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NON EXISTENCE OF HOMOGENEOUS CONTACT METRIC MANIFOLDS OF NONPOSITIVE CURVATURE

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Abstract. We prove that there exist no simply connected homogeneous contact metric manifolds having nonpositive sectional curvature.

1. Introduction. Let (M, η) be a contact manifold of dimension $N = 2n+1$, $n \geq 1$. We recall that η is a 1-form satisfying

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. It is well known that there exists a unique globally defined vector field ξ, called the Reeb vector field, which is transverse to the contact distribution $\mathrm{Ker}(\eta)$, such that $\eta(\xi) = 1$ and

$$d\eta(X, \xi) = 0 \tag{1}$$

for every smooth vector field X on M. There is an extensive literature concerning the Riemannian geometry of *associated metrics* on (M, η), starting from the investigations of Sasaki [11], see [4] and the references therein. An associated metric g is a Riemannian metric for which there exists a (1, 1) tensor field φ such that

$$\varphi^2 = -\mathrm{Id} + \eta \otimes \xi\,, \quad \eta(X) = g(X, \xi)\,, \quad d\eta(X, Y) = g(X, \varphi Y)$$

for every X, Y vector fields on M. The tensors (φ, ξ, η, g) make up a *contact metric structure* on M. By means of a polarization process, one can show that every contact form admits associated metrics (cf. [4, Theorem 4.4]). This paper focuses on the curvature of associated metrics. It was proved by Blair in [6] that no flat associated metric can exist on a contact manifold of dimension $N \geq 5$. On the contrary, three-dimensional flat contact metric manifolds do exist (see [10]). More generally, if $N \geq 5$ and one asks for constant sectional curvature c, then c must be equal to 1, and the structure must be Sasakian (see [8]). It is also relevant that a *compact* contact manifold cannot admit any associated metric of *negative* curvature; this is settled by a result of Zeghib on geodesic plane fields [12], because for every associated metric the integral curves of ξ are geodesics (cf. [10] or [4, Theorem 7.4]).

In the light of these facts, Blair conjectured the non existence of contact metric manifolds having nonpositive curvature, with the exception of the flat 3-dimensional case (see [4] and also [5]).

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In this direction, in the present note we deal with the homogeneous case. Namely, we prove the following result.

THEOREM 1.1. *Let $(M, \varphi, \xi, \eta, g)$ be a homogeneous simply connected contact metric manifold having nonpositive sectional curvature. Then M is 3-dimensional, flat, and it is equivalent to the Lie group $\tilde{E}(2)$, universal covering of the group of Euclidean motions of $\boldsymbol{R}^2$, endowed with a left invariant contact metric structure.*

Here a contact metric manifold is defined to be *homogeneous* if it admits a transitive Lie group of diffeomorphisms preserving the structure tensor fields (φ, ξ, η, g).

We actually prove a more general result. Let (M, η) be a contact manifold. A Riemannian metric g on M will be called *admissible* if the Reeb vector field ξ is orthogonal to the contact distribution $\mathrm{Ker}(\eta)$ with respect to g. Of course, every associated metric is admissible. If (M, η) is homogeneous, by a homogeneous admissible metric we mean an admissible metric such that $I_g(M) \cap \mathrm{Aut}(M, \eta)$ is transitive on M, where $I_g(M)$ denotes the isometry group of g, while $\mathrm{Aut}(M, \eta)$ is the group of diffeomorphisms $f : M \to M$ such that $f^*\eta = \eta$.

THEOREM 1.2. *Let (M, η) be a homogeneous simply connected contact manifold of dimension $N \geq 5$. Then M does not admit any admissible homogeneous Riemannian metric g having nonpositive curvature.*

We remark that Theorem 1.2 implies Theorem 1.1; indeed, the 3-dimensional simply connected homogeneous contact metric manifolds have been completely classified by Perrone [9]. According to this classification, up to equivalence there is a unique such manifold having nonpositive curvature, namely the group $\tilde{E}(2)$ endowed with a standard left invariant contact metric structure, which is flat.

2. Proof of Theorem 1.2. We argue by contradiction. Assume g is an admissible homogeneous Riemannian metric of nonpositive curvature and let $G = I_g(M) \cap \mathrm{Aut}(M, \eta)$. Then according to [2, Corollary 2.6], G contains a solvable simply transitive Lie subgroup S; we can therefore identify M with the group S, and transfer η and g to left invariant tensor fields on S. We shall denote by $\langle\, ,\, \rangle$ the scalar product induced canonically by g on the Lie algebra $\mathfrak{s}$ of S. Observe that the Reeb vector field ξ is invariant under the action of G, so it also yields a left invariant vector field on S. We shall prove that (S, g) is flat, which gives the desired contradiction, according to a result of Diatta which excludes the existence of flat left invariant metrics on contact Lie groups of dimension at least five (cf. [7, Theorem 3]).

According to the structure theory of Azencott and Wilson [2, 3] (cf. also [1]), the metric Lie algebra $(\mathfrak{s}, \langle\, ,\, \rangle)$ admits an orthogonal decomposition

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}\,,$$

where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is the derived ideal, and $\mathfrak{a}$ is an abelian subalgebra. Moreover, let $S \cong S_o \times S_+$ be the de Rham decomposition of the Riemannian manifold (S, g), where S_o is an Euclidean space and S_+ is a homogeneous Riemannian manifold of nonpositive curvature and having no Euclidean factor. Both S_o and S_+ are isometric to two solvmanifolds, with correspondig metric Lie algebras $\mathfrak{s}_o$ and $\mathfrak{s}_+$, which are Lie subalgebras of $\mathfrak{s}$, such that:

$$\mathfrak{s} = \mathfrak{s}_o \oplus \mathfrak{s}_+ \,. \tag{2}$$

Moreover, we have orthogonal decompositions

$$\mathfrak{s}_o = \mathfrak{a}_o \oplus \mathfrak{n}_o \,, \quad \mathfrak{s}_+ = \mathfrak{a}_+ \oplus \mathfrak{n}_+ \,, \tag{3}$$

where $\mathfrak{a}_o$ and $\mathfrak{n}_o$ are subalgebras of $\mathfrak{a}$ and $\mathfrak{n}$, while $\mathfrak{a}_+$, $\mathfrak{n}_+$ are the respective othogonal complements in $\mathfrak{a}$ and in $\mathfrak{n}$ (see [3, Theorem 4.6]). We have the following characterization of $\mathfrak{a}_o$ (cf. [3, 3.8]):

$$\mathfrak{a}_o = \{A \in \mathfrak{a} \,;\ \mathrm{ad}_{A|\mathfrak{n}} \ \text{is skew symmetric}\} \,.$$

It is also known that $\mathfrak{n}_o$ is central in $\mathfrak{n}$ (cf. [3, 3.4]); moreover

$$\mathrm{ad}_{A|\mathfrak{n}_o} \ \text{is skew symmetric for all } A \in \mathfrak{a} \tag{4}$$

(see [3, 3.8]). From these facts it follows that $\mathfrak{n}_+$ is an ideal of $\mathfrak{s}$.

By definition of an admissible Riemannian metric and using (1), we see that

$$\langle [X, \xi],\, \xi \rangle = 0 \quad \text{for all} \ \ X \in \mathfrak{s}. \tag{5}$$

We decompose $\xi = \xi_o + \xi_+ = A_o + N_o + \xi_+$ according to (2) and (3). Then from (5), for every $A \in \mathfrak{a}_+$ we get

$$\begin{aligned} 0 &= \langle [A, \xi], \xi \rangle \\ &= \langle [A, \xi_o], \xi \rangle + \langle [A, \xi_+], \xi_+ \rangle \\ &= \langle [A, N_o], \xi \rangle + \langle [A, \xi_+], \xi_+ \rangle \\ &= \langle [A, N_o], N_o \rangle + \langle [A, \xi_+], \xi_+ \rangle \\ &= \langle [A, \xi_+], \xi_+ \rangle \,, \end{aligned}$$

where we have used (4). Now, since S_+ has no Euclidean factor, it is known that there exists $A \in \mathfrak{a}_+$ such that the symmetric part of $\mathrm{ad}_A : \mathfrak{n}_+ \to \mathfrak{n}_+$ is positive definite (cf. [13, Lemma 3.1]). From this and the computation above, it follows that $\xi_+ \in \mathfrak{a}_+$. Let $X \in \mathfrak{s}$ and let $Y \in \mathfrak{n}_+$. Then we have $[X, Y] \in \mathfrak{n}_+$, so that:

$$\langle [X, Y], \xi \rangle = \langle [X, Y], \xi_+ \rangle = 0 \,.$$

This implies that $d\eta(X, Y) = 0$ for every $X \in \mathfrak{s}$ and $Y \in \mathfrak{n}_+$. Accordingly, we must have $\mathfrak{n}_+ \subset \mathrm{span}(\xi)$, and thus actually $\mathfrak{n}_+ = \{0\}$ which in turn forces $\mathfrak{s}_+ = \{0\}$, i.e., forces (S, g) to be flat as claimed.

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