A TOPOLOGICAL SPLITTING THEOREM
FOR WEIGHTED ALEXANDROV SPACES

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Abstract. Under an infinitesimal version of the Bishop-Gromov relative volume comparison condition for a measure on an Alexandrov space, we prove a topological splitting theorem of Cheeger-Gromoll type. As a corollary, we prove an isometric splitting theorem for Riemannian manifolds with singularities of nonnegative (Bakry-Emery) Ricci curvature.

1. Introduction. A main purpose of this paper is to prove a splitting theorem of Cheeger-Gromoll type for singular spaces. Since it is impossible to define the Ricci curvature tensor on Alexandrov spaces, we consider an infinitesimal version of the Bishop-Gromov volume comparison condition as a candidate of the conditions of the Ricci curvature bounded below. Under the volume comparison condition for a measure on an Alexandrov space, we prove a topological splitting theorem. As a corollary, we prove an isometric splitting theorem for an Alexandrov space whose regular part is a smooth Riemannian manifold of nonnegative (Bakry-Emery) Ricci curvature.

Let us present the volume comparison condition. For a real number κ, we set

$$
s_\kappa(r) := \begin{cases} \sin(\sqrt{\kappa}r)/\sqrt{\kappa} & \text{if } \kappa > 0, \\ r & \text{if } \kappa = 0, \\ \sinh(\sqrt{|\kappa|}r)/\sqrt{|\kappa|} & \text{if } \kappa < 0. \end{cases}
$$

The function s_κ is the solution of the Jacobi equation $s_\kappa''(r) + \kappa s_\kappa(r) = 0$ with initial condition $s_\kappa(0) = 0, s_\kappa'(0) = 1$.

Let M be an Alexandrov space of curvature bounded from below locally and set $r_p(x) := d(p, x)$ for $p, x \in M$, where d is the distance function. For $p \in M$ and $0 < t \le 1$, we define a subset $W_{p,t} \subset M$ and a map $\Phi_{p,t} : W_{p,t} \to M$ as follows. We first set $\Phi_{p,t}(p) := p \in W_{p,t}$. A point $x \ (\ne p)$ belongs to $W_{p,t}$ if and only if there exists $y \in M$ such that $x \in py$ and $r_p(x) : r_p(y) = t : 1$, where py is a minimal geodesic from p to y. Since a geodesic does not branch on an Alexandrov space, for a given point $x \in W_{p,t}$ such a point y is unique and we set $\Phi_{p,t}(x) := y$. The triangle comparison condition implies the local Lipschitz continuity of the map $\Phi_{p,t} : W_{p,t} \to M$. We call $\Phi_{p,t}$ the *radial expansion map*.

Let μ be a positive Radon measure on M with full support, $N \geq 1$ a real number, and $\Omega \subset M$ a subset. The following is an infinitesimal version of the Bishop-Gromov volume comparison condition for μ corresponding to the condition of the lower Ricci curvature bound $\mathrm{Ric} \geq (N-1)\kappa$ with dimension N.

INFINITESIMAL BISHOP-GROMOV CONDITION $\mathrm{BG}(\kappa, N)$ FOR μ ON Ω. For any $p \in M$, $t \in (0, 1]$, and any measurable function $f : M \to [0, +\infty)$ with the property $(*)$ below, we have

$$\int_{W_{p,t}} f \circ \Phi_{p,t}(y) \, d\mu(y) \geq \int_M \frac{t \, s_\kappa(t \, r_p(x))^{N-1}}{s_\kappa(r_p(x))^{N-1}} \, f(x) \, d\mu(x).$$

$(*)$ f has a compact support in $\Omega \setminus \{p\}$ and, if $\kappa > 0$, the support is contained in the open metric ball $B(p, \pi/\sqrt{\kappa})$ centered at p of radius $\pi/\sqrt{\kappa}$.
We say that μ satisfies $\mathrm{BG}(\kappa, N)$ if it satisfies $\mathrm{BG}(\kappa, N)$ on $\Omega = M$.

For an n-dimensional complete Riemannian manifold, the Riemannian volume measure satisfies $\mathrm{BG}(\kappa, n)$ if and only if the Ricci curvature satisfies $\mathrm{Ric} \geq (n-1)\kappa$ (see [27, Theorem 3.2] for the 'only if' part). We see some studies on similar (or same) conditions to $\mathrm{BG}(\kappa, N)$ in [6, 13, 39, 18, 19, 37, 27, 45, 21] etc. $\mathrm{BG}(\kappa, N)$ is sometimes called the Measure Contraction Property and is weaker than the curvature-dimension (or lower N-Ricci curvature) condition $\mathrm{CD}((N-1)\kappa, N)$ introduced by Sturm [40] and Lott-Villani [24] in terms of mass transportation. For a measure on an Alexandrov space, $\mathrm{BG}(\kappa, N)$ is equivalent to the $(\kappa/(N-1), N)$-measure contraction property introduced by Ohta [27]. For an n-dimensional Alexandrov space of curvature $\geq \kappa$, the n-dimensional Hausdorff measure $\mathcal{H}^n$ on M satisfies $\mathrm{BG}(\kappa, n)$ (see [21]). Note that we do not necessarily assume M to be of curvature *uniformly* bounded below. We assume the Alexandrov curvature condition just for the local regularity of the space. If an Alexandrov space M has a measure μ satisfying $\mathrm{BG}(\kappa, N)$, then the dimension of M is less than or equal to N [27, Corollary 2.7]. The infinitesimal Bishop-Gromov condition is stable under the measured Gromov-Hausdorff convergence (cf. [6, Appendix 2] and [13, §5. I_+]).

One of our main theorems is stated as follows.

THEOREM 1.1 (Topological splitting theorem). *Let M be an Alexandrov space of curvature bounded from below locally and μ a positive Radon measure on M with full support. Assume that, for any relatively compact open subset $\Omega \subset M$, there exists a real number $N_\Omega \geq 1$ such that μ satisfies $\mathrm{BG}(0, N_\Omega)$ on Ω. If, in addition, M contains a straight line, then M is homeomorphic to $M' \times \boldsymbol{R}$ for some metric space M'.*

Note that $\mathrm{BG}(0, N_\Omega)$ in this theorem can be replaced with the curvature dimension condition.

This theorem is new even if M is a complete Riemannian manifold. We do not know if the isometric splitting in the theorem is true, i.e., if M is isometric to $M' \times \boldsymbol{R}$ for some Alexandrov space M', even in the case where μ is the n-dimensional Hausdorff measure. If we replace '$\mathrm{BG}(0, N_\Omega)$' with 'curvature ≥ 0', then the isometric splitting was proved by

Milka [26], Grove-Petersen [14] and Yamaguchi [47], as a generalization of the well-known Toponogov splitting theorem. For n-dimensional Riemannian manifolds with Riemannian volume measure, BG$(0, n)$ is equivalent to Ric ≥ 0 and the isometric splitting under Ric ≥ 0 was proved by Cheeger-Gromoll [7]. In our case, we do not have the Weitzenböck formula, so that we cannot obtain the isometric splitting at present.

A rough idea of our proof came from that of Cheeger-Gromoll [7]. One of essential points in our proof is to prove a generalized version of the Laplacian comparison theorem (Theorem 4.1), where our discussion is much different from the Riemannian case. We also prove the maximum principle for subharmonic functions, by using the result of the first named author [16] and Cheeger's theory [5].

If the metric of M has enough smooth part, we can prove the isometric splitting. For that, we consider the following.

DEFINITION 1.2 (Singular Riemannian space). M is called a *singular Riemannian space* if the following (1), (2) and (3) are satisfied.
  (1) M is an Alexandrov space of curvature bounded below locally.
  (2) The set S_M of singular points is a closed set in M.
  (3) The set $M \setminus S_M$ of non-singular points is an (incomplete) C^2 Riemannian manifold.

Note that any complete Riemannian orbifold is a singular Riemannian space.

COROLLARY 1.3. *Let M be an n-dimensional singular Riemannian space. If the Ricci curvature satisfies* Ric ≥ 0 *on $M \setminus S_M$, then M is isometric to $M' \times \mathbf{R}^k$, where M' is a singular Riemannian space containing no straight line and $k := n - \dim M'$.*

If M is a complete Riemannian orbifold, then Corollary 1.3 was proved by Borzellino-Zhu [3].

We next consider the Bakry-Emery Ricci curvature. Let n be an integer with $n \geq 1$, and N a real number with $N > n$, or $N = +\infty$. On an n-dimensional C^2 Riemannian manifold with a measure $d\mu(x) = e^{-V(x)} d\,\mathrm{vol}(x)$, where V is a C^2 function and vol the Riemannian volume measure, the *N-dimensional Bakry-Emery Ricci curvature tensor* $\mathrm{Ric}_{N,\mu}$ is defined by

$$\mathrm{Ric}_{N,\mu} := \begin{cases} \mathrm{Ric} + \mathrm{Hess}\, V - (N-n)^{-1} dV \otimes dV & \text{if } n < N < +\infty, \\ \mathrm{Ric} + \mathrm{Hess}\, V & \text{if } N = +\infty. \end{cases}$$

COROLLARY 1.4. *Let M be an n-dimensional singular Riemannian space, N a number with $n < N \leq +\infty$, and $V : M \to \mathbf{R}$ a continuous function which is of C^2 on $M \setminus S_M$. We assume that $\sup_M V < +\infty$ if $N = +\infty$. If the Bakry-Emery Ricci curvature satisfies* $\mathrm{Ric}_{N,\mu} \geq 0$ *on $M \setminus S_M$ for $d\mu(x) := e^{-V(x)} d\,\mathrm{vol}(x)$, then M is isometric to $M' \times \mathbf{R}^k$ and V is constant on $\{x\} \times \mathbf{R}^k$ for each $x \in M'$, where M' is a singular Riemannian space containing no straight line and $k := n - \dim M'$.*

Corollary 1.4 is an extension of the result of Lichnerowicz [22] (see also [46] and [11]) for complete Riemannian manifolds. In the case where $N = +\infty$, the assumption $\sup_M V < +\infty$ is necessary as was pointed out by Lott [23] (see also [46]).

REMARK 1.5. (1) All the results in this paper are true even in the case where M has non-empty boundary. We implicitly assume the Neumann boundary condition on the boundary of M when we consider the Laplacian on M. In particular, the results hold for any convex subset of M.

(2) We can apply Corollaries 1.3 and 1.4 to get some results for the fundamental group of a singular Riemannian space of nonnegative (Bakry-Emery) Ricci curvature (cf. [7]). However, we do not know if we can obtain the same results for an Alexandrov space satisfying the infinitesimal Bishop-Gromov condition. One of the problems is that we cannot prove that a covering space inherits the infinitesimal Bishop-Gromov condition. Another problem is that the splitting is only homeomorphic. If the space splits as $M' \times \mathbf{R}$ homeomorphically, then we do not know if M' is an Alexandrov space or not, and we cannot apply our splitting theorem to M'. This is not enough to investigate the fundamental group.

(3) In our previous paper [20], we proved a Laplacian comparison theorem and a splitting theorem weaker than those in this paper. The proof in this paper is much easier than that in [20]. We will not publish [20] from any journal.

(4) Recently, H.-C. Zhang and X.-P. Zhu [48] have proved a version of an isometric splitting theorem under a new condition corresponding to the nonnegativity of Ricci curvature. Their condition implies the curvature-dimension condition and the infinitesimal Bishop-Gromov condition for the Hausdorff measure.

2. Preliminaries. A *geodesic space* is defined to be a metric space in which any two points x and y can be joined by a length-minimizing curve whose length is equal to the distance between x and y. Let M be a proper geodesic space, where '*proper*' means that any bounded subset of M is relatively compact. We call a locally (resp. globally) length-minimizing curve in M a *geodesic* (resp. a *minimal geodesic*). Denote by $M^2(\kappa)$ a complete simply connected 2-dimensional space form of constant curvature κ. For three different points $x, y, z \in M$ and a real number κ, we denote by $\tilde{\angle}_\kappa xyz$ the angle between a minimal geodesic from $\tilde{y}$ to $\tilde{x}$ and a minimal geodesic from $\tilde{y}$ to $\tilde{z}$ for three points $\tilde{x}, \tilde{y}, \tilde{z} \in M^2(\kappa)$ such that $d(\tilde{x}, \tilde{y}) = d(x, y), d(\tilde{y}, \tilde{z}) = d(y, z)$ and $d(\tilde{z}, \tilde{x}) = d(z, x)$, where d is the distance function. $\tilde{\angle}_\kappa xyz$ is uniquely determined if either the following (1) or (2) is satisfied.

(1) $\kappa \leq 0$.
(2) $\kappa > 0$ and $d(x, y) + d(y, z) + d(z, x) < 2\pi/\sqrt{\kappa}$.

A proper geodesic space M is said to be an *Alexandrov space* (*of curvature bounded below locally*) if for any point $x \in M$ there exists a neighborhood U of x and a real number κ such that, for any different four points $p, q_1, q_2, q_3 \in U$, we have

(T) $\tilde{\angle}_\kappa q_i p q_j, i, j = 1, 2, 3$, are all defined and satisfy

$$\tilde{\angle}_\kappa q_1 p q_2 + \tilde{\angle}_\kappa q_2 p q_3 + \tilde{\angle}_\kappa q_3 p q_1 \leq 2\pi .$$

For a given point $x \in M$, we denote by $\underline{\kappa}(x)$ the supremum of such κ's. Then, $\underline{\kappa}(x)$ is lower semi-continuous in $x \in M$, so that $\underline{\kappa}$ is bounded from below on any compact subset of an Alexandrov space. The globalization theorem states that, for any compact subset Ω o1f an

Alexandrov space M, there exists a compact set $\Omega' \supset \Omega$ such that (T) holds for any mutually different $p, q_1, q_2, q_3 \in \Omega$ and for any real number κ with $\kappa \leq \inf_{x \in \Omega'} \underline{\kappa}(x)$, provided that M is not a 1-dimensional Riemannian manifold. For a constant κ, we say that M is of *curvature* $\geq \kappa$ if (T) holds for any mutually different four points $p, q_1, q_2, q_3 \in M$. In the case where M is not a 1-dimensional Riemannian manifold, the globalization theorem implies that M is of curvature $\geq \kappa$ if and only if $\underline{\kappa} \geq \kappa$ on M. For a 1-dimensional complete Riemannian manifold M and for $\kappa > 0$, M is of curvature $\geq \kappa$ if and only if the diameter of M is at most $\pi/\sqrt{\kappa}$, i.e., M is isometric to either a segment of length at most $\pi/\sqrt{\kappa}$ or a circle of length at most $2\pi/\sqrt{\kappa}$.

In this paper, we always assume that all Alexandrov spaces have finite Hausdorff dimensions. Refer to [4, 29, 17] for the basics of the geometry and analysis on Alexandrov spaces, such as, the space of directions, the tangent cone, etc.

Let M be an Alexandrov space of Hausdorff dimension $n < +\infty$. Then, n coincides with the covering dimension of M, which is a nonnegative integer. Take any point $p \in M$ and fix it. Denote by $\Sigma_p M$ the space of directions at p, and by $K_p M$ the tangent cone at p. $\Sigma_p M$ is an $(n-1)$-dimensional compact Alexandrov space of curvature ≥ 1 and $K_p M$ an n-dimensional Alexandrov space of curvature ≥ 0.

DEFINITION 2.1 (Singular point, δ-singular point). A point $p \in M$ is called a *singular point of M* if $\Sigma_p M$ is not isometric to the unit sphere S^{n-1}. For $\delta > 0$, we say that a point $p \in M$ is δ-*singular* if $\mathcal{H}^{n-1}(\Sigma_p M) \leq \mathrm{vol}(S^{n-1}) - \delta$. Let us denote the set of singular points of M by S_M and the set of δ-singular points of M by S_δ.

Note that a point $p \in M$ is non-singular if and only if the tangent cone $K_p M$ is isometric to $\mathbf{R}^n$. We have $S_M = \bigcup_{\delta > 0} S_\delta$. Since the map $M \ni p \mapsto \mathcal{H}^n(\Sigma_p M)$ is lower semicontinuous, the set S_δ of δ-singular points in M is a closed set. The following lemma is sometimes useful.

LEMMA 2.2 (cf. [35]). *Let γ be a minimal geodesic joining two points p and q in M. Then, the spaces of directions $\Sigma_x M$ at all points $x \in \gamma \setminus \{p, q\}$ are isometric to each other. In particular, any minimal geodesic joining two non-singular (resp. non-δ-singular) points is contained in the set of non-singular (resp. non-δ-singular) points (for any $\delta > 0$).*

DEFINITION 2.3 (Boundary). The boundary of an Alexandrov space M is defined inductively. If M is one-dimensional, then M is a complete Riemannian manifold and the *boundary of M* is defined as usual. Assume that M has dimension ≥ 2. A point $p \in M$ is said to be a *boundary point of M* if $\Sigma_p M$ has non-empty boundary.

Any boundary point of M is a singular point. More strongly, the boundary of M is contained in S_δ for a sufficiently small $\delta > 0$, which follows from the Morse theory in [30, 32].

The doubling theorem (cf. [30, §5], [4, 13.2]) states that if M has non-empty boundary, then the *double* dbl(M) *of M* (i.e., the gluing of two copies of M along their boundaries) is an Alexandrov space without boundary and each copy of M is convex in the double.

Denote by $\hat{S}_M$ (resp. $\hat{S}_\delta$) the set of singular (resp. δ-singular) points of $\mathrm{dbl}(M)$ contained in M, where M is identified with a copy in $\mathrm{dbl}(M)$. We agree that $\hat{S}_M = S_M$ and $\hat{S}_\delta = S_\delta$ provided M has no boundary.

The following shows the existence of differentiable and Riemannian structure on M.

THEOREM 2.4.   *For an n-dimensional Alexandrov space M, we have the following*:

(1)   *There exists a number $\delta_n > 0$ depending only on n such that $M^* := M \setminus \hat{S}_{\delta_n}$ is a manifold (with boundary ∂M^*) [4, 30, 32] and has a natural C^∞ differentiable structure (even on the boundary ∂M^*) [17].*

(2)   *The Hausdorff dimension of S_M is at most $n-1$ [4, 29], and that of $\hat{S}_M$ is at most $n-2$ [4]. We have $S_M = \hat{S}_M \cup \partial M^*$.*

(3)   *We have a unique continuous Riemannian metric g on $M \setminus S_M \subset M^*$ such that the distance function induced from g coincides with the original one of M [29]. The tangent space at each point in $M \setminus S_M$ is isometrically identified with the tangent cone [29]. The volume measure on M^* induced from g coincides with the n-dimensional Hausdorff measure $\mathcal{H}^n$ [29].*

REMARK 2.5.   In [17] we construct a C^∞ structure only on $M \setminus B(S_{\delta_n}, \varepsilon)$. However this is independent of ε and extends to M^*. The C^∞ structure is a refinement of the structures of [29, 28, 31] and is compatible with the DC structure of [31].

Note that the metric g is defined only on $M^* \setminus S_M$ and does not continuously extend to any other point of M. In general, the set of non-singular points $M^* \setminus S_M$ is not a manifold. There is an example of an Alexandrov space M such that S_M is dense in M (see [29]).

DEFINITION 2.6 (Cut-locus).   Let $p \in M$ be a point. We say that a point $x \in M$ is a *cut point of p* if no minimal geodesic from p contains x as an interior point. We agree that p is not a cut point of p. The set of cut points of p is called the *cut-locus of p* and denoted by Cut_p.

Note that Cut_p is not necessarily a closed set. For the $W_{p,t}$ defined in §1, it follows that $\bigcup_{0<t<1} W_{p,t} = M \setminus \mathrm{Cut}_p$. The cut-locus Cut_p is a Borel subset and satisfies $\mathcal{H}^n(\mathrm{Cut}_p) = 0$ [29, Proposition 3.1].

By [29, Lemma 4.1], the function $r_p = d(p, \cdot)$ is differentiable on $M \setminus (S_M \cup \mathrm{Cut}_p \cup \{p\})$. At any differentiable point x of r_p, $-\nabla r_p(x)$ is tangent to a unique minimal geodesic from p to x, where $\nabla r_p(x)$ denotes the gradient vector of r_p at x. This implies that the gradient vector field ∇r_p is continuous at all differentiable points of r_p.

3.   Sobolev spaces and maximum principle.   A main purpose of this section is to prove the maximum principle for subharmonic functions on a weighted Alexandrov space. To prove it, we apply the maximum principle in the setting of a Dirichlet form which was proved by the first named author [16]. For that, we need to investigate Sobolev spaces on a weighted Alexandrov space by using Cheeger's theory [5]. We refer [12] for the basic terminologies of Dirichlet forms and [5] for those of Cheeger's Sobolev spaces.

Let (X, d) be a proper geodesic space and μ a positive Radon measure on X with full support. We take two open subsets $\Omega, \Omega' \subset X$ with $\bar{\Omega} \subset \Omega'$, where $\bar{\Omega}$ is the closure of Ω in X. Assume that (X, d, μ) satisfies the volume doubling condition for metric balls contained in Ω' and a $(1, p)$-Poincaré inequality on any metric ball contained in Ω' for a fixed number $p > 1$ in the sense of upper gradient (cf. [5]). $L^p(\Omega; \mu)$ denotes the Banach space of L^p functions on Ω with respect to μ, and $W^{1,p}(\Omega; \mu)$ the $(1, p)$-Sobolev space of (Ω, d, μ) defined by Cheeger [5]. We denote by g_u a minimal generalized upper gradient for $u \in W^{1,p}(\Omega; \mu)$, which is unique up to modification on sets of μ-measure zero (see [5, Theorem 2.10]). Let $W_0^{1,p}(\Omega; \mu)$ be the $W^{1,p}$-closure of the set of functions $u \in W^{1,p}(\Omega; \mu)$ such that the support of u is compact and contained in Ω. Denote by $W_{0,\mathrm{loc}}^{1,p}(\Omega; \mu)$ the *localization of* $W_0^{1,p}(\Omega; \mu)$, i.e., the set of functions $u : \Omega \to \mathbf{R}$ such that, for any relatively compact open subset $U \subset X$ with $\bar{U} \subset \Omega$, there is a function $u_U \in W_0^{1,p}(\Omega; \mu)$ satisfying that $u = u_U$ μ-a.e. on U. For $u \in W_{0,\mathrm{loc}}^{1,p}(\Omega; \mu)$, we define $g_u : \Omega \to \mathbf{R}$ to be g_{u_U} on each U. By [5, Corollary 2.25], the function g_u is defined uniquely up to modification on sets of μ-measure zero. We also call g_u a minimal generalized upper gradient for u.

REMARK 3.1.   In Cheeger's paper [5], the statements of the theorems are described under $\Omega' = X$. However, all the discussions in the proofs are local and valid for $\Omega' \subset X$.

The following lemma is needed for the proof of the maximum principle.

LEMMA 3.2.   *For a function* $u : \Omega \to \mathbf{R}$, *the following* (1) *and* (2) *are equivalent to each other.*

(1)   *We have* $u \in W_{0,\mathrm{loc}}^{1,p}(\Omega; \mu) \cap C(\Omega)$ *and* $g_u \leq 1$ μ-a.e., *where* $C(\Omega)$ *denotes the set of continuous functions on* Ω.

(2) u *is locally* 1-*Lipschitz on* Ω, *i.e., for any point* $x_0 \in \Omega$, *there exists a neighborhood* B *of* x_0 *in* Ω *such that* $u|_B$ *is* 1-*Lipschitz.*

PROOF.   The implication (2) $\Rightarrow$ (1) follows from a standard discussion.

We prove (1) $\Rightarrow$ (2). Assume that $u \in W_{0,\mathrm{loc}}^{1,p}(\Omega; \mu) \cap C(\Omega)$ and $g_u \leq 1$ μ-a.e. We fix a point $x_0 \in \Omega$ and take a closed ball B' centered at x_0 and contained in Ω. There is a closed ball B centered at x_0 such that all minimal geodesics joining two points in B are entirely contained in B'. We have $B \subset B'$ and $u|_{B'} \in W^{1,p}(B'; \mu) \cap C(B')$. Denote by $C^{\mathrm{Lip}}(B')$ the set of Lipschitz functions on B'. By [5, Theorem 5.3] (see also the proof of [5, Theorem 5.1]), there are functions $u_i \in C^{\mathrm{Lip}}(B')$ and $g_i \in L^p(B'; \mu) \cap C(B')$, $i = 1, 2, \ldots$, such that $u_i \to u$, $g_i \to g_u$ in $L^p(B'; \mu)$ as $i \to \infty$, $\limsup_{i \to \infty} g_i \leq g_u$ μ-a.e. on B', and each g_i is an upper gradient for u_i. Since $g_u \leq 1$ μ-a.e. on Ω and since each g_i is continuous, there is a number i_ε for each $\varepsilon > 0$ such that $g_i \leq 1 + \varepsilon$ on B' for all $i \geq i_\varepsilon$. For any $x, y \in B$, we take a minimal geodesic $\gamma : [0, d(x, y)] \to X$ joining x to y with arclength parameter. Since γ is contained in B', we have

$$|u_i(x) - u_i(y)| \leq \int_0^{d(x,y)} g_i \circ \gamma(s) \, ds \leq (1 + \varepsilon) \, d(x, y),$$

namely u_i for $i \geq i_\varepsilon$ is $(1 + \varepsilon)$-Lipschitz on B for any $\varepsilon > 0$. By the Arzelá-Ascoli theorem, $\{u_i|_B\}$ has a subsequence which uniformly converges to a 1-Lipschitz function v on B. Since u is continuous, we have $u = v$ on B and u is 1-Lipschitz on B. This completes the proof. $\square$

From now on, we consider an Alexandrov space. Let M be an Alexandrov space, μ a positive Radon measure on M with full support, and $\Omega \subset M$ an open subset. We assume that μ satisfies $\mathrm{BG}(\kappa, N)$ on some neighborhood Ω' of $\bar{\Omega}$ for two real numbers $N \geq 1$ and κ. According to the result of Ranjbar-Motlagh [37], we have a $(1, 1)$-Poincaré inequality on any ball in Ω' in the sense of upper gradient, which implies a $(1, p)$-Poincaré inequality on any ball in Ω' for any $p \geq 1$. Since the infinitesimal Bishop-Gromov condition implies the volume doubling condition for balls in Ω', we can apply Cheeger's theory [5] of Sobolev spaces on the metric measure space (Ω, d, μ).

LEMMA 3.3. *The set* $\mathrm{Cut}_p \cap \Omega$ *is of μ-measure zero for any point $p \in M$.*

PROOF. Assume that $\mu(\mathrm{Cut}_p \cap \Omega) > 0$ for some point $p \in M$. Then, for such a point p, there is a small number $\delta > 0$ such that the set

$$A := \{x \in \mathrm{Cut}_p \cap \Omega \,;\, d(x, \partial\Omega) > \delta, \ \delta < d(x, p) < 1/\delta\}$$

has positive μ-measure. The closure of A is compact and contained in $\Omega \setminus \{p\}$. Applying $\mathrm{BG}(\kappa, N_\Omega)$ on Ω to the indicator function of A yields that $\mu(\Phi_{p,t}^{-1}(A)) > 0$ for any $t \in (0, 1)$. It follows from $A \subset \mathrm{Cut}_p$ that $\Phi_{p,t}^{-1}(A) \cap \Phi_{p,t'}^{-1}(A) = \emptyset$ for any mutually different $t, t' \in (0, 1)$. Since μ is a Radon measure, this is a contradiction. $\square$

LEMMA 3.4. *The set $S_M \cap \Omega$ is of μ-measure zero.*

PROOF. We find a dense countable subset $\{p_i\}_{i=1}^\infty$ of M. Lemma 3.3 implies that $\bigcup_{i=1}^\infty \mathrm{Cut}_{p_i} \cap \Omega$ is of μ-measure zero. Thus, it suffices to prove that $S_M \subset \bigcup_{i=1}^\infty \mathrm{Cut}_{p_i}$. Take any point $x \in M \setminus \bigcup_{i=1}^\infty \mathrm{Cut}_{p_i}$. We are going to prove that x is non-singular. Since x is not a cut point of p_i, there is a minimal geodesic γ_i from p_i passing through x for each i. Since the tangent cone K_x is isometric to the magnification limit of M around x, as the limit of each γ_i we have a straight line $\bar{\gamma}_i$ in K_x passing through the vertex of K_x. Since $\{p_i\}$ is dense in M, the union of the images of all $\bar{\gamma}_i$'s is dense in K_x. By using the splitting theorem of Toponogov type (cf. [26]), K_x is isometric to $\mathbf{R}^n$, i.e., x is non-singular. This completes the proof. $\square$

PROPOSITION 3.5. *Any locally Lipschitz function on Ω is differentiable μ-a.e. on Ω.*

PROOF. By [5, Theorem 10.2], any locally Lipschitz function on Ω is infinitesimally generalized linear μ-a.e. By Lemma 3.4, it suffices to consider only non-singular points. At a non-singular point, any infinitesimally generalized linear function is differentiable by [5, Theorem 8.11]. This completes the proof. $\square$

REMARK 3.6. We can improve Proposition 3.5 as that any locally Lipschitz function on Ω' is differentiable μ-a.e. on Ω'. This is because the proposition holds for any Ω with

$\bar{\Omega} \subset \Omega'$ and there is an increasing sequence Ω_i, $i = 1, 2, \dots$, such that $\bar{\Omega}_i \subset \Omega$ and $\bigcup_i \Omega_i = \Omega$.

With the help of Proposition 3.5, we define a Dirichlet form. Denote by $C_0^{\mathrm{Lip}}(\Omega)$ the set of Lipschitz functions with compact support in Ω. We define a bilinear form by

$$\mathcal{E}^\mu(u, v) := \int_\Omega \langle \nabla u, \nabla v \rangle \, d\mu, \quad u, v \in C_0^{\mathrm{Lip}}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on $M \setminus S_M$. Note that $\langle \nabla u, \nabla v \rangle$ is μ-a.e. defined by Proposition 3.5.

LEMMA 3.7.   *The bilinear form $(\mathcal{E}^\mu, C_0^{\mathrm{Lip}}(\Omega))$ is closable in $L^2(\Omega; \mu)$ and its closure, say $(\mathcal{E}^\mu, W_0^{1,2}(\Omega; \mu))$, coincides with Cheeger's $(1, 2)$-Sobolev space with the Dirichlet boundary condition and is a strongly local regular Dirichlet form on $L^2(\Omega; \mu)$ in the sense of* [12].

PROOF.   By [5, Theorem 5.1], the minimal upper gradient of a locally Lipschitz function $u : \Omega \to \mathbf{R}$ coincides with the local Lipschitz constant of u, which is equal to $\|\nabla u\|$ at any differentiable point of u, where $\| \cdot \|$ denotes the norm induced from the Riemannian metric on $M \setminus S_M$. Therefore, by Proposition 3.5, the Cheeger's energy of $u \in C_0^{\mathrm{Lip}}(\Omega)$ coincides with $\mathcal{E}^\mu(u, u)$, so that, by [5, Theorem 4.24], $(\mathcal{E}^\mu, C_0^{\mathrm{Lip}}(\Omega))$ is closable and its closure $(\mathcal{E}^\mu, W_0^{1,2}(\Omega; \mu))$ coincides with Cheeger's Sobolev space with Dirichlet boundary condition. The strong locality, the regularity and the Markovian property of $(\mathcal{E}^\mu, W_0^{1,2}(\Omega; \mu))$ are all obvious. This completes the proof.   □

Let ρ_Ω denote the *intrinsic metric on Ω induced from the Dirichlet form* $(\mathcal{E}^\mu, W_0^{1,2}(\Omega; \mu))$ (cf. [2]), i.e., for $x, y \in \Omega$,

$$\rho_\Omega(x, y) := \sup\{u(x) - u(y) \, ; \, u \in W_{0, \mathrm{loc}}^{1,2}(\Omega; \mu) \cap C(\Omega), \ d\mu_{\langle u \rangle} \leq d\mu\},$$

where $\mu_{\langle u \rangle}$ is the energy measure of u (cf. [12, §3.2]). It is known that ρ_Ω is a pseudo-metric in general.

PROPOSITION 3.8.   *For any $x_0 \in \Omega$, there exists a neighborhood B of x_0 in Ω such that $\rho_\Omega = d$ on $B \times B$. In particular, ρ_Ω is a metric which induces the same topology of d on Ω.*

PROOF.   It is easy to prove that $\mu_{\langle u \rangle} = g_u^2 \, d\mu$ for any $u \in W_{0, \mathrm{loc}}^{1,2}(\Omega; \mu)$, so that $d\mu_{\langle u \rangle} \leq d\mu$ is equivalent to $g_u \leq 1$ μ-a.e. By Lemma 3.2 we have

$$\rho_\Omega(x, y) = \sup\{u(x) - u(y) \, ; \, u : \Omega \to \mathbf{R} \text{ is locally 1-Lipschitz}\}.$$

Let us prove that $d(x, y) \leq \rho_\Omega(x, y)$ for any $x, y \in \Omega$. We fix $x, y \in \Omega$ and set $u(z) := d(y, z)$, $z \in \Omega$. Then, u is 1-Lipschitz and $u(x) - u(y) = d(x, y)$, which imply $d(x, y) \leq \rho_\Omega(x, y)$.

Fix a point $x_0 \in \Omega$. There is a neighborhood B of x_0 in Ω such that all minimal geodesics joining two points in B are entirely contained in Ω. Let us prove that $d(x, y) \geq \rho_\Omega(x, y)$

for any $x, y \in B$. Take any locally 1-Lipschitz function $u : \Omega \to \mathbf{R}$. It follows from the condition for B that u is (globally) 1-Lipschitz on B, so that, for any $x, y \in B$, we have $u(x) - u(y) \leq d(x, y)$, which implies $\rho_\Omega(x, y) \leq d(x, y)$. This completes the proof. $\square$

DEFINITION 3.9 (μ-subharmonicity).   A function $u \in W^{1,2}_{0,\mathrm{loc}}(\Omega; \mu)$ is said to be μ-*subharmonic* if

$$\int_\Omega \langle \nabla u, \nabla f \rangle \, d\mu \leq 0$$

for any nonnegative function $f \in C^{\mathrm{Lip}}_0(\Omega)$.

Using Lemma 3.7 and Proposition 3.8, we prove the maximum principle.

THEOREM 3.10 (Maximum principle).   *Assume that Ω is connected. If a continuous μ-subharmonic function $u \in W^{1,2}_{0,\mathrm{loc}}(\Omega; \mu)$ attains its maximum in Ω, then u is constant on Ω.*

PROOF.   As is mentioned before, we have the volume doubling condition for balls in Ω' and a $(1, p)$-Poincaré inequality on any ball in Ω' for any $p \geq 1$. By [15, Theorem 5.1 and Corollary 9.8], we have a $(2, 2)$-Poincaré inequality on Ω, which together with Proposition 3.8 implies a parabolic Harnack inequality on Ω (see [41, Theorem 3.5]). Therefore, the same proof as in [39, Theorem 7.4] works to obtain a strictly positive locally Hölder continuous heat kernel $p_\Omega(t, x, y)$, $(t, x, y) \in (0, \infty) \times \Omega \times \Omega$, associated to $(\mathcal{E}^\mu, W^{1,2}_0(\Omega; \mu))$ on $L^2(\Omega; \mu)$. The method of the proof of [39, Proposition 7.5] also works to obtain the strong Feller property of the semigroup $T^\Omega_t f(x) := \int_\Omega p_\Omega(t, x, y) f(y) \, \mu(dy)$, where f is a bounded Borel function on Ω. Owing to the strong maximum principle due to the first named author [16, Theorems 1.3 and 8.5 with the last remark in Example 8.2], we have the theorem. $\square$

4. Laplacian comparison theorem.   The purpose of this section is to prove the following theorem. We set $\cot_\kappa(r) := s'_\kappa(r)/s_\kappa(r)$ for the function s_κ defined in §1.

THEOREM 4.1 (Laplacian comparison theorem).   *Let M be an Alexandrov space, μ a positive Radon measure on M with full support, and $\Omega \subset M$ an open subset. If μ satisfies $\mathrm{BG}(\kappa, N_\Omega)$ on Ω for two real numbers $N_\Omega \geq 1$ and κ, then we have*

$$(4.1) \qquad \int_M \langle \nabla r_p, \nabla f \rangle \, d\mu \geq \int_M \{-(N_\Omega - 1) \cot_\kappa \circ r_p\} f \, d\mu$$

for any point $p \in M$ and for any nonnegative function $f \in C^{\mathrm{Lip}}_0(\Omega \setminus \{p\})$.

Note that r_p and f above are differentiable μ-a.e. on Ω by Proposition 3.5 and Remark 3.6.

Let V be a function on M with a certain regularity condition. For the measure $d\mu(x) = e^{-V(x)} \, d\mathcal{H}^n(x)$, we define

$$\Delta_\mu := \Delta + \nabla V = -e^V \, \mathrm{div}(e^{-V} \nabla \cdot) \,,$$

where Δ is the nonnegative Laplacian and ∇V is the gradient vector field of V, considered to be a directional derivative. The inequality (4.1) is a weak form of the formal inequality

$$(4.2) \qquad \Delta_\mu r_p \geq -(N_\Omega - 1) \cot_\kappa \circ r_p$$

on $\Omega \setminus \{p\}$, and $-(N_\Omega - 1) \cot_\kappa \circ r_p$ is the Laplacian of the distance function on an N_Ω-dimensional complete simply connected space form of constant curvature κ, provided that N_Ω is an integral number with $N_\Omega \geq 2$. We do not know if the pointwise inequality (4.2) implies the weak form (4.1) in general. However, if M is a singular Riemannian space and if V is a C^2 function, then (4.2) implies (4.1). For the proof of this, we first prove (4.2) in the sense of barrier by the same way as in [10] and then prove (4.1). The details are omitted here.

Since, for an n-dimensional Alexandrov space of curvature $\geq \kappa$, the n-dimensional Hausdorff measure $\mathcal{H}^n$ satisfies $\mathrm{BG}(\kappa, n)$ (see [21]), the above Laplacian Comparison Theorem (Theorem 4.1) leads us to the following.

COROLLARY 4.2. *If M is an n-dimensional Alexandrov space of curvature $\geq \kappa$, then for any $p \in M$ we have $\Delta r_p \geq -(n-1) \cot_\kappa \circ r_p$ on $M \setminus \{p\}$ in the weak sense.*

Since the Riemannian metric on an Alexandrov space is not continuous on any singular point, a standard proof of the Laplacian comparison theorem for Riemannian manifolds does not work. Renesse [43] proved Corollary 4.2 under some additional condition. In the case of $\mu = \mathcal{H}^n$ with $\mathrm{BG}(\kappa, n)$, another different proof using a version of Green formula can be seen in our previous paper [20].

PROOF OF THEOREM 4.1. Let $f \in C_0^{\mathrm{Lip}}(\Omega \setminus \{p\})$ be any nonnegative function. By Proposition 3.5 and Remark 3.6, f and r are differentiable μ-a.e. on Ω. It follows from $t\,r_p(\Phi_{p,t}(x)) = r_p(x)$ that, for μ-almost all $x \in \Omega$,

$$\frac{d}{dt} \Phi_{p,t}(x)\Big|_{t=1} = -r_p(x)\nabla r_p(x)$$

and so

$$\langle r_p \nabla r_p, \nabla f \rangle = -\left\langle \frac{d}{dt} \Phi_{p,t}(x)\Big|_{t=1}, \nabla f \right\rangle = -\frac{d}{dt} f \circ \Phi_{p,t}(x)\Big|_{t=1}.$$

For $0 < t < 1$ we define a function $F_t : M \to \mathbf{R}$ by

$$F_t(x) := \begin{cases} (f \circ \Phi_{p,t}(x) - f(x))/(1-t) & \text{for } x \in W_{p,t}, \\ 0 & \text{for } x \in M \setminus W_{p,t}. \end{cases}$$

Then we have

$$\lim_{t \to 1-0} F_t(x) = -\frac{d}{dt} f \circ \Phi_{p,t}(x)\Big|_{t=1}$$

for μ-almost all $x \in \Omega$. It follows from $d(x, \Phi_{p,t}(x)) = (1-t)r_p(x)/t$, $x \in W_{p,t}$, that $|F_t(x)| \leq L\,r_p(x)/t$ for all $t \in (0, 1)$ and $x \in M$, where L is a Lipschitz constant of f. Thus, the dominated convergence theorem implies

$$\int_M \langle r_p \nabla r_p, \nabla f \rangle \, d\mu = -\int_M \frac{d}{dt} f \circ \Phi_{p,t}(x)\Big|_{t=1} d\mu(x)$$

$$= \lim_{t \to 1-0} \int_M F_t(x) \, d\mu(x)$$

$$= \lim_{t \to 1-0} \left[\int_{W_{p,t}} \frac{f \circ \Phi_{p,t}(x)}{1-t} \, d\mu(x) - \int_{W_{p,t}} \frac{f(x)}{1-t} \, d\mu(x) \right]$$

and by BG(κ, N_Ω) and $f(x)/(1-t) \geq 0$,

$$\geq \liminf_{t \to 1-0} \left[\int_M \frac{t \, s_\kappa(t \, r_p(x))^{N_\Omega - 1} f(x)}{(1-t) s_\kappa(r_p(x))^{N_\Omega - 1}} \, d\mu(x) - \int_M \frac{f(x)}{1-t} \, d\mu(x) \right]$$

$$\geq \int_M \liminf_{t \to 1-0} \frac{t \, s_\kappa(t \, r_p(x))^{N_\Omega - 1} - s_\kappa(r_p(x))^{N_\Omega - 1}}{(1-t) s_\kappa(r_p(x))^{N_\Omega - 1}} \, f(x) \, d\mu(x)$$

$$= -\int_M \frac{d}{dt} \{t \, s_\kappa(t \, r_p(x))^{N_\Omega - 1}\} \bigg|_{t=1} \frac{f(x)}{s_\kappa(r_p(x))^{N_\Omega - 1}} \, d\mu(x)$$

$$= \int_M \{-1 - (N_\Omega - 1) r_p(x) \cot_\kappa(r_p(x))\} f(x) \, d\mu(x) \,.$$

By $\nabla(r_p f) = f \nabla r_p + r_p \nabla f$, we see that

$$\langle r_p \nabla r_p, \nabla f \rangle = \langle \nabla r_p, \nabla(r_p f) \rangle - f \quad \mu\text{-a.e.}$$

and therefore,

$$(4.3) \qquad \int_M \langle \nabla r_p, \nabla(r_p f) \rangle \, d\mu \geq \int_M \{-(N_\Omega - 1) \cot_\kappa(r_p(x))\} r_p(x) f(x) \, d\mu(x) \,.$$

We now give any nonnegative function $\hat{f} \in C_0^{\mathrm{Lip}}(\Omega \setminus \{p\})$. Set $f(x) := \hat{f}(x)/r_p(x)$ for $x \neq p$ and $f(p) := 0$. Then, $f : M \to \mathbf{R}$ is a nonnegative function which belongs to $C_0^{\mathrm{Lip}}(\Omega \setminus \{p\})$. (4.3) implies (4.1) for $\hat{f}$. This completes the proof. $\qquad \square$

   In our paper [17], we proved for an Alexandrov space M the existence of the heat kernel of M and the discreteness of the spectrum of the Laplacian (the generator of the Dirichlet energy form) on a relatively compact domain in M. As applications to Theorem 4.1, we have the following heat kernel and first eigenvalue comparison results, which generalize the results of Cheeger-Yau [8] and Cheng [9].

   $B(p, r)$ denotes the metric ball centered at p and of radius r and $M^n(\kappa)$ an n-dimensional complete simply connected space form of curvature κ.

   COROLLARY 4.3. *Let M be an n-dimensional Alexandrov space and assume that $\mathcal{H}^n$ satisfies* BG(κ, n). *Let $\Omega \subset M$ be an open subset containing $B(p, r)$ for a number $r > 0$. Denote by $h_t : \Omega \times \Omega \to \mathbf{R}, t > 0$, the heat kernel on Ω with Dirichlet boundary condition, and by $\bar{h}_t : B(\bar{p}, r) \times B(\bar{p}, r) \to \mathbf{R}$ that on $B(\bar{p}, r)$ for a point $\bar{p} \in M^n(\kappa)$. Then, for any $t > 0$ and $q \in B(p, r)$, we have*

$$h_t(p, q) \geq \bar{h}_t(\bar{p}, \bar{q}) \,,$$

where $\bar{q} \in M^n(\kappa)$ is a point such that $d(\bar{p}, \bar{q}) = d(p, q)$.

COROLLARY 4.4. *Let M be an n-dimensional Alexandrov space and $r > 0$ a real number. Assume that $\mathcal{H}^n$ satisfies* BG(κ, n). *Denote by $\lambda_1(B(p, r))$ the first eigenvalue of the Laplacian on $B(p, r)$ with Dirichlet boundary condition, and by $\lambda_1(B(\bar{p}, r))$ that on $B(\bar{p}, r)$ for a point $\bar{p} \in M^n(\kappa)$. Then we have*

$$\lambda_1(B(p, r)) \le \lambda_1(B(\bar{p}, r)) \,.$$

Once we have Theorem 4.1, the proofs of Corollaries 4.3 and 4.4 are the same as of Renesse [43, Theorem II and Corollary 1]. We verify that the local $(L^1, 1)$-volume regularity is not needed in the proof of [43, Theorem II]. We also obtain a Brownian motion comparison theorem in the same way as in [43].

5. Splitting theorem. We prove the Topological Splitting Theorem (Theorem 1.1) following the idea of Cheeger-Gromoll [7]. However, we still need some extra lemmas to fit the discussions of [7] to Alexandrov spaces.

Let M be a non-compact Alexandrov space and γ a *ray* in M, i.e., a geodesic defined on $[0, +\infty)$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for any $s, t \ge 0$.

DEFINITION 5.1 (Busemann function). The *Busemann function $b_\gamma : M \to \mathbf{R}$ for γ* is defined by

$$b_\gamma(x) := \lim_{t \to +\infty} \{t - d(x, \gamma(t))\}, \quad x \in M \,.$$

It follows from the triangle inequality that $t - d(x, \gamma(t))$ is monotone non-decreasing in t, so that the limit above exists. b_γ is a 1-Lipschitz function.

DEFINITION 5.2 (Asymptotic relation). We say that a ray σ in M is *asymptotic to γ* if there exist a sequence $t_i \to +\infty$, $i = 1, 2, \ldots$, and minimal geodesics $\sigma_i : [0, l_i] \to M$ with $\sigma_i(l_i) = \gamma(t_i)$ such that σ_i converges to σ as $i \to \infty$, (i.e., $\sigma_i(t) \to \sigma(t)$ for each t).

For any point in M, there is a ray asymptotic to γ from the point. Any subray of a ray asymptotic to γ is asymptotic to γ. By the same proof as for Riemannian manifolds (cf. [38, Theorem 3.8.2(3)]), for any ray σ asymptotic to γ, we have

(5.1) $$b_\gamma \circ \sigma(s) = s + b_\gamma \circ \sigma(0) \quad \text{for any } s \ge 0 \,.$$

LEMMA 5.3. *Let $f : M \to \mathbf{R}$ be a 1-Lipschitz function and $u, v \in \Sigma_p M$ two directions at a point $p \in M$. If the directional derivative of f to u is equal to 1 and that to v equal to -1, then the angle between u and v is equal to π.*

PROOF. There are points $x_t, y_t \in M$, $t > 0$, such that $d(p, x_t) = d(p, y_t) = t$ for all $t > 0$ and that the direction at p of px_t (resp. py_t) converges to u (resp. v) as $t \to 0$. The assumption for f tells us that

$$\lim_{t \to 0} \frac{f(x_t) - f(p)}{t} = 1 \quad \text{and} \quad \lim_{t \to 0} \frac{f(y_t) - f(p)}{t} = -1 \,,$$

which imply

$$\lim_{t \to 0} \frac{d(x_t, y_t)}{t} \ge \lim_{t \to 0} \frac{f(x_t) - f(y_t)}{t} = 2 \,.$$

This completes the proof.   □

LEMMA 5.4.   *Assume that a ray* $\sigma : [0, +\infty) \to M$ *is asymptotic to a ray* $\gamma :$ $[0, +\infty) \to M$, *and let s be a given positive number.*

(1)   *If* $\sigma(s)$ *is a non-singular point, then* b_γ *is differentiable at* $\sigma(s)$ *and* $\nabla b_\gamma(\sigma(s))$ *is tangent to* σ.

(2)   *Among all rays emanating from* $\sigma(s)$, *only the subray* $\sigma|_{[s,+\infty)}$ *of* σ *is asymptotic to* γ.

PROOF.   (1) follows from the same discussion as for Riemannian manifolds (see [38, Theorem 3.8.2]), in which we need the total differentiability of the distance function from a compact subset of M. This is obtained in the same way as in [29, Theorem 3.5 and Lemma 4.1].

(2)   Take any ray τ from $\sigma(s)$ asymptotic to γ. By (5.1), the derivative of $b_\gamma \circ \tau$ is equal to 1. Therefore, using Lemma 5.3 yields that the angle between $\sigma|_{[0,s]}$ and τ is equal to π, so that $\sigma'(s) = \tau'(0)$. This completes the proof.   □

Note that if $\sigma(s)$ is a non-singular point, then Lemma 5.4(1) implies (2).

LEMMA 5.5.   *Let* γ *be a straight line in* M. *Denote by* b_+ *the Busemann function for* $\gamma_+ := \gamma|_{[0,+\infty)}$ *and by* b_- *that for* $\gamma_- := \gamma|_{(-\infty,0]}$. *If* $b_+ + b_- \equiv 0$ *holds, then* M *is covered by disjoint straight lines bi-asymptotic to* γ. *In particular,* $b_+^{-1}(t)$ *for all* $t \in \mathbf{R}$ *are homeomorphic to each other and* M *is homeomorphic to* $b_+^{-1}(t) \times \mathbf{R}$.

PROOF.   Take any point $p \in M$ and a ray $\sigma : [0, +\infty) \to M$ from p asymptotic to γ_+. For any $s > 0$, the directional derivatives of b_+ to the two opposite directions at $\sigma(s)$ tangent to σ are -1 and 1, respectively. Since $b_- = -b_+$ and by Lemma 5.3, a ray from $\sigma(s)$ asymptotic to γ_- is unique and contains $\sigma([0, s])$. By the arbitrariness of $s > 0$, σ extends to a straight line bi-asymptotic to γ. Namely, for a given point $p \in M$, we have a straight line σ_p passing through p and bi-asymptotic to γ. By Lemma 5.4(2), any ray from a point in σ_p asymptotic to $\gamma_\pm$ is a subray of σ_p. In particular, σ_p is unique (up to parameters) for a given p, and for any two points $p, q \in M$ the images of σ_p and σ_q either coincide or do not intersect each other. M is covered by $\{\sigma_p\}_{p \in M}$ and this completes the proof.   □

LEMMA 5.6.   *Let* μ *be a positive Radon measure on* M *with full support and let* $\Omega \subset M$ *be an open subset. Assume that* μ *satisfies* $BG(0, N_\Omega)$ *on* Ω *for a real number* $N_\Omega \geq 1$. *Then, the Busemann function* b_γ *for any ray* γ *in* M *is* μ-*subharmonic on* Ω *in the sense of Definition* 3.9.

PROOF.   We take a sequence $t_i \to +\infty$, $i = 1, 2, \ldots$. Since $r_{\gamma(t_i)}$, b_γ are both Lipschitz, they are differentiable μ-a.e. on Ω by Proposition 3.5 and Remark 3.6. Let $x \in \Omega$ be any non-singular point where $r_{\gamma(t_i)}$ and b_γ are all differentiable. We have a unique minimal geodesic $\sigma_{x,i}$ from x to $\gamma(t_i)$ and $-\nabla r_{\gamma(t_i)}(x)$ is tangent to it. A ray σ_x from x asymptotic to γ is unique and $\nabla b_\gamma(x)$ is tangent to it. Since $\sigma_{x,i} \to \sigma_x$ as $i \to \infty$, we have

$-\nabla r_{\gamma(t_i)}(x) \to \nabla b_\gamma(x)$. For any nonnegative function $f \in C_0^{\mathrm{Lip}}(\Omega)$, the dominated convergence theorem and Laplacian Comparison Theorem (Theorem 4.1) show that

$$\int_\Omega \langle \nabla b_\gamma, \nabla f \rangle \, d\mu = -\lim_{i \to \infty} \int_\Omega \langle \nabla r_{\gamma(t_i)}, \nabla f \rangle \, d\mu$$

$$\leq (N_\Omega - 1) \lim_{i \to \infty} \int_{\mathrm{supp}\, f} \frac{f}{r_{\gamma(t_i)}} \, d\mu = 0 \,,$$

where $\mathrm{supp}\, f$ is the support of f. This completes the proof. $\qquad\square$

PROOF OF THEOREM 1.1.   Let $\Omega \subset M$ be any connected, relatively compact and open subset. The assumption of the theorem yields that there is a number $N_\Omega \geq 1$ such that μ satisfies BG$(0, N_\Omega)$ on a neighborhood of $\bar\Omega$. By Lemma 5.6, $b := b_+ + b_-$ is μ-subharmonic on Ω, where b_+ and b_- are the Busemann functions as in Lemma 5.5 for a straight line γ in M. It follows from the triangle inequality that $b \leq 0$. We have $b \circ \gamma \equiv 0$ by the definition of $b_\pm$. The maximum principle (Theorem 3.10) proves that $b \equiv 0$ on Ω if Ω intersects γ. By the aribitrariness of such Ω, we have $b \equiv 0$ on M. Lemma 5.5 proves the theorem. $\qquad\square$

PROOF OF COROLLARIES 1.3 AND 1.4.   Let M, N and V be as in Corollary 1.4. In Corollary 1.3, we assume that $N = n$ and $V = 1$, in which case we have $\mathrm{Ric}_{N,\mu} = \mathrm{Ric}$ and $\Delta_\mu = \Delta + \nabla V = \Delta$. For the corollaries, it suffices to prove that if M contains a straight line, then M is isometric to $M' \times \mathbf{R}$ and V is constant on $\{x\} \times \mathbf{R}$ for each $x \in M'$. This is because if $M' \times \mathbf{R}$ is a singular Riemannian space, then so is M'.

We first assume that $N < +\infty$. Since any geodesic joining two points in $M \setminus S_M$ is contained in $M \setminus S_M$ (see Lemma 2.2), the condition $\mathrm{Ric}_{N,\mu} \geq 0$ on $M \setminus S_M$ implies BG$(0, N)$ for μ on $M \setminus S_M$ (see [1] and also [40, 24]). By $\mathcal{H}^n(S_M) = 0$, μ satisfies BG$(0, N)$ on M (an easy discussion proves that for any convergent sequence $p_i \to p_\infty$ in M, BG$(0, N)$ for $p = p_i$ implies BG$(0, N)$ for $p = p_\infty$). We then apply Theorem 1.1 to M and μ for $N_\Omega = N$. In the proof of the theorem, we obtain that b_+ and b_- are both μ-subharmonic and $b_+ + b_- = 0$. Therefore, $b_\pm$ is μ-harmonic, i.e., a weak solution of $\Delta_\mu b_\pm = 0$ on $M \setminus S_M$. By the regularity theorem of elliptic differential equations, b_+ is of C^2 on $M \setminus S_M$ and satisfies $\Delta_\mu b_+ = 0$ pointwise on $M \setminus S_M$. We use the generalized Weitzenböck formula for $\mathrm{Ric}_{N,\mu}$:

$$-\Delta_\mu \left(\frac{\|\nabla f\|^2}{2} \right) + \langle \nabla \Delta_\mu f, \nabla f \rangle$$

$$= \frac{(\Delta_\mu f)^2}{N} + \mathrm{Ric}_{N,\mu}(\nabla f, \nabla f) + \left\| \mathrm{Hess}\, f + \frac{\Delta f}{n} I_n \right\|_{HS}^2$$

$$+ \frac{n}{N(N-n)} \left(-\frac{N-n}{n} \Delta f + \langle \nabla V, \nabla f \rangle \right)^2$$

for any C^2 function $f : M \setminus S_M \to \mathbf{R}$ (see [42, (14.46)]), where I_n denotes the identity operator and $\| \cdot \|_{HS}$ the Hilbert-Schmidt norm. Since $\|\nabla b_+\| = 1$ and $\mathrm{Ric}_{N,\mu}(\nabla b_+, \nabla b_+) \geq 0$, putting $f := b_+$ in the above formula yields that $\mathrm{Hess}\, b_+ = -(\Delta b_+/n)I_n$ and $((N -$

$n)/n)\Delta b_+ = \langle \nabla V, \nabla b_+ \rangle$. Since

$$0 = \Delta_\mu b_+ = \Delta b_+ + \langle \nabla V, \nabla b_+ \rangle = \frac{N}{n} \Delta b_+ ,$$

we have $\operatorname{Hess} b_+ = 0$ and $\langle \nabla V, \nabla b_+ \rangle = 0$ on $M \setminus S_M$. Thus, b_+ is a linear function along any geodesic in $M \setminus S_M$. Since any geodesic segments in M can be approximated by geodesic segments in $M \setminus S_M$, b_+ is linear along any geodesic in M. Since M is covered by straight lines bi-asymptotic to γ, b_+ is averaged D^2 in the sense of [25]. The isometric splitting follows from [25, Theorem A]. Since $\langle \nabla V, \nabla b_+ \rangle = 0$ on $M \setminus S_M$, V is constant along each straight line bi-asymptotic to γ. This proves the corollaries in the case of $N < +\infty$.

We next consider the case where $N = +\infty$. By $\operatorname{Ric}_{\infty,\mu} \geq 0$, the same discussion as in [22], [11, (1)] and [46, (2.21)] leads to

$$\Delta_\mu r_p(x) \geq -\frac{n-1}{r_p(x)} + \frac{2V(x)}{r_p(x)} - \frac{2}{r_p(x)^2} \int_0^{r_p(x)} V(\gamma(s))\, ds$$
$$\geq -\frac{n - 1 + 2\sup_M V - 2V(x)}{r_p(x)}$$

for any $p \in M$ and $x \in M \setminus (S_M \cup \operatorname{Cut}_p \cup \{p\})$, where γ is a unique unit speed geodesic joining p to x. Therefore, for a given compact subset $\Omega \subset M$, setting $N_\Omega := n + 2\sup_M V - 2\inf_\Omega V$, we have

$$\Delta_\mu r_p \geq -\frac{N_\Omega - 1}{r_p}$$

on $\Omega \setminus (S_M \cup \operatorname{Cut}_p \cup \{p\})$, which together with a standard discussion (cf. the proof of [27, Theorem 3.2]) yields $BG(0, N_\Omega)$ on Ω for μ. (Also, we directly obtain the Laplacian comparison (Theorem 4.1) as was stated under (4.2)). By applying Theorem 1.1, M splits as $M' \times \mathbf{R}$ homeomorphically. We have $\Delta_\mu b_+ = 0$ on $M \setminus S_M$. Apply the generalized Weitzenböck formula for $\operatorname{Ric}_{\infty,\mu}$:

$$-\Delta_\mu \left(\frac{\|\nabla f\|^2}{2} \right) + \langle \nabla \Delta_\mu f, \nabla f \rangle = \operatorname{Ric}_{\infty,\mu}(\nabla f, \nabla f) + \| \operatorname{Hess} f \|_{HS}^2 .$$

By setting $f := b_+$, the left-hand side vanishes, so that, by $\operatorname{Ric}_{\infty,\mu} \geq 0$, we have $\operatorname{Ric}_{\infty,\mu}(\nabla b_+, \nabla b_+) = 0$ and $\operatorname{Hess} b_+ = 0$ on $M \setminus S_M$. In the same way as in the case of $N < +\infty$, we obtain that M is isometric to $M' \times \mathbf{R}$. Since $\operatorname{Ric}(\nabla b_+, \nabla b_+) = 0$, we have

$$0 = \operatorname{Ric}_{\infty,\mu}(\nabla b_+, \nabla b_+) = \operatorname{Hess} V(\nabla b_+, \nabla b_+) = \frac{\partial^2}{\partial t^2} V$$

in the coordinate $(x, t) \in M' \times \mathbf{R} = M$, so that V is linear along each line $\{x\} \times \mathbf{R}$, $x \in M'$. Since V is bounded, it is constant along each line $\{x\} \times \mathbf{R}$, $x \in M'$. $\square$

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