GEOMETRIC INVARIANTS ASSOCIATED WITH PROJECTIVE STRUCTURES AND UNIVALENCE CRITERIA

SEONG-A KIM AND TOSHIYUKI SUGAWA

Abstract. For a nonconstant holomorphic map between projective Riemann surfaces with conformal metrics, we consider invariant Schwarzian derivatives and projective Schwarzian derivatives of general virtual order. We show that these two quantities are related by the "Schwarzian derivative" of the metrics of the surfaces (at least for the case of virtual orders 2 and 3). As an application, we give univalence criteria for a meromorphic function on the unit disk in terms of the projective Schwarzian derivative of virtual order 3.

1. Introduction. The (classical) Schwarzian derivative

$$(1.1) \qquad Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

of a nonconstant meromorphic function f on a plane domain was introduced by Schwarz to construct a conformal mapping of a Jordan domain bounded by finitely many circular arcs. The reason why the Schwarzian derivative is so useful is that it satisfies the invariance relation $S(M \circ f \circ L) = Sf \circ L \cdot (L')^2$ for Möbius transformations L and M. In particular, the quantity $\Sigma f = \lambda^{-2} Sf$ for a function on the unit disk $\boldsymbol{D} = \{z \in \boldsymbol{C}; |z| < 1\}$ is invariant in the sense that $\Sigma(M \circ f \circ T) = (\Sigma f) \circ T \cdot (T'/|T'|)^2$ for a Möbius transformation M and an analytic automorphism T of $\boldsymbol{D}$. Here, $\lambda(z)|dz| = |dz|/(1 - |z|^2)$ is the hyperbolic metric of $\boldsymbol{D}$. Due to these invariance properties, the Schwarzian derivative has found many applications in complex analysis, Teichmüller theory, 1-dimensional dynamical systems, and so on.

It is thus a natural desire to seek for more quantities analogous to the Schwarzian derivative. Indeed, Schwarzian derivatives of higher order were proposed in [1], [24] and [21]. Those Schwarzians certainly enjoy several interesting properties but they do not find many applications so far. One reason perhaps comes from the lack of invariance. For instance, as we will see in Section 5, Schwarzians of Aharonov and Tamanoi are invariant under the post-composition with a Möbius transformation but not under the pre-composition (as differentials), in general.

The authors proposed in [13] the *invariant* Schwarzian derivative $\Sigma^n f$ of virtual order n for a nonconstant holomorphic map f from a Riemann surface with conformal metric into another. This derivative satisfies the invariance relation $\Sigma^n(M \circ f \circ T) = (\Sigma^n f) \circ T \cdot$

$(T'/|T'|)^n$ for local isometries M and T (see Lemma 3.2 for a more precise formulation) at the expense of analyticity. It involves conformal metrics of both the source and target surfaces and therefore has a complicated form in general. Note that $\Sigma^2 f$ is nothing but the above Σf when the source and target surfaces are D and $\widehat{C} = C \cup \{\infty\}$ equipped with the hyperbolic and spherical metrics, respectively.

In the present paper, we will introduce yet another kind of Schwarzian derivatives, denoted by $V^n f$ and called the *projective* Schwarzian derivative of virtual order n, for a nonconstant holomorphic map f from a projective surface with a conformal metric into a projective surface. Here, we note that a projective structure is finer than a complex structure and that a plane domain has a natural projective structure (see the next section for details). This satisfies the invariance property that $V^n(M \circ f \circ T) = (V^n f) \circ T \cdot (T')^n$ for a projective map M and a projective local isometry T (see Lemma 5.1 below for details). Since $V^n f$ does not involve a conformal metric of the target surface, the form of $V^n f$ is much simpler than that of $\Sigma^n f$.

One of the most important applications of the Schwarzian derivative is a univalence criterion due to Nehari [20].

THEOREM A (Nehari).   *Let f be a nonconstant meromorphic function on the unit disk D. If f is univalent in D, then $\|Sf\|_2 \le 6$. Conversely, if $\|Sf\|_2 \le 2$, then f must be univalent in D. The numbers 6 and 2 are both sharp.*

Here, we set

$$\|\varphi\|_c = \sup_{z \in D}(1 - |z|^2)^c |\varphi(z)|$$

for a $\widehat{C}$-valued function φ on D and a real number c. We notice the invariance property that

$$(1.2) \qquad\qquad \|\varphi \circ T \cdot |T'|^c\|_c = \|\varphi\|_c$$

holds for each analytic automorphism T of D because of the formula $|T'(z)|(1 - |z|^2) = 1 - |T(z)|^2$.

The first assertion in Theorem A was indeed found by Wilhelm Kraus as early as in 1932, and re-discovered by Nehari later. Therefore, it is sometimes called the Kraus-Nehari theorem. Theorem A constitutes a basis of the theory of Teichmüller spaces. See [16] for details.

A similar result for the pre-Schwarzian derivative f''/f' is also known and it is utilized to construct another model of the universal Teichmüller space (see [3] or [23]).

In the present paper, as a by-product of our investigation, we give a univalence criterion for a function f on D in terms of the projective Schwarzian

$$(1.3) \qquad\qquad Vf(z) = V^3 f(z) = (Sf)'(z) - \frac{4\bar{z}}{1 - |z|^2} Sf(z)$$

of virtual order 3.

THEOREM 1.1.   *Let f be a nonconstant meromorphic function on the unit disk D. If f is univalent in D, then $\|Vf\|_3 \le 16$. The number 16 is sharp. Conversely, if $\|Vf\|_3 \le 3/2$, then f is univalent in D.*

It appears that the constant $3/2$ in the theorem is not sharp. On the other hand, the constant cannot be replaced by a number greater than $16\sqrt{3}/9 \approx 3.0792$ as we will see in Example 6.4. In the proof, we will see that the theorem is not stronger than the Nehari univalence criterion in Theorem A. We, however, expect that this quantity Vf would open a new window to a family of univalence criteria, as a paper by Duren, Shapiro and Shields [7] led to Becker's univalence criterion [4].

Let us summarize the contents of the paper. We recall the definition and basic properties of the Schwarzian derivative of a holomorphic map between projective (Riemann) surfaces in Section 2. In Section 3, we review basics of the Peschl-Minda derivatives and Schwarzian derivatives of higher order due to Aharonov [1], Tamanoi [24] and the authors [13] for a nonconstant holomorphic map between Riemann surfaces with conformal metrics. Section 4 is devoted to a relation between Σf and Sf when the surfaces are projective and have conformal metrics (Theorem 4.6). To this end, we introduce the Schwarzian derivative of a conformal metric. This result has several applications as we will see there.

The higher-order Schwarzians of Aharonov and Tamanoi cannot be extended to holomorphic maps between projective Riemann surfaces unlike the classical Schwarzian. In Section 5, we introduce *projective* Schwarzian derivatives. We will then try to generalize Theorem 4.6 for projective Schwarzians of order 3. Our future task is to extend this result to the case of general order. The last section will be devoted to the proof of Theorem 1.1 and to the computation for a concrete example.

2. Projective structures. Let us briefly recall basic properties of the Schwarzian derivative Sf, given in (1.1), of a nonconstant meromorphic function f. It is well known that $Sf \equiv 0$ if and only if f is (a restriction of) a Möbius transformation and that the formula

$$(2.1) \qquad S(g \circ f) = (Sg) \circ f \cdot (f')^2 + Sf$$

holds for the composite map $g \circ f$. In particular, $S(M \circ f \circ L) = Sf \circ L \cdot (L')^2$ for Möbius transformations L and M as we already mentioned in Introduction.

We are tempted to define the Schwarzian derivative for a holomorphic map f between Riemann surfaces. The above formula, however, tells us that the value Sf may depend on the choice of local coordinates. Hence, we are naturally led to the idea to restrict the local coordinates so that the transition functions are Möbius, that is, the idea of projective structures.

Though the notion of projective structures is standard, we describe its basics in some detail in order to clarify the formulations below. The notion of projective structures is obtained by replacing holomorphic maps by Möbius maps in the definition of complex structure (see [8, §9]). More precisely, a *projective structure* on a surface R is the equivalence class of an atlas $\{z_\alpha : U_\alpha \to U'_\alpha\}_{\alpha \in A}$, where $U_\alpha \subset R$, $U'_\alpha \subset \mathbf{C}$ are open sets and $z_\alpha : U_\alpha \to U'_\alpha$ is a

homeomorphism for $\alpha \in A$ such that the transition function $z_\beta \circ z_\alpha^{-1}$ is a Möbius map on each connected component of $z_\alpha(U_\alpha \cap U_\beta)$ for $\alpha, \beta \in A$. Such an atlas will be called a *projective atlas*. Two atlases are defined to be equivalent if the union of the two is again a projective atlas.

A *projective surface* will mean a surface with a projective structure. A map $z_\alpha : U_\alpha \rightarrow U'_\alpha$ in a projective atlas of a projective surface R will be called a *projective coordinate.* Note that a projective surface has the canonical complex structure, which is called the underlying complex structure. In other words, a projective structure is finer than a complex structure, and thus a projective surface can be regarded as a Riemann surface in a canonical way. A continuous map f from a projective surface R into another projective surface R' is called *projective* if $w \circ f \circ z^{-1}$ is either Möbius or constant whenever z and w are projective coordinates of R and R', respectively.

For instance, a plane domain Ω has the natural atlas $\{\text{id} : \Omega \rightarrow \Omega\}$, which gives rise to a projective structure on Ω. In the sequel, unless otherwise stated, a plane domain will be endowed with this *natural* projective structure. The uniformization theorem states that the universal covering surface of a Riemann surface R is conformally equivalent to one (and only one) of the standard surfaces; the Riemann sphere $\widehat{C} = C \cup \{\infty\} = C_{+1}$, the complex plane $C = C_0$, and the unit disk $D = \{z \in C; |z| < 1\} = C_{-1}$. Here the notation C_δ, $\delta = +1, 0, -1$, is introduced to handle with these three at once. According to the cases $\delta = +1, 0, -1$, the surface R is called elliptic, parabolic, or hyperbolic, respectively. Let $h : C_\delta \rightarrow R$ be a holomorphic universal covering projection of C_δ onto R. Since the group of conformal automorphisms of C_δ is contained in the group of Möbius transformations, the local inverses of h give rise to a projective structure on R. This projective structure will be called *standard.* The standard projective structure is characterized by the property that a holomorphic universal covering projection of the standard domain C_δ onto R is projective.

For projective coordinates z of R and w of R', we define a meromorphic quadratic differential on R by $S_{R,R'} f = S(w \circ f \circ z^{-1})dz^2$ for a nonconstant holomorphic map $f : R \rightarrow R'$. Then the meromorphic quadratic differential $S_{R,R'} f = S_{R,R'} f(z)dz^2$ does not depend on the choice of the projective coordinates, thus it is well defined. In other words, the system of functions $S_{R,R',\alpha} f = S(w \circ f \circ z_\alpha^{-1})$ for projective coordinates z_α of R and w of R', we have the relations $S_{R,R',\beta} f \circ g_{\beta,\alpha} \cdot (g'_{\beta,\alpha})^2 = S_{R,R',\alpha} f$ for $g_{\beta,\alpha} = z_\beta^{-1} \circ z_\alpha$. Note that $S_{R,R'} f \equiv 0$ if and only if f is a nonconstant projective map. If R and R' are plane domains (with natural projective structures), then obviously $S_{R,R'} f$ coincides with the usual Schwarzian derivative $Sf(z)dz^2$. If we do not need to refer to the projective structures of R and R', we write Sf for $S_{R,R'} f$ simply. For basic information about projective structures and applications to Teichmüller spaces, see [19] and references therein.

3. Invariant Schwarzian derivative. We first recall the Peschl-Minda derivatives. See [12] or [22] for details.

For the sake of simplicity, we begin with the case of plane domains. Let Ω and Ω' be plane domains with conformal metrics $\rho = \rho(z)|dz|$ and $\sigma = \sigma(w)|dw|$, respectively. Throughout the present paper, a conformal metric will be always smooth.

The ρ-derivative of a smooth function φ on Ω is defined by

$$\partial_\rho \varphi = \frac{\partial \varphi}{\rho} = \frac{1}{\rho(z)} \frac{\partial \varphi(z)}{\partial z}.$$

For a holomorphic map $f : \Omega \to \Omega'$, we define the Peschl-Minda derivative $D^n f$ of order n with respect to ρ and σ inductively by

$$D^1 f = \frac{\sigma \circ f}{\rho} f'$$

$$D^{n+1} f = [\partial_\rho - n\partial_\rho(\log \rho) + (\partial_\sigma \log \sigma) \circ f \cdot D^1 f]D^n f \quad (n \geq 1).$$

We further set

$$Q^n f = \frac{D^{n+1} f}{D^1 f}, \quad n \geq 1.$$

Since $D^n f$ and $Q^n f$ depend on the metrics, we often write $D^n_{\sigma,\rho} f$ and $Q^n_{\sigma,\rho} f$ for them.

LEMMA 3.1 ([13]). *In the above situation, the following formula is valid:*

$$\partial_\rho(Q^n f) = Q^{n+1} f - [Q^1 f - n\partial_\rho \log \rho]Q^n f.$$

We next recall the definitions of Aharonov invariants and Tamanoi's Schwarzian derivatives. Let f be a meromorphic function on a domain D in the complex plane. For $z \in D$ with $f(z) \neq \infty$ and $f'(z) \neq 0$, we expand

$$\frac{f'(z)}{f(z+w) - f(z)} = \frac{1}{w} - \sum_{n=1}^{\infty} \psi_n[f](z)w^{n-1}$$

for small enough w. The quantities $\psi_n[f](z)$ were introduced by Aharonov [1] and called the *Aharonov invariants* by Harmelin [9]. Independently, Tamanoi [24] defined the Schwarzian derivative $S_n[f]$ of virtual order n by

$$\frac{f'(z)(f(z+w) - f(z))}{\frac{1}{2}f''(z)(f(z+w) - f(z)) + f'(z)^2} = \sum_{n=0}^{\infty} S_n[f](z)\frac{w^{n+1}}{(n+1)!}.$$

Note that $S_0[f] = 1$ and $S_1[f] = 0$. The Aharonov invariants and Tamanoi's Schwarzian derivatives are essentially same in the sense that the following relation holds:

$$\sigma_n[f] = \psi_n[f] + \sum_{k=2}^{n-2} \psi_k[f]\sigma_{n-k}[f], \quad n \geq 2,$$

where $\sigma_n[f] = S_n[f]/(n+1)!$. In particular, $3!\psi_2[f] = S_2[f] = Sf$ and $4!\psi_3[f] = S_3[f] = (Sf)'$. We also have the following recursive relations:

$$(3.1) \qquad S_n[f] = S_{n-1}[f]' + \frac{1}{2}S_2[f]\sum_{k=1}^{n-1}\binom{n}{k}S_{k-1}[f]S_{n-k-1}[f], \quad n \geq 3$$

(see [13] for details). For example, $S_4[f] = S_3[f]' + 4S_2[f]^2$.

We note that $S_n[M \circ f] = S_n[f]$ holds for a Möbius transformation M. However, unlike $Sf = S_2[f]$, the higher-order Schwarzian derivative $S_n[f]$ does not behave nicely with pre-composition with Möbius transformations. For instance,

$$S_3[f \circ L] = (S_2[f \circ L])' = S_3[f] \circ L \cdot (L')^3 + 2S_2[f] \circ L \cdot L'L''$$

for a Möbius transformation L. Therefore, in general, $\sigma_n[f]dz^n$ and $S_n[f]dz^n$ are not invariant under the change of projective coordinates. Moreover, it is unlikely that a result similar to the second half of Theorem A holds for these Schwarzians. For instance, we have no constant $c \geq 0$ such that $\|S_3[f]\|_3 \leq c$ implies univalence of f. Indeed, the function $f(z) = e^{az}$ is not univalent in $\boldsymbol{D}$ for $|a| > \pi$ but $S_3[f] = 0$.

In the sequel, we mainly consider Tamanoi's Schwarzian derivatives. We now give another description of them. Define a sequence of polynomials $P_n = P_n(x_1, \ldots, x_n)$ of n indeterminates $x_1, \ldots, x_n$ inductively by $P_0 = 1$, $P_1 = 0$, $P_2 = x_2 - 3x_1^2/2$, and

$$P_n = \sum_{k=1}^{n-1}(x_{k+1} - x_1 x_k)\frac{\partial P_{n-1}}{\partial x_k} + \frac{1}{2}P_2 \sum_{k=1}^{n-1}\binom{n}{k}P_{k-1}P_{n-k-1}, \quad n \geq 3.$$

For instance, $P_3 = x_3 - 4x_1x_2 + 3x_1^3$. Then, by letting $q_n[f] = f^{(n+1)}/f'$, we have (see [13])

$$S_n[f] = P_n(q_1[f], q_2[f], \ldots, q_n[f]), \quad n \geq 0.$$

By using the above expression of $S_n[f]$, we define the higher-order invariant Schwarzian derivatives $\Sigma^n f$ for a nonconstant holomorphic map $f : \Omega \to \Omega'$ between plane domains Ω and Ω' with conformal metrics ρ and σ, respectively, by

$$\Sigma^n f = P_n(Q^1 f, \ldots, Q^n f), \quad n \geq 0.$$

We sometimes write $\Sigma_{\sigma,\rho}^n f$ for $\Sigma^n f$ to indicate the metrics involved. For brevity, we also write Σf for $\Sigma^2 f$. More concretely,

$$(3.2) \qquad \Sigma f = Q^2 f - \frac{3}{2}(Q^1 f)^2 = \frac{D^3 f}{D^1 f} - \frac{3}{2}\left(\frac{D^2 f}{D^1 f}\right)^2.$$

We can deduce the following formula from (3.1) (see [13]):

$$(3.3) \qquad \Sigma^n f = (\partial_\rho - (n-1)\partial_\rho \log \rho)\Sigma^{n-1}f + \frac{1}{2}\Sigma^2 f \sum_{k=1}^{n-1}\binom{n}{k}\Sigma^{k-1}f\Sigma^{n-k-1}f$$

for $n \geq 3$.

We record an invariance property of these quantities in the following form.

LEMMA 3.2.   *Let $\Omega, \hat{\Omega}, \Omega', \hat{\Omega}'$ be plane domains with conformal metrics $\rho, \hat{\rho}, \sigma, \hat{\sigma}$, respectively. Suppose that locally isometric holomorphic maps $g : \hat{\Omega} \to \Omega$ and $h : \Omega' \to \hat{\Omega}'$*

are given. Then, for a nonconstant holomorphic map $f : \Omega \to \Omega'$, *the following transformation rule is valid*:

$$Q^n_{\hat{\sigma},\hat{\rho}}(h \circ f \circ g) = (Q^n_{\sigma,\rho} f) \circ g \cdot \left(\frac{g'}{|g'|}\right)^n,$$

$$\Sigma^n_{\hat{\sigma},\hat{\rho}}(h \circ f \circ g) = (\Sigma^n_{\sigma,\rho} f) \circ g \cdot \left(\frac{g'}{|g'|}\right)^n.$$

See [12, Lemma 3.6] for the proof of the relation for Q^n. For the proof of the relation for Σ^n, we observe that P_n is of weight n, that is, P_n is a linear combination of monomials of weight n. Here, the weight of a monomial $x_{j_1} \cdots x_{j_k}$ is defined to be the number $j_1 + \cdots + j_k$. Therefore, $\Sigma^n f$ obeys the same transformation rule as that of $Q^n f$. See [13, Lemma 4.1] for details.

By the last lemma, $Q^n f$ and $\Sigma^n f$ can be defined for a nonconstant holomorphic map f between Riemann surfaces with conformal metrics as a suitable differential form. In particular, if $f : R \to R'$ is a locally isometric holomorphic map, then the local coordinates can be chosen so that $f = \mathrm{id}$ and $\sigma = \rho$, and hence, $D^n f = 0$ for $n \geq 2$ and $Q^n f = \Sigma^n f = 0$ for $n \geq 1$. Note that the Riemann surfaces are *not* required to be projective for the definition of $Q^n f$ and $\Sigma^n f$.

4. Relationship between invariant and classical Schwarzians. It is fundamental to have a relation between the invariant Schwarzian $\Sigma f = \Sigma^2 f$ and the classical Schwarzian Sf. To this end, we introduce a few quantities associated with a conformal metric.

Let $\rho = \rho(z)|dz|$ be a (smooth) conformal metric on a Riemann surface R. We recall that the Gaussian curvature of ρ is defined as

(4.1)
$$\kappa_\rho = -\frac{\Delta \log \rho}{\rho^2} = -4\frac{\partial \bar{\partial} \log \rho}{\rho^2}.$$

Note that κ_ρ does not depend on the particular choice of local coordinates.

For the hyperbolic metric λ_Ω of a plane domain Ω, Minda [18] considered the Schwarzian derivative $2\partial^2 \log \lambda_\Omega - 2(\partial \log \lambda_\Omega)^2$ and showed that this behaves like the Schwarzian derivative. Indeed, suppose that we are given plane domains Ω and $\tilde{\Omega}$ with general conformal metrics $\rho(z)|dz|$ and $\tilde{\rho}(w)|dw|$ respectively, and a conformal mapping g of Ω onto Ω' such that $\tilde{\rho} \circ g \cdot |g'| = \rho$. Then, by following the computation in [18], we obtain

$$\partial \log \rho = (\partial \log \hat{\rho}) \circ g \cdot g' + \frac{g''}{2g'}$$

and therefore,

$$2\partial^2 \log \rho - 2(\partial \log \rho)^2 = 2[(\partial^2 \log \hat{\rho}) \circ g - (\partial \log \hat{\rho})^2 \circ g](g')^2 + Sg.$$

This can be regarded as an analog of Cayley's formula for Schwarzian derivatives.

Noting that $Sg = 0$ when g is Möbius, as a consequence of the last formula, we extend this quantity for a conformal metric on a projective surface.

LEMMA 4.1.   *For a projective surface R with a conformal metric ρ, let*

$$\Theta_{R,\rho}(z) = 2\frac{\partial^2 \rho}{\rho} - 4\left(\frac{\partial \rho}{\rho}\right)^2 = 2\partial^2 \log \rho - 2(\partial \log \rho)^2$$
$$= 2\left(\frac{\partial}{\partial z}\right)^2 \log \rho(z) - 2\left(\frac{\partial \log \rho(z)}{\partial z}\right)^2.$$

Then $\Theta_{R,\rho}(z)dz^2$, evaluated with projective coordinates z, are pieced together to a smooth quadratic differential $\Theta_{R,\rho}$ on R.

REMARK 4.2.   Here and in the sequel, we adopt this redundant-looking notation $\Theta_{R,\rho}$ because this quantity depends not only on the metric ρ, but also on the projective structure of R. Thus the reader should note that R in the subscript notation indicates rather the projective structure than the underlying surface.

The following well-known fact is important in the sense that the metric of constant curvature yields a holomorphic quadratic differential. By solving the Schwarzian differential equation, in turn, we can reproduce the metric (see [15] for details). For convenience of the reader, we supply a short proof as well.

LEMMA 4.3.   *Let R be a projective surface with a smooth conformal metric ρ. Then $\Theta_{R,\rho}$ is a holomorphic quadratic differential on R if and only if ρ has a constant Gaussian curvature.*

PROOF.   By (4.1), we have $\bar{\partial}\partial \log \rho = -\kappa_\rho \rho^2/4$ and therefore $\bar{\partial}\Theta_{R,\rho} = -\rho^2 \partial\kappa_\rho/2$. Thus we see now that $\Theta_{R,\rho}$ is holomorphic if and only if $\partial\kappa_\rho = 0$. Since κ_ρ is real-valued, the last condition is equivalent to that κ_ρ is constant. $\square$

EXAMPLE 4.4.   The standard domain $\boldsymbol{C}_\delta$ has the complete metric $\lambda_\delta = (1 + \delta|z|^2)^{-1}$ $|dz|$ of constant Gaussian curvature 4δ and has the natural projective structure. These metrics are called *spherical*, *Euclidean*, and *hyperbolic* according to the cases when $\delta = +1, 0$, and -1. The metric λ_δ will be called *standard*. Since $\partial \log \lambda_\delta = -\delta\bar{z}/(1 + \delta|z|^2)$, we easily see that $\Theta_{\boldsymbol{C}_\delta, \lambda_\delta} = 0$ for $\delta = +1, 0, -1$.

Let $h : \boldsymbol{C}_\delta \rightarrow R$ be a holomorphic universal covering projection of $\boldsymbol{C}_\delta$ onto a Riemann surface R. Since the covering transformations are contained in the group $\text{Isom}^+(\boldsymbol{C}_\delta)$ of isometries on $(\boldsymbol{C}_\delta, \lambda_\delta)$, the metric λ_δ projects to a metric λ_R, which will be called the standard metric of R. Thus, λ_R is a smooth conformal metric on R of constant Gaussian curvature 4δ such that $h^*(\lambda_R) = \lambda_\delta$. Here we record the following observation.

LEMMA 4.5.   *Let R be a Riemann surface with standard metric λ_R and standard projective structure. Then $\Theta_{R,\lambda_R} = 0$.*

The following result connects the invariant Schwarzian derivative Σf with the classical one Sf.

THEOREM 4.6. *Let R, R' be projective surfaces with smooth conformal metrics ρ, σ, respectively, and let $f : R \to R'$ be a nonconstant holomorphic map. Then*

$$(4.2) \qquad \Sigma_{\rho,\sigma} f = \rho^{-2}[S_{R,R'} f + f^* \Theta_{R',\sigma} - \Theta_{R,\rho}],$$

where $f^*\Theta_{R',\sigma}$ is the pull-back $(\Theta_{R',\sigma} \circ f)(f')^2$ as a quadratic differential.

PROOF. By taking projective coordinates, we may assume that R and R' are plane domains. First, by Lemma 3.1, we have

$$Q^2 f = \partial_\rho(Q^1 f) + \big[Q^1 f - \partial_\rho \log \rho\big] Q^1 f .$$

Substituting the last formula into (3.2), we also have

$$\Sigma f = \partial_\rho Q^1 f - \frac{1}{2}(Q^1 f)^2 - (\partial_\rho \log \rho) Q^1 f .$$

Since

$$(4.3) \qquad \begin{aligned} Q^1 f = \frac{D^2 f}{D^1 f} &= 2\partial_\rho \log(\sigma \circ f) + \partial_\rho \log f' - 2\partial_\rho \log \rho \\ &= \rho^{-1}[2(\partial \log \sigma) \circ f \cdot f' + f''/f' - 2\partial \log \rho], \end{aligned}$$

the relation

$$\begin{aligned} \partial_\rho Q^1 f = &-(\partial_\rho \log \rho) Q^1 f \\ &+ \rho^{-2}[2(\partial^2 \log \sigma) \circ f \cdot (f')^2 + 2(\partial \log \sigma) \circ f \cdot f'' + (f''/f')' - 2\partial^2 \log \rho] \end{aligned}$$

can be obtained. Therefore, we have

$$\begin{aligned} \rho^2 \Sigma f = &\, 2(\partial^2 \log \sigma) \circ f \cdot (f')^2 + 2(\partial \log \sigma) \circ f \cdot f'' + (f''/f')' - 2\partial^2 \log \rho \\ &- \frac{1}{2}(\rho Q^1 f)^2 - 2(\partial \log \rho)(\rho Q^1 f) . \end{aligned}$$

In view of (4.3) we compute

$$\begin{aligned} \frac{1}{2}(\rho Q^1 f)^2 + 2(\partial \log \rho)(\rho Q^1 f) &= \frac{1}{2}(\rho Q^1 f + 4\partial \log \rho)(\rho Q^1 f) \\ &= \frac{1}{2}\big[2(\partial \log \sigma) \circ f \cdot f' + f''/f'\big]^2 - 2(\partial \log \rho)^2 \\ &= 2(\partial \log \sigma)^2 \circ f \cdot (f')^2 + 2(\partial \log \sigma) \circ f \cdot f'' + \frac{1}{2}(f''/f')^2 - 2(\partial \log \rho)^2 , \end{aligned}$$

and substitute it into the last expression of $\rho^2 \Sigma f$ to get the required relation. □

COROLLARY 4.7. *Let R be a Riemann surface with conformal metric ρ and let $\varepsilon \in \{+1, 0, -1\}$. For a holomorphic map $f : R \to \boldsymbol{C}_\varepsilon$ and a Möbius transformation M with $M(f(R)) \subset \boldsymbol{C}_\varepsilon$,*

$$\Sigma^n_{\rho,\lambda_\varepsilon}(M \circ f) = \Sigma^n_{\rho,\lambda_\varepsilon} f , \quad n \geq 0 .$$

PROOF.   In view of the formula (3.3), it suffices to show the relation for $n = 0, 1, 2$ by induction. The relation trivially holds for $n = 0, 1$. Thus we may assume that $n = 2$. We assign a projective structure (e.g., the standard one) to R so that we regard R as a projective surface. Since $\Theta_{C_\varepsilon, \lambda_\varepsilon} = 0$ (see Example 4.4), we have

$$\Sigma^2(M \circ f) = \rho^{-2}[S_{R, C_\varepsilon}(M \circ f) - \Theta_{R, \rho}] = \rho^{-2}[S_{R, C_\varepsilon} f - \Theta_{R, \rho}] = \Sigma^2 f$$

by Theorem 4.6.   □

Also, by Example 4.4, we obtain the following.

COROLLARY 4.8.   *Let $\delta, \varepsilon \in \{-1, 0, +1\}$. For a nonconstant holomorphic map $f : C_\delta \to C_\varepsilon$, the following relation holds*:

$$\Sigma f = \lambda_\delta^{-2} S f \,.$$

This relation was observed from time to time for various combinations of standard metrics (see [10], [11] and [17]). Note that a nonconstant holomorphic map $f : C_\delta \to C_\varepsilon$ exists if and only if $\delta \le \varepsilon$.

Let R be a projective surface equipped with the standard metric λ_R. The quadratic differential $\theta_R = \Theta_{R, \lambda_R}$ is called the *uniformizing connection* of a projective surface R (see [14] for the case when R is a hyperbolic plane domain). We remark that, when the projective structure of R is standard, we have $\theta_R = 0$ by Lemma 4.5. The following result gives a way of computing the universal covering projection of a given surface R once we have an explicit form of θ_R. This idea traces back to Henri Poincaré.

COROLLARY 4.9.   *Let R be a projective surface and let $h : C_\delta \to R$ be a holomorphic universal covering projection. Then the uniformizing connection θ_R is a holomorphic quadratic differential on R and related to h by*

$$h^* \theta_R \equiv (\theta_R \circ h) \cdot (h')^2 = -S h \,.$$

PROOF.   By Lemma 4.3, we can see that θ_R is holomoprhic. Note that $\Theta_{C_\delta, \lambda_\delta} = 0$ by Example 4.4 and that $\Sigma h = 0$ because h is a local isometry. We now apply Theorem 4.6 to obtain $S_{C_\delta, R} h + h^* \theta_R = 0$, which is nothing but the required relation.   □

5. Projective Schwarzian derivatives of higher order.

As we noted, Tamanoi's Schwarzian derivatives are not well defined for holomorphic maps between projective Riemann surfaces. However, if the source surface is equipped with a conformal metric, it is possible to define another sort of Schwarzian derivatives of higher order. We begin by recalling a differential-geometric tool to do so. Let $\varphi = \varphi(z) dz^n$ be a smooth n-differential on a Riemann surface R with conformal metric ρ. Then

$$\Lambda_\rho(\varphi) = [\partial \varphi - 2n(\partial \log \rho) \varphi] dz^{n+1}$$

is a well-defined $(n + 1)$-differential on R. Indeed, $\Lambda_\rho(\varphi)$ is nothing but the covariant derivative of φdz^n in z-direction with respect to the Levi-Civita connection of ρ (see [12, §3] for details).

Based on (3.1), we can express $S_n[f]$ in terms of Sf and its higher derivatives in the same way as before. Define a sequence of polynomials $T_n = T_n(x_2, \ldots, x_n)$ of $n-1$ indeterminates with integer coefficients, inductively, by $T_2 = x_2$ and

$$T_n = \sum_{k=2}^{n-1} \frac{\partial T_{n-1}}{\partial x_k} \cdot x_{k+1} + \frac{x_2}{2} \sum_{k=1}^{n-1} \binom{n}{k} T_{k-1} T_{n-k-1}, \quad n \geq 3.$$

Here, we also set $T_0 = 1$ and $T_1 = 0$. For instance, $T_3 = x_3$, $T_4 = x_4 + 4x_2^2$ and $T_5 = x_5 + 13x_2 x_3$. We can also easily verify that T_n is of weight n. Then, we have

(5.1) $$S_n[f] = T_n(Sf, (Sf)', \ldots, (Sf)^{(n-2)}), \quad n \geq 3.$$

Let R and R' be projective surfaces and let ρ be a conformal metric on R. For a non-constant holomorphic map $f : R \to R'$, we define differentials $\mathfrak{D}^n_{R,\rho,R'} f\, dz^n$ $(n \geq 2)$ on R inductively by $\mathfrak{D}^2_{R,\rho,R'} f = S_{R,R'} f$ and

$$\mathfrak{D}^n_{R,\rho,R'} f\, dz^n = \Lambda_\rho(\mathfrak{D}^{n-1}_{R,R'} f\, dz^{n-1}) \quad (n \geq 3),$$

where differentiations are performed with respect to projective coordinates. Namely, $\mathfrak{D}^n_{R,\rho,R'} f\, dz^n = \Lambda_\rho^{n-2}(S_{R,R'} f\, dz^2)$. Furthermore, we define $V^n_{R,\rho,R'} f$ by

$$V^n_{R,\rho,R'} f = T_n(\mathfrak{D}^2_{R,\rho,R'} f, \ldots, \mathfrak{D}^n_{R,\rho,R'} f)$$

for $n \geq 2$. Here, the products of the substituted $\mathfrak{D}^k_{R,\rho,R'}$'s are understood as the tensor products. Since T_n is of weight n, $V^n_{R,\rho,R'} f$ can be regarded as an n-differential on R. We call $V^n_{R,\rho,R'} f$ the *projective Schwarzian derivative* of virtual order n for a nonconstant holomorphic map $f : R \to R'$. When we do not need to indicate the projective structures and/or the conformal metric, we simply write $\mathfrak{D}^n f$ or $\mathfrak{D}^n_\rho f$ for $\mathfrak{D}^n_{R,\rho,R'} f$. We do the same thing for V. Note that $V^2 f$ is nothing but the classical Schwarzian Sf, which is independent of the metric ρ. Furthermore, we have

$$V^3 f = S_3[f] - 4\frac{\partial \rho}{\rho} S_2[f] = (Sf)' - 4\frac{\partial \rho}{\rho} Sf$$

and

$$V^4 f = S_4[f] - 10\frac{\partial \rho}{\rho} S_3[f] + 4\left[7\left(\frac{\partial \rho}{\rho}\right)^2 - \frac{\partial^2 \rho}{\rho}\right] S_2[f].$$

If R and R' are plane domains with the Euclidean metric $|dz|$, we obviously have $\mathfrak{D}^n f = (Sf)^{(n-2)}$ and, by (5.1), $V^n f = S_n[f]$ for $n \geq 2$.

For $R = C_{-1}$ and $R' = C_{+1}$ with standard metrics and standard projective structures, $\mathfrak{D}^3 f = V^3 f$ is same as in (1.3).

It is convenient for future reference to rephrase explicitly the fact that $V^n f$ is a *well-defined n-differential* in the following way.

LEMMA 5.1. *Let Ω be a plane domain with conformal metric ρ. For a Möbius transformation g, set $\hat{\Omega} = g^{-1}(\Omega)$ and $\hat{\rho} = g^*\rho$. For a nonconstant meromorphic function f on*

Ω and a Möbius transformation h, the following relation holds:

$$V_\rho^n(h \circ f \circ g) = (V_\rho^n f) \circ g \cdot (g')^n, \quad n \geq 2.$$

Since every analytic automorphism of $C_{-1} = D$ is Möbius and isometric with respect to λ_{-1}, the following result can be derived immediately (see also (1.2)).

COROLLARY 5.2. *Let f be a nonconstant meromorphic map on D. For an analytic automorphism T of D and a Möbius transformation M,*

$$V^n(M \circ f \circ T) = V^n f \circ T \cdot (T')^n.$$

In particular, $\|V^n(M \circ f \circ T)\|_n = \|V^n f\|_n$, $n \geq 2$. Here V^n is defined for the hyperbolic metric on D.

We also need to consider the "derivatives" of $\Theta_{R,\rho}$. For a projective surface R with conformal metric $\rho = \rho(z)|dz|$, set $\Theta_{R,\rho}^n dz^n = \Lambda_\rho^{n-2}(\Theta_{R,\rho} dz^2)$ for $n \geq 2$. It is a basic problem to find a relation between $\Sigma^n f$ and $V^n f$ for a nonconstant holomorphic map f between projective surfaces with conformal metrics. We treat, however, with the case when $n = 3$ only. Compare with Theorem 4.6.

THEOREM 5.3. *Let R, R' be projective surfaces with smooth conformal metrics ρ, σ, respectively, and let $f : R \to R'$ be a nonconstant holomorphic map. Then*

$$\Sigma_{\sigma,\rho}^3 f = \rho^{-3}[V_{R,\rho,R'}^3 f + f^*\Theta_{R',\sigma}^3 - \Theta_{R,\rho}^3] + 2\rho^{-2} f^*\Theta_{R',\sigma}^2 Q_{\sigma,\rho}^1 f,$$

where $f^*\Theta_{R',\sigma}^n$ is the pull-back $(\Theta_{R',\sigma}^n \circ f)(f')^n$ as an n-differential.

PROOF. By (3.1) with $n = 3$, we have

$$\Sigma^3 f = \partial_\rho \Sigma f - 2\partial_\rho \log \rho \cdot \Sigma f.$$

Letting $U = \rho^2 \Sigma f$, we now see that

$$\rho^3 \Sigma^3 f = \rho^3 \partial_\rho(\rho^{-2} U) - 2\rho \partial_\rho \log \rho \cdot U = \partial U - 4\partial \log \rho \cdot U.$$

Substituting (4.2) into the last formula yields

$$\begin{aligned}
\rho^3 \Sigma^3 f &= [(Sf)' + (\partial \Theta_{R',\sigma}) \circ f \cdot (f')^3 + \Theta_{R',\sigma} \circ f \cdot (2f'f'') - \partial \Theta_{R,\rho}] \\
&\quad -4\partial \log \rho [Sf + \Theta_{R',\sigma} \circ f \cdot (f')^2 - \Theta_{R,\rho}] \\
&= V_{R,\rho,R'}^3 f + f^*\Theta_{R',\sigma}^3 - \Theta_{R,\rho}^3 \\
&\quad +[4(\partial \log \sigma) \circ f \cdot (f')^3 + 2f'f'' - 4(\partial \log \rho)(f')^2]\Theta_{R',\sigma} \circ f.
\end{aligned}$$

The required relation now follows from (4.3). □

In conjunction with Lemma 4.5, we have the following.

COROLLARY 5.4. *For a nonconstant holomorphic map $f : C_\delta \to C_\varepsilon$, the following relation holds:*

$$\Sigma^3 f = \lambda_\delta^{-3} V_{\lambda_\delta}^3 f.$$

6. Applications to univalence criteria. Aharonov [1] (see also [9]) showed that a nonconstant meromorphic function f on the unit disk $\boldsymbol{D}$ is univalent if and only if

$$\sum_{n=1}^{\infty} n \left| \sum_{k=1}^{n} \binom{n-1}{k-1} (-\bar{z})^{n-k} (1-|z|^2)^{k+1} \psi_{k+1}[f](z) \right|^2 \leq 1, \quad z \in \boldsymbol{D}.$$

In particular, the first term gives the Kraus-Nehari theorem (the first half of Theorem A). The second term gives the inequality

$$\left| (1-|z|^2)^3 \psi_3[f](z) - \bar{z}(1-|z|^2)^2 \psi_2[f](z) \right| = \frac{(1-|z|^2)^3 |Vf(z)|}{24} \leq \frac{1}{\sqrt{2}}$$

for f univalent in $\boldsymbol{D}$. Thus we have $\|Vf\|_3 \leq 12\sqrt{2} \approx 16.97$, which is close to the estimate in Theorem 1.1. Here, $Vf = V_{\lambda_{-1}}^3 f$ with the notation introduced in the previous section.

We are now in a position to prove Theorem 1.1. First let f be univalent in $\boldsymbol{D}$. For an arbitrary $a \in \boldsymbol{D}$, letting $T(z) = (z+a)/(1+\bar{a}z)$, by Corollary 5.2 we have the relation

$$(1-|a|^2)^3 |Vf(a)| = |(Vf) \circ T(0)||T'(0)|^3 = |V(f \circ T)(0)|.$$

Since $\|V(f \circ T)\|_3 = \|Vf\|_3$ by Corollary 5.2, it is enough to show that $|Vf(0)| \leq 16$ for univalent meromorphic function f on $\boldsymbol{D}$. We may further assume that $f(0) \neq \infty$. Then we look at the Laurent expansion

$$\frac{f'(0)}{f(w) - f(0)} = \frac{1}{w} - \sum_{n=0}^{\infty} c_n w^n = \frac{1}{w} - \sum_{n=1}^{\infty} \psi_n[f](0) w^{n-1}$$

to have $\psi_3[f](0) = c_2$. It is known that $|c_2| \leq 2/3$ and equality holds if and only if

$$\frac{f'(0)}{f(w) - f(0)} + \frac{f''(0)}{2 f'(0)} = \frac{(1 - e^{i\theta} w^3)^{2/3}}{w}$$

for a real constant θ (see, for instance, [6, Theorem 4.6, p. 135]). Therefore, we have

$$|Vf(0)| = |S_3[f](0)| = 24|\psi_3[f](0)| = 24|c_2| \leq 16,$$

where equality holds if and only if f is a Möbius transformation of the function

$$f_0(w) = \frac{w}{(1 - e^{i\theta} w^3)^{2/3}}, \quad w \in \boldsymbol{D},$$

for a real constant θ.

The latter part of Theorem 1.1 follows from the next proposition together with Theorem A.

PROPOSITION 6.1. *For a locally univalent meromorphic function f on the unit disk, the inequalities*

$$\frac{16}{25\sqrt{5}} \|Vf\|_3 \leq \|Sf\|_2 \leq \frac{4}{3} \|Vf\|_3$$

hold. Here, the constant $16/25\sqrt{5}$ is sharp.

We are left to prove Proposition 6.1 only. The following representation formula will be the main tool for the proof.

LEMMA 6.2.   *Let f be a locally univalent meromorphic function on the unit disk with* $\|Vf\|_3 < \infty$. *Then*

$$Sf(z) = -\frac{1}{\pi} \iint_{|\zeta|<1} \frac{(1-|\zeta|^2)^4 Vf(\zeta)}{(1-|z|^2)^4(\bar{\zeta}-\bar{z})} d\xi d\eta \quad (\zeta = \xi + i\eta).$$

PROOF.   Let $\rho(z) = 1/(1-|z|^2)$ as before. Then, by definition, we have

$$(6.1) \qquad\qquad Vf = (Sf)' - 4\frac{\partial\rho}{\rho}Sf = \rho^4\partial(\rho^{-4}Sf) \quad \text{on } \mathbf{D}.$$

Define functions ψ and φ on $\mathbf{C}$ by $\psi = \rho^{-4}Sf$, $\varphi = \rho^{-4}Vf$ on $\mathbf{D}$ and $\psi = \varphi = 0$ on $\mathbf{C} \setminus \mathbf{D}$. Since $\rho^{-3}Vf$ is bounded on $\mathbf{D}$ by assumption, we see that $\varphi \in C_0^0(\mathbf{C})$, where $C_0^k(\mathbf{C})$ denotes the class of continuous functions on $\mathbf{C}$ with compact support which have continuous partial derivatives up to order k.

We show now that ψ is in $C_0^1(\mathbf{C})$. Consider the function $y(t) = Sf(t\zeta)$ in $0 \le t < 1$ for a fixed $\zeta \in \partial\mathbf{D}$. By (1.3), we see that $y(t)$ satisfies the linear differential equation

$$y'(t) - \frac{4t}{1-t^2}y(t) = v(t),$$

where $v(t) = \zeta \cdot Vf(t\zeta)$. Thus

$$(1-t^2)^2 y(t) = \int_0^t (1-s^2)^2 v(s)ds + y(0).$$

Since $|v(s)| \le \|Vf\|_3(1-s^2)^{-3}$, we obtain

$$(1-t^2)^2|y(t)| \le \|Vf\|_3 \int_0^t (1-s^2)^{-1}ds + |Sf(0)| = \frac{1}{2}\|Vf\|_3 \log\frac{1+t}{1-t} + |Sf(0)|.$$

In view of (1.3), we thus have

$$Sf(z) = O\left(\frac{1}{(1-|z|^2)^2}\log\frac{1+|z|}{1-|z|}\right) \quad \text{and} \quad (Sf)'(z) = O\left(\frac{1}{(1-|z|^2)^3}\log\frac{1+|z|}{1-|z|}\right)$$

as $|z| \to 1^-$. In particular, $\rho(z)^{-3}Sf(z) \to 0$ and $\rho(z)^{-4}(Sf)'(z) \to 0$ when $|z| \to 1^-$. It is now easy to verify that $\psi \in C_0^1(\mathbf{C})$. Hence, by (6.1) it is confirmed that $\partial\psi = \varphi$ on $\mathbf{C}$ and, equivalently, $\bar{\partial}\bar{\psi} = \bar{\varphi}$ on $\mathbf{C}$.

Consider now the integral transform

$$h(z) = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\overline{\varphi(\zeta)}}{\zeta - z} d\xi d\eta$$

of $\bar{\varphi}$. It is known (see [2]) that h is Hölder continuous with exponent less than 1 and satisfies $\bar{\partial}h = \bar{\varphi}$ (in the sense of distribution). Therefore, $\bar{\partial}(h-\bar{\psi}) = \bar{\partial}h - \bar{\varphi} = 0$ and, by Weyl's lemma, $g = h - \bar{\psi}$ is holomorphic on $\mathbf{C}$. Since $g(z) \to 0$ as $z \to \infty$, we conclude that $g = 0$ by the Liouville theorem. Thus, we have $h = \bar{\psi}$, which is equivalent to the required relation.   □

Let $c \geq 0$ and $0 \leq r < 1$. Wirths [25, Satz 1] gave the sharp upper bound of $|g'(z)|$, $|z| = r$, in terms of c and r for holomorphic functions g on $\mathbf{D}$ with $\|g\|_c \leq 1$. His result specialized for $c = 2$ and $r = 0$ can be stated as follows.

LEMMA 6.3. *Every holomorphic function g on the unit disk $\mathbf{D}$ with $\|g\|_2 \leq 1$ satisfies*

$$|g'(0)| \leq \frac{25\sqrt{5}}{16},$$

where equality holds when $g(z) = (25\sqrt{5}/16)z$.

Proof of Proposition 6.1. As in the proof of Theorem 1.1, we can reduce the proof to the assertions $|Vf(0)| \leq (25\sqrt{5}/16)\|Sf\|_2$ and $|Sf(0)| \leq (4/3)\|Vf\|_3$.

Since $Vf(0) = (Sf)'(0)$, the first inequality and its sharpness follow from Lemma 6.3. Indeed, a function f with $Sf(z) = cz$ for a constant c (e.g., $f(z) = F(az)$ for a constant $a \neq 0$, where $F(z) = \mathrm{Bi}(z)/\mathrm{Ai}(z)$ for linearly independent Airy functions $\mathrm{Ai}(z), \mathrm{Bi}(z)$) satisfies the equality.

We next show the second inequality. By Lemma 6.2, we have

$$
\begin{aligned}
|Sf(0)| &\leq \frac{1}{\pi} \iint_{|\zeta|<1} \frac{(1-|\zeta|^2)^4 |Vf(\zeta)|}{|\zeta|} d\xi d\eta \\
&\leq \frac{1}{\pi} \iint_{|\zeta|<1} \frac{(1-|\zeta|^2)\|Vf\|_3}{|\zeta|} d\xi d\eta \\
&= 2\|Vf\|_3 \int_0^1 (1-r^2) dr = \frac{4}{3}\|Vf\|_3.
\end{aligned}
$$

The proof is now complete. □

EXAMPLE 6.4. Meromorphic functions f on the unit disk with the property that $Sf(z) = c(1 - z^2)^{-2}$ for a constant c are sometimes very important (see for example [5]). The function $l(z) = \log \frac{1+z}{1-z}$ has the least norm $\|Sl\|_2 = 2$ within those which have no quasiconformal extension to the Riemann sphere, and it satisfies $Sl(z) = 2(1 - z^2)^{-2}$. Indeed, it is known that f is never univalent in $\mathbf{D}$ if $Sf(z) = c(1 - z^2)^{-2}$ for a constant $c > 2$. On the other hand, the Koebe function $k(z) = z/(1 - z)^2$ has the maximal norm $\|Sk\|_2 = 6$ within univalent meromorphic functions on $\mathbf{D}$ and again satisfies $Sk(z) = -6(1 - z^2)^{-2}$. By the following lemma, we see that $\|Vf\|_3 = 8\sqrt{3}|c|/9$ whenever $Sf(z) = c(1 - z^2)^{-2}$ for a constant c. In particular, $\|Vl\|_3 = 16\sqrt{3}/9$ and $\|Vk\|_3 = 16\sqrt{3}/3$.

LEMMA 6.5. *Suppose that $Sf(z) = (1 - z^2)^{-2}$ in $z \in \mathbf{D}$. Then $\|Vf\|_3 = 8\sqrt{3}/9$.*

PROOF. By (1.3), we have

$$Vf(z) = \frac{4z}{(1-z^2)^3} - \frac{4\bar{z}}{1-|z|^2} \cdot \frac{1}{(1-z^2)^2} = \frac{8i \, \mathrm{Im}\, z}{(1-|z|^2)(1-z^2)^3}.$$

Fix $r \in (0, 1)$ for a while and set $z = re^{i\theta}$. Then

$$|Vf(z)| = \frac{8}{1-r^2}\left(\frac{r^2 \sin^2 \theta}{(1-2r^2 \cos 2\theta + r^4)^3}\right)^{1/2} = \frac{8}{1-r^2} F(r, \theta)^{1/2}.$$

Since $F(r, \theta) = F(r, -\theta) = F(r, \pi - \theta)$, it is enough to consider the case when $0 \le \theta \le \pi/2$. We now have

$$\frac{\partial F}{\partial \theta} = \frac{2r^2 \cos\theta \sin\theta (1 - 6r^2 + r^4 + 4r^2 \cos 2\theta)}{(1 - 2r^2 \cos 2\theta + r^4)^4}.$$

Note that $1 - 6r^2 + r^4 + 4r^2 \cos 2\theta = 0$ if and only if $\sin\theta = (1 - r^2)/(2\sqrt{2}r)$ and that there exists $\theta = \theta(r) \in (0, \pi/2]$ satisfying this relation only when $\sqrt{3} - \sqrt{2} \le r < 1$. In this case, $F(r, \theta)$ takes its maximum value at $\theta = \theta(r)$ within $0 \le \theta \le \pi/2$. Since $1 - 2r^2 \cos(2\theta(r)) + r^4 = 3(1 - r^2)^2/2$, we have

$$F(r, \theta(r)) = \frac{(1 - r^2)^2/8}{27(1 - r^2)^6/8} = \frac{1}{27(1 - r^2)^4}.$$

Thus

$$(1 - r^2)^3 \max_{|z|=r} |Vf(z)| = \frac{8(1 - r^2)^3}{3\sqrt{3}(1 - r^2)^3} = \frac{8}{3\sqrt{3}}$$

for $\sqrt{3} - \sqrt{2} \le r < 1$. When $0 < r < \sqrt{3} - \sqrt{2}$, $F(r, \theta)$ is increasing in $0 < \theta < \pi/2$ and thus $F(r, \theta) \le F(r, \pi/2) = r^2/(1 + r^2)^6 < 1/[27(1 - r^2)^4]$. Therefore, we conclude that $\|Vf\|_3 = 8/(3\sqrt{3}) = 8\sqrt{3}/9$.   □

References

[1]   D. Aharonov, A necessary and sufficient condition for univalence of a meromorphic function, Duke Math. J. 36 (1969), 599–604.

[2]   L. V. Ahlfors, Lectures on quasiconformal mappings, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966.

[3]   K. Astala and F. W. Gehring, Injectivity, the BMO norm and the universal Teichmüller space, J. Analyse Math. 46 (1986), 16–57.

[4]   J. Becker, Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen, J. Reine Angew. Math. 255 (1972), 23–43.

[5]   M. Chuaqui and B. Osgood, Sharp distortion theorems associated with the Schwarzian derivative, J. London Math. Soc. 48 (1993), 289–298.

[6]   P. L. Duren, Univalent functions, Grundlehren Math. Wiss. 259, Springer-Verlag, New York, 1983.

[7]   P. L. Duren, H. S. Shapiro and A. L. Shields, Singular measures and domains not of Smirnov type, Duke Math. J. 33 (1966), 247–254.

[8]   R. C. Gunning, Lectures on Riemann surfaces, Princeton Univ. Press, Princeton, N.J., 1966.

[9]   R. Harmelin, Aharonov invariants and univalent functions, Israel J. Math. 43 (1982), 244–254.

[10]   S. Kim and D. Minda, The hyperbolic and quasihyperbolic metrics in convex regions, J. Anal. 1 (1993), 109–118.

[11]   S. Kim and D. Minda, The hyperbolic metric and spherically convex regions, J. Math. Kyoto Univ. 41 (2001), 297–314.

[12]   S. Kim and T. Sugawa, Invariant differential operators associated with a conformal metric, Michigan Math. J. 55 (2007), 459–479.

[13]   S. Kim and T. Sugawa, Invariant Schwarzian derivatives of higher order, to appear in Complex Anal. Oper. Theory.

[14]   I. Kra, Accessory parameters for punctured spheres, Trans. Amer. Math. Soc. 313 (1989), 589–617.

[15]   D. Kraus and O. Roth, Conformal metrics, arXiv:0805.2235.

[16] O. LEHTO, Univalent functions and Teichmüller spaces, Grad. Texts in Math. 109, Springer-Verlag, New York, 1987.

[17] W. MA AND D. MINDA, Two-point distortion theorems for bounded univalent functions, Ann. Acad. Sci. Fenn. Math. 22 (1997), 425–444.

[18] D. MINDA, The density of the hyperbolic metric near an isolated boundary point, Complex Variables Theory Appl. 32 (1997), 331–340.

[19] S. NAG, The complex analytic theory of Teichmüller spaces, Wiley, New York, 1988.

[20] Z. NEHARI, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc. 55 (1949), 545–551.

[21] E. SCHIPPERS, Distortion theorems for higher order Schwarzian derivatives of univalent functions, Proc. Amer. Math. Soc. 128 (2000), 3241–3249.

[22] E. SCHIPPERS, The calculus of conformal metrics, Ann. Acad. Sci. Fenn. Math. 32 (2007), 497–521.

[23] T. SUGAWA, The universal Teichmüller space and related topics, Proceedings of the International Workshop on Quasiconformal Mappings and their Applications (India) (T. Sugawa, S. Ponnusamy and M. Vuorinen, eds.), 261–289, Narosa Publishing House, 2007.

[24] H. TAMANOI, Higher Schwarzian operators and combinatorics of the Schwarzian derivative, Math. Ann. 305 (1996), 127–151.

[25] K.-J. WIRTHS, Über holomorphe Funktionen, die einer Wachstumsbeschränkung unterliegen, Arch. Math. 30 (1978), 606–612.

DEPARTMENT OF MATHEMATICS EDUCATION
DONGGUK UNIVERSITY
GYEONGJU 780–714
KOREA

E-mail address: sakim@dongguk.ac.kr

GRADUATE SCHOOL OF INFORMATION SCIENCES
TOHOKU UNIVERSITY
AOBA-KU, SENDAI 980–8579
JAPAN

E-mail address: sugawa@math.is.tohoku.ac.jp