

STOCHASTIC RANKING PROCESS WITH TIME DEPENDENT INTENSITIES

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Abstract. We consider the stochastic ranking process with the jump times of the particles determined by Poisson random measures. We prove that the joint empirical distribution of scaled position and intensity measure converges almost surely in the infinite particle limit. We give an explicit formula for the limit distribution and show that the limit distribution function is a unique global classical solution to an initial value problem for a system of a first order non-linear partial differential equations with time dependent coefficients.

1. Introduction. Let $\mathcal{M}(\mathbf{R}_+)$ be the space of Radon measures ρ on the Borel σ -algebra $\mathcal{B}(\mathbf{R}_+)$ of non-negative reals \mathbf{R}_+ . Let N be a positive integer, and let $v_i^{(N)}$, $i = 1, 2, \dots, N$, be independent Poisson random measures (Poisson point processes) on \mathbf{R}_+ , defined on a probability space $(\mathbf{P}, \mathcal{F}, \Omega)$. For each i , denote the intensity measure of $v_i^{(N)}$ by $\rho_i^{(N)}$;

$$(1) \quad \mathbf{E}[v_i^{(N)}(A)] = \rho_i^{(N)}(A), \quad A \in \mathcal{B}(\mathbf{R}_+).$$

Throughout this paper, we assume $\rho_i^{(N)} \in \mathcal{M}(\mathbf{R}_+)$ and that $\rho_i^{(N)}$ is continuous (i.e., $\rho_i^{(N)}(\{t\}) = 0$, $t \geq 0$) for all N and i .

Let $x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)}$ be a permutation of $1, 2, \dots, N$, and define a process

$$X^{(N)} = (X_1^{(N)}, \dots, X_N^{(N)})$$

by

$$(2) \quad \begin{aligned} X_i^{(N)}(t) = & x_i^{(N)} + \sum_{k=1}^N \int_0^t \mathbf{1}_{X_k^{(N)}(s-0) > X_i^{(N)}(s-0)} v_k^{(N)}(ds) \\ & + \int_0^t (1 - X_i^{(N)}(s-0)) v_i^{(N)}(ds), \quad i = 1, 2, \dots, N, \quad t \geq 0, \end{aligned}$$

where, $\mathbf{1}_A$ is the indicator function of event A .

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Denote the unit measure concentrated on c by δ_c . With probability 1 we can write

$$(3) \quad v_i^{(N)} = \sum_{j=1}^{\infty} \delta_{\tau_{i,j}^{(N)}}, \quad i = 1, 2, \dots, N,$$

where, with probability 1, $\tau_{i,j}^{(N)}$'s are random variables satisfying $0 < \tau_{i,1}^{(N)} < \tau_{i,2}^{(N)} < \dots$, $i = 1, 2, \dots, N$, and $\tau_{i,j}^{(N)} \neq \tau_{i',j'}^{(N)}$ if $(i, j) \neq (i', j')$. In the following, we work on the event that these inequalities hold.

The right-hand side of (2) is a simple function in t . At $t = \tau_{i,j}$ we see

$$X_i^{(N)}(\tau_{i,j}) - X_i^{(N)}(\tau_{i,j}-) = 1 - X_i^{(N)}(\tau_{i,j}-),$$

which implies

$$(4) \quad X_i^{(N)}(\tau_{i,j}) = 1.$$

With similar consideration, we see that the process $X^{(N)}$ is uniquely determined by (2): Explicitly, we have, for $i = 1, \dots, N$,

$$(5) \quad X_i^{(N)}(t) = \begin{cases} x_i^{(N)} + \sum_{i'; x_{i'}^{(N)} > x_i^{(N)}} \mathbf{1}_{\tau_{i',1}^{(N)} \leq t} & 0 \leq t < \tau_{i,1}^{(N)}, \\ 1 + \sum_{i'=1}^N \mathbf{1}_{\exists j' \in N; \tau_{i,j}^{(N)} < \tau_{i',j'}^{(N)} \leq t} & \tau_{i,j}^{(N)} \leq t < \tau_{i,j+1}^{(N)}, \quad j = 1, 2, 3, \dots \end{cases}$$

In the case of the (homogeneous) Poisson process (i.e., the case $\rho_i^{(N)}((0, t]) = w_i^{(N)}t$, $t \geq 0$, for positive constants $w_i^{(N)}$), a discrete time version of the process (5) has been known for a long time [26, 23, 16, 6, 22, 20] and is called move-to-front (MTF) rules. The process has, in particular, been extensively studied as a model of least-recently-used (LRU) caching in the field of information theory [24, 8, 4, 7, 5, 25, 9, 11, 10, 17, 18, 19], and also is noted as a time-reversed process of top-to-random shuffling. With a great advance in the internet technologies, a new application of the process appeared [13, 15]. The ranking numbers such as those found in the web pages of online bookstores are found to follow the predictions of the model.

In [12], the case where $v_i^{(N)}$'s are (homogeneous) Poisson processes with $\rho_i^{(N)}((0, t]) = w_i^{(N)}t$ is considered, and the joint empirical distribution of jump rate $w_i^{(N)}$ and normalized position

$$(6) \quad Y_i^{(N)}(t) = \frac{1}{N}(X_i^{(N)}(t) - 1),$$

given by $\mu_t^{(N)} = \frac{1}{N} \sum_{i=1}^N \delta_{(w_i^{(N)}, Y_i^{(N)}(t))}$, is studied. (We will abuse notation slightly and denote a unit measure on any space by δ_c .) It is proved in [12] that a scaling limit

$$(7) \quad \mu_t = \lim_{N \rightarrow \infty} \mu_t^{(N)}$$

exists (under reasonable assumptions), and an explicit formula for μ_t , which is a deterministic distribution on $\mathbf{R}_+ \times [0, 1)$, is given. In [13], it is proved that, if the scaling limit of the jump

rate distribution is a discrete distribution, the limit μ_t is the unique time global solution to an initial value problem for a system of first order non-linear partial differential equations (inviscid Burgers equations with a term representing evaporation). The structure of the explicit formula for μ_t is naturally explained by a standard method of characteristic curves for the solution to the partial differential equations.

In the present paper, we will generalize the main results of [12, 13] to the case where $\nu_i^{(N)}$'s are Poisson random measures. We shall call the process $X^{(N)}$ defined by (2), or equivalently by (5), a stochastic ranking processes after [12, 13, 14].

Put

$$(8) \quad X_C^{(N)}(t) = \sum_{i=1}^N \mathbf{1}_{\tau_{i,1}^{(N)} \leq t}, \quad t \geq 0.$$

$X_C^{(N)}(t)$ is a random variable which denotes the position of the boundary between the top side $x \leq X_C^{(N)}(t)$ and the tail side $x > X_C^{(N)}(t)$, where each particle in the top side (i.e., i which satisfies $X_i^{(N)}(t) \leq X_C^{(N)}(t)$) has experienced jump to the top by time t (i.e., $\tau_{i,1}^{(N)} \leq t$), and the particles in the tail side are those particles which have not jumped to the top by time t .

PROPOSITION 1.1. *Let $t \geq 0$, and assume that a sequence of distributions $\{\lambda_t^{(N)}; N \in \mathbf{N}\}$ on \mathbf{R}_+ defined by*

$$(9) \quad \lambda_t^{(N)} = \frac{1}{N} \sum_{i=1}^N \delta_{\rho_i^{(N)}((0,t])}$$

converges weakly as $N \rightarrow \infty$ to a probability distribution λ_t . Then the scaled position of the boundary

$$(10) \quad Y_C^{(N)}(t) = \frac{1}{N} X_C^{(N)}(t) = \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\tau_{i,1}^{(N)} \leq t}$$

converges almost surely as $N \rightarrow \infty$ to

$$(11) \quad y_C(t) = 1 - \int_0^\infty e^{-s} \lambda_t(ds).$$

PROOF. The definition (10) implies that $Y_C^{(N)}(t) - \mathbb{E}[Y_C^{(N)}(t)]$ is an arithmetic mean of independent variables

$$Z_i^{(N)} = \mathbf{1}_{\tau_{i,1}^{(N)} \leq t} - \mathbb{P}[\tau_{i,1}^{(N)} \leq t], \quad i = 1, 2, \dots, N,$$

with bounded 4th order moment. (In fact, $|Z_i^{(N)}| \leq 1$, for all N and i .) Hence,

$$\mathbb{E} \left[\sum_{N=1}^{\infty} (Y_C^{(N)}(t) - \mathbb{E}[Y_C^{(N)}(t)])^4 \right] = \sum_{N=1}^{\infty} \mathbb{E}[(Y_C^{(N)}(t) - \mathbb{E}[Y_C^{(N)}(t)])^4] < \infty,$$

which implies

$$Y_C^{(N)}(t) - \mathbb{E}[Y_C^{(N)}(t)] \rightarrow 0, \quad \text{a.e., as } N \rightarrow \infty.$$

On the other hand, definition of Poisson random measure implies

$$\mathbb{E}[Y_C^{(N)}(t)] = \frac{1}{N} \sum_{i=1}^N \mathbb{P}[\tau_{i,1}^{(N)} \leq t] = \frac{1}{N} \sum_{i=1}^N (1 - e^{-\rho_i^{(N)}((0,t])}) = 1 - \int_0^\infty e^{-s} \lambda_t^{(N)}(ds),$$

which converges to (11) by assumption. \square

Since by Proposition 1.1 we have almost sure convergence at each time t , we have almost sure convergence for all rational number times simultaneously. By definition, $y_C(t)$ and $Y_C^{(N)}(\omega)(t)$, $\omega \in \Omega$, are non-decreasing in t . Hence, if $y_C(t)$ is continuous, we have almost sure convergence as a function in t .

COROLLARY 1.2. *In addition to the assumptions in Proposition 1.1, assume that λ_t is continuous in t with respect to the topology of weak convergence. Then for almost all sample $\omega \in \Omega$, $Y_C^{(N)}(\omega) : \mathbf{R}_+ \rightarrow [0, 1)$ defined by (10) converges pointwise in t as $N \rightarrow \infty$ to a deterministic function $y_C : \mathbf{R}_+ \rightarrow [0, 1)$ defined by (11).*

Proposition 1.1 is a generalization to inhomogeneous case of [12, Proposition 2] for the (homogeneous) Poisson process. The correspondence with λ_t in Proposition 1.1 and λ in [12] is given by $\lambda_t((0, ct]) = \lambda((0, c])$. (9) implies that λ_t is the asymptotic distribution of the expectation of number of jumps to rank 1 for each particle in the time interval $(0, t]$.

Consider a joint empirical distribution $\mu^{(N)}$ of intensity measure $\rho_i^{(N)}$ and scaled position $Y_i^{(N)}$ of the stochastic ranking process:

$$(12) \quad \mu_t^{(N)} = \frac{1}{N} \sum_{i=1}^N \delta_{(\rho_i^{(N)}, Y_i^{(N)}(t))}, \quad t \geq 0.$$

$\mu_t^{(N)}$, $N \in \mathbf{N}$, are random variables whose samples are distributions on the product space $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$ of space of Radon measures $\mathcal{M}(\mathbf{R}_+)$ and an interval $[0, 1) \subset \mathbf{R}_+$.

We consider the standard vague topology on $\mathcal{M}(\mathbf{R}_+)$, that is, a sequence $\{\rho_n\} \subset \mathcal{M}(\mathbf{R}_+)$ converges to $\rho \in \mathcal{M}(\mathbf{R}_+)$ if and only if

$$(13) \quad \lim_{n \rightarrow \infty} \int_{\mathbf{R}_+} f(s) \rho_n(ds) = \int_{\mathbf{R}_+} f(s) \rho(ds),$$

for all continuous function f with compact support. Since \mathbf{R}_+ is a Polish space, i.e., complete and separable metric space, so is $\mathcal{M}(\mathbf{R}_+)$ [2, Theorem 31.5], and consequently, $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$ is also a Polish space [2, Example 26.2].

Assume that a sequence of initial configurations

$$\mu_0^{(N)} = \frac{1}{N} \sum_{i=1}^N \delta_{(\rho_i^{(N)}, (x_i^{(N)} - 1)/N)}, \quad N = 1, 2, \dots,$$

converges weakly as $N \rightarrow \infty$ to a probability distribution μ_0 on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$. Then, in particular,

$$(14) \quad \Lambda^{(N)}(d\rho) := \mu_0^{(N)}(d\rho \times [0, 1)) = \frac{1}{N} \sum_{i=1}^N \delta_{\rho_i^{(N)}}(d\rho) \rightarrow \Lambda(d\rho) := \mu_0(d\rho \times [0, 1)),$$

weakly, as $N \rightarrow \infty$.

Note also that $\lambda_t^{(N)}$ in (9) has an expression

$$(15) \quad \lambda_t^{(N)} = \int_{\mathcal{M}(\mathbf{R}_+)} \delta_{\rho((0,t])} \Lambda^{(N)}(d\rho).$$

We shall generalize (15) and define, for $0 \leq s \leq t$,

$$(16) \quad \lambda_{s,t}^{(N)} = \int_{\mathcal{M}(\mathbf{R}_+)} \delta_{\rho((s,t])} \Lambda^{(N)}(d\rho).$$

THEOREM 1.3. *Assume that $\mu_0^{(N)} \rightarrow \mu_0$ weakly as $N \rightarrow \infty$ for a probability distribution μ_0 on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$. Assume that for each (s, t) satisfying $t \geq s \geq 0$,*

$$(17) \quad \lambda_{s,t}^{(N)} \rightarrow \lambda_{s,t} := \int_{\mathcal{M}(\mathbf{R}_+)} \delta_{\rho((s,t])} \Lambda(d\rho), \quad \text{weakly as } N \rightarrow \infty,$$

where Λ is as in (14). Then for any $t > 0$, and for almost all sample $\omega \in \Omega$, the distribution $\mu_t^{(N)}(\omega)$ converges weakly to a non-random probability distribution μ_t on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$.

μ_t has a following expression in terms of $U(d\rho, y, t) := \mu_t(d\rho \times [y, 1))$.

$$(18) \quad U(d\rho, y, t) := \mu_t(d\rho \times [y, 1)) = \begin{cases} e^{-\rho((t-t_0(y,t),t])} \Lambda(d\rho) & 0 \leq y \leq y_C(t), \\ e^{-\rho((0,t])} U(d\rho, \hat{y}(y, t), 0) & y_C(t) \leq y < 1. \end{cases}$$

Here, $t_0(y, t)$ is the inverse function with respect to t_0 of

$$(19) \quad y_A(t_0, t) = 1 - \int_{\mathcal{M}(\mathbf{R}_+)} e^{-\rho((t-t_0,t])} \Lambda(d\rho), \quad 0 \leq t_0 \leq t,$$

namely,

$$(20) \quad t_0(y, t) = \inf\{s \in [0, t] ; y_A(s, t) \geq y\},$$

and $\hat{y}(y, t)$ is the inverse function with respect to y of

$$(21) \quad y_B(y, t) = 1 - \int_{\mathcal{M}(\mathbf{R}_+)} e^{-\rho((0,t])} \mu_0(d\rho \times [y, 1)), \quad t \geq 0, 0 \leq y < 1,$$

namely,

$$(22) \quad \hat{y}(y, t) = \inf\{x \in [0, 1) ; y_B(x, t) \geq y\}.$$

Note that $y_C(t) = y_A(t, t) = y_B(0, t)$. Note also that, as will be evident from the proof of Theorem 1.3 in Section 2 for $0 \leq y \leq y_C(t)$, the assumption $\mu_0^{(N)} \rightarrow \mu_0$ can be replaced by a weaker assumption $\Lambda_0^{(N)} \rightarrow \Lambda$ for $0 \leq y \leq y_C(t)$.

In contrast to Proposition 1.1, we do not have a result analogous to Corollary 1.2 for Theorem 1.3, because we can expect no monotonicity for $\mu_t^{(N)}$. If we impose additional

conditions, we may go further and prove almost sure convergence as sequences of processes on a finite time interval $[0, T]$, both for $Y_C^{(N)} \rightarrow y_C$ and $\mu^{(N)} \rightarrow \mu$. See Section 4 for statement (Theorem 4.1) and proof.

The structure of the explicit limit formula (18), in particular, the appearance of the inverse functions t_0 of y_A and \hat{y} of y_B , can mathematically be understood through a system of partial differential equations, which is a generalization of that in [13]. To avoid notational complication, consider the case that the limit distribution Λ is supported on a discrete set: $\Lambda = \sum_{\alpha} r_{\alpha} \delta_{\rho_{\alpha}}$. Then (18) implies, for $U_{\alpha}(y, t) := \mu_t(\{\rho_{\alpha}\} \times [y, 1))$,

$$(23) \quad U_{\alpha}(y, t) = \begin{cases} r_{\alpha} e^{-\rho_{\alpha}((t-t_0(y,t), t))} & 0 \leq y \leq y_C(t), \\ U_{\alpha}(\hat{y}(y, t), 0) e^{-\rho_{\alpha}((0, t))} & y_C(t) \leq y < 1, \end{cases}$$

where t_0 and \hat{y} are inverse functions, respectively, of

$$(24) \quad y_A(t_0, t) = 1 - \sum_{\alpha} r_{\alpha} e^{-\rho_{\alpha}((t-t_0, t))},$$

and

$$(25) \quad y_B(y, t) = 1 - \sum_{\alpha} U_{\alpha}(y, 0) e^{-\rho_{\alpha}((0, t))},$$

defined by (20) and (22).

THEOREM 1.4. *Let k be a positive integer, and for each $\alpha = 1, 2, \dots, k$, let r_{α} be a positive constant, $w_{\alpha} : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ a measurable function satisfying $w_{\alpha}(t) > 0$, $t \geq 0$, and $u_{\alpha} : [0, 1) \rightarrow \mathbf{R}_+$ a non-negative smooth strictly decreasing function, satisfying*

$$(26) \quad \sum_{\beta=1}^k r_{\beta} = 1, \quad \sum_{\beta=1}^k r_{\beta} w_{\beta}(t) < \infty, \quad t \geq 0, \quad \text{and} \quad \sum_{\beta=1}^k u_{\beta}(y) = 1 - y, \quad 0 \leq y < 1.$$

Then an initial value problem for a system of partial differential equations

$$(27) \quad \frac{\partial U_{\alpha}}{\partial t}(y, t) + \sum_{\beta=1}^k w_{\beta}(t) U_{\beta}(y, t) \frac{\partial U_{\alpha}}{\partial y}(y, t) = -w_{\alpha}(t) U_{\alpha}(y, t),$$

$$(y, t) \in [0, 1) \times \mathbf{R}_+, \quad \alpha = 1, 2, \dots, k,$$

with a boundary condition

$$(28) \quad U_{\alpha}(0, t) = r_{\alpha}, \quad t \geq 0, \quad \alpha = 1, 2, \dots, k,$$

and initial data

$$(29) \quad U_{\alpha}(\cdot, 0) = u_{\alpha}, \quad \alpha = 1, 2, \dots, k,$$

has a unique time global classical solution, whose formula is given by (23) with

$$(30) \quad \rho_{\alpha}((s, t)) = \int_s^t w_{\alpha}(u) du \quad \text{and} \quad U_{\alpha}(y, 0) = u_{\alpha}(y).$$

As in [13, §2], (27) is solved by a method of characteristic curves, and y_A , y_B , and y_C turn out to be the characteristic curves for (27), which mathematically explains how the inverse functions of these functions appear in the solutions.

For the homogeneous case ($\rho_i^{(N)}((0, t]) = w_i^{(N)}t$), Theorem 1.3 reduces to [12, Theorem 5] (with slightly weaker assumption on μ_0 , Λ , and λ_t , and with stronger convergence in $(\Omega, \mathcal{F}, \mathbb{P})$, thanks to technical refinement in the proof), and Theorem 1.4 reduces to [13, Theorem 1]. Motivation for extending the previous results to the present case arises both from mathematical and application point of view.

MATHEMATICAL. The model is a natural extension of [12], with (homogeneous) Poisson processes in the formulation of [12] generalized to (inhomogeneous) Poisson random measures in (2) or (5). Also, as seen from Theorem 1.4, the system of PDE corresponding to the limit distribution is a natural extension of that considered in [13], with constant coefficients w_α in [13] generalized to time dependent coefficients $w_\alpha(t)$ in (27). On the other hand, the space on which μ_t is defined becomes large; μ_t considered in [12] is a distribution on $\mathbf{R}_+ \times [0, 1)$, whereas μ_t in Theorem 1.3 is on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$. Hence it is necessary to extend the definition of the model, compared to [12, 13].

APPLICATION. The model has successfully been applied to statistical explanation of ranking data at an online bookstore Amazon.co.jp [14, 13] and data of list of subject titles at a collected bulletin board 2ch.net [13]. These data arise as results of social activities, hence it is inevitable that the data have day-night difference in their time dependence. This motivates considering the inhomogeneous cases from an application side.

Note that we directly see from (2), the Markov property

$$\begin{aligned} X_i^{(N)}(t+u) &= X_i^{(N)}(u) + \sum_{k=1}^N \int_0^t \mathbf{1}_{X_k^{(N)}(s+u-0) > X_i^{(N)}(s+u-0)} \tilde{v}_k^{(N)}(ds) \\ &\quad + \int_0^t (1 - X_i^{(N)}(s+u-0)) \tilde{v}_i^{(N)}(ds), \end{aligned}$$

where we put $\tilde{v}_i^{(N)}(A) = v_i^{(N)}(A+u)$. In practical application, this property enables us to shift the time origin $t = 0$ to the time that a particle we observe jumps to the top, namely, we may set $X_i^{(N)}(0) = x_i^{(N)} = 1$, by adjusting the ‘clock’ for the intensity measure accordingly. This motivates our formulating the model in terms of Poisson random measures, even though in Proposition 1.1 we apparently do not use Markov properties.

Note also that if $x_i^{(N)} = 1$, then up to the first jump of i to the top, namely, for $t < \tau_{i,1}^{(N)}$, comparison of (5) and (8) leads to

$$X_i^{(N)}(t) = X_C^{(N)}(t) + 1,$$

because, if $x_i^{(N)} = 1$ then, $x_{i'}^{(N)} > x_i^{(N)}$ for all $i' \neq i$. Therefore, in practical application, we may proceed with observing a trajectory (time development) of a single particle, putting the time of its first jump to top as $t = 0$ and observing until its next jump to top, and then apply Proposition 1.1 or Corollary 1.2 [13, 14].

The plan of the paper and a brief description of the role of the authors are as follows. In Section 2 we prove Theorem 1.3, and we prove Theorem 1.4 in Section 3. In Section 4, we state and prove Theorem 4.1, time-uniform results corresponding to Proposition 1.1 and Theorem 1.3. The core structure of the present work, including basic properties of the stochastic ranking process which are essential for the proofs of these results, are based on collaboration of K. Hattori and T. Hattori. In extending the previous results for the convergence of empirical distribution on $\mathbf{R}_+ \times [0, 1)$ to $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$, where $\mathcal{M}(\mathbf{R}_+)$ is a space of Borel measures, we have to reformulate the process using Poisson random measures and provide abstract measure theory result Lemma 2.1, for which collaboration with Hariya is crucial. Convergence result as measure valued processes developed in Section 4 is achieved by collaboration with Nagahata. Also, various technical refinements, implying in particular stronger convergence with less assumptions for the uniform intensity case [12], are results of the collaboration of these 4 authors. In Section 5 we consider a simple case where the intensities of the Poisson random measures have a common time dependence, and prove another scaling limit for the particle trajectory, corresponding to a time change with respect to the intensity. This is a result of collaboration of T. Hattori, Hariya, Kobayashi, and Takeshima at Tohoku University, and provides a mathematical result of scaling limit with time changes, as well as a practically useful formula in applying the present results to online rankings. A practical method based on this mathematical result is partly checked by actual data obtained at 2ch.net in the master theses of Kobayashi and Takeshima (unpublished). In Appendix, we give remarks to be kept in mind when applying our results to practical data through statistical analysis.

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2. Proof of Theorem 1.3. Throughout this section, we assume that the assumptions of Theorem 1.3 hold.

We first note the following rather technical generality.

LEMMA 2.1. *Let $t > 0$. If, for each $y \in [0, 1)$ and for each bounded continuous function $g : \mathcal{M}(\mathbf{R}_+) \rightarrow \mathbf{R}$, there exists $\tilde{\mathcal{Q}}$ with $\mathbb{P}[\tilde{\mathcal{Q}}] = 1$ such that*

$$(31) \quad \begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{Y_i^{(N)}(t) \geq y}(\omega) \\ &= \begin{cases} \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((t-t_0(y,t), t))} \Lambda(d\rho) & 0 \leq y \leq y_C(t), \\ \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t))} \mu_0(d\rho \times [\hat{y}(y, t), 1)) & y_C(t) \leq y < 1, \end{cases} \end{aligned}$$

holds on $\tilde{\mathcal{Q}}$, then the claim of Theorem 1.3 holds for this t .

The point here is that $\tilde{\mathcal{Q}}$ may depend on y and g , while Theorem 1.3 claims the existence of a sample set, independently of test functions.

We make use of the results in [2, Exercises 30.3, 31.2] for a proof of Lemma 2.1. Note that $\mathcal{M}(\mathbf{R}_+)$ is not locally compact, while local compactness is assumed in the relevant results of the reference. We prepare the next Lemma to fill the gap.

LEMMA 2.2. *There exists a countable subset $\mathcal{T} = \{f_n ; n \in \mathbf{N}\}$ of uniformly continuous functions $f_n : \mathcal{M}(\mathbf{R}_+) \times [0, 1) \rightarrow \mathbf{R}$, such that if for each $f_n \in \mathcal{T}$*

$$(32) \quad \lim_{N \rightarrow \infty} \int_{\mathcal{M}(\mathbf{R}_+) \times [0, 1)} f_n(\rho, y) v_N(d\rho \times dy) = \int_{\mathcal{M}(\mathbf{R}_+) \times [0, 1)} f_n(\rho, y) v(d\rho \times dy)$$

holds for a sequence of Borel probability measures v_N and a Borel probability measure v on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$, then $v_N \rightarrow v$, weakly as $N \rightarrow \infty$.

PROOF. We noted below (13) that $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$ is a Polish space. Note also that there exists a countable set of continuous functions $\{e_n : \mathbf{R}_+ \rightarrow \mathbf{R} ; n \in \mathbf{N}\}$ of compact support, such that

$$(33) \quad \begin{aligned} & d((\rho_1, y_1), (\rho_2, y_2)) \\ &= |y_1 - y_2| + \sum_{n \in \mathbf{N}} 2^{-n} \left(1 \wedge \left| \int_{\mathbf{R}_+} e_n(s) \rho_1(ds) - \int_{\mathbf{R}_+} e_n(s) \rho_2(ds) \right| \right), \\ & (\rho_i, y_i) \in \mathcal{M}(\mathbf{R}_+) \times [0, 1), i = 1, 2, \end{aligned}$$

defines a metric d compatible with the topology we are considering [2, (31.4)].

Denote a set of sequences by $\mathbf{R}^\infty = \{x = (x_1, x_2, \dots)\}$, and define a metric d' on $\mathbf{R}^\infty \times [0, 1)$ by

$$(34) \quad d'((x_1, y_1), (x_2, y_2)) = |y_1 - y_2| + \sum_{n \in \mathbf{N}} 2^{-n} (1 \wedge |x_{1,n} - x_{2,n}|)$$

where $x_i = (x_{i,1}, x_{i,2}, \dots)$, $i = 1, 2$. We have a natural one-to-one map $\iota = (\iota_1, \iota_2, \dots, \iota_0) : \mathcal{M}(\mathbf{R}_+) \times [0, 1) \rightarrow \mathbf{R}^\infty \times [0, 1)$ defined by

$$(35) \quad \iota(\rho, y)_n = \int_{\mathbf{R}_+} e_n(s) \rho(ds), \quad n \in \mathbf{N}, \quad \text{and} \quad \iota(\rho, y)_0 = y.$$

Put

$$(36) \quad E' = \iota(\mathcal{M}(\mathbf{R}_+) \times [0, 1)) \subset \mathbf{R}^\infty \times [0, 1).$$

Then (33), (34) and (35) imply that $\iota : \mathcal{M}(\mathbf{R}_+) \times [0, 1) \rightarrow E'$ is a one-to-one onto isometric map. Since $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$ is complete, E' is a closed set in $\mathbf{R}^\infty \times [0, 1)$.

Let $F \subset \mathcal{M}(\mathbf{R}_+) \times [0, 1)$ be a closed set. Since ι is isometric, $\iota(F)$ is a closed subset of E' , and since E' is a closed set in $\mathbf{R}^\infty \times [0, 1)$, $\iota(F)$ is a closed set in $\mathbf{R}^\infty \times [0, 1)$. Hence, if a sequence of probability measures $v_N \circ \iota^{-1}$ on $\mathbf{R}^\infty \times [0, 1)$ converges weakly as $N \rightarrow \infty$ to $v \circ \iota^{-1}$, then

$$\overline{\lim}_{N \rightarrow \infty} v_N(F) = \overline{\lim}_{N \rightarrow \infty} v_N \circ \iota^{-1}(\iota(F)) \leq v \circ \iota^{-1}(\iota(F)) = v(F),$$

which implies $v_N \rightarrow v$, weakly as $N \rightarrow \infty$. Thus the conclusion of Lemma 2.2 is reduced to a weak convergence $v_N \circ \iota^{-1} \rightarrow v \circ \iota^{-1}$ on $\mathbf{R}^\infty \times [0, 1)$.

For each $k \in N$ define a projection to finite dimensional space $\pi_k : \mathbf{R}^\infty \times [0, 1) \rightarrow \mathbf{R}^k \times [0, 1)$ by

$$(37) \quad \pi_k(x) = (x_1, x_2, \dots, x_k, y), \quad x = (x_1, x_2, \dots, y) \in \mathbf{R}^\infty \times [0, 1).$$

Then $\nu_N \circ \iota^{-1} \circ \pi_k^{-1}$ and $\nu \circ \iota^{-1} \circ \pi_k^{-1}$ are probability measures on $\mathbf{R}^k \times [0, 1)$. Note that a Borel probability measure on Polish space is a Radon measure [2, Theorem 26.3], and that the vague convergence of probability measures to a probability measure on \mathbf{R}^k is equivalent to the weak convergence [2, Theorem 30.8]. Since $\mathbf{R}^k \times [0, 1)$ is a locally compact Polish space, there exists a *countable* subset $\mathcal{T}_k = \{f_{k,i} ; i \in N\}$ of continuous functions $f_{k,i} : \mathbf{R}^k \times [0, 1) \rightarrow \mathbf{R}$ with compact support, such that if for each $f_{k,i} \in \mathcal{T}_k$

$$(38) \quad \lim_{N \rightarrow \infty} \int_{\mathbf{R}^k \times [0, 1)} f_{k,i}(z) \nu_N \circ \iota^{-1} \circ \pi_k^{-1}(dz) = \int_{\mathbf{R}^k \times [0, 1)} f_{k,i}(z) \nu \circ \iota^{-1} \circ \pi_k^{-1}(dz)$$

holds, then $\nu_N \circ \iota^{-1} \circ \pi_k^{-1} \rightarrow \nu \circ \iota^{-1} \circ \pi_k^{-1}$, weakly as $N \rightarrow \infty$ [2, Exercises 30.3, 31.2].

Let

$$\mathcal{T} = \bigcup_{k \in N} \{f_{k,i} \circ \pi_k \circ \iota : \mathcal{M}(\mathbf{R}_+) \times [0, 1) \rightarrow \mathbf{R} ; f_{k,i} \in \mathcal{T}_k\},$$

be the \mathcal{T} in the assumption of Lemma 2.2. Since $f_{k,i}$, π , ι are continuous, the functions in \mathcal{T} are continuous. Note further that since $f_{k,i}$ is of bounded support, the functions in \mathcal{T} are uniformly continuous. Since a countable union of countable sets is countable, \mathcal{T} so defined is a countable set. With this choice of \mathcal{T} , the assumption (32), with a change in integration variable $z = \pi_k \circ \iota(\rho, y)$, implies

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_{\mathbf{R}^k \times [0, 1)} f_{k,i}(z) \nu_N \circ \iota^{-1} \circ \pi_k^{-1}(dz) \\ &= \lim_{N \rightarrow \infty} \int_{\mathcal{M}(\mathbf{R}_+) \times [0, 1)} f_{k,i} \circ \pi_k \circ \iota(\rho, y) \nu_N(d\rho \times dy) \\ &= \int_{\mathcal{M}(\mathbf{R}_+) \times [0, 1)} f_{k,i} \circ \pi_k \circ \iota(\rho, y) \nu(d\rho \times dy) \\ &= \int_{\mathbf{R}^k \times [0, 1)} f_{k,i}(z) \nu \circ \iota^{-1} \circ \pi_k^{-1}(dz), \end{aligned}$$

for all k and i , which, as noted below (38), implies $\nu_N \circ \iota^{-1} \circ \pi_k^{-1} \rightarrow \nu \circ \iota^{-1} \circ \pi_k^{-1}$, weakly as $N \rightarrow \infty$, for all k . This implies that as measures on $\mathbf{R}^\infty \times [0, 1)$, $\nu_N \circ \iota^{-1} \rightarrow \nu \circ \iota^{-1}$, weakly as $N \rightarrow \infty$ [3, §2 Example 2.4]. As noted in the paragraph between (36) and (37), this further implies $\nu_N \rightarrow \nu$, weakly as $N \rightarrow \infty$. \square

REMARK. We could alternatively make use of separability of $\mathcal{M}(\mathbf{R}_+)$ directly to obtain a countable set \mathcal{T} , following the discussion in [21, §1, Remark 4.17, and remark after Corollary 9.3].

PROOF OF LEMMA 2.1. Let \mathcal{T} be as in Lemma 2.2. If there exists, for each $n \in N$, $\tilde{\Omega}_n \subset \Omega$ such that (32) holds for $\omega \in \tilde{\Omega}_n$ and $\mathbb{P}[\tilde{\Omega}_n] = 1$ holds, then $\Omega' := \bigcap_{n=1}^\infty \tilde{\Omega}_n$

satisfies $\mathbb{P}[\Omega'] = 1$ and (32) holds for all $\omega \in \Omega'$ and $f_n \in \mathcal{T}$, which, with Lemma 2.2, implies Theorem 1.3.

Let d be the metric on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$ as in the proof of Lemma 2.2. Let $f_n \in \mathcal{T}$. Since f_n is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\rho_1, \rho_2 \in \mathcal{M}(\mathbf{R}_+)$ and $y_1, y_2 \in [0, 1)$, $d((\rho_1, y_1), (\rho_2, y_2)) < \delta$ implies $|f_n(\rho_1, y_1) - f_n(\rho_2, y_2)| < \varepsilon$. Let k be a positive integer greater than $1/\delta$ and put

$$(39) \quad f_{n,k}(\rho, y) = \sum_{l=0}^{k-1} f_n(\rho, l/k) \chi_{[l/k, (l+1)/k)}(y),$$

where $\chi_{[a,b)}(y) = 1$ if $a \leq y < b$ and 0 otherwise. Then for each $\rho \in \mathcal{M}(\mathbf{R}_+)$ we have

$$\sup_{y \in [0,1)} |f_n(\rho, y) - f_{n,k}(\rho, y)| < \varepsilon.$$

Therefore, $\lim_{k \rightarrow \infty} f_{n,k} = f_n$ uniformly on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$. Noting that

$$\chi_{[l/k, (l+1)/k)} = \chi_{[l/k, 1)} - \chi_{[(l+1)/k, 1)},$$

we see from (39) that $f_{n,k}$ has an expression

$$f_{n,k}(\rho, y) = \sum_{l=0}^{k-1} g_{n,k,l}(\rho) \chi_{[l/k, 1)}(y),$$

where $g_{n,k,l} : \mathcal{M}(\mathbf{R}_+) \rightarrow \mathbf{R}$ is bounded continuous.

Therefore, if (31) holds, then using the definition (12) and the explicit formula (18) claimed in Theorem 1.3, we see that there exists $\tilde{\Omega}_{n,k}$ satisfying $\mathbb{P}[\tilde{\Omega}_{n,k}] = 1$ and

$$\lim_{N \rightarrow \infty} \int_{\mathcal{M}(\mathbf{R}_+) \times [0,1)} f_{n,k}(\rho, y) \mu_t^{(N)}(d\rho \times dy)(\omega) = \int_{\mathcal{M}(\mathbf{R}_+) \times [0,1)} f_{n,k}(\rho, y) \mu_t(d\rho \times dy)$$

if $\omega \in \tilde{\Omega}_{n,k}$. Hence, $\tilde{\Omega}_n = \bigcap_{k=1}^{\infty} \tilde{\Omega}_{n,k}$ satisfies $\mathbb{P}[\tilde{\Omega}_n] = 1$ and (32) holds for $\omega \in \tilde{\Omega}_n$. \square

In view of Lemma 2.1, we fix (y, t) and a bounded continuous function g , in the remainder of this section. Since g is bounded, there exists a constant $M > 0$ such that

$$(40) \quad |g(\rho)| \leq M, \quad \rho \in \mathcal{M}(\mathbf{R}_+).$$

Since the jump times $\{\tau_{i,1}^{(N)}\}$ are independent, Proposition 1.1 is proved in a straightforward way. In contrast, $\{Y_i^{(N)}\}$ appearing in the left-hand side of (31) are dependent, and moreover, the non-linearity in (27) indicates that the dependence cannot be neglected in the limit $N \rightarrow \infty$. A strategy, inherited from the proof in [12], is to (i) choose a nice quantity defined as a sum of independent random variables in such a way that the quantity converges to the right-hand side of (31), and (ii) show that the difference between the chosen quantity and the left-hand side of (31) can be shown to disappear in the limit, using the properties of the model. We state these two steps explicitly in the following two Lemmas, respectively.

LEMMA 2.3. *The following hold.*

(i) For $0 \leq y \leq y_C(t)$,

$$(41) \quad \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{v_i^{(N)}((t-t_0(y,t),t])>0} \rightarrow \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) (1 - e^{-\rho((t-t_0(y,t),t])}) \Lambda(d\rho),$$

almost surely as $N \rightarrow \infty$.

(ii) For $y_C(t) \leq y < 1$,

$$(42) \quad \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{(x_i^{(N)}-1)/N \geq \hat{y}(y,t), \tau_{i,1}^{(N)} > t} \\ \rightarrow \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t])} \mu_0(d\rho \times [\hat{y}(y,t), 1]),$$

almost surely as $N \rightarrow \infty$.

LEMMA 2.4. *The following hold.*

(i) For $0 \leq y \leq y_C(t)$,

$$(43) \quad \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{Y_i^{(N)}(t) < y} - \mathbf{1}_{v_i^{(N)}((t-t_0(y,t),t])>0}| \rightarrow 0,$$

almost surely as $N \rightarrow \infty$.

(ii) For $y_C(t) \leq y < 1$,

$$(44) \quad \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{Y_i^{(N)}(t) \geq y} - \mathbf{1}_{(x_i^{(N)}-1)/N \geq \hat{y}(y,t), \tau_{i,1}^{(N)} > t}| \rightarrow 0,$$

almost surely as $N \rightarrow \infty$.

PROOF OF (31) ASSUMING LEMMA 2.3 AND LEMMA 2.4. For the case $0 \leq y \leq y_C(t)$, (40), (14), (41), and (43) imply

$$\begin{aligned} & \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{Y_i^{(N)}(t) \geq y} - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((t-t_0(y,t),t])} \Lambda(d\rho) \right| \\ &= \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) (1 - \mathbf{1}_{Y_i^{(N)}(t) < y}) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((t-t_0(y,t),t])} \Lambda(d\rho) \right| \\ &= \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) (-\mathbf{1}_{Y_i^{(N)}(t) < y} + \mathbf{1}_{v_i^{(N)}((t-t_0(y,t),t])>0}) \right. \\ & \quad \left. + \left(\frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda(d\rho) \right) \right. \\ & \quad \left. + \left(-\frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{v_i^{(N)}((t-t_0(y,t),t])>0} \right) \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) (1 - e^{-\rho((t-t_0(y,t),t)})} \Lambda(d\rho) \right) \\
\leq & M \frac{1}{N} \sum_{i=1}^N | \mathbf{1}_{Y_i^{(N)}(t) < y} - \mathbf{1}_{v_i^{(N)}((t-t_0(y,t),t)) > 0} | \\
& + \left| \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda^{(N)}(d\rho) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda(d\rho) \right| \\
& + \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{v_i^{(N)}((t-t_0(y,t),t)) > 0} - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) (1 - e^{-\rho((t-t_0(y,t),t)})} \Lambda(d\rho) \right) \\
\rightarrow & 0, \quad \text{a.s., as } N \rightarrow \infty,
\end{aligned}$$

which proves (31) for $0 \leq y \leq y_C(t)$.

Similarly, for the case $y_C(t) \leq y < 1$, (40), (42), and (44) imply

$$\begin{aligned}
& \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{Y_i^{(N)}(t) \geq y} - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t))} \mu_0(d\rho \times [\hat{y}(y,t), 1]) \right) \\
& = \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) (\mathbf{1}_{Y_i^{(N)}(t) \geq y} - \mathbf{1}_{(x_i^{(N)}-1)/N \geq \hat{y}(y,t), \tau_{i,1}^{(N)} > t}) \right. \\
& \quad + \left(\frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{(x_i^{(N)}-1)/N \geq \hat{y}(y,t), \tau_{i,1}^{(N)} > t} \right. \\
& \quad \left. \left. - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t))} \mu_0(d\rho \times [\hat{y}(y,t), 1]) \right) \right) \\
& \leq M \frac{1}{N} \sum_{i=1}^N | \mathbf{1}_{Y_i^{(N)}(t) \geq y} - \mathbf{1}_{(x_i^{(N)}-1)/N \geq \hat{y}(y,t), \tau_{i,1}^{(N)} > t} | \\
& \quad + \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{(x_i^{(N)}-1)/N \geq \hat{y}(y,t), \tau_{i,1}^{(N)} > t} \right. \\
& \quad \left. - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t))} \mu_0(d\rho \times [\hat{y}(y,t), 1]) \right) \rightarrow 0, \quad \text{a.s., as } N \rightarrow \infty,
\end{aligned}$$

which proves (31) for $y_C(t) \leq y < 1$. \square

Before proving Lemma 2.3 and Lemma 2.4, we prepare a couple of random variables which converge as $N \rightarrow \infty$ to y_A in (19) and y_B in (21). The following Lemma 2.5 is used in the proof of Lemma 2.4, and the proof of Lemma 2.3 is similar to that of Lemma 2.5.

LEMMA 2.5. (i) For $0 \leq t_0 \leq t$ define

$$(45) \quad Y_A^{(N)}(t_0, t) = \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{v_i^{(N)}((t-t_0,t)) > 0}.$$

Then $Y_A^{(N)}(t_0, t) \rightarrow y_A(t_0, t)$, almost surely as $N \rightarrow \infty$.

(ii) For $t \geq 0$ and $0 \leq y_0 < 1$ define

$$(46) \quad Y_B^{(N)}(y_0, t) = y_0 + \frac{1}{N} \sum_{i; (x_i^{(N)} - 1)/N \geq y_0} \mathbf{1}_{\tau_{i,1}^{(N)} \leq t}.$$

Then $Y_B^{(N)}(y_0, t) \rightarrow y_B(y_0, t)$, almost surely as $N \rightarrow \infty$.

PROOF. As in the proof of Proposition 1.1, a strong law of large numbers implies, almost surely as $N \rightarrow \infty$

$$Y_A^{(N)}(t_0, t) - \mathbb{E}[Y_A^{(N)}(t_0, t)] \rightarrow 0 \quad \text{and} \quad Y_B^{(N)}(y_0, t) - \mathbb{E}[Y_B^{(N)}(y_0, t)] \rightarrow 0.$$

On the other hand, (16) and (17) imply

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E}[Y_A^{(N)}(t_0, t)] &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (1 - \mathbb{P}[v_i^{(N)}((t-t_0, t]) = 0]) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (1 - e^{-\rho_i^{(N)}((t-t_0, t])}) = 1 - \lim_{N \rightarrow \infty} \int_{\mathcal{M}(\mathbf{R}_+)} e^{-\rho((t-t_0, t])} \Lambda^{(N)}(d\rho) \\ &= 1 - \lim_{N \rightarrow \infty} \int_0^\infty e^{-s} \lambda_{t-t_0, t}^{(N)}(ds) = 1 - \int_{\mathcal{M}(\mathbf{R}_+)} e^{-\rho((t-t_0, t])} \Lambda(d\rho) = y_A(t_0, t). \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E}[Y_B^{(N)}(y_0, t)] &= y_0 + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i; (x_i^{(N)} - 1)/N \geq y_0} \mathbb{P}[\tau_{i,1}^{(N)} \leq t] \\ &= y_0 + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i; (x_i^{(N)} - 1)/N \geq y_0} (1 - e^{-\rho_i^{(N)}((0, t])}) \\ &= y_0 + 1 - y_0 - \lim_{N \rightarrow \infty} \int_{\mathcal{M}(\mathbf{R}_+)} e^{-\rho((0, t])} \mu_0^{(N)}(d\rho \times [y_0, 1]) \\ &= 1 - \int_{\mathcal{M}(\mathbf{R}_+)} e^{-\rho((0, t])} \mu_0(d\rho \times [y_0, 1]) = y_B(y_0, t). \end{aligned}$$

□

PROOF OF LEMMA 2.3. The proof is a repetition of the proof of Lemma 2.5, by replacing $\mathbf{1}_{v_i^{(N)}((t-t_0, t]) > 0}$ with $g(\rho_i^{(N)}) \mathbf{1}_{v_i^{(N)}((t-t_0(y, t), t]) > 0}$ for (41), and $\mathbf{1}_{\tau_{i,1}^{(N)} \leq t}$ with $g(\rho_i^{(N)}) \mathbf{1}_{\tau_{i,1}^{(N)} > t}$ for (42). □

The proof of (31) now will be complete if we prove Lemma 2.4, which is proved in a similar way as the corresponding part in [12].

PROOF OF (43) FOR $0 \leq y \leq y_C(t)$. Note that $y_A(t_0, t)$ of (19) is non-decreasing in t_0 and t , with $y_A(0, t) = 0$ and $y_A(t, t) = y_C(t)$, and by assumption of the Theorem 1.3, is

continuous. Hence

$$(47) \quad y_A(t_0(y, t), t) = y, \quad 0 \leq y \leq y_C(t), \quad t \geq 0.$$

Lemma 2.5 therefore implies that there exists $\Omega_A \subset \Omega$, satisfying $P[\Omega_A] = 1$, such that

$$(48) \quad \lim_{N \rightarrow \infty} Y_A^{(N)}(t_0(y, t), t)(\omega) = y, \quad \omega \in \Omega_A.$$

Fix $\omega \in \Omega_A$ arbitrarily. The definition of the stochastic ranking process and (45) imply that $v_i^{(N)}((t - t_0(y, t), t))(\omega) > 0$, if and only if $Y_i^{(N)}(t)(\omega)$ is on the top side of $Y_A^{(N)}(t_0(y, t), t)(\omega)$; $Y_i^{(N)}(t)(\omega) < Y_A^{(N)}(t_0(y, t), t)(\omega)$. Therefore

$$(49) \quad \begin{aligned} & \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{Y_i^{(N)}(t) < y}(\omega) - \mathbf{1}_{v_i^{(N)}((t-t_0(y,t),t)) > 0}(\omega)| \\ &= \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{Y_i^{(N)}(t) < y}(\omega) - \mathbf{1}_{Y_i^{(N)}(t) < Y_A^{(N)}(t_0(y,t),t)}(\omega)|. \end{aligned}$$

Note that the definition of $Y_i^{(N)}(t)$ in (6) implies that it takes values in $\{k/N; k = 0, 1, \dots, N-1\}$. Hence (48) implies

$$(50) \quad \begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{Y_i^{(N)}(t) < y}(\omega) - \mathbf{1}_{Y_i^{(N)}(t) < Y_A^{(N)}(t_0(y,t),t)}(\omega)| \\ & \leq \lim_{N \rightarrow \infty} \frac{1}{N} \times (N|Y_A^{(N)}(t_0(y, t), t)(\omega) - y| + 1) = 0. \end{aligned}$$

The relations (49) and (50) imply (43). \square

PROOF OF (44) FOR $y_C(t) \leq y < 1$. $y_B(y, t)$ of (21) is non-decreasing in y and t , with $y_B(0, t) = y_C(t)$ and $y_B(1-, t) = 1 - 0$, and by assumption of the Theorem 1.3, is continuous. Hence

$$(51) \quad y_B(\hat{y}(y, t), t) = y, \quad y_C(t) \leq y < 1, \quad t \geq 0.$$

Lemma 2.5 therefore implies that there exists $\Omega_B \subset \Omega$, satisfying $P[\Omega_B] = 1$, such that

$$(52) \quad \lim_{N \rightarrow \infty} Y_B^{(N)}(\hat{y}(y, t), t)(\omega) = y, \quad \omega \in \Omega_B.$$

Fix $\omega \in \Omega_B$ arbitrarily. The definition of the stochastic ranking process and (46) imply that $(x_i^{(N)} - 1)/N \geq \hat{y}(y, t)$ and $\tau_{i,1}^{(N)}(\omega) > t$ hold together, if and only if $Y_i^{(N)}(t)(\omega)$ is on the tail side of $Y_B^{(N)}(\hat{y}(y, t), t)(\omega)$; $Y_i^{(N)}(t)(\omega) \geq Y_B^{(N)}(t_0(y, t), t)(\omega)$. Therefore

$$(53) \quad \begin{aligned} & \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{Y_i^{(N)}(t) \geq y}(\omega) - \mathbf{1}_{(x_i^{(N)} - 1)/N \geq \hat{y}(y,t), \tau_{i,1}^{(N)} > t}(\omega)| \\ &= \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{Y_i^{(N)}(t) \geq y}(\omega) - \mathbf{1}_{Y_i^{(N)}(t) \geq Y_B^{(N)}(\hat{y}(y,t),t)}(\omega)|. \end{aligned}$$

As in the proof of (43), $Y_i^{(N)}(t)$ takes values in $\{k/N ; k = 0, 1, \dots, N - 1\}$, which implies, with (52),

$$(54) \quad \begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{Y_i^{(N)}(t) \geq y}(\omega) - \mathbf{1}_{(x_i^{(N)} - 1)/N \geq \hat{y}(y, t), \tau_{i,1}^{(N)} > t}(\omega)| \\ & \leq \lim_{N \rightarrow \infty} \frac{1}{N} \times (N|Y_B^{(N)}(\hat{y}(y, t), t)(\omega) - y| + 1) = 0. \end{aligned}$$

The relations (53) and (54) imply (44). \square

This completes the proof of Lemma 2.4, hence of Theorem 1.3.

3. Proof of Theorem 1.4. To prove Theorem 1.4, we apply a standard method of characteristic curves.

First, assume $0 \leq y \leq y_C(t) = y_A(t, t)$. Let $t_1 \geq 0$, and consider an ordinary differential equation for a characteristic curve intersecting $(0, t_1)$, defined by

$$(55) \quad \begin{aligned} \frac{dy}{dt}(t) &= \sum_{\beta=1}^k w_\beta(t) U_\beta(y(t), t), \quad \alpha = 1, 2, \dots, k, t \geq t_1, \\ y(t_1) &= 0. \end{aligned}$$

Put

$$(56) \quad \varphi_\alpha(t) = U_\alpha(y(t), t), \quad \alpha = 1, 2, \dots, k, t \geq t_1.$$

Then (56), (27), and (55) imply

$$(57) \quad \frac{d\varphi_\alpha}{dt}(t) = -w_\alpha(t)U_\alpha(y(t), t) = -w_\alpha(t)\varphi_\alpha(t),$$

which, with $y(t_1) = 0$ in (55), has a unique solution

$$(58) \quad \varphi_\alpha(t) = U_\alpha(0, t_1) \exp\left(-\int_{t_1}^t w_\alpha(u)du\right) = r_\alpha \exp\left(-\int_{t_1}^t w_\alpha(u)du\right),$$

where we also used (28). Substituting (56) and (58) in (55), we have

$$\frac{dy}{dt}(t) = \sum_{\beta=1}^k w_\beta(t) r_\beta \exp\left(-\int_{t_1}^t w_\beta(u)du\right),$$

which, with $y(t_1) = 0$, has a unique solution

$$(59) \quad y(t) = \sum_{\beta=1}^k r_\beta \left(1 - \exp\left(-\int_{t_1}^t w_\beta(u)du\right)\right) = y_A(t - t_1, t),$$

where we also used $\sum_{\beta=1}^k r_\beta = 1$ in (26) and (24) with (30), in the last equality. The assumptions for w_α in Theorem 1.4 imply that $y_A(t_0, t)$ is strictly increasing and differentiable in t_0 , satisfying $y_A(0, t) = 0$ and $y_A(t, t) = y_C(t)$. Hence there exists a unique, strictly increasing, differentiable inverse function $t_0 = t_0(y, t)$, taking values in $[0, t]$, satisfying

$$y_A(t_0(y, t), t) = y, \quad 0 \leq y \leq y_C(t), \quad t \geq 0.$$

This, with (56), (58), and (59), implies

$$U_\alpha(y, t) = r_\alpha \exp\left(-\int_{t-t_0(y,t)}^t w_\alpha(u)du\right),$$

which proves (23) for $0 \leq y \leq y_C(t)$.

Next, assume $y_C(t) = y_B(0, t) \leq y < 1$. Let $0 \leq y_0 < 1$, and consider an ordinary differential equation for a characteristic curve intersecting $(y_0, 0)$, defined by

$$(60) \quad \begin{aligned} \frac{dy}{dt}(t) &= \sum_{\beta=1}^k w_\beta(t) U_\beta(y(t), t), \quad \alpha = 1, 2, \dots, k, \quad t \geq 0, \\ y(0) &= y_0. \end{aligned}$$

Put

$$(61) \quad \varphi_\alpha(t) = U_\alpha(y(t), t), \quad \alpha = 1, 2, \dots, k, \quad t \geq t_1.$$

Then (61), (27), and (60) imply, exactly as for the case $y \leq y_C(t)$,

$$(62) \quad \frac{d\varphi_\alpha}{dt}(t) = -w_\alpha(t)\varphi_\alpha(t),$$

which, with $y(0) = y_0$, has a unique solution

$$(63) \quad \varphi_\alpha(t) = u_\alpha(y_0) \exp\left(-\int_0^t w_\alpha(u)du\right),$$

where we also used (29). Substituting (61) and (63) in (60), we have another differential equation for $y(t)$, which, with $y(0) = y_0$, has a unique solution

$$(64) \quad y(t) = y_B(y_0, t),$$

where we used $\sum_{\beta=1}^k u_\beta(y) = 1 - y$ in (26) and (25) with (30). The assumptions for u_α in Theorem 1.4 imply that $y_B(y, t)$ is strictly increasing and differentiable in y , satisfying $y_B(0, t) = y_C(t)$ and $y_B(1-, t) = 1-$. Hence there exists a unique, strictly increasing, differentiable inverse function $\hat{y}(y, t)$, taking values in $[0, 1)$, satisfying

$$y_B(\hat{y}(y, t), t) = y, \quad y_C(t) \leq y < 1, \quad t \geq 0.$$

As in the proof for $y \leq y_C(t)$, this, with (61), (63), and (64), implies (23) for $y_C(t) \leq y < 1$.

This completes a proof of Theorem 1.4. \square

4. Scaling limit results uniform in time. Let $T > 0$ and

$$(65) \quad \mathcal{I} = \{r_i^{(N)} : [0, T] \rightarrow \mathbf{R}_+ ; i = 1, 2, \dots, N, N \in \mathbf{N}\}$$

be a set of continuous functions on $[0, T]$ defined by $r_i^{(N)}(t) = \rho_i^{(N)}((0, t], t) \geq 0$. Note that since we assumed in the beginning that $\rho_i^{(N)}$ is continuous, $r_i^{(N)}$ is continuous. In this section, we prove the following.

THEOREM 4.1. *Let $T > 0$. In addition to the assumptions in Proposition 1.1, assume that a set of continuous functions \mathcal{I} defined by (65) is uniformly equicontinuous; namely,*

$$(66) \quad \limsup_{\delta \downarrow 0} \sup_{r \in \mathcal{I}, s, t \in [0, T]; |s-t| \leq \delta} |r(s) - r(t)| = 0.$$

Then, $Y_C^{(N)}$ of (10) converges almost surely to y_C of (11) as $N \rightarrow \infty$, as a sequence in the space of continuous functions on $[0, T]$ with supremum norm:

$$(67) \quad \mathbb{P} \left[\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} |Y_C^{(N)}(t) - y_C(t)| = 0 \right] = 1.$$

Assume next that all the assumptions of Theorem 1.3 and (66) hold. Assume also that a set of functions

$$\mathcal{J} = \{r : [0, T] \rightarrow \mathbf{R}_+ ; r(t) = \rho((0, t]), \quad t \in [0, T], \quad \rho \in \text{suppt } \Lambda\}$$

is uniformly equicontinuous, and that for y_A of (19) and y_B of (21), $y_A(t - t_1, t)$ and $y_B(y, t)$ are equicontinuous in (t_1, t) and (y, t) , respectively. Then, $\mu^{(N)}$ of (12) converges almost surely to μ of (18) as $N \rightarrow \infty$, as a sequence in the space of probability measure valued functions $\mu : t \mapsto \mu_t$ with supremum norm.

PROOF. First we assume that the assumptions of Proposition 1.1 and (66) hold. Note that (1) implies that, for $i = 1, 2, \dots, N$, $N = 1, 2, \dots$,

$$v_i^{(N)}((0, t]) - r_i^{(N)}(t) = v_i^{(N)}((0, t]) - \rho_i^{(N)}((0, t]), \quad t \geq 0,$$

is a martingale up to fixed time T . Note also that (3) implies

$$(68) \quad v_i^{(N)}((0, t \wedge \tau_{i,1}^{(N)}]) = \mathbf{1}_{\tau_{i,1}^{(N)} \leq t}.$$

Hence

$$(69) \quad W_i^{(N)}(t) := \mathbf{1}_{\tau_{i,1}^{(N)} \leq t} - r_i^{(N)}(t \wedge \tau_{i,1}^{(N)}), \quad t \in [0, T],$$

is a bounded martingale. This with (10) further implies that

$$Y_C^N(t) - \frac{1}{N} \sum_{i=1}^N r_i^{(N)}(t \wedge \tau_{i,1}^{(N)}) = \frac{1}{N} \sum_{i=1}^N (\mathbf{1}_{\tau_{i,1}^{(N)} \leq t} - r_i^{(N)}(t \wedge \tau_{i,1}^{(N)})) = \frac{1}{N} \sum_{i=1}^N W_i^{(N)}(t)$$

is also a bounded martingale. Using Doob's inequality, independence of $\{\tau_{i,1}^{(N)} ; i = 1, 2, \dots, N\}$, and $|W_i^{(N)}(T)| \leq 1$, we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\frac{1}{N} \sum_{i=1}^N W_i^{(N)}(t) \right)^4 \right] \leq \frac{4^4}{3^4} \mathbb{E} \left[\left(\frac{1}{N} \sum_{i=1}^N W_i^{(N)}(T) \right)^4 \right] \leq \frac{4^4}{3^3 N^2}.$$

With an argument similar to that in the proof of Proposition 1.1,

$$(70) \quad \sup_{0 \leq t \leq T} \left| \frac{1}{N} \sum_{i=1}^N W_i^{(N)}(t) \right| \rightarrow 0, \quad \text{a.e., as } N \rightarrow \infty.$$

On the other hand, for each $0 \leq t \leq T$, as in the proof of Proposition 1.1, independence and boundedness of $r_i^{(N)}(t \wedge \tau_{i,1}^{(N)})$, $i = 1, 2, \dots, N$, imply

$$(71) \quad \frac{1}{N} \sum_{i=1}^N r_i^{(N)}(t \wedge \tau_{i,1}^{(N)}) - \frac{1}{N} \sum_{i=1}^N \mathbb{E}[r_i^{(N)}(t \wedge \tau_{i,1}^{(N)})] \rightarrow 0, \quad \text{a.e., as } N \rightarrow \infty,$$

and

$$\mathbb{E}[r_i^{(N)}(t \wedge \tau_{i,1}^{(N)})] = \mathbb{E}[\mathbf{1}_{\tau_{i,1}^{(N)} \leq t}] - \mathbb{E}[W_i^{(N)}(t)] = \mathbb{E}[\mathbf{1}_{\tau_{i,1}^{(N)} \leq t}]$$

implies

$$(72) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E}[r_i^{(N)}(t \wedge \tau_{i,1}^{(N)})] = \lim_{N \rightarrow \infty} \mathbb{E}[Y_C^{(N)}(t)] = y_C(t).$$

Since $Y_C^{(N)}(t)$ is non-decreasing in t , and $y_C(t)$ is its pointwise limit, it is also non-decreasing. As in the case of Corollary 1.2, (71) and (72) imply that, with probability one,

$$(73) \quad \frac{1}{N} \sum_{i=1}^N r_i^{(N)}(t \wedge \tau_{i,1}^{(N)}) \rightarrow y_C(t), \quad t \in \mathcal{Q} \cap [0, T], \quad \text{as } N \rightarrow \infty.$$

Since $\{r_i^{(N)}\}$ is equicontinuous, (73) implies that y_C is continuous on rationals, and the monotonicity of y_C proves that it is continuous on $[0, T]$.

By assumption of equicontinuity and the convergence (73) on a dense subset of $[0, T]$, it follows that the convergence is uniform:

$$(74) \quad \sup_{t \in [0, T]} \left| \frac{1}{N} \sum_{i=1}^N r_i^{(N)}(t \wedge \tau_{i,1}^{(N)}) - y_C(t) \right| \rightarrow 0, \quad \text{a.e., as } N \rightarrow \infty.$$

The equations (10), (69), (70), (74) prove (67).

In the remainder of this section, we assume that the assumptions of Theorem 4.1 hold. To prove uniform convergence of $\mu_t^{(N)}$, we first prepare t -uniform version of Lemma 2.1.

LEMMA 4.2. *If, for each $y \in [0, 1)$ and for each bounded continuous function $g : \mathcal{M}(\mathbf{R}_+) \rightarrow \mathbf{R}$, there exists $\tilde{\Omega}$ with $\mathbb{P}[\tilde{\Omega}] = 1$ such that, for each $\omega \in \tilde{\Omega}$,*

$$(75) \quad \lim_{N \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \mu_t^{(N)}(d\rho \times [y, 1))(\omega) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \mu_t(d\rho \times [y, 1)) \right| = 0,$$

then $\mu_t^{(N)}$ converges to μ_t uniformly in $t \in [0, T]$ as $N \rightarrow \infty$, almost surely.

PROOF. Let $\mathcal{T} = \{f_n ; n \in \mathbf{N}\}$ be as in the proof of Lemma 2.1, and for probability measures μ and ν on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$, put

$$\begin{aligned} \pi(\mu, \nu) := & \sum_{n=1}^{\infty} 2^{-n} \left(\left| \int_{\mathcal{M}(\mathbf{R}_+) \times [0, 1)} f_n(\rho, y) \mu(d\rho \times dy) \right. \right. \\ & \left. \left. - \int_{\mathcal{M}(\mathbf{R}_+) \times [0, 1)} f_n(\rho, y) \nu(d\rho \times dy) \right| \wedge 1 \right). \end{aligned}$$

Then π is a metric on the space of probability measures on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$, and the convergence with respect to π is equivalent to convergence (32) for each $f_n \in \mathcal{T}$. Hence, as noted just below (32), it is equivalent to weak convergence of the probability measures on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$.

Now assume that (75) holds. Then following the arguments of the proof of Lemma 2.1, replacing (31) by (75), we see that there exists $\Omega' \subset \Omega$ such that $\mathbb{P}[\Omega'] = 1$ and

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_{\mathcal{M}(\mathbf{R}_+) \times [0, 1)} f_n(\rho, y) \mu_t^{(N)}(d\rho \times dy)(\omega) - \int_{\mathcal{M}(\mathbf{R}_+) \times [0, 1)} f_n(\rho, y) \mu_t(d\rho \times dy) \right| = 0,$$

for all $n \in N$ and $\omega \in \Omega'$. Therefore,

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} \pi(\mu_t^{(N)}(\omega), \mu_t) = 0, \quad \omega \in \Omega',$$

which, by the equivalence of convergence in π and the convergence in the weak topology of the space of probability measures on $\mathcal{M}(\mathbf{R}_+) \times [0, 1)$, implies the almost sure uniform convergence in $t \in [0, T]$, of $\mu_t^{(N)}$ to μ_t . \square

In view of Lemma 4.2, we fix $y \in [0, 1)$ and a bounded continuous function g , in the remainder of this section. Note that (40) holds. The assumption $\Lambda^{(N)} \rightarrow \Lambda$ in (14) further implies that for any $K > 0$ there exists a positive integer N_0 such that, for $N > N_0$,

$$(76) \quad \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda(d\rho) \right| = \left| \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda^{(N)}(d\rho) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda(d\rho) \right| < \frac{M}{K}.$$

The following Lemma corresponds to Lemma 2.5.

LEMMA 4.3. For each $t_1 \in [0, T]$, $Y_A^{(N)}$ of (45) and y_A of (19) satisfy

$$(77) \quad \sup_{t \in [t_1, T]} |Y_A^{(N)}(t - t_1, t) - y_A(t - t_1, t)| \rightarrow 0, \quad a.e., \text{ as } N \rightarrow \infty,$$

and for each $y_0 \in [0, 1)$, $Y_B^{(N)}$ of (46) and y_B of (21) satisfy

$$(78) \quad \sup_{t \in [0, T]} |Y_B^{(N)}(y_0, t) - y_B(y_0, t)| \rightarrow 0, \quad a.e., \text{ as } N \rightarrow \infty.$$

PROOF. Define, for $i = 1, 2, \dots, N$, $N = 1, 2, \dots$,

$$\tilde{\tau}_i^{(N)} = \tau_{i, k_i}^{(N)},$$

where $k_i := \inf\{j ; \tau_{i, j}^{(N)} > t_1\}$. Then just as in the proof of (67), we see that

$$V_i^{(N)}(t) := \mathbf{1}_{\tilde{\tau}_i^{(N)} \leq t} - \rho_i^{(N)}((t_1, t \wedge \tilde{\tau}_i^{(N)}]), \quad t \in [t_1, T]$$

and, with (45), accordingly,

$$(79) \quad Y_A^N(t - t_1, t) - \frac{1}{N} \sum_{i=1}^N \rho_i^{(N)}((t_1, t \wedge \tilde{\tau}_i^{(N)}]) = \frac{1}{N} \sum_{i=1}^N V_i^{(N)}(t), \quad t \in [t_1, T]$$

are bounded martingales, and we have

$$(80) \quad \sup_{t_1 \leq t \leq T} \left| \frac{1}{N} \sum_{i=1}^N V_i^{(N)}(t) \right| \rightarrow 0, \quad t \in \mathcal{Q} \cap [0, T], \quad \text{a.e., as } N \rightarrow \infty.$$

On the other hand, we have with probability one,

$$(81) \quad \frac{1}{N} \sum_{i=1}^N \rho_i^{(N)}((t_1, t \wedge \tilde{\tau}_i^{(N)}]) = y_A(t - t_1, t), \quad t \in \mathcal{Q} \cap [0, T], \quad \text{as } N \rightarrow \infty.$$

By assumptions of equicontinuity and the convergence (81) on a dense subset of $[0, T]$, it follows that the convergence is uniform:

$$(82) \quad \sup_{t \in [0, T]} \left| \frac{1}{N} \sum_{i=1}^N \rho_i^{(N)}((t_1, t \wedge \tilde{\tau}_i^{(N)}]) - y_A(t - t_1, t) \right| \rightarrow 0, \quad \text{a.e., as } N \rightarrow \infty.$$

The equations (79), (80) and (82) lead to (77).

A proof of (78) goes in exact correspondence with that of (67), if we directly use the assumption of continuity of y_B in place of monotonicity of y_C . \square

COROLLARY 4.4. *For each $t_1 \in [0, T]$,*

$$(83) \quad \sup_{t \in [t_1, T]} \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{v_i^{(N)}((t_1, t]) > 0} - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) (1 - e^{-\rho((t_1, t])}) \Lambda(d\rho) \right| \rightarrow 0,$$

almost surely as $N \rightarrow \infty$, and for each $y_0 \in [0, 1)$,

$$(84) \quad \sup_{t \in [0, T]} \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{(x_i^{(N)} - 1)/N \geq y_0, \tau_{i,1}^{(N)} > t} - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0, t])} \mu_0(d\rho \times [y_0, 1]) \right| \rightarrow 0,$$

almost surely as $N \rightarrow \infty$.

PROOF. This is proved as in the proof of Lemma 4.3, if one notes (40). \square

Fix a positive integer K arbitrarily. By the assumptions of Theorem 4.1 of uniform equicontinuity of \mathcal{J} , y_A and y_B , and noting that $\mu_0(\mathcal{M}(\mathbf{R}_+) \times [y, 1]) = 1 - y$, there exist a positive integer L and sequences $0 = t_{1,0} < t_{1,1} < \dots < t_{1,L} = T$ and $0 = y_{0,0} < y_{0,1} < \dots < y_{0,L} = 1$ such that

(i) for $j = 0, 1, 2, \dots, M$ and $s \in [t_{1,j-1}, t_{1,j+1}]$,

$$(85) \quad \int_{\mathcal{M}(\mathbf{R}_+)} |e^{-\rho((t_1, j, t])} - e^{-\rho((s, t])}| \Lambda(d\rho) < \frac{1}{K},$$

where, for convenience we put $t_{1,j} = 0$ if $j \leq 0$, and $t_{1,j} = T$ if $j \geq L$, and also for $j = 0, 1, 2, \dots, M$ and $z \in [y_{0,j-1}, y_{0,j+1}]$,

$$(86) \quad \left| \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t))} \mu_0(d\rho \times [y_{0,j}, 1)) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t))} \mu_0(d\rho \times [z, 1)) \right| \\ = \left| \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t))} \mu_0(d\rho \times [y_{0,j} \wedge z, y_{0,j} \vee z)) \right| < \frac{M}{K},$$

where, we put $y_{0,j} = 0$ if $j \leq 0$, and $y_{0,j} = 1$ if $j \geq L$,

(ii) the sequences of functions $y_{A,j}(t) = y_A((t-t_{1,j}) \vee 0, t)$, $j = 0, 1, 2, \dots, L$, which is decreasing in j , and $y_{B,j}(t) = y_B(y_{0,j}, t)$, $j = 0, 1, 2, \dots, L-1$, which is increasing in j , satisfy

$$(87) \quad 0 \leq y_{A,j}(t) - y_{A,j+1}(t) < \frac{1}{K}, \quad j = 0, 1, 2, \dots, L-1, \quad t \in [0, T],$$

and

$$(88) \quad 0 \leq y_{B,j+1}(t) - y_{B,j}(t) < \frac{1}{K}, \quad j = 0, 1, 2, \dots, L-2, \quad t \in [0, T].$$

Lemma 4.3 and Corollary 4.4 imply that there exists $\tilde{\Omega}_K \subset \Omega$, satisfying $\mathbb{P}[\tilde{\Omega}_K] = 1$, such that for all $\omega \in \tilde{\Omega}_K$ there exists an integer $N_0 = N_0(\omega)$ such that if $N > N_0$ then

$$(89) \quad |Y_A^{(N)}(t - t_{1,j}, t)(\omega) - y_{A,j}(t)| < \frac{1}{K}, \quad t \in [t_{1,j}, T], \quad j = 0, 1, \dots, L,$$

$$(90) \quad |Y_B^{(N)}(y_{0,j}, t)(\omega) - y_{B,j}(t)| < \frac{1}{K}, \quad j = 0, 1, \dots, L, \quad t \in [0, T],$$

$$(91) \quad \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{v_i^{(N)}((t_{1,j}, t)) > 0}(\omega) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) (1 - e^{-\rho((t_{1,j}, t))}) \Lambda(d\rho) \right| \\ < \frac{M}{K}, \quad t \in [t_{1,j}, T], \quad j = 0, 1, \dots, L,$$

and

$$(92) \quad \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{(x_i^{(N)} - 1)/N \geq y_{0,j}, \tau_{i,1}^{(N)} > t}(\omega) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0,t))} \right. \\ \left. \times \mu_0(d\rho \times [y_{0,j}, 1)) \right| < \frac{M}{K}, \quad j = 0, 1, \dots, L-1, \quad t \in [0, T].$$

Now, we shall consider the case $y_C(t) \geq y$ and the case $y_C(t) \leq y$ separately. First, let $y_C(t) \geq y$, and let $j = j(t)$ be the integer such that

$$(93) \quad y_{A,j}(t) \leq y < y_{A,j-1}(t).$$

Note that $y_C(t) \geq y$ implies $y = y_A(t_0(y, t), t)$ (see (47)), with which $y_{A,0}(t) = y_A(t, t) = y_C(t)$, $y_{A,L}(t) = y_A(0, t) = 0$, and monotonicity of $y_A(t_0, t)$ with respect to t_0 imply that

such an integer $j = j(t)$ exists if $y_C(t) \geq y$. Since $y_A(t_0, t)$ is increasing in t_0 , (93) also implies

$$(94) \quad t_{1,j-1} < t - t_0(y, t) \leq t_{1,j}.$$

Since (87) implies

$$0 \leq y - y_{A,j}(t) \leq y_{A,j-1}(t) - y_{A,j}(t) < \frac{1}{K},$$

with (89) and a similar argument as for (50), we have

$$(95) \quad \begin{aligned} & \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) (\mathbf{1}_{Y_i^{(N)}(t) < y} - \mathbf{1}_{Y_i^{(N)}(t) < Y_A^{(N)}(t-t_{1,j},t)}) (\omega) \right| \\ & \leq \frac{1}{N} \sum_{i=1}^N |g(\rho_i^{(N)})| \left| \mathbf{1}_{Y_i^{(N)}(t) < y} - \mathbf{1}_{Y_i^{(N)}(t) < y_{A,j}(t)} \right| (\omega) \\ & \quad + \frac{1}{N} \sum_{i=1}^N |g(\rho_i^{(N)})| \left| \mathbf{1}_{Y_i^{(N)}(t) < y_{A,j}(t)} - \mathbf{1}_{Y_i^{(N)}(t) < Y_A^{(N)}(t-t_{1,j},t)} \right| (\omega) \\ & \leq M(y - y_{A,j}(t)) + M|Y_A^{(N)}(t - t_{1,j}, t)(\omega) - y_{A,j}(t)| < \frac{2M}{K}. \end{aligned}$$

Note also that, as in the argument for (49),

$$(96) \quad \mathbf{1}_{v_i^{(N)}((t_{1,j},t]) > 0} = \mathbf{1}_{Y_i^{(N)}(t) < Y_A^{(N)}(t-t_{1,j},t)}.$$

Adding up (76), (95), (91) and (85), and using (96) and triangular inequality, we arrive at

$$(97) \quad \begin{aligned} & \sup_{t \in [0, T]; y_C(t) \geq y} \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{Y_i^{(N)}(t) \geq y} (\omega) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((t-t_0(y,t),t])} \Lambda(d\rho) \right| \\ & \leq \sup_{t \in [0, T]; y_C(t) \geq y} \left| \left(\frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda(d\rho) \right) \right. \\ & \quad \left. - \left(\frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{Y_i^{(N)}(t) < y} (\omega) \right) \right. \\ & \quad \left. - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) (1 - e^{-\rho((t-t_0(y,t),t])}) \Lambda(d\rho) \right) \Big| \leq \frac{5M}{K}, \end{aligned}$$

for $\omega \in \tilde{\Omega}_K$ and $N > N_0(\omega)$.

Next, let $y_C(t) \leq y$, and let $j = j(t)$ be the integer such that

$$(98) \quad y_{B,j}(t) \leq y < y_{B,j+1}(t).$$

With an argument similar as that below (93), such an integer $j = j(t)$ exists if $y_C(t) \leq y$. Since $y_B(y_0, t)$ is increasing in y_0 , (98) also implies

$$(99) \quad y_{0,j} < \hat{y}(y, t) \leq y_{0,j+1}.$$

Since (88) implies

$$0 \leq y - y_{B,j}(t) \leq y_{B,j+1}(t) - y_{B,j}(t) < \frac{1}{K},$$

with (90) and a similar argument as for (54), we have

$$(100) \quad \begin{aligned} & \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) (\mathbf{1}_{Y_i^{(N)}(t) \geq y} - \mathbf{1}_{Y_i^{(N)}(t) \geq Y_B^{(N)}(y_{0,j}, t)}) (\omega) \right| \\ & \leq \frac{1}{N} \sum_{i=1}^N |g(\rho_i^{(N)})| \left| \mathbf{1}_{Y_i^{(N)}(t) \geq y} - \mathbf{1}_{Y_i^{(N)}(t) \geq y_{B,j}(t)} \right| (\omega) \\ & \quad + \frac{1}{N} \sum_{i=1}^N |g(\rho_i^{(N)})| \left| \mathbf{1}_{Y_i^{(N)}(t) \geq y_{B,j}(t)} - \mathbf{1}_{Y_i^{(N)}(t) \geq Y_B^{(N)}(y_{0,j}, t)} \right| (\omega) \\ & \leq M(y - y_{B,j}(t)) + M|Y_B^{(N)}(y_{0,j}, t)(\omega) - y_{B,j}(t)| < \frac{2M}{K}. \end{aligned}$$

Note also that, as in the argument for (53),

$$(101) \quad \mathbf{1}_{(x_i^{(N)} - 1)/N \geq y_{0,j}, \tau_{i,1}^{(N)} > t} = \mathbf{1}_{Y_i^{(N)}(t) \geq Y_B^{(N)}(y_{0,j}, t)}.$$

Adding up (100), (92) and (86), and using (101) and triangular inequality, we arrive at

$$(102) \quad \sup_{t \in [0, T]; y_{C(t)} \leq y} \left| \frac{1}{N} \sum_{i=1}^N g(\rho_i^{(N)}) \mathbf{1}_{Y_i^{(N)}(t) \geq y} (\omega) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) e^{-\rho((0, t])} \times \mu_0(d\rho \times [\hat{y}(y, t), 1]) \right| \leq \frac{4M}{K},$$

for $\omega \in \tilde{\Omega}_K$ and $N > N_0(\omega)$.

Combining (97) and (102), we have

$$\sup_{t \in [0, T]} \left| \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \mu_t^{(N)}(d\rho \times [y, 1]) (\omega) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \mu_t(d\rho \times [y, 1]) \right| \leq \frac{5M}{K},$$

$N > N_0(\omega)$, $\omega \in \tilde{\Omega}_K$.

Finally, put $\tilde{\Omega} = \bigcap_{K=1}^{\infty} \tilde{\Omega}_K$. Then $\mathbb{P}[\tilde{\Omega}] = 1$. Let $\omega \in \tilde{\Omega}$. For any $\varepsilon > 0$ take an integer K such that $K > 5M/\varepsilon$. Then $\omega \in \tilde{\Omega} \subset \tilde{\Omega}_K$ implies

$$\sup_{t \in [0, T]} \left| \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \mu_t^{(N)}(d\rho \times [y, 1]) (\omega) - \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \mu_t(d\rho \times [y, 1]) \right| \leq \frac{5M}{K} < \varepsilon,$$

for $N > N_0(\omega)$, which implies (75), and therefore Lemma 4.2 implies the Theorem. \square

5. Case when the intensities have common time dependence. To consider the case where the intensity measure ρ has a density, denote the set of locally integrable functions on \mathbf{R}_+ by $L_{\text{loc}}^1(\mathbf{R}_+)$. $L_{\text{loc}}^1(\mathbf{R}_+)$ is a complete separable metric space. Let ι be a map $\iota :$

$L^1_{\text{loc}}(\mathbf{R}_+) \rightarrow \mathcal{M}(\mathbf{R}_+)$ which maps $\tilde{w} \in L^1_{\text{loc}}(\mathbf{R}_+)$ to the measure on \mathbf{R}_+ with density \tilde{w} determined by

$$(103) \quad \iota(\tilde{w})([s, t]) = \int_s^t \tilde{w}(u) du, \quad 0 \leq s < t.$$

PROPOSITION 5.1. *Assume that $\tilde{w}_i^{(N)} \in L^1_{\text{loc}}(\mathbf{R}_+)$, $i = 1, 2, \dots, N$, $N = 1, 2, \dots$, and for each N , put*

$$\tilde{\Lambda}^{(N)} = \frac{1}{N} \sum_{i=1}^N \delta_{\tilde{w}_i^{(N)}}.$$

If there exists a probability distribution $\tilde{\Lambda}$ on $L^1_{\text{loc}}(\mathbf{R}_+)$ such that $\tilde{\Lambda}^{(N)}$ converges weakly to $\tilde{\Lambda}$ as $N \rightarrow \infty$, then the sequence of distribution $\Lambda^{(N)}$, $N = 1, 2, \dots$, on the set of intensity measures $\mathcal{M}(\mathbf{R}_+)$ defined by $\Lambda^{(N)} = \tilde{\Lambda}^{(N)} \circ \iota^{-1}$, with ι as in (103), converges weakly as $N \rightarrow \infty$ to $\Lambda := \tilde{\Lambda} \circ \iota^{-1}$. Moreover, for all $0 \leq s < t$, $\lambda_{s,t}^{(N)}$ defined by (16) converges weakly as $N \rightarrow \infty$ to $\lambda_{s,t}$ defined by (17).

PROOF. Let $g : \mathcal{M}(\mathbf{R}_+) \rightarrow \mathbf{R}$ be a bounded continuous function on $\mathcal{M}(\mathbf{R}_+)$. Then the definitions imply

$$\int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda^{(N)}(d\rho) = \int_{L^1_{\text{loc}}(\mathbf{R}_+)} g(\iota(\tilde{w})) \tilde{\Lambda}^{(N)}(d\tilde{w}).$$

Let $\{\tilde{w}_n\}$ be a sequence converging in $L^1_{\text{loc}}(\mathbf{R}_+)$ to \tilde{w} , and let $f : \mathbf{R}_+ \rightarrow \mathbf{R}$ be a continuous function with compact support: $f(u) = 0$, $u \geq k$, for some integer k . Then f is bounded: $|f(u)| \leq M$, $u \in \mathbf{R}_+$, for some M . Hence

$$\begin{aligned} \left| \int_{\mathbf{R}_+} f(u) \tilde{w}_n(u) du - \int_{\mathbf{R}_+} f(u) \tilde{w}(u) du \right| &= \left| \int_0^k f(u) \tilde{w}_n(u) du - \int_0^k f(u) \tilde{w}(u) du \right| \\ &\leq M \int_0^k |\tilde{w}_n(u) - \tilde{w}(u)| du \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

This holds for all continuous function f with compact support, hence $\lim_{n \rightarrow \infty} \iota(\tilde{w}_n) = \iota(\tilde{w})$ in vague topology, which further implies

$$\lim_{n \rightarrow \infty} g(\iota(\tilde{w}_n)) = g(\iota(\tilde{w})).$$

This proves that $g \circ \iota : L^1_{\text{loc}}(\mathbf{R}_+) \rightarrow \mathbf{R}$ is a bounded continuous function, hence the assumption $\tilde{\Lambda}^{(N)} \rightarrow \tilde{\Lambda}$ implies

$$\lim_{N \rightarrow \infty} \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda^{(N)}(d\rho) = \int_{\mathcal{M}(\mathbf{R}_+)} g(\rho) \Lambda(d\rho).$$

This holds for any bounded continuous function g , which proves $\Lambda^{(N)} \rightarrow \Lambda$, weakly as $N \rightarrow \infty$.

Let $t > s > 0$ and put $b[\tilde{w}] = \int_s^t \tilde{w}(u) du$. In a similar way as above, the definitions imply

$$\lambda_{s,t}^{(N)} = \int_{L_{\text{loc}}^1(\mathbf{R}_+)} \delta_{b[\tilde{w}]} \tilde{\Lambda}^{(N)}(d\tilde{w}) \quad \text{and} \quad \lambda_{s,t} = \int_{L_{\text{loc}}^1(\mathbf{R}_+)} \delta_{b[\tilde{w}]} \tilde{\Lambda}(d\tilde{w}).$$

Let $h : \mathbf{R}_+ \rightarrow \mathbf{R}$ be a bounded continuous function. Then the map

$$L_{\text{loc}}^1(\mathbf{R}_+) \ni \tilde{w} \mapsto h(b[\tilde{w}]) \in \mathbf{R}$$

is bounded and continuous, hence the assumption $\tilde{\Lambda}^{(N)} \rightarrow \tilde{\Lambda}$ implies

$$\begin{aligned} \int_{\mathbf{R}_+} h(w) \lambda_{s,t}^{(N)}(dw) &= \int_{L_{\text{loc}}^1(\mathbf{R}_+)} h(b[\tilde{w}]) \tilde{\Lambda}^{(N)}(d\tilde{w}) \\ &\rightarrow \int_{L_{\text{loc}}^1(\mathbf{R}_+)} h(b[\tilde{w}]) \tilde{\Lambda}(d\tilde{w}) = \int_{\mathbf{R}_+} h(w) \lambda_{s,t}(dw), \quad N \rightarrow \infty, \end{aligned}$$

hence $\lambda_{s,t}^{(N)} \rightarrow \lambda_{s,t}$, weakly as $N \rightarrow \infty$. □

Proposition 5.1 implies that the assumption (17) in Theorem 1.3 is redundant if the intensity measures have densities.

For the rest of this section, we further assume a common time dependence for all $\tilde{w}_i^{(N)}$ in Proposition 5.1. Namely, we assume that there exist $\tilde{a} \in L_{\text{loc}}^1(\mathbf{R}_+)$ and positive constants

$$w_i^{(N)} > 0, \quad i = 1, 2, \dots, N, \quad N = 1, 2, \dots,$$

such that the intensity measure of the Poisson random measures $v_i^{(N)}$ in the stochastic ranking process (2) is given by

$$(104) \quad \rho_i^{(N)}((s, t]) = w_i^{(N)} \int_s^t \tilde{a}(u) du, \quad i = 1, 2, \dots, N, \quad N = 1, 2, \dots$$

As in the proof of Proposition 5.1, we have

COROLLARY 5.2. *Let $\tilde{a} \in L_{\text{loc}}^1(\mathbf{R}_+)$. If there exists a probability distribution λ on \mathbf{R}_+ such that*

$$(105) \quad \lambda^{(N)} := \frac{1}{N} \sum_{i=1}^N \delta_{w_i^{(N)}} \rightarrow \lambda, \quad \text{weakly, as } N \rightarrow \infty,$$

then a sequence of probability distributions $\tilde{\Lambda}^{(N)}$, $N = 1, 2, \dots$, on $L_{\text{loc}}^1(\mathbf{R}_+)$ defined by

$$\tilde{\Lambda}^{(N)} = \int_{\mathbf{R}_+} \delta_{w\tilde{a}} \lambda_i^{(N)}(dw) = \frac{1}{N} \sum_{i=1}^N \delta_{w_i^{(N)}\tilde{a}}$$

converges weakly to a probability distribution $\tilde{\Lambda} = \int_{\mathbf{R}_+} \delta_{w\tilde{a}} \lambda(dw)$, as $N \rightarrow \infty$.

In particular, Proposition 1.1 holds with $\rho_i^{(N)}((s, t]) = w_i^{(N)} \int_s^t \tilde{a}(u) du$, and $y_C(t)$ of (11) is given by

$$(106) \quad y_C(t) = 1 - \int_{\mathbf{R}_+} e^{-w A(t)} \lambda(dw),$$

where

$$(107) \quad A(t) = \int_0^t \tilde{a}(u) du .$$

The formula (106) is to be compared with the case of the (homogeneous) Poisson process in [12, Proposition 2], where we have

$$(108) \quad y_C(t) = 1 - \int_{\mathbf{R}_+} e^{-wt} \lambda(dw) .$$

λ in (108) is the (infinite particle limit asymptotic) distribution of jump rates, while λ in the case of common time dependence (106) is the distribution of relative jump rates.

To study a time change according to the common intensity measure, let us first make a heuristic observation. Suppose we could trace the trajectories of $n \leq N$ particles j_1, j_2, \dots, j_n . The total number of jumps of the n particles in the time interval $(0, t]$ is given by

$$(109) \quad S^{(N,n)}(t) = \sum_{i=1}^n v_{j_i}^{(N)}((0, t]) .$$

If n is large ($n \gg 1$), we expect as a consequence of the law of large numbers, as in Proposition 1.1,

$$(110) \quad S^{(N,n)}(t) \simeq \sum_{i=1}^n \rho_{j_i}^{(N)}((0, t]) = A(t) Z(N, n) ,$$

where we put

$$(111) \quad Z(N, n) = \sum_{i=1}^n w_{j_i}^{(N)} ,$$

and also used (104) and (107). Using (110) in (106), we have

$$(112) \quad y_C(t) \simeq 1 - \int_{\mathbf{R}_+} e^{-w S^{(N,n)}(t)/Z(N, n)} \lambda(dw) .$$

The approximate formula (112) suggests that, if we perform a time change $t' = S^{(N,n)}(t)$, then modulo scaling constant $Z(N, n)$, we recover a formula (108) for the homogeneous case.

We can put the heuristic consideration which lead to (112) in a mathematically precise form. For $t \geq 0$, let

$$(113) \quad S^{(N)}(t) = \sum_{i=1}^N v_i^{(N)}((0, t])$$

and denote its right continuous inverse by

$$(114) \quad s^{(N)}(t) = \inf\{s \geq 0 ; S^{(N)}(s) > t\} .$$

Let $\tilde{a} \in L^1_{\text{loc}}(\mathbf{R}_+)$. For simplicity, assume further that

$$(115) \quad \tilde{a}(t) > 0, \quad t \geq 0 .$$

Then $A(t)$ of (107) is strictly increasing, and the inverse function A^{-1} is also continuous.

THEOREM 5.3. *Let $\tilde{a} \in L^1_{\text{loc}}(\mathbf{R}_+)$, and assume (115). Put*

$$(116) \quad Z(N) = \sum_{i=1}^N w_i^{(N)}$$

and assume

$$(117) \quad \lim_{N \rightarrow \infty} Z(N) = \infty.$$

If, as in Corollary 5.2, there exists a probability distribution λ on \mathbf{R}_+ such that (105) holds, then for each $t \geq 0$

$$(118) \quad Y_C^{(N)}(s^{(N)}(Z(N)t)) \rightarrow y_C(A^{-1}(t)) = 1 - \int_{\mathbf{R}_+} e^{-wt} \lambda(dw),$$

in probability, as $N \rightarrow \infty$,

where $Y_C^{(N)}$ is defined in (10).

To prove Theorem 5.3, we first provide a rigorous version of (110).

LEMMA 5.4. *For $t \geq 0$,*

$$(119) \quad \frac{1}{Z(N)} S^{(N)}(t) \rightarrow A(t), \quad \text{in probability, as } N \rightarrow \infty,$$

and

$$(120) \quad s^{(N)}(Z(N)t) \rightarrow A^{-1}(t), \quad \text{in probability, as } N \rightarrow \infty.$$

PROOF. Since by definition $\nu_i^{(N)}((0, t])$ follows the Poisson distribution with expectation $\rho_i^{(N)}((0, t])$, we have

$$(121) \quad \mathbb{E}[S^{(N)}(t)] = \mathbb{V}[S^{(N)}(t)] = A(t) Z(N),$$

where $\mathbb{V}[\cdot]$ denotes variance. For $\varepsilon > 0$, (121), (116), and Chebyshev's inequality imply

$$[|S^{(N)}(t) - \mathbb{E}[S^{(N)}(t)]| > Z(N)\varepsilon] \leq (\varepsilon Z(N))^{-2} \mathbb{V}[S^{(N)}(t)] = \frac{A(t)}{\varepsilon^2 Z(N)},$$

which, with (117), implies

$$\frac{1}{Z(N)} (S^{(N)}(t) - \mathbb{E}[S^{(N)}(t)]) \rightarrow 0, \quad \text{in probability, as } N \rightarrow \infty.$$

This, with (121), implies (119).

Next, noting that $S^{(N)}(t)$ is non-decreasing in t , (114) implies

$$(122) \quad \left\{ \begin{aligned} &\omega \in \Omega ; s^{(N)}(Z(N)t)(\omega) \geq A^{-1}(t) + \varepsilon \\ &\subset \left\{ \omega \in \Omega ; \frac{1}{Z(N)} S^{(N)}\left(A^{-1}(t) + \frac{\varepsilon}{2}\right)(\omega) \leq t \right\}. \end{aligned} \right.$$

The assumption (115) implies that A is strictly increasing, hence, $\delta = A(A^{-1}(t) + \varepsilon/2) - t > 0$, and

$$\begin{aligned} & \left\{ \omega \in \Omega ; \frac{1}{Z(N)} S^{(N)} \left(A^{-1}(t) + \frac{\varepsilon}{2} \right) \leq t \right\} \\ &= \left\{ \omega \in \Omega ; \frac{1}{Z(N)} S^{(N)} \left(A^{-1}(t) + \frac{\varepsilon}{2} \right) \leq A \left(A^{-1}(t) + \frac{\varepsilon}{2} \right) - \delta \right\} \\ &\subset \left\{ \omega \in \Omega ; \left| \frac{1}{Z(N)} S^{(N)} \left(A^{-1}(t) + \frac{\varepsilon}{2} \right) - A \left(A^{-1}(t) + \frac{\varepsilon}{2} \right) \right| \geq \delta \right\}. \end{aligned}$$

This and (119) and (122) imply

$$(123) \quad \lim_{N \rightarrow \infty} \mathbb{P}[s^{(N)}(Z(N)t) \geq A^{-1}(t) + \varepsilon] = 0.$$

Similarly, $\delta' = t - A(A^{-1}(t) - \varepsilon/2) > 0$, and

$$\begin{aligned} & \left\{ \omega \in \Omega ; s^{(N)}(Z(N)t) \leq A^{-1}(t) - \varepsilon \right\} \\ &\subset \left\{ \omega \in \Omega ; \frac{1}{Z(N)} S^{(N)} \left(A^{-1}(t) - \frac{\varepsilon}{2} \right) \geq t \right\} \\ &\subset \left\{ \omega \in \Omega ; \left| \frac{1}{Z(N)} S^{(N)} \left(A^{-1}(t) - \frac{\varepsilon}{2} \right) - A \left(A^{-1}(t) - \frac{\varepsilon}{2} \right) \right| \geq \delta' \right\}, \end{aligned}$$

which implies

$$(124) \quad \lim_{N \rightarrow \infty} \mathbb{P}[s^{(N)}(Z(N)t) \leq A^{-1}(t) - \varepsilon] = 0.$$

(123) and (124) prove (120). □

PROOF OF THEOREM 5.3. By triangular inequality, we have

$$\begin{aligned} & |Y_C^{(N)}(s^{(N)}(Z(N)t)) - y_C(A^{-1}(t))| \\ &\leq |Y_C^{(N)}(s^{(N)}(Z(N)t)) - Y_C^{(N)}(A^{-1}(t))| + |Y_C^{(N)}(A^{-1}(t)) - y_C(A^{-1}(t))|. \end{aligned}$$

Corollary 5.2 implies that the second term in the right-hand side converges to 0 in probability as $N \rightarrow \infty$, so it suffices to prove that, for all $\varepsilon > 0$,

$$(125) \quad \lim_{N \rightarrow \infty} \mathbb{P}[|Y_C^{(N)}(s^{(N)}(Z(N)t)) - Y_C^{(N)}(A^{-1}(t))| \geq \varepsilon] = 0$$

holds.

For $\delta > 0$ put

$$(126) \quad \Omega_\delta^{(N)} := \{ \omega \in \Omega ; |s^{(N)}(Z(N)t)(\omega) - A^{-1}(t)| < \delta \}.$$

Then (120) implies

$$(127) \quad \lim_{N \rightarrow \infty} \mathbb{P}[\Omega_\delta^{(N)c}] = 0.$$

The definition (10) of $Y_C^{(N)}$ implies

$$(128) \quad \begin{aligned} & |Y_C^{(N)}(s^{(N)}(Z(N)t)) - Y_C^{(N)}(A^{-1}(t))| \\ &= \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{s^{(N)}(Z(N)t) < \tau_{i,1}^{(N)} \leq A^{-1}(t)} + \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{A^{-1}(t) < \tau_{i,1}^{(N)} \leq s^{(N)}(Z(N)t)}. \end{aligned}$$

Combining (126) and (128), we have

$$\begin{aligned} & \mathbb{P}[|Y_C^{(N)}(s^{(N)}(Z(N)t)) - Y_C^{(N)}(A^{-1}(t))| \geq \varepsilon, \Omega_\delta] \\ & \leq \mathbb{P}\left[\sum_{i=1}^N \mathbf{1}_{\tau_{i,1}^{(N)} \in (A^{-1}(t)-\delta, A^{-1}(t)+\delta)} \geq N\varepsilon\right]. \end{aligned}$$

Applying Chebyshev's inequality, we further have

$$\begin{aligned} & \mathbb{P}[|Y_C^{(N)}(s^{(N)}(Z(N)t)) - Y_C^{(N)}(A^{-1}(t))| \geq \varepsilon, \Omega_\delta] \\ & \leq \frac{1}{N\varepsilon} \sum_{i=1}^N \mathbb{E}[\mathbf{1}_{\tau_{i,1}^{(N)} \in (A^{-1}(t)-\delta, A^{-1}(t)+\delta)}] \\ & = \frac{1}{N\varepsilon} \sum_{i=1}^N (e^{-A(A^{-1}(t)-\delta)w_i^{(N)}} - e^{-A(A^{-1}(t)+\delta)w_i^{(N)}}) \\ & = \frac{1}{\varepsilon} \int_{\mathbf{R}_+} (e^{-A(A^{-1}(t)-\delta)w} - e^{-A(A^{-1}(t)+\delta)w}) \lambda^{(N)}(dw). \end{aligned}$$

This, with (127) and the assumption (105), implies

$$\begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \mathbb{P}[|Y_C^{(N)}(s^{(N)}(Z(N)t)) - Y_C^{(N)}(A^{-1}(t))| \geq \varepsilon] \\ & \leq \overline{\lim}_{N \rightarrow \infty} \mathbb{P}[\Omega_\delta^{(N)c}] + \overline{\lim}_{N \rightarrow \infty} \mathbb{P}[|Y_C^{(N)}(s^{(N)}(Z(N)t)) - Y_C^{(N)}(A^{-1}(t))| \geq \varepsilon, \Omega_\delta^{(N)}] \\ & \leq \frac{1}{\varepsilon} \int_{\mathbf{R}_+} (e^{-A(A^{-1}(t)-\delta)w} - e^{-A(A^{-1}(t)+\delta)w}) \lambda(dw). \end{aligned}$$

This holds for all $\delta > 0$, hence the bounded convergence theorem and the continuity of $A(t)$ imply

$$\begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \mathbb{P}[|Y_C^{(N)}(s^{(N)}(Z(N)t)) - Y_C^{(N)}(A^{-1}(t))| \geq \varepsilon] \\ & \leq \inf_{\delta > 0} \frac{1}{\varepsilon} \int_{\mathbf{R}_+} (e^{-A(A^{-1}(t)-\delta)w} - e^{-A(A^{-1}(t)+\delta)w}) \lambda(dw) \\ & \leq \frac{1}{\varepsilon} \int_{\mathbf{R}_+} \lim_{\delta \downarrow 0} (e^{-A(A^{-1}(t)-\delta)w} - e^{-A(A^{-1}(t)+\delta)w}) \lambda(dw) = 0. \end{aligned}$$

This proves (125), hence Theorem 5.3 is proved. \square

As an explicit example to $Z(N)$ and λ , consider, as in [13, 14], the Zipf's law, which is

$$(129) \quad w_i^{(N)} = a \left(\frac{N}{i} \right)^{1/b}, \quad i = 1, 2, \dots, N,$$

for positive constants a and b . For this choice,

$$(130) \quad Z(N) = \sum_{i=1}^N w_i^{(N)} = (1+o(N^0)) \times \begin{cases} aN \int_0^1 x^{-1/b} dx = \frac{aNb}{b-1} & b > 1, \\ aN \int_{1/N}^1 x^{-1} dx = aN \log N & b = 1, \\ aN^{1/b} \sum_{i=1}^{\infty} \frac{1}{i^{1/b}} = aN^{1/b} \zeta(1/b) & 0 < b < 1. \end{cases}$$

The corresponding $N \rightarrow \infty$ weak limit is the (generalized) Pareto distribution, defined by

$$(131) \quad \lambda([w, \infty)) = \begin{cases} \left(\frac{a}{w}\right)^b & w \geq a, \\ 1 & w < a. \end{cases}$$

With the Pareto distribution (131) for λ , (112) is (for $N = n$)

$$(132) \quad \begin{aligned} x_C(t) &= N y_C(t) + 1 \simeq N - N \int_{\mathbf{R}_+} e^{-w S^{(N)}(t)/Z(N)} \lambda(dw) \\ &= N - b \left(\frac{S^{(N)}(t)}{\zeta_N(1/b)} \right)^b \Gamma \left(-b, \frac{S^{(N)}(t)}{N^{1/b} \zeta_N(1/b)} \right) \\ &= N - N e^{-S^{(N)}(t)/(N^{1/b} \zeta_N(1/b))} \\ &\quad + \left(\frac{S^{(N)}(t)}{\zeta_N(1/b)} \right)^b \Gamma \left(1 - b, \frac{S^{(N)}(t)}{N^{1/b} \zeta_N(1/b)} \right) =: x_b^{(N)}(S^{(N)}(t)), \end{aligned}$$

where $\zeta_N(z) = \sum_{i=1}^N i^{-z}$. The last line in (132) is obtained by integration by parts from the second line, as in [14], and is suitable for $0 < b < 1$. Note that the parameter a in the Pareto distribution (131) disappears in the time changed formula (132).

A. Remarks on practical application. In [13, 14], the mathematical results on the stochastic ranking processes has been successfully applied to practical data, such as ranking data of books at an online bookstore Amazon.co.jp [14, 13] and list of subject titles at a collected bulletin board 2ch.net [13].

One may wonder why such a simple rule as the move-to-front rule could be observed in actual social activities. An explanation is that the ranking numbers on the web (such as those representing the books, in the case of online bookstores) usually seek to align the web pages in the order of *current popularity* of the pages. A social impact of the development of web-based activities is that it has become possible to catalog a huge amount of unpopular items [1]. In fact, a majority of books catalogued on an online bookstore are sold less than one copy a month. For such books, any reasonable order reflecting the current popularity would be equal to the order of the time of most recent sales, because the second recent sale of such

book would be long ago, hence would not reflect current popularity. Thus the move-to-front rule will provide a simple but *universal* model in the rankings on the web.

A ranking of a book at Amazon.co.jp jumps close to top of the ranking whenever the book is sold at Amazon.co.jp [14], and a subject title in the web page for the list of 2ch.net jumps to the top whenever a comment (a ‘response’) concerning the subject is written [13]. Ordering a book and responding to a subject are social activities which naturally are expected to contain day-night difference in the intensity.

Explicit time dependence, reflecting day-night difference of social activities, are observed in actual data. Let us regard such time dependence as the non-uniformity of intensity measures $\rho_i^{(N)}$. $\rho_i^{(N)}$ are usually unknown quantities to be determined statistically from observed data. We then have to consider both particle dependence and time dependence in the statistical analysis of the practical data. The assumption of common time dependence (104) developed in Section 5 provides a simple way to take day-night-difference of social activity into account, in applying the stochastic ranking process with inhomogeneous intensity.

A.1. Factorization of day-night social activity difference. In [14], a data taken during the period of about 3 months at Amazon.co.jp is used to statistically obtain λ , based on (108). The data was taken manually in the year 2007, at 21:00 each day. We can show that in the case of common time dependence assumption (104), we can ‘factorize’ periodic time dependence of \tilde{a} , and that the use of (108) in [14, 13] is justified in obtaining λ from data with periodic time dependence. In fact, assume that there exists a positive constant T such that

$$(133) \quad \tilde{a}(t + T) = \tilde{a}(t), \quad t \geq 0.$$

We may normalize $w_i^{(N)}$'s in (104) so that

$$(134) \quad \frac{1}{T} \int_0^T \tilde{a}(u) du = 1$$

holds. Then (133) and (134) imply $\int_t^{t+T} (\tilde{a}(u) - 1) du = 0$, so that

$$(135) \quad A_p(t) := A(t) - t = \int_0^t (\tilde{a}(u) - 1) du$$

is a periodic function with period T , and (106) is

$$(136) \quad y_C(t) = 1 - \int_{\mathbf{R}_+} e^{-w(t+A_p(t))} \lambda(dw).$$

If we collect data at each fixed time of the day, at $t_n = t_0 + nT$, $n = 0, 1, 2, \dots$, then (136) implies

$$(137) \quad y_C(t_n) = 1 - \int_{\mathbf{R}_+} e^{-w(nT+t_0+A_p(t_0))} \lambda(dw).$$

Hence the effect of day-night difference in \tilde{a} is absorbed in the translation of origin of time $t_0 \mapsto t_0 + A_p(t_0)$, and the use of formula (108) for the constant intensity is justified.

A consideration of this subsection is of practical use when one has a data much longer than 24 hours, as in the case of [14].

A.2. Time change according to intensity measure. In [13], a data of list of subject ('thread') titles at a collected bulletin board 2ch.net is statistically analyzed using stochastic ranking process. In [13] the data was collected from a short period in the daytime, and the problem of day-night activity difference was not serious, hence a fit to the formula (108) for the constant jump rate (homogeneous intensity) was possible [13]. However, to study data of longer periods for sharper statistical results, effects of day-night activity difference need to be taken into account.

In applying (104) to the obtained data to extract time dependence (day-night difference), we need to estimate the function \tilde{a} in (104) or A in (107). This is accomplished by making use of (110) and (112). In the case of 2ch.net [13], N in (110) or (112) is about 700, and since full records of transaction are accessible at 2ch.net, it is possible to put n in (112) equal to N and count all the threads' jumps. In the case of Amazon.co.jp, N is of order million, and $n = N$ is unrealistic. Even in such cases, if we observe sufficiently large number of books ($n \gg 1$), we can apply the idea introduced here.

Note that the series $Z(N)$ are approaching their asymptotics in (130) rather slowly for the Pareto distribution. Therefore in practical application of (118) with the Pareto distribution for λ , if one takes $N = O(10^3)$ as in 2ch.net [13], one should avoid using the asymptotic formula in the right-hand side of (130), and calculate the finite sums (116) or (111).

We announce that we actually collected a 24 hours data of size $n_d = 70140$ from 2ch.net, and performed a statistical fit of the data to (132), with $N = 697$, and obtained $b = 0.872 \pm 0.002$. (The error is 90% confidence level. See [14] for details.) Apparently, we have a good single parameter fit to the data, which suggests that the practical assumption (104) is good. Details may be reported elsewhere.

We note that in [13], a value of $b = 0.6145$ was obtained for 2ch.net (with different set of data). This is much smaller than the present result. The data used in [13] was small in size, because the data was collected manually in those times, and also, to avoid influence of day-night difference in the total activity, the data was for a short time period in [13], so that the result in [13] is less reliable compared to the present result.

We also note that we have $b < 1$, consistently with previous observation [14] for Amazon.co.jp, where we obtained $b = 0.809$. This shows that, as in Amazon.co.jp, the popularity of subjects is concentrated to a relatively small number of threads in 2ch.net.

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