol{R}$, $\lambda > 0$, *and any positive local supersolution* u *of Equation* (3) *in* $(s-\tau r^2, s) \times B$, *we have*

$$\begin{aligned}
&\bar{\mu}(\{(t, z) \in (s-\tau r^2, s-\varepsilon\tau r^2) \times \delta B; \log u(t, z) > \lambda - c_u\}) \\
&\qquad \le 4\frac{C_5(\nu, k, \beta)(1-\delta) + (1-\varepsilon)^2\tau^2}{(1-\delta)^2}\Theta(C_2(k)r)^{C_6(k)}\Upsilon(C_4(k)r)r^2\frac{\mu(B)}{\lambda}\,,
\end{aligned}$$

where

$$c_u := \frac{\int_B -\log u(s-\varepsilon\tau r^2, y)\,(1-d(x, y)/r)^2\,\mu(dy)}{\int_B (1-d(x, y)/r)^2\,\mu(dy)}\,,$$

and $C_5(\nu, k, \beta)$, $C_2(k)$, $C_6(k)$, *and* $C_4(k)$ *are the constants given by Corollary* 2.3.

We will need the following lemma proved in [S02] from an original idea of [BG72].

LEMMA 2.11. *Let* $(X, \mathcal{B}, \mu)$ *a measure space*, $\{U_\sigma; 0 < \sigma \le 1\}$ *a family of measurable subsets such that* $\mu(U_1) > 0$ *and* $U_{\sigma'} \subset U_\sigma$ *if* $\sigma' \le \sigma$, $0 < \delta < 1$, γ *and* C *positive constants*, $0 < p_0 \le +\infty$, *and* $f : U_1 \to (0, +\infty)$ *a measurable function satisfying the following properties*:

(i) *for all* σ, σ' *such that* $\delta \le \sigma' \le \sigma$ *and* $0 < p \le \min\{1, p_0/2\}$

$$\|f\|_{L^{p_0}(U_{\sigma'})} \le \big(C(\sigma-\sigma')^{-\gamma}\mu(U_1)^{-1}\big)^{1/p-1/p_0}\|f\|_{L^p(U_\sigma)}\,;$$

(ii) $\mu(\log f > \lambda) \le C\mu(U)\lambda^{-1}$ *for all* $\lambda > 0$.

Setting

$$\begin{aligned}
&A_1 := \min\left\{x \in (0, +\infty); \frac{1}{y}\log\left(\frac{y}{C}\right) \le \min\left\{1, \frac{p_0}{2}\right\} \text{ for all } y \ge x\right\} \quad \text{and} \\
&A_2 := \max\left\{\log 2, \frac{1}{2}C^3 + \frac{1}{4}A_1\right\},
\end{aligned}$$

then

$$\|f\|_{L^{p_0}(U_\delta)} \le \sum_{n \in N}\left(\frac{3}{4}\right)^{n-1}\left(\frac{1-\delta}{n(1+n)}\right)^{-2\gamma}A_2\,\mu(U_1)^{1/p_0}\,.$$

We can now prove a weak form of the Harnack Inequality. The main argument is the same as for the case of the Laplacian on Riemannian manifolds (see [S02]).

PROPOSITION 2.12. *Let* (X, d, μ), $\nu > 2$, $\Omega \subset X$, $\Theta : (0, +\infty) \to [1, +\infty)$, $(\mathcal{E}, \mathcal{F})$, $k > 1$, *and* $\Upsilon : (0, +\infty) \to [1, +\infty)$ *be satisfying Assumption* 1.2. *Let* $\tau > 0$,

$0 < p_0 < 2/\nu$, *and* $0 < \varepsilon < \eta < 1$ *and* $\xi \in (0,1)$ *such that*

$$\xi\eta - \varepsilon \le (1-\eta)(1-\xi)\,. \tag{8}$$

Then, there exist two constants $H_3 := H_3(\nu, k, p_0, \tau, \eta, \varepsilon, \xi)$, *depending only on* ν, k, p_0, τ, η, ε *and* ξ, *and* $H_4 := H_4(\nu, k)$, *depending only on* ν *and* k, *such that for any admissible open ball* $B(x,r)$, *with* $B(x, 3r/2) \subset \Omega$, *any* $s \in \boldsymbol{R}$, *and any positive local supersolution* u *of Equation* (3) *in* $(s - \tau r^2, s) \times B(x,r)$, *we have*

$$\left(\int_{Q'_-} u^{p_0} d\bar{u}\right)^{1/p_0} \le \left(\varepsilon\tau r^2 \mu(B(x,\xi r))\right)^{1/p_0} H_3 \left(\Theta(C_2 r)\Upsilon(C_4 r)\right)^{H_4} \operatorname*{ess\,inf}_{Q_+} u\,,$$

with

$$Q'_- := (s - \tau r^2, s - \eta\tau r^2) \times B(x, \xi r) \quad \text{and} \quad Q_+ := (s - \varepsilon\tau r^2, s) \times B(x, \xi r)\,,$$

where $C_2(k)$ *and* $C_4(k)$ *are the constants given in Proposition* 2.2.

PROOF. Let C_5, C_2, C_6, C_4, S_1, C_2, and S_3 be the constants given by Corollary 2.3 and Proposition 2.4. Fix $0 < \varepsilon < \eta < \delta < 1$ and $\xi \in (0,1)$. We first choose α so that $1 > \alpha \le (1-\eta)(1-\xi)/(\xi(\eta-\varepsilon))$. Set $v := e^{c(\alpha)}u$, where $c(\alpha)$ is defined by

$$c(\alpha) := \frac{\int_B -\log u\left(s - ((1-\alpha)\eta + \alpha\varepsilon)\tau r^2, y\right)(1 - d(x,y)/r)^2\,\mu(dy)}{\int_B (1 - d(x,y)/r)^2\,\mu(dy)}\,,$$

and, for $0 < \sigma \le 1$, $U_\sigma := (\bar{s}, \bar{s} + \sigma\bar{\tau}\bar{r}^2) \times B(x, \sigma\bar{r})$, where

$$\bar{s} := s - \tau r^2, \bar{r} := \frac{\xi(1 - (1-\alpha)\eta - \alpha\varepsilon)}{1-\eta} r \le r\,, \quad \text{and} \quad \bar{\tau} := \frac{(1-\eta)^2}{\xi^2(1 - (1-\alpha)\eta - \alpha\varepsilon)}\tau\,.$$

Notice that $U_{\sigma_0} = (s - \tau r^2, s - \eta\tau r^2)$ for $\sigma_0 := (1-\eta)(1 - (1-\alpha)\eta - \alpha\varepsilon)^{-1}$. By Proposition 2.9

$$\|v\|_{L^{p_0}(U_{\sigma'})} \le \left(\frac{C(\bar{\tau}, p_0, \nu)^{4\nu}\left(S_1\Theta(C_2\bar{r})^{S_3+1}\Upsilon(C_4\bar{r})\right)^{2\nu^2}}{(\sigma - \sigma')^{4\nu(2+\nu)}\bar{r}^2\mu(B(x,\bar{r}))}\right)^{1/p - 1/p_0} \|v\|_{L^p(U_\sigma)}\,,$$

for all $0 < \sigma_0 \le \sigma' < \sigma \le 1$ and $0 < p < p_0(1 + 2/\nu)$. Hence,

$$\|v\|_{L^{p_0}(U_{\sigma'})} \le \left(\frac{\bar{\tau}C(\bar{\tau}, p_0, \nu)^{4\nu}\left(S_1\Theta(C_2 r)^{S_3+1}\Upsilon(C_4 r)\right)^{2\nu^2}}{(\sigma - \sigma')^{4\nu(2+\nu)}\bar{\mu}(U_1)}\right)^{1/p - 1/p_0} \|v\|_{L^p(U_\sigma)}.$$

Moreover, applying Lemma 2.10 to v we obtain, for any $\lambda > 0$,

$$\begin{aligned}
&\bar{\mu}(\{(t,z) \in U_1; \log v(t,z) > \lambda\}) \\
&\qquad \le \frac{8C_5 \max\{1,\tau\}^2(1-\eta)^2}{((1-\eta) - \xi(1 - (1-\alpha)\eta - \alpha\varepsilon))^2}\Theta(C_2 r)^{C_6}\Upsilon(C_4 r) r^2 \frac{\mu(B)}{\lambda} \\
&\qquad \le F_1(\nu, \tau, \eta, \varepsilon, \xi, \alpha)\Theta(C_2 r)^{C_6+1}\Upsilon(C_4 r)\frac{\bar{\mu}(U_1)}{\lambda}\,,
\end{aligned}$$

where F_1 is an opportune constant. Thus, applying Lemma 2.11 we obtain

$$\|e^{c(\alpha)}u\|_{L^{p_0}(Q'_-)} \le G_1\Theta(C_2 r)^{G_2}\Upsilon(C_4 r)^{G_3}\mu(U_1)^{1/p_0}\,, \tag{9}$$

where G_1 is a constant depending only on ν, p_0, τ, η, ε, ξ, and α, and G_2 and G_3 are constants depending only on k and ν. Choose now $\tilde{\alpha}$ so that $0 < \tilde{\alpha} \geq (\xi\eta - \varepsilon)/(\xi(\eta - \varepsilon))$. Set $v := e^{-c(\tilde{\alpha})}u^{-1}$, and, for $0 < \sigma \leq 1$, $\tilde{U}_\sigma := (s - \sigma\tilde{\tau}\tilde{r}^2, s) \times B(x, \sigma\tilde{r})$, where

$$\tilde{r} := \frac{\xi((1-\tilde{\alpha})\eta + \tilde{\alpha}\varepsilon)}{\varepsilon} r \leq r \quad \text{and} \quad \tilde{\tau} := \frac{\varepsilon^2}{\xi^2((1-\tilde{\alpha})\eta + \tilde{\alpha}\varepsilon)}\tau\,.$$

Notice that $\tilde{U}_{\tilde{\sigma}_0} = Q_+$ for $\tilde{\sigma}_0 := \varepsilon((1-\tilde{\alpha})\eta + \tilde{\alpha}\varepsilon)^{-1}$. By Proposition 2.7 (see also Remark 2.8)

$$\|v\|_{L^\infty(\tilde{U}_{\sigma'})} \leq \left(\frac{B(\tilde{\tau}, \nu)\left(S_1\Theta(C_2r)^{S_3+1}\Upsilon(C_4r)\right)^{\nu/2}}{(\sigma - \sigma')^{2+\nu}\tilde{r}^2\mu(B(x,\tilde{r}))}\right)^{1/p} \|v\|_{L^p(\tilde{U}_\sigma)}\,,$$

for all $0 < \tilde{\sigma}_0 \leq \sigma' < \sigma \leq 1$ and $0 < p < +\infty$. Hence,

$$\|v\|_{L^\infty(\tilde{U}_{\sigma'})} \leq \left(\frac{\tilde{\tau}B(\tilde{\tau}, \nu)\left(S_1\Theta(C_2r)^{S_3+1}\Upsilon(C_4r)\right)^{\nu/2}}{(\sigma - \sigma')^{2+\nu}\mu(\tilde{U}_1)}\right)^{1/p} \|v\|_{L^p(\tilde{U}_\sigma)}\,.$$

As above, applying again Lemma 2.10 to v and (14) we obtain, for any $\lambda > 0$,

$$\begin{aligned}
&\bar{\mu}(\{(t,z) \in U_1; \log v(t,z) > \lambda\}) \\
&\quad \leq \frac{8C_5\max\{1,\tau\}^2\varepsilon^2}{(\varepsilon - \xi((1-\tilde{\alpha})\eta + \tilde{\alpha}\varepsilon))^2}\Theta(C_2r)^{C_6}\Upsilon(C_4r)r^2\frac{\mu(B)}{\lambda} \\
&\quad \leq \tilde{F}_1(n, \tau, \eta, \varepsilon, \xi, \tilde{\alpha})\Theta(C_2r)^{C_6+1}\Upsilon(C_4r)\frac{\bar{\mu}(U_1)}{\lambda}\,,
\end{aligned}$$

where $\tilde{F}_1$ is an opportune constant. Thus, we can apply Lemma 2.11 and obtain

$$\|e^{-c(\tilde{\alpha})}u^{-1}\|_{L^\infty(Q_+)} \leq G_4\Theta(C_2r)^{G_5}\Upsilon(C_4r)^{G_6}\,, \tag{10}$$

where G_4 is a constant depending only on ν, p_0, τ, η, ε, ξ, and α, and G_5 and G_6 are constants depending only on k and ν. Because of the assumption on η, ε, and ξ, we may choose $\alpha = \tilde{\alpha}$ and the desired inequality follows from (9) and (10). □

REMARK 2.13. Setting $\eta := (3-\xi)/4$ and $\varepsilon := (1+\xi)/4$, (8) is satisfied for any $\xi \in (0,1)$ even strictly:

$$\xi\eta - \varepsilon < (1-\eta)(1-\xi)\,.$$

THEOREM 2.14. *Let (X, d, μ), $\nu > 2$, $\Omega \subset X$, $\Theta : (0,+\infty) \to [1,+\infty)$, $(\mathcal{E}, \mathcal{F})$, $k > 1$, and $\Upsilon : (0,+\infty) \to [1,+\infty)$ be satisfying Assumption* 1.2. *Let $\tau > 0$ and $0 < \varepsilon < \eta < \delta \leq 1$ and $\xi \in (0,1)$ such that*

$$\xi\eta - \varepsilon < (1-\eta)(1-\xi)\,. \tag{11}$$

Then, there exist two constants $H_1(\nu, k, \tau, \delta, \eta, \varepsilon, \xi)$, depending only on ν, k, τ, δ, η, ε and ξ, and $H_2(\nu, k)$, depending only on ν and k, such that for any admissible open ball $B(x,r)$, with $B(x, 3r/2) \subset \Omega$, any $s \in \boldsymbol{R}$, and any nonnegative local solution u of Equation (3) *in $(s - \tau r^2, s) \times B(x,r)$, we have*

$$\operatorname*{ess\,sup}_{Q_-} u \leq H_1(\nu, k, \tau, \delta, \eta, \varepsilon, \xi)\left(\Theta(C_2r)\Upsilon(C_4r)\right)^{H_2(\nu,k)}\operatorname*{ess\,inf}_{Q_+} u\,,$$

with

$$Q_- := (s - \delta\tau r^2, s - \eta\tau r^2) \times B(x, \xi r) \quad \textit{and} \quad Q_+ := (s - \varepsilon\tau r^2, s) \times B(x, \xi r)\,,$$

where $C_2(k)$ *and* $C_4(k)$ *are the constants given in Proposition* 2.2.

PROOF. First assume that u is positive and $\delta < 1$. There exist a constant $\gamma := \gamma(\eta, \varepsilon, \xi) > \xi$ depending only on η, ε, and ξ such that

$$\frac{\gamma\eta - \varepsilon}{\gamma(\eta - \varepsilon)} \leq \frac{(1-\eta)(1-\gamma)}{\gamma(\eta - \varepsilon)}\,.$$

Set for $0 < \sigma \leq 1$, $U_\sigma := (\bar{s} - \sigma\bar{\tau}\bar{r}^2, \bar{s}) \times B(x, \sigma\bar{r})$, where

$$\bar{s} := s - \eta\tau r^2, \bar{r} := \xi\frac{1-\eta}{\delta-\eta}r\,, \quad \text{and} \quad \bar{\tau} := \frac{(\delta-\eta)^2}{\xi^2(1-\eta)}\tau\,.$$

Notice that $U_{\sigma_0} = Q_-$ for $\sigma_0 := (\delta - \eta)(1-\eta)^{-1}$. By Proposition 2.7, there exist a constant $G_1(\nu, k, \bar{\tau})$, depending only on ν, k and $\bar{\tau}$, and a constant $G_2(\nu, k)$, depending only on ν and k, such that

$$\operatorname*{ess\,sup}_{Q_-} u \leq \frac{G_1(\nu, k, \bar{\tau})\,(\Theta(C_2 r)\Upsilon(C_4 r))^{G_2(\nu,k)}}{(\bar{\sigma}_0 - \sigma_0)^{2+\nu}\bar{r}^2\mu(B(x,\bar{r}))}\int_{U_{\bar{\sigma}_0}} u\, d\bar{\mu}\,,$$

where $\bar{\sigma}_0 := \min\{1, (\gamma/\xi)(\delta-\eta)(1-\eta)^{-1}\}$. On the other side, an application of Proposition 2.12 (with γ instead of ξ) yields

$$\int_{U_{\bar{\sigma}_0}} u\, d\bar{\mu} \leq \varepsilon\tau r^2\mu(B(x,\gamma r))H_3 e^{H_4 r}\operatorname*{ess\,inf}_{Q_+} u\,.$$

Combining these two inequalities completes the proof. The case $\delta = 1$ is obtained by tending δ to 1. If u is nonnegative, then first consider the function $u + \bar{\varepsilon}$, $\bar{\varepsilon} > 0$, and then let $\bar{\varepsilon}$ tend to 0. □

3. Some applications for Alexandrov spaces. In this section we prove some Liouville theorems and a two-sided heat kernel bound for Alexandrov spaces. The following notation was first introduced in [KS10] in order to define the infinitesimal Bishop-Gromov condition on Alexandrov spaces (see Definition 3.2).

NOTATION 3.1. Let $k \in \boldsymbol{R}$. In the sequel, we shall denote by $s_k : \boldsymbol{R} \to \boldsymbol{R}$ the function defined by

$$s_k(r) := \begin{cases} \frac{1}{\sqrt{k}}\sin(\sqrt{k}r) & \text{if } k > 0\,, \\ r & \text{if } k = 0\,, \\ \frac{1}{\sqrt{-k}}\sinh(\sqrt{-k}r) & \text{if } k < 0\,. \end{cases}$$

Let (X, d) be a complete, locally compact Alexandrov length space of curvature locally bounded below. Fix a point $x \in X$ and $t \in (0, 1]$. We define a function $\Phi_{x,t} : W_{x,t} \to X$ where $W_{x,t} \subset X$ is the set including x and the set of points z that have the following property: if $z \neq x$, there is a point $y_z \in X$ for that there exists a geodesic segment $[x, y_z]$ joining x and

y_z such that $z \in [x, y_z]$ and $d(x,z) = t\,d(x, y_z)$. If such a point y_z exists, then this point is unique and we set $\Phi_{x,t}(z) := y_z$ and $\Phi_{x,t}(x) := x$. It is not so difficult to prove that $W_{x,t}$ is closed and $\Phi_{x,t}$ is locally Lipschitz, that is, for every open or closed ball B, there is a constant $k_B > 0$ such that $d(\Phi_{x,t}(z_1), \Phi_{x,t}(z_2)) \leq k_B d(z_1, z_2)$ for all $z_1, z_2 \in W_{x,t} \cap B$ (see the proof of [OS94, Proposition 3.1]).

DEFINITION 3.2. Let (X, d) be a complete locally compact Alexandrov space of curvature locally bounded below, $\mu : \mathcal{B}(X) \to [0, +\infty]$ a locally finite measure on the Borel σ-algebra $\mathcal{B}(X)$ of (X, d) such that $\mu(X) > 0$, $\kappa \in \boldsymbol{R}$ and $n \in [1, +\infty)$. (X, d, μ) is said to satisfy the *infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\kappa, n)$ on an open subset $\Omega \subset X$ if

$$\int_A d\Phi_{x,t\,*}\mu(y) \geq \int_A t\left(\frac{s_\kappa(t\,d(x,y))}{s_\kappa(d(x,y))}\right)^{n-1} d\mu(y) \tag{12}$$

for any $x \in \Omega$ and $t \in (0,1]$ and for any measurable set $A \subset \Omega$ if $\kappa \leq 0$ and any measurable set $A \subset B(x, \pi/\sqrt{\kappa}) \cap \Omega$ if $\kappa > 0$.

REMARK 3.3. If (X, d, μ) satisfies the infinitesimal Bishop-Gromov condition $\mathrm{BG}(\kappa, n)$ on the open set Ω, then

$$\int_M f(y)\,\Phi_{x,t\,*}\mu(dy) \geq \int_M t f(y)\left(\frac{s_\kappa(t\,d(x,y))}{s_\kappa(d(x,y))}\right)^{n-1} \mu(dy) \tag{13}$$

for any $x \in \Omega$, for every $t \in (0,1]$, any measurable set $M \subset \Omega$, if $\kappa \leq 0$, $M \subset B(x, \pi/\sqrt{\kappa}) \cap \Omega$ if $\kappa > 0$, and any measurable nonnegative function $f : X \to \boldsymbol{R}$.

The following proposition follows easily from the above definition.

PROPOSITION 3.4. *Let (X, d, μ) be a complete locally compact Alexandrov measure space of curvature locally bounded below satisfying the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\kappa, n)$ *on the open set $\Omega \subset X$. Then, (X, d, μ) satisfies the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\bar{\kappa}, n)$ *on Ω for all $\bar{\kappa} \leq \kappa$. Moreover, if $\kappa \leq 0$, then (X, d, μ) satisfies the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\bar{\kappa}, \bar{n})$ *on Ω for all $\bar{\kappa} \leq \kappa$ and $\bar{n} \geq n$.*

The following elementary Lemma will be useful in the sequel.

LEMMA 3.5. *Let $0 < r_1 \leq r_2$. Then*

$$\frac{\sinh(r_2)}{\sinh(r_1)} \leq \frac{r_2}{r_1} e^{r_2}.$$

PROOF. The above inequality follows from the fact that

$$e^{r_2} - e^{-r_2} \leq \frac{r_2}{r_1} e^{r_2}(e^{r_1} - e^{-r_1}),$$

since, fixed a constant $t \geq 1$, the function $h(r) := te^r(e^{r/t} - e^{-r/t}) - (e^r - e^{-r})$, $r \in (0, +\infty)$, is increasing and $h(0) = 0$. Indeed, for the first derivative of the function h we have

$$h'(r) = (t+1)e^{(t+1)r/t} - (t-1)e^{(t-1)r/t} - e^r - e^{-r} > 0. \qquad \square$$

The main justification of the definition of the infinitesimal Bishop-Gromov condition is the following proposition that gives a kind of doubling property of the volume of balls.

PROPOSITION 3.6. *Let (X, d, μ) be a complete locally compact Alexandrov measure space of curvature locally bounded below satisfying the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\kappa, n)$ *on the open subset $\Omega \subset X$. Then, for $0 < r_1 \leq r_2$ and $x \in \Omega$ such that $B(x, r_2) \subset \Omega$, we have*

$$\mu(B(x, r_2)) \leq \left(\frac{r_2}{r_1}\right)^n e^{(n-1)\sqrt{-\min\{\kappa,0\}}r_2}\mu(B(x, r_1))\,. \tag{14}$$

PROOF. From Proposition 3.4 we may assume that (X, d, μ) satisfies the infinitesimal Bishop-Gromov condition $\mathrm{BG}(-|\kappa|, n)$ on Ω. Let $t := r_1/r_2$ and $A := B(x, r_2)$ in (12). Notice that $\Phi_{x,t}^{-1}(B(x, r_2)) \subset B(x, r_1)$. Thus, we have

$$\int_{B(x,r_2)} \frac{r_1}{r_2}\left(\frac{s_{-|\kappa|}((r_1/r_2)d(x, y))}{s_{-|\kappa|}(d(x, y))}\right)^{n-1} \mu(dy) \leq \mu(B(x, r_1))\,.$$

Moreover, from Lemma 3.5, we obtain (the case $\kappa = 0$ is also easily verified)

$$\begin{aligned}\int_{B(x,r_2)} \frac{r_1}{r_2}\left(\frac{s_{-|\kappa|}(t\, d(x, y))}{s_{-|\kappa|}(d(x, y))}\right)^{n-1} \mu(dy) &\geq \int_{B(x,r_2)} \left(\frac{r_1}{r_2}\right)^n e^{-(n-1)\sqrt{|\kappa|}d(x,y)}\mu(dy)\\ &\geq \left(\frac{r_1}{r_2}\right)^n e^{-(n-1)\sqrt{|\kappa|}r_2}\mu(B(x, r_2))\,.\end{aligned}$$

Inequality (14) follows from the above two inequalities. □

Under the same assumption of Proposition 3.6, with a similar argument we can easily prove that, for any $0 < r_1 \leq r_2$, if $\kappa \leq 0$, for $0 < r_1 \leq r_2 < \pi/\sqrt{\kappa}$, if $\kappa > 0$, and $x \in \Omega$ such that $B(x, r_2) \subset \Omega$, we have

$$\frac{\mu(B(x, r_2))}{\mu(B(x, r_1))} \leq \frac{r_2}{r_1} \sup_{\gamma\in[0,1]} \left(\frac{s_\kappa(r_2\gamma)}{s_\kappa(r_1\gamma)}\right)^{n-1}.$$

By means of this inequality we can follow the argument of [Oh07, Theorem 5.1] and prove the following Bishop-Gromov volume comparison theorem.

PROPOSITION 3.7. *Let (X, d, μ) be a complete locally compact Alexandrov measure space of curvature locally bounded below satisfying the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\kappa, n)$ *on the open subset $\Omega \subset X$. Then, for any $x \in \Omega$ the function*

$$f_x(r) := \mu(B(x, r))\left(\int_0^r s_\kappa(t)^{n-1}dt\right)^{-1},$$

defined on $(0, d(x, \partial\Omega))$, if $k \leq 0$, on $(0, \min\{\pi/\sqrt{\kappa}, d(x, \partial\Omega)\})$, if $\kappa > 0$, is a non-increasing function (we set $d(x, \partial\Omega) := +\infty$ if the topological boundary $\partial\Omega$ of Ω is empty).

We recall the following important result essentially proved in [KS10] which gives a sufficient condition in terms of curvature for an Alexandrov space to satisfy the infinitesimal Bishop-Gromov condition.

THEOREM 3.8. *Let* (X, d, μ) *be an* m*-dimensional, complete, locally compact Alexandrov measure space of curvature bounded below and* μ *the* m*-dimensional Hausdorff measure. If* (X, d) *is an Alexandrov space of curvature bounded below by* κ *on an open subset* $\Omega \subset X$, *then* (X, d, μ) *satisfies the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\kappa, m)$ *on* Ω.

ASSUMPTION 3.9. (X, d, μ) is an m-dimensional, complete, locally compact Alexandrov measure space of curvature bounded below and μ is a Radon measure on the Borel σ-algebra of (X, d) with support equal to X. Moreover, there exists $n \in \boldsymbol{N}$ such that for any relatively compact open subset $\Omega \subset X$ there is $\kappa \in \boldsymbol{R}$ such that (X, d, μ) satisfies the infinitesimal Bishop-Gromov condition $\mathrm{BG}(\kappa, n)$ on Ω.

NOTATION 3.10. Let (X, d, μ) be satisfying Assumption 3.9. Following [KS11] we have a regular, strongly local symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$ with core $C_0^{\mathrm{Lip}}(X)$ such that:

$$\mathcal{E}(f_1, f_2) := \int_X g(\nabla f_1, \nabla f_2) d\mu ,$$

for any $f_1, f_2 \in C_0^{\mathrm{Lip}}(X)$, where g is the μ-a.e. defined canonical Riemannian metric (see [OS94] and [KS11, Lemma 3.3]) and ∇f is the gradient of a function f. The generator of $(\mathcal{E}, \mathcal{F})$ is called Laplacian and is denoted by Δ. Notice that all the requirements of Assumption 1.2 are satisfied (see also [KS11, Proposition 3.3]) and for the energy measure we have $\Gamma(f_1, f_2) = g(\nabla f_1, \nabla f_2)\mu$ for any $f_1, f_2 \in C_0^{\mathrm{Lip}}(X)$. Finally, there is a measurable map $\gamma : [0, 1] \times X \times X \to X$ such that $\gamma_{x,t}(y) := \gamma(t, x, y)$ is the point z of a geodesic segment $[x, y]$ such that $d(x, z) = t\, d(x, y)$ (see the proof of [KS01, Proposition 6.1] where the key argument is the fact that the cut-locus of any point has measure zero with respect to μ, see [KS11, Lemma 3.2]).

We need the following Lemma to prove a weak Poincaré type inequality.

LEMMA 3.11. *Let* (X, d, μ) *be satisfying Assumption* 3.9. *If* (X, d, μ) *satisfies the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\kappa, n)$ *on an open subset* $\Omega \subset X$, *then for any* $x \in \Omega$, *any open ball* $B(y, r)$ *including* x, $x \in B(y, r)$, *with* $B(y, 2r) \subset \Omega$, *and any measurable nonnegative function* $f : X \to \boldsymbol{R}$, *we have*

$$\int_{B(y,r)} f(\gamma_{x,t}(z))\, \mu(dz) \le \frac{1}{t} \left(\frac{s_{-|\kappa|}(r)}{s_{-|\kappa|}(tr)} \right)^{n-1} \int_{B(y,2r)} f(z)\mu(dz) , \tag{15}$$

where $\gamma_{x,t} : X \to X$ *is the function defined in Notation* 3.10.

PROOF. By Proposition 3.4 we may assume $\kappa < 0$. Since

$$\frac{s_\kappa(tr)}{s_\kappa(r)} \le \frac{s_\kappa(t\, d(y, z))}{s_\kappa(d(y, z))}$$

for all $z \in B(y, r)$, and $t \in (0, 1]$, we obtain from (13)

$$\int_{B(y,r)} f(\gamma_{x,t}(z))\, \mu(dz) \le \frac{1}{t} \left(\frac{s_\kappa(r)}{s_\kappa(tr)} \right)^{n-1} \int_{B(y,r)} f(\gamma_{x,t}(z)) \Phi_{x,t\,*}\mu(dz) ,$$

for every $t \in (0,1]$. If $z \in B(y,r)$, then any geodetic segment $[x,z]$ joining x and z is included in $B(y,2r)$. In particular, $\Phi_{x,t}^{-1}(B(y,r)) \subset B(y,2r)$. Remarking that $\gamma_{x,t}(\Phi_{x,t}(z)) = z$, we obtain (15). $\square$

Doing a similar calculation as in [S02, Theorem 5.6.6] or [H09, Lemma 3.2], we are able to prove the following weak Poincaré-type inequality.

PROPOSITION 3.12. *Let (X,d,μ) be satisfying Assumption 3.9. If (X,d,μ) satisfies the infinitesimal Bishop-Gromov condition* BG(κ,n) *on an open subset $\Omega \subset X$, then for any open ball $B(x,r)$, with $B(x,2r) \subset \Omega$, and any $f \in \mathcal{F}$, we have*

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 \, d\mu \le 8r^2 \left(\int_{1/2}^{1} \frac{\left(s_{-|\kappa|}(r)\right)^{n-1}}{t\left(s_{-|\kappa|}(tr)\right)^{n-1}} \, dt \right) \int_{B(x,2r)} d\Gamma(f,f)\,. \tag{16}$$

PROOF. It suffices to consider a Lipschitz function f. In the following, we implicitly use the Fubini Theorem. Writing $B := B(x,r)$, $|B| := \mu(B)$, and $|\nabla f| := g(f,f)^{1/2}$, we have by Hölder Inequality

$$\begin{aligned}
\int_B |f(y) - f_B|^2 \, d\mu(y) &\le \frac{1}{|B|^2} \int_B \left(\int_B |f(y) - f(z)| \, d\mu(z) \right)^2 d\mu(y) \\
&\le \frac{|B|}{|B|^2} \int_B \left(\int_B |f(y) - f(z)|^2 d\mu(z) \right) d\mu(y) \\
&\le \frac{(2r)^2}{|B|} \int_B \int_B \left(\int_0^1 |\nabla f(\gamma_{y,t}(z))| \, dt \right)^2 d\mu(y) d\mu(z) \\
&\le \frac{(2r)^2}{|B|} \int_B \int_B \int_0^1 |\nabla f(\gamma_{y,t}(z))|^2 dt d\mu(y) d\mu(z)\,,
\end{aligned}$$

where $\gamma_{y,t} : B(x,r) \to X$ is the function defined in Notation 3.10. Since $\gamma_{y,t}(z) = \gamma_{z,1-t}(y)$, we have

$$\int_0^{1/2} |\nabla f(\gamma_{y,t}(z))|^2 dt = \int_0^{1/2} |\nabla f(\gamma_{z,1-t}(y))|^2 dt = \int_{1/2}^1 |\nabla f(\gamma_{z,t}(y))|^2 dt.$$

We would like to remark that this trick was first used in [KoS97]. Since the integral

$$\frac{(2r)^2}{|B|} \int_B \int_B \int_0^1 |\nabla f(\gamma_{y,t}(z))|^2 dt d\mu(y) d\mu(z)$$

is symmetric with respect to the variables y and z, using Fubini's Theorem and (15), we get

$$\begin{aligned}
&\int_B |f(y) - f_B|^2 d\mu(y) \\
&\quad \le \frac{(2r)^2}{|B|} \int_B \int_B \int_0^1 |\nabla f(\gamma_{y,t}(z))|^2 dt d\mu(y) d\mu(z) \\
&\quad \le \frac{2(2r)^2}{|B|} \int_B \int_B \int_{1/2}^1 |\nabla f(\gamma_{y,t}(z))|^2 dt d\mu(y) d\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{8r^2}{|B|}\int_B d\mu(y)\int_{1/2}^1 dt\int_B |\nabla f(\gamma_{y,t}(z))|^2 d\mu(z)\\
&\leq \frac{8r^2}{|B|}\int_B d\mu(y)\int_{1/2}^1 \frac{1}{t}\left(\frac{s_{-|\kappa|}(r)}{s_{-|\kappa|}(tr)}\right)^{n-1} dt\int_{B(x,2r)} |\nabla f(z)|^2 d\mu(z)\\
&\leq 8r^2\int_{1/2}^1 \frac{1}{t}\left(\frac{s_{-|\kappa|}(r)}{s_{-|\kappa|}(tr)}\right)^{n-1} dt\int_{B(x,2r)} |\nabla f(z)|^2 d\mu(z)\,.
\end{aligned}
$$

□

COROLLARY 3.13. *Let (X, d, μ) be satisfying Assumption* 3.9. *If (X, d, μ) satisfies the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\kappa, n)$ *on an open subset $\Omega \subset X$, then for any open ball $B(x, r)$, with $B(x, 2r) \subset \Omega$, and any $f \in \mathcal{F}$, we have*

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \leq C_1(n)e^{C_2(n,\kappa)r} r^2 \int_{B(x,2r)} d\Gamma(f, f)\,, \tag{17}$$

where $C_2(n, \kappa) := n - 1$ if $\kappa < 0$, and $C_2(n, \kappa) := 0$ if $\kappa \geq 0$.

PROOF. For $\kappa < 0$, we have

$$\int_{1/2}^1 \frac{(s_\kappa(r))^{n-1}}{t\,(s_\kappa(tr))^{n-1}}\,dt \leq \left(\frac{e^r - e^{-r}}{e^{r/2} - e^{-r/2}}\right)^{n-1}\int_{1/2}^1 \frac{1}{t}\,dt \leq 2e^{(n-1)r}\log 2\,,$$

where the second inequality follows from Lemma 3.5. For $\kappa \geq 0$, we may assume $\kappa = 0$ and perform a similar calculation. □

As applications we now prove two Liouville theorems for Alexandrov spaces. Their corresponding results for Riemannian manifolds were proved in [LS84].

DEFINITION 3.14. With the same notation as in 3.10, a function $u \in \mathcal{F}$ is said to be subharmonic (resp. superharmonic) if $\mathcal{E}(u, \phi) \leq 0$ (resp. $\mathcal{E}(u, \phi) \geq 0$) for any $\phi \in \mathcal{F}$ with compact support. A function $u \in \mathcal{F}$ is said to be harmonic if u is subharmonic and superharmonic.

REMARK 3.15. It is easy to verify that a subharmonic (resp. superharmonic) function $u \in \mathcal{F}$ is a subsolution (resp. supersolution) of Equation (3) on $\boldsymbol{R} \times X$. Hence, from Proposition 2.7 we deduce the following mean value inequalities for subharmonic functions.

PROPOSITION 3.16. *Let (X, d, μ) be satisfying Assumption* 3.9. *Let $\kappa := \kappa(x, 18r)$ be a constant, (in general) depending only on x and r, such that (X, d, μ) satisfies the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(\kappa, n)$ *on the open ball $B(x, 18r)$. If $\kappa \geq 0$, there exists a constant $C_0(p, n)$, depending only on p and n, such that for all $0 < \delta' < \delta \leq 1$, we have*

$$\|u^p\|_{L^\infty(B(x,\delta' r),\mu)} \leq \frac{C_0(p, n)}{(\delta - \delta')^{2+\max\{n,3\}}\mu(B(x, r))}\int_{B(x,\delta r)} u^p d\mu\,,$$

whenever $0 < p < +\infty$ and u is a nonnegative subharmonic function. If $\kappa < 0$, there exist two constants $C_1(p, n)$, depending only on p and n, and $C_2(n)$, depending only on n, such *that, for all* $0 < \delta' < \delta \le 1$, *we have*

$$\|u^p\|_{L^\infty(B(x,\delta' r),\mu)} \le \frac{C_1(p,n)e^{C_2(n)(1+\sqrt{-\kappa})r}}{(\delta-\delta')^{2+\max\{n,3\}}\mu(B(x,r))}\int_{B(x,\delta r)} u^p d\mu\,,$$

whenever $0 < p < +\infty$ *and* u *is a nonnegative subharmonic function.*

REMARK 3.17. From Proposition 3.16 it follows easily that if (X, d, μ) is an n-dimensional, complete, locally compact Alexandrov measure space of curvature bounded below by $\kappa \ge 0$ on X and μ is the n-dimensional Hausdorff measure, then every subharmonic function u, such that $u \in L^p(X, \mu)$, for some $p > 0$, is constant. This result could have been easily deduced also from [St95], since under this assumption the classical volume doubling property holds. We also remember that, if (X, d, μ) satisfies Assumption 3.9 and u is a subharmonic function, such that $u \in L^p(X, \mu)$, for some $p > 1$, then u is constant (see [St94]).

The following two propositions are straight extensions of [LS84, Theorems 2.4 and 2.5].

PROPOSITION 3.18. *Let* (X, d, μ) *be satisfying Assumption* 3.9*. Assume that* (X, d) *is noncompact. If there exist a point* $x_0 \in X$ *and two constants* $C > 0$ *and* $\alpha > 0$ *such that, for any* $r > 0$, (X, d, μ) *satisfies the infinitesimal Bishop-Gromov condition* $\mathrm{BG}(-C(1+r^2)(\log(1+r^2))^{-\alpha}, n)$ *on the open ball* $B(x_0, r)$ *for some* $n \in [1, +\infty)$, *then every nonnegative subharmonic function* $u \in L^1(X, \mu)$ *is constant.*

PROOF. Let u be a nonnegative subharmonic function such that $u \in L^1(X, \mu)$. Set $t_l := 2^{2^l}$, $l \in \boldsymbol{N}$. Notice that, for every $l \in \boldsymbol{N}$, $[t_l, t_{l+1}]$ is the disjoint union of these 2^l intervals: $[a_{l,0}, a_{l,1}), [a_{l,1}, a_{l,2}), \ldots, [a_{l,2^l-1}, a_{l,2^l}]$, where $a_{l,i} := 2^i t_l$, $i = 0, \ldots, 2^l$. Thus, there is an $i_l \in \{0, \ldots, 2^l - 1\}$ such that, if we set $r_l := a_{l,i}$ and $B_l := B(x_0, r_l)$, we have

$$\int_{2B_l\setminus B_l} u d\mu \le 2^{-l}\int_{B(x_0,t_{l+1})\setminus B(x_0,t_l)} u d\mu\,.$$

Since $r_l < t_{l+1}$ and, hence, $\log r_l < 2^{l+1}\log 2$, we get

$$\log r_l \int_{2B_l\setminus B_l} u\, d\mu \le (2\log 2)\int_{B(x_0,t_{l+1})\setminus B(x_0,t_l)} u d\mu\,.$$

From $u \in L^1(X, \mu)$ we obtain

$$\lim_{l\to+\infty}(\log r_l)\int_{2B_l\setminus B_l} u\, d\mu = 0\,.$$

Let $f : [e, +\infty) \to [0, +\infty)$ be the function $f(x) := (\log\log x)^\alpha$. f is a nondecreasing function with bounded and continuous derivative such that $\limsup_{t\to a} f^2(t)/f'(t) < +\infty$. It is easy to verify that $\phi^2 f(u \vee a) \in \mathcal{F}$ for any nonnegative $\phi \in \mathcal{F} \cap C_0(X)$. Hence, since u is subharmonic,

$$\int_X d\Gamma(\phi^2 f(u \vee e), u) \le 0\,.$$

Using the Leibniz rule and the chain rule, we easily have

$$\Gamma(\phi^2 f(u \vee e), u) = \phi^2 f'(u \vee e)\Gamma(u \vee e, u) + 2\phi f(u \vee e)\Gamma(\phi, u).$$

From the locality of the energy measure and the Cauchy-Schwarz inequality, we get

$$\int_{\{u>e\}} \phi^2 f'(u)\Gamma(u,u) \le \int_{\{u>e\}} -2\phi f(u) d\Gamma(\phi,u)$$
$$\le \frac{1}{2}\int_{\{u>e\}} \phi^2 f'(u)\Gamma(u,u) + 2\int_{\{u>e\}} \frac{f^2(u)}{f'(u)}\, d\Gamma(\phi,\phi)\,,$$

that is

$$\int_{\{u>e\}} \phi^2 f'(u)\Gamma(u,u) \le 4\int_{\{u>e\}} \frac{f^2(u)}{f'(u)}\, d\Gamma(\phi,\phi)\,.$$

For $l \in \boldsymbol{N}$ set $\phi := ((2/r_l)d_{x_0,3r_l/2}) \wedge 1$ (see Notation 2.5). Notice that $\phi(x) = 1$ for $x \in B_l$ and $\phi(x) = 0$ for $x \in X\backslash(2B_l)$, where $2B_l$ is the open ball of double radius with the same center as B_l. Inserting ϕ in the last inequality and using the truncation property, we obtain

$$\int_{\{u>e\}\cap B_l} f'(u)\Gamma(u,u) \le \frac{16}{r_l^2}\int_{\{u>e\}\cap(2B_l\backslash B_l)} \frac{f^2(u)}{f'(u)}\, d\mu$$
$$\le \frac{8}{\alpha r_l^2 \log r_l}\|(\log\log u)^{\alpha+1}\log u\|_{L^\infty(2B_l,\mu)}(\log r_l)\int_{2B_l\backslash B_l} u\, d\mu.$$

Since (X,d,μ) satisfies the infinitesimal Bishop-Gromov condition $\mathrm{BG}(-C(1+36^2r^2)(\log(1+36^2r^2))^{-\alpha}, n)$ on the open ball $B(x_0, 36r_l)$, from Proposition 3.16 we easily obtain that

$$\|u\|_{L^\infty(2B_l,\mu)} \le e^{C_3\sqrt{-\kappa}r_l} \le e^{C_4 r_l^2/(\log r_l)^{\alpha/2}}\,,$$

for some constants C_3 and C_4, depending only on the dimension n of X, $\|u\|_{L^1(X,\mu)}$, and $\mu(B_1)$. Using this in the above inequality (notice that $(\log\log u)^{\alpha+1}\log u$ is nondecreasing) and letting l tend to $+\infty$, we get

$$\int_{\{u>e\}} f'(u)\Gamma(u,u) \le 0\,.$$

Since f' is positive on $\{u>e\}$, we have either $\{u>e\} = \emptyset$ or u is constant on $\{u>e\}$. In any case $u \in L^2(X,\mu)$ and hence is constant. □

PROPOSITION 3.19. *Let (X,d,μ) be satisfying Assumption 3.9. Assume that (X,d) is non-compact. If there exist a point $x_0 \in X$, a constant $C > 0$ and $r_0 > 0$ such that, for any $r > r_0$, (X,d,μ) satisfies the infinitesimal Bishop-Gromov condition $\mathrm{BG}(-C^2r^{-2}, n)$ on the open set $X\backslash\bar{B}(x_0,r)$ for some $n \in [1,+\infty)$, then every nonnegative subharmonic function $u \in L^p(X,\mu)$, $p \in (0,+\infty)$, is constant. Moreover, if $p \in (0,1)$ and u is a nonnegative subharmonic function such that $u \in L^p(X,\mu)$, then $u = 0$.*

PROOF. In view of Proposition 3.18 and Remark 3.17, one only needs to consider the case $p \in (0,1)$. We want to prove that if u is a nonnegative subharmonic function such that $u \in L^p(X,\mu)$ for some $p \in (0,1)$, then for any $\varepsilon > 0$ there is an $r_\varepsilon > 0$ such that $\|u\|_{L^\infty(X\backslash B(x_0,r_\varepsilon),\mu)} < \varepsilon$. This implies $u = 0$. In fact, from Proposition 3.16 we have that $u \in L^\infty(B(x_0,r),\mu)$ for all $r > 0$, since (X,d) satisfies the infinitesimal Bishop-Gromov condition on $B(x_0, r)$. Thus, $u \in L^\infty(X, \mu)$. Hence, $u \in L^2(X, \mu)$ and must be constant. Since u vanishes at infinity in the above sense, u must be zero. Fix a $\beta > 2/(\sqrt[n]{2} - 1)$ and define $t_i := 2\sum_{j=0}^{i} \beta^j - 1 - \beta^i$, $i \in \boldsymbol{N}$. Let $x \in X \backslash \bar{B}(x_0, 1 + \beta)$ and consider a geodesic $\gamma : [0, d(x, x_0)] \to X$ with $\gamma(0) = x_0$ and $\gamma(d(x, x_0)) = x$. Let $k_x \geq 1$ the largest integer such that $t_{k_x} \leq d(x, x_0)$. It is easy to verify that $d(\gamma(t_{k_x}), x) < \beta^{k_x} + \beta^{k_x+1}$ and $d(\gamma(t_i), \gamma(t_{i+1})) = \beta^i + \beta^{i+1}$ and $B(\gamma(t_i), \beta^i) \cap B(\gamma(t_{i+1}), \beta^{i+1}) = \emptyset$ for all $i \in \{0, \ldots, k_x - 1\}$. Denoting $\kappa_i < 0$ a constant such that (X, d) satisfies the (κ_i, n)-contraction property on the open ball $B(\gamma(t_{i+1}), \beta^{i+1} + 2\beta^i)$, using the Gromov-Bishop inequality (see Proposition 3.7), we can repeat the argument of [CGT82, Proposition 4.1] and deduce

$$\mu(B(\gamma(t_{i+1}), \beta^{i+1}) \geq T_i \mu(B(\gamma(t_i), \beta^i)$$

for all $i \in \{0, \ldots, k_x - 1\}$, where

$$T_i := \frac{\displaystyle\int_0^{\beta^{i+1}\sqrt{-\kappa_i}} \sinh^{n-1} t \, dt}{\displaystyle\int_{\beta^{i+1}\sqrt{-\kappa_i}}^{(\beta^{i+1}+2\beta^i)\sqrt{-\kappa_i}} \sinh^{n-1} t \, dt}.$$

Iterating this inequality, we have

$$\mu(B(\gamma(t_{k_x}), \beta^{k_x}) \geq \prod_{i=0}^{k_x-1} T_i \mu(B(x_0, 1)) .$$

Notice that

$$\begin{aligned} T_i &\geq \frac{\displaystyle\int_0^{\beta^{i+1}\sqrt{-\kappa_i}} t^{n-1} \, dt}{\left(\dfrac{\sinh((\beta^{i+1} + 2\beta^i)\sqrt{-\kappa_i})}{(\beta^{i+1} + 2\beta^i)\sqrt{-\kappa_i}}\right)^{n-1} \displaystyle\int_{\beta^{i+1}\sqrt{-\kappa_i}}^{(\beta^{i+1}+2\beta^i)\sqrt{-\kappa_i}} t^{n-1} \, dt} \\ &= \left(\frac{(\beta^{i+1} + 2\beta^i)\sqrt{-\kappa_i}}{\sinh((\beta^{i+1} + 2\beta^i)\sqrt{-\kappa_i})}\right)^{n-1} \frac{\beta^n}{(\beta + 2)^n - \beta^n} . \end{aligned}$$

Since the ball $B(\gamma(t_{i+1}), \beta^{i+1} + 2\beta^i)$ is contained in $X \backslash \bar{B}(x_0, t_{i+1} - \beta^{i+1} - 2\beta^i)$, there is a constant $C_1(\beta) > 0$, depending only on β, such that we may choose

$$\kappa_i := -C_1(\beta)^2 \left(\frac{\beta - 1}{2\beta^i - \beta - 1}\right)^2 ,$$

for all $i \in \{0, \ldots, k_x - 1\}$. Hence, there exists a constant $C_2(\beta) > 0$, depending only on β, such that

$$(\beta^{i+1} + 2\beta^i)\sqrt{-\kappa_i} \leq C_2(\beta)$$

for all $i \in \{0, \ldots, k_x - 1\}$. We deduce that there exists a constant $C_3(\beta, n) < 1$, depending only on β and n, so that

$$T_i \geq C_3(\beta, n) \frac{\beta^n}{(\beta+2)^n - \beta^n} .$$

Hence,

$$\mu(B(\gamma(t_k), \beta^{k_x}) \geq C_3(\beta, n) \left(\frac{\beta^n}{(\beta+2)^n - \beta^n} \right)^{k_x} \mu(B(x_0, 1)) .$$

Setting $A_k := \bar{B}(x_0, t_k + \beta^k/10) \backslash B(x_0, t_k)$ and $B_k := B(x_0, t_{k+1}) \backslash \bar{B}(x_0, t_k + \beta^k/10)$, since, for $k_0 \in \boldsymbol{N}$, $X \backslash B(x_0, t_{k_0}) = \bigcup_{k \geq k_0} (A_k \cup B_k)$, to prove our claim it suffices to prove that $\|u\|_{L^\infty(A_k,\mu)}$ and $\|u\|_{L^\infty(B_k,\mu)}$ can be made arbitrarily small for sufficiently large k. Let $x_k \in \partial B(x_0, t_k)$. Notice that x_k is the point $\gamma(t_k)$ of a geodetic path γ joining x_0 and x_k. Then, from Proposition 3.16 and (14) we have

$$\begin{aligned}
\|u^p\|_{L^\infty(B(x_k,\beta^k/10),\mu)} &\leq C_4(p, n) e^{C_5(n) C_2(\beta)} \frac{1}{\mu(B(x_k, \beta^k/5))} \int_X u^p \, d\mu \\
&\leq C_6(p, n, \beta) \frac{1}{\mu(B(x_k, \beta^k))} \int_X u^p \, d\mu \\
&\leq C_7(p, n, \beta) \left(\frac{(\beta+2)^n - \beta^n}{\beta^n} \right)^k \frac{1}{\mu(B(x_0, 1))} \int_X u^p \, d\mu .
\end{aligned}$$

For our choice of β, we have $((\beta+2)^n - \beta^n)/\beta^n < 1$. An application of a standard separability arguments leads to the fact that $\|u\|_{L^\infty(A_k,\mu)}$ can be made arbitrarily small for sufficiently large k. Let $x \in B_k$. Since $B(x, t_k/10) \subset B(\gamma(t_k), 2\beta^k + \beta^{k+1})$, where γ is a geodetic path joining x_0 and x, we have as above

$$\begin{aligned}
\|u^p\|_{L^\infty(B(x,\beta^k/40),\mu)} &\leq C_8(p, n) e^{C_5(n) C_2(\beta)} \frac{1}{\mu(B(x, \beta^k/20))} \int_X u^p \, d\mu \\
&\leq C_9(p, n, \beta) \frac{1}{\mu(B(\gamma(t_k), \beta^k/20))} \int_X u^p \, d\mu \\
&\leq C_{10}(p, n, \beta) \left(\frac{(\beta+2)^n - \beta^n}{\beta^n} \right)^k \frac{1}{\mu(B(x_0, 1))} \int_X u^p \, d\mu .
\end{aligned}$$

Hence, as above, we conclude that that $\|u\|_{L^\infty(B_k,\mu)}$ can be made arbitrarily small for sufficiently large k. □

ASSUMPTION 3.20. (X, d, μ) is an m-dimensional, complete, locally compact Alexandrov length space of curvature locally bounded below and μ is a Radon measure on the Borel σ-algebra of (X, d) with support equal to X. Moreover, (X, d, μ) satisfies the infinitesimal Bishop-Gromov condition $\mathrm{BG}(\kappa, n)$ on X.

The existence of the heat kernel associated to the canonical Dirichlet form is proved in [KMS01]. The formal statement is as follow.

THEOREM 3.21. *Let (X, d, μ) be satisfying Assumption 3.20 with its canonical Dirichlet form $(\mathcal{E}, \mathcal{F})$ (see Notation 3.10) and let $(T_t)_{t\geq 0}$ be the semigroup associated to $(\mathcal{E}, \mathcal{F})$. There exists a unique, measurable, and locally Hölder continuous function $h : (0, +\infty) \times X \times X \to (0, +\infty)$ satisfying the following* (i) *and* (ii):

(i) *For any $t > 0$ and $f \in L^2(X)$ we have*

$$T_t f(x) = \int_X h(t, x, y) f(y)\, d\mu(y),$$

for any μ-a.e. $x \in X$. Hence, for any $y \in X$, $h(\cdot, \cdot, y)$ is a local solution of Equation (3) *in $(0, +\infty) \times X$.*

(ii) *For any $s, t > 0$, $x, z \in X$, we have $h(t, x, z) = h(t, z, x)$,*

$$h(s + t, x, z) = \int_X h(s, x, y) h(t, z, y)\, d\mu(y)\,,$$

$$\text{and} \int_X h(t, x, y)\, d\mu(y) \leq 1\,.$$

In particular, for any $(x, y) \in X \times X$ and $0 < s < t$, we have $h(s, x, y) \geq h(t, x, y)$.

DEFINITION 3.22. The function h given by Theorem 3.21 is called the heat kernel associated to the canonical (regular, strongly local) Dirichlet form $(\mathcal{E}, \mathcal{F})$.

LEMMA 3.23. *Let (X, d, μ) be satisfying Assumption 3.20. There exist two constants $G_5(n)$ and $G_7(n)$, depending only on n, and two constants $G_6(n, \kappa)$ and $G_8(n, \kappa)$, depending only on n and κ, where $G_6(n, \kappa) = G_8(n, \kappa) = 0$ if $n = 1$ or $\kappa \geq 0$, such that for the heat kernel h associated to the canonical Dirichlet form we have, for any $x \in X$ and $t > 0$,*

$$\frac{G_5(n) e^{-G_6(n,\kappa)\sqrt{t}}}{\mu(B(x, \sqrt{t}))} \leq h(t, x, x) \leq \frac{G_7(n) e^{G_8(n,\kappa)\sqrt{t}}}{\mu(B(x, \sqrt{t}))}\,.$$

PROOF. Applying Theorem 2.14 to the function $u := h(\cdot, \cdot, x)$ with $r := 2\sqrt{t}$, $s := 3t$, $\tau := 3/4$, $\delta := 1$, $\eta := 7/12$, $\varepsilon := 5/12$, and $\xi := 1/2$, it follows that there exist two constants $K_1(n)$, depending only on n, and $K_2(n, \kappa)$, depending only on n and κ, where $K_2(n, \kappa) = 0$ if $n = 1$ or $\kappa \geq 0$, such that, for any $x \in X$, $t > 0$, and $y \in B(x, \sqrt{t})$, we have

$$h(t, x, x) \leq K_1(n) e^{K_2(n,\kappa)\sqrt{t}} h(2t, x, y)\,.$$

Integrating over $B(x, \sqrt{t})$, by Theorem 3.21 (ii), we obtain

$$h(t, x, x) \leq \frac{K_1(n) e^{K_2(n,\kappa)\sqrt{t}}}{\mu(B(x, \sqrt{t}))}\,.$$

Now fix $x \in X$ and $t > 0$, and set $\tilde{u}(y) := 1 \wedge (d_{x,2\sqrt{x}}(y)/\sqrt{t})$ (see Notation 2.5) and

$$u(s, y) := \begin{cases} \tilde{u}(y) & \text{if } (s, y) \in]-\infty, 0[\times B(x, \sqrt{t})\,, \\ \int_X h(s, y, z) \tilde{u}(z)\, d\mu(z) & \text{if } (s, y) \in (0, +\infty) \times B(x, \sqrt{t})\,. \end{cases}$$

Notice that $u(s, y)$ is a nonnegative solution of Equation (3) in $\boldsymbol{R} \times B(x, \sqrt{t})$. Applying again Theorem 2.14 to the function u and then to the heat kernel we have

$$\begin{aligned} 1 = u(0, x) &\le K_3(n) e^{K_4(n,\kappa)\sqrt{t}} u(t/2, x) \\ &= K_3(n) e^{K_4(n,\kappa)\sqrt{t}} \int_X h(t/2, x, z) \tilde{u}(z)\, d\mu(z) \\ &\le K_3(n) e^{K_4(n,\kappa)\sqrt{t}} \int_{B(x, 2\sqrt{t})} h(t/2, x, z)\, d\mu(z) \\ &\le K_5(n) e^{K_6(n,\kappa)\sqrt{t}} h(t, x, x) \mu(B(x, 2\sqrt{t})) \\ &\le K_7(n) e^{K_8(n,\kappa)\sqrt{t}} h(t, x, x) \mu(B(x, \sqrt{t}))\,, \end{aligned}$$

where in the last inequality we use (14). As usual, $K_3(n)$, $K_5(n)$, and $K_7(n)$ are constants depending only on n, and $K_4(n, \kappa)$, $K_6(n, \kappa)$, and $K_8(n, \kappa)$ are constants depending only on n and κ. Hence, we obtain

$$h(t, x, x) \ge \frac{K_1(n)^{-1} e^{-K_8(n,\kappa)\sqrt{t}}}{\mu(B(x, \sqrt{t}))}\,.$$

□

By means of Theorem 2.14 and a standard chain argument (see [S02, Corollary 5.4.4]), it is easy to prove the following Lemma.

LEMMA 3.24. *Let (X, d, μ) be satisfying Assumption 3.20. There exist two constants $G_1(n)$, depending only on n, and $G_2(n, \kappa)$, depending only on n and κ, where $G_2(n, \kappa) = 0$ if $n = 1$ or $\kappa \ge 0$, such that for any positive local solution of Equation (3) in $Q := (0, +\infty) \times X$, there exist a version of u (which is still denoted by u) so that*

$$\log \frac{u(s, x)}{u(t, y)} \le \big(G_1(n) + G_2(n, \kappa)\sqrt{t - s}\big) \left(1 + \frac{t - s}{s} + \frac{d(x, y)^2}{t - s}\right),$$

for any $0 < s < t < +\infty$ and $x, y \in X$.

THEOREM 3.25. *Let (X, d, μ) be satisfying Assumption 3.20. There exist constants $B_1(n)$ and $B_3(n)$, depending only on n, and constants $B_2(n, \kappa)$ and $B_4(n, \kappa)$, depending only on n and κ, where $B_2(n, \kappa) = B_4(n, \kappa) = 0$ if $n = 1$ or $\kappa \ge 0$, such that for the heat kernel h associated to the canonical Dirichlet form we have, for any $x, y \in X$ and $t > 0$,*

$$h(t, x, y) \ge \frac{B_1(n) e^{-B_2(n,\kappa)\sqrt{t}} e^{-\left(B_3(n)/t + B_4(n,\kappa)/\sqrt{t}\right) d(x,y)^2}}{\mu(B(x, \sqrt{t}))}\,.$$

PROOF. Applying Lemma 3.24 to the function $u := h(\cdot, \cdot, x)$, we have

$$h(t, x, y) \ge h(t/2, x, x) \exp\left(-2\left(G_1(n) + G_2(n, \kappa)\frac{\sqrt{t}}{\sqrt{2}}\right)\left(1 + \frac{d(x, y)^2}{t}\right)\right).$$

The lower bound follows now by an application of Lemma 3.23. □

Finally, with the same argument of [S02, Theorem 5.2.10], we are able to prove some upper bounds for the heat kernel. The proofs are omitted.

THEOREM 3.26. *Let* (X, d, μ) *be satisfying Assumption* 3.20. *There exist two constants* $K_1(n)$, *depending only on* n, *and* $K_2(n,\kappa)$, *depending only on* n *and* κ, *where* $K_2(n,\kappa) = 0$ *if* $n = 1$ *or* $\kappa \geq 0$, *such that, for any open balls* $B(x, r_1)$ *and* $B(y, r_2)$, *we have*

$$h(t,x,y) \leq \frac{K_1(n) e^{K_2(n,\kappa)(r_1+r_2)}}{\mu(B(x,r_1))^{1/2}\mu(B(y,r_2))^{1/2}} e^{-d(x,y)^2/4t+\varepsilon d(x,y)/\sqrt{t}},$$

for any $\varepsilon > 0$ *and any* $t \geq \varepsilon^{-2}\max\{r_1^2, r_2^2\}$.

COROLLARY 3.27. *Let* (X, d, μ) *be satisfying Assumption* 3.20. *There exist two constants* $K_3(n)$, *depending only on* n, *and* $K_4(n,\kappa)$, *depending only on* n *and* κ, *where* $K_4(n,\kappa) = 0$ *if* $n = 1$ *or* $\kappa \geq 0$, *such that, for any* $x, y \in X$, *and* $t > 0$, *we have*

$$h(t,x,y) \leq \frac{K_3(n)}{\mu(B(x, t/(\sqrt{t}+d(x,y))))^{1/2}\mu(B(y, t/(\sqrt{t}+d(x,y))))^{1/2}} \times e^{-d(x,y)^2/4t+K_4(n,\kappa)t/(\sqrt{t}+d(x,y))}.$$

COROLLARY 3.28. *Let* (X, d, μ) *be satisfying Assumption* 3.20. *There exist two constants* $K_3(n)$, *depending only on* n, *and* $K_6(n,\kappa)$, *depending only on* n *and* κ, *where* $K_6(n,\kappa) = 0$ *if* $n = 1$ *or* $\kappa \geq 0$, *such that, for any* $x, y \in X$, *and* $t > 0$, *we have*

$$h(t,x,y) \leq \frac{K_3(n)}{\mu(B(x,\sqrt{t}))^{1/2}\mu(B(y,\sqrt{t}))^{1/2}}\left(1+\frac{d(x,y)}{\sqrt{t}}\right)^n \times e^{-d(x,y)^2/4t+K_6(n,\kappa)t/(\sqrt{t}+d(x,y))}.$$

COROLLARY 3.29. *Let* (X, d, μ) *be satisfying Assumption* 3.20. *There exist two constants* $K_3(n)$, *depending only on* n, *and* $K_8(n,\kappa)$, *depending only on* n *and* κ, *where* $K_8(n,\kappa) = 0$ *if* $n = 1$ *or* $\kappa \geq 0$, *such that, for any* $x, y \in X$, *and* $t > 0$, *we have*

$$h(t,x,y) \leq \frac{K_3(n)}{\mu(B(x,\sqrt{t}))}\left(1+\frac{d(x,y)}{\sqrt{t}}\right)^{3n/2} e^{-d(x,y)^2/4t+K_8(n,\kappa)\sqrt{t}}.$$

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DEPARTMENT OF MATHEMATICS AND ENGINEERING
GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY
KUMAMOTO UNIVERSITY
KUMAMOTO, 860–8555
JAPAN

E-mail address: giacomo.deleva@gmail.com