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SYMMETRIC CANTOR MEASURE, COIN-TOSSING AND SUM SETS

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Abstract. Construct a probability measure μ on the circle by successive removal of middle third intervals with redistributions of the existing mass at the nth stage being determined by probability p_n applied uniformly across that level. Assume that the sequence $\{p_n\}$ is bounded away from both 0 and 1. Then, for sufficiently large N, (estimates are given) the Lebesgue measure of any algebraic sum of Borel sets $E_1, E_2, \dots, E_N$ exceeds the product of the corresponding $\mu(E_i)^\alpha$, where α is determined by N and $\{p_n\}$. It is possible to replace 3 by any integer $M \ge 2$ and to work with distinct measures $\mu_1, \mu_2, \dots, \mu_N$.

This substantially generalizes work of Williamson and the author (for powers of single-coin coin-tossing measures in the case $M = 2$) and is motivated by the extension to $M = 3$.

We give also a simple proof of a result of Yin and the author for random variables whose binary digits are determined by coin-tossing.

1. Introduction. When working with convolutions of probability measures we find it natural to consider sum sets and related measure estimates. Curiously, perhaps, the latter were slow to develop inasmuch as, in 1947, Marshall Hall Jr, [6] proved that under certain conditions the sums of two Cantor type sets may contain an interval, but the first metrical result involving measure estimates appears to have been given in 1983 by Moran and the present author [2] (and independently in 1985 by Hajela and Seymour [5]). Further historical comments are given in [1].

To be precise we define the sum set $E_1 + E_2 + \cdots + E_N$ of subsets E_j, $j = 1, \dots, N$ of $[0, 1)$ by

$$E_1 + E_2 + \cdots + E_N = \{x_1 + x_2 + \cdots + x_N \pmod 1\,;\, x_j \in E_j\}\,.$$

The early result, just mentioned, is that for Borel sets E and F,

$$\lambda(E + F) \ge \mu(E)^\alpha \mu(F)^\alpha\,,$$

for $\alpha = \log 3/\log 4$, where λ is the Lebesgue measure on $[0, 1)$ and μ is Lebesgue's singular measure on the Cantor middle third set. There is a good account in Yin's thesis [7] of further developments, which typically take summands non-null with respect to singular measures uniformly distributed over sets of numbers missing certain digits in their base M expansion (M being a positive integer, typically 3 or 4).

Williamson and the present author [3] considered different μ, viz. the distribution of random variables whose binary digits are generated by infinitely many tosses of a single biased coin, giving 0 with probability p and 1 with probability $1 - p$. Their result is that

$$\lambda(E_1 + E_2 + \cdots + E_N) \ge \mu(E_1)^\alpha \mu(E_2)^\alpha \cdots \mu(E_N)^\alpha,$$

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for $N \geq \log 2/\log a$, $\alpha = N^{-1}\log 2/\log a$, where $a^{-1} = \max(p, 1-p)$.

In this paper we combine coin-tossing and missing digits, but are still able to obtain metrical results. Let $\delta(s)$ denote the discrete probability measure located at the point s. Let $*$ denote convolution, so that $\delta(s+t) = \delta(s) * \delta(t)$. The original motivation was to consider measures of the type $\mu = *_{n=1}^{\infty}(p_n\delta(0) + (1-p_n)\delta(3^{-n}))$, but, in fact, we can replace 3 by any integer $M \geq 2$. Moreover, in the case $M = 3$, the existence of the isomorphism $t \to 2t$ of $\mathbf{Z}_3$ makes it clear that the proof applies equally well to measures of the type, $\mu = *_{n=1}^{\infty}(p_n\delta(0) + (1-p_n)\delta(2\cdot 3^{-n}))$. This relates more precisely to the classical middle-third construction. In fact, Lebesgue's singular measure on the Cantor set may be constructed by removing the middle third of $[0, 1)$ and then redistributing mass one uniformly over the remaining intervals; then at stage 2 removing the middle third of both remaining intervals and again redistributing mass evenly. In out case the distribution of mass between the two new intervals created by removing the middle third of an interval from stage $n-1$ is governed by a biased coin with probabilities p_n, $(1-p_n)$. This p_n is uniform across stage n. The fact that we obtain precise metrical estimates on the basis of such a weak condition seems interesting from a fractal perspective.

We now state the main result.

THEOREM 1.1. *Let M be an integer greater than one. Let $0 < p_{j,n} < 1$ for $j = 1, 2, \ldots, N$, $n = 1, 2, \ldots$. Suppose that $1 > a^{-1} = \sup_n \max_j(p_{j,n}, (1-p_{j,n}))$, let the integer N be greater than $2M - 4 + (M-1)\log M/\log a$ and let α equal $(N - 2M + 4)^{-1}(\log M/\log a)$. Then, for arbitrary Borel sets E_j,*

$$\lambda(E_1 + E_2 + \cdots + E_N) \geq \mu_1(E_1)^{\alpha} \cdots \mu(E_n)^{\alpha}$$

where λ is the Lebesgue measure on $[0, 1)$ and $\mu_j = *_{n=1}^{\infty}(p_{j,n}\delta(0) + (1-p_{j,n})\delta(M^{-n}))$.

In the next section this will be reduced first to a discrete problem, then to a combinatorial inequality. In Section 3 the combinatorial theorem will be established and, in Section 4, we will give a very simple proof of a basic coin-tossing result of Yin and the author which is not captured by Theorem 1.

These matters were researched while the author held a University Professorship at Tohoku University and he is deeply grateful for an atmosphere of encouragement and generous hospitality.

2. Reduction process. We adopt the definitions of Theorem 1 and show how to reduce the problem, first to a purely discrete one, then to a combinatorial one. The first step is by now relatively standard so we'll not labour the proof but we do require some notation.

Let

$$S_n = \left\{ \sum_{k=1}^{n} \varepsilon_k M^{-k};\, \varepsilon_k \in \{0, 1, \ldots, M-1\} \right\},$$

$$\mu_j^{(n)} = \mathop{*}_{k=1}^{n} (p_{j,n}\delta(0) + (1-p_{j,n})\delta(M^{-k})),$$

and let $\lambda^{(n)}$ be the measure which assigns mass M^{-n} to each point of S_n.

Our aim is to show that, if

$$\lambda^{(k)}(A_1 + A_2 + \cdots + A_N) \geq \mu_1^{(k)}(A_1)^\alpha \mu_2^{(k)}(A_2)^\alpha \cdots \mu_N^{(k)}(A_N)^\alpha , \tag{1}$$

for all subsets $A_1, A_2, \ldots, A_N$ of S_k and $k = 1, 2, \ldots$, then we obtain

$$\lambda(E_1 + E_2 + \cdots + E_N) \geq \mu_1(E_1)^\alpha \mu_2(E_2)^\alpha \cdots \mu_N(E_N)^\alpha , \tag{2}$$

for all Borel subsets $E_1, E_2, \ldots E_N$ of $[0, 1)$.

By regularity of λ, μ_j we may assume all E_j are closed. Let us write $E_{k,j} = A_{k,j} + [0, M^{-k}]$, where $A_{k,j}$ is the subset of S_k which corresponds to the first k terms of the base M expansion of each number in E_j.

We note that

$$E_j = \bigcap_{k=1}^{\infty} E_{k,j} , \qquad E_1 + E_2 + \cdots + E_N = \bigcap_{k=1}^{\infty} (E_{k,1} + E_{k,2} + \cdots + E_{k,N})$$

and that

$$\mu_j(E_j) = \lim_{k\to\infty} \mu_j(E_{k,j}) = \lim_{k\to\infty} \mu_j^{(k)}(A_{k,j}) \tag{3}$$

and

$$\begin{aligned}
\lambda(E_1 + E_2 + \cdots + E_N) &= \lim_{k\to\infty} \lambda(E_{k,1} + E_{k,2} + \cdots + E_{k,N}) \\
&\geq \lim_{k\to\infty} \lambda(A_{k,1} + A_{k,2} + \cdots + A_{k,N} + [0, M^{-k}]) \\
&\geq \lim_{k\to\infty} \lambda^{(k)}(A_{k,1} + A_{k,2} + \cdots + A_{k,N}) \\
&\geq \lim_{k\to\infty} \mu_1^{(k)}(A_k, 1)^\alpha \mu_2^{(k)}(A_k, 2)^\alpha \cdots \mu_N^{(k)}(A_k, N)^\alpha ,
\end{aligned}$$

provided that (1) holds. Applying (3) we find that (1) does indeed imply (2).

Now we will set about proving (1) by induction. This has similarities with the proof in [3] with the added complication of missing digits.

We must check (1) for $k = 1$. Let A_j be non-empty subsets of S_1 and write $A = A_1 + A_2 + \cdots + A_N$. As $\mu_j^{(n)}(M^{-1}) = 0$ for $2 \leq i \leq M - 1$ (the case does not arise when $M = 2$), we may assume that no A_j contains any iM^{-1} for $2 \leq i \leq M - 1$. Now suppose that at least $M - 1$ A_j's have cardinality two. A must then contain $\{0, 1, \ldots, M - 1\} + t$, for some t, so $\lambda(A) = 1$ and (1) certainly holds. In the remaining case there are at least $N - M + 2$ singletons amongst the A_j. Each of these contributes a factor not greater than $\max(p_{j,1}, (1 - p_{j,1}))^\alpha$ to the right hand side of (1). From the definition of a, it will suffice to prove that

$$\lambda(A) \geq a^{-(N-M+2)\alpha} . \tag{4}$$

Certainly $\lambda(A)$ is not less than M^{-1}, so (4) is true provided

$$(N - M + 2)\alpha \log a \geq \log M ,$$

in other words

$$N - M + 2 \geq N - 2M + 4\,,$$

which follows because $M \geq 2$.

Now let us suppose that (1) holds for some k. Consider subsets A_j, $j = 1, \ldots, N$ of S_{k+1}, and let $A = A_1 + \cdots + A_N$. We seek to prove

$$\lambda^{(k+1)}(A) \geq \mu_1^{(k+1)}(A_1)^\alpha \mu_2^{(k+1)}(A_2)^\alpha \cdots \mu_N^{(k+1)}(A_N)^\alpha\,. \tag{5}$$

For any subset B of S_{k+1} and $i = 0, 1, \ldots, M-1$, let

$$B^i = \left\{ \sum_{j=1}^{k} \varepsilon_j M^{-j} ; \sum_{j=1}^{k+1} \varepsilon_j M^{-j} \in B, \varepsilon_{k+1} = i \right\}.$$

Then

$$B = B^0 \cup (B^1 + M^{-k-1}) \cup \cdots \cup (B^{M-1} + (M-1)M^{-k-1})\,, \tag{6}$$

where the union is disjoint. Taking $B = A$, we find

$$\lambda^{(k+1)}(A) = M^{-1}(\lambda^{(k)}(A^0) + \lambda^{(k)}(A^1) + \cdots + \lambda^{(k)}(A^N))\,. \tag{7}$$

Note next that $\mu_j^{(k+1)}(B^i + iM^{-k-1})$ equals zero for $i \geq 2$ for each subset B of S_{k+1}. Applying this to each A_j and noting the form of (5) and (6) we may assume that $A_j^i = (A_j)^i$ is empty for each $i \geq 2$. Thus for $s = 0, 1, \ldots, M-1$, we see that the set $A^s + sM^{-k-1}$ is the union of all these sets of the form

$$(A_1^{i_1} + i_1 M^{-k-1}) + (A_2^{i_2} + i_2 M^{-k-1}) + \cdots + (A_N^{i_N} + i_N M^{-k-1})\,,$$

where $\sum_{j=1}^{N} i_j \equiv s \pmod{M}$ and $i_j \in \{0, 1\}$.

It follows that, for $s = 0, 1, \ldots, M-1$, A^s is a union of translates (by members of S_k) of sets of the form

$$A_1^{i_1} + A_2^{i_2} + \cdots + A_N^{i_N}, \quad \sum_{j=1}^{N} i_j \equiv s \pmod{M}, \quad i_j \in \{0, 1\}\,.$$

Therefore, from (7), we find

$$\begin{aligned}\lambda^{(k+1)}(A) \geq M^{-1} \sum_{s=0}^{M-1} \max \Bigg\{ & \lambda^{(k)}(A_1^{i_1} + A_2^{i_2} + \cdots + A_N^{i_N}) \\ & ; \sum_{j=1}^{N} i_j \equiv s \pmod{M}, \quad i_j \in \{0, 1\} \Bigg\}\end{aligned} \tag{8}$$

Applying the inductive hypothesis we obtain

$$\lambda^{(k+1)} \geq M^{-1} \sum_{s=0}^{M-1} \max \left\{ \mu_1^{(k)}(A_1^{i_1})^{\alpha} \mu_2^{(k)}(A_2^{i_2})^{\alpha} \cdots \mu_N^{(k)}(A_N^{i_N})^{\alpha} \right.$$
(9)
$$\left. ; \sum_{j=1}^{N} i_j \equiv s \pmod{M}, i_j \in \{0,1\} \right\}. $$

Now recall that $A_j = A_j^0 \cup (A_j^1 + M^{-k-1})$ for each j, where the union is disjoint. Therefore

$$\mu_j^{(k+1)}(A_j^0) = x_j \mu_j^{(k+1)}(A_j)\,, \quad \mu_j^{(k+1)}(A_j^1 + M^{-k-1}) = (1-x_j)\mu_j^{(k+1)}(A_j)\,,$$

for some $0 \leq x_j \leq 1$. Also, writing $p_j = p_{j,k+1}$, we have

$$\mu_j^{(k+1)}(A_j^0) = p_j \mu_j^{(k)}(A_j^0)\,, \quad \mu_j^{(k+1)}(A_j^1 + M^{-k-1}) = (1-p_j)\mu_j^{(k)}(A_j^1)\,.$$

Thus

$$\mu_j^{(k)}(A_j^0) = (x_j/p_j)\mu_j^{(k+1)}(A_j)\,, \quad \mu_j^{(k)}(A_j^1) = ((1-x_j)/(1-p_j))\mu_j^{(k+1)}(A_j)\,.$$

Combining this with (9) we obtain

$$\lambda^{k+1}(A) \geq C \mu_1^{(k+1)}(A_1)^{\alpha} \mu_2^{(k+1)}(A_2)^{\alpha} \cdots \mu_N^{(k+1)}(A_N)^{\alpha},$$

where

$$C = M^{-1} \sum_{s=0}^{M-1} \max \left\{ \prod_{j=1}^{N} (x_j/p_j)^{(1-i_j)\alpha} ((1-x_j)/(1-p_j))^{i_j \alpha} \right.$$
$$\left. ; \sum_{j=1}^{N} i_j \equiv s \pmod{M}, i_j \in \{0,1\} \right\}$$

and $0 \leq x_j \leq 1$, $j = 1, \dots, N$.
It will therefore suffice to prove that $C \geq 1$, and this is what we have formulated as Theorem 2 (where we exchanged i_j and $1 - i_j$ for minor convenience).

3. Basic combinatorial result.

THEOREM 3.1. *Let* $0 < p_j < 1$, $j = 1, \dots, N$, *and* $1 > a^{-1} = \max(p_j, (1-p_j))$. *Suppose that* $N \geq 2M - 4 + (M-1)(\log M/\log a)$, *for an integer* $M \geq 2$. *Let* $\alpha = (N - 2M + 4)^{-1}(\log M/\log a)$. *Then, for arbitrary* $0 \leq x_j \leq 1$, $j = 1, \dots, N$,

$$\sum_{s=0}^{M-1} \max \left\{ \prod_{j=1}^{N} (x_j/p_j)^{\alpha i_j} ((1-x_j)/(1-p_j))^{\alpha(1-i_j)} \right.$$
$$\left. ; \sum_{j=1}^{N} i_j \equiv s \pmod{M}, i_j \in \{0,1\} \right\} \geq M\,.$$

PROOF. First we clear away a very special case viz. $M = 2$, $N = 1$. This forces $a = 2$ and hence all $p_j = 1/2$ and $\alpha = 1$. The statement of the theorem becomes

$$2x + 2(1 - x) \geq 2 \quad \text{for all } 0 \leq x \leq 1\,,$$

which is obviously true. Thus we may assume henceforth that $N \geq 2$.

By rearrangement of labels we may assume, for the moment, that (x_j/p_j) is non-increasing. Let us defer consideration of the case where $p_1 > x_1$. This allows us to choose k to be the largest index j such that $x_j \geq p_j$. Note that $1 - x_j > 1 - p_j$, for $j > k$. (Of course it is possible that $k = N$.)

Now let us write

$$y_j = 1 - x_{N-j+1}\,, \quad q_j = 1 - p_{N-j+1}\,, \quad j = 1, \ldots, N\,. \tag{10}$$

Set $l = N-k$ and, for $l \geq 1$, rearrange the set $\{y_j/q_j;\ j = 1, \ldots, l\}$ so that y_j/q_j is non-increasing in this range. (This may disrupt the order of growth of x_j/p_j for $j = k+1, \ldots, N$.)

Let us now return to the deferred case, $p_1 > x_1$, and write $l = N$, rearranging all the y_j/q_j to be non-inceasing. To retain formal symmetry, let us write $k = 0$ when $l = N$, and recall that we already set $l = 0$ when $k = N$.

We now have

$$\begin{aligned} k + l = N\,; \quad x_1/p_1 \geq \cdots \geq x_k/p_k \geq 1 \ \text{ for } \ k \geq 1\,; \\ y_1/q_1 \geq \cdots \geq y_l/q_l \geq 1\,, \quad l \geq 1\,. \end{aligned} \tag{11}$$

The first step is to deal with the case where $l \leq M - 2$. Note at the outset that, when $M = 2$, this gives $k = N \geq 2$ and , when $M \geq 3$, this gives $k \geq N - M + 2 \geq M + 1$.

We choose M products of the type

$$\prod_{j=1}^{N} (x_j/p_j)^{\alpha i_j} ((1 - x_j)/(1 - p_j))^{\alpha(1-i_j)}\,. \tag{12}$$

In fact we choose the first product so that, in (12), $i_j = 1$ for $j = 1, \ldots, k$, and $i_j = 0$ for $j > k$. We choose the next product so that $i_j = 1$ for $j = 1, \ldots, k - 1$, and $i_j = 0$ for $j > k - 1$; and so on until we reach the Mth product when we choose $i_j = 1$ for $j = 1, \ldots, k - (M - 1)$, and $i_j = 0$ for $j > k - M + 1$.

For the successive products $\sum_{j=1}^{N} i_j$ takes values $k, k - 1, \ldots, k - M + 1$, giving a sequence of M distinct residues modulo M. To verify the theorem in the case under discussion, it will suffice to show that the sum of the M products in not less than M.

Let us choose r so that $(1-x_j)/(1-p_j)$ attains its minimum for $j = k-(M-2), \ldots, k$ when $j = r$. Write $u = x_r/p_r$.

In each of the chosen products, consider those factors of the form $((1 - x_j)/(1 - p_j))^{\alpha}$ where $k - M + 2 \leq j \leq k$. The first product we chose has no such term, the next has one factor of this type that is not less than $((1 - p_r u)/(1 - p_r))^{\alpha}$ and so on until the Mth product consists solely of factors of this type for the range $k - M + 2 \leq j \leq k$ and the sub-product of these is not less than $((1 - p_r u)/(1 - p_r))^{(M-1)\alpha}$. Factors of the form $(x_j/p_j)^{\alpha}$ in the same range $k - M + 2 \leq j \leq k$ are not less than one, but the first chosen product certainly has u^{α}

as a factor and we elect to retain that. All the terms $(x_j/p_j)^\alpha$ for $1 \le j \le k - M + 1$ are not less than u^α and all the terms of the form $((1 - x_j)/(1 - p_j))^\alpha$ for $j > k$ are not less than one.

This implies that a lower bound for the sum of the chosen products is

$$u^{(k-M+1)\alpha}\left(u^\alpha + \sum_{t=1}^{M-1}((1 - p_r u)/(1 - p_r))^{t\alpha}\right). \tag{13}$$

Recall that $k \ge N - M + 2$, so out task is reduced to proving that

$$u^\alpha + \sum_{t=1}^{M-1}((1 - p_r u)/(1 - p_r))^{t\alpha} - Mu^{-(N-2M+3)\alpha} \ge 0\,. \tag{14}$$

Because $\alpha \le (M - 1)^{-1}$, the left side of (14) is a concave function of u, and so we need only check the inequality at the end-points $u = 1$, $u = p_r^{-1}$. At $u = 1$, we have equality and, at $u = p_r^{-1}$, we must check that $p_r^{-(N-2M+4)\alpha} \ge M$. Because $a \le p_r^{-1}$, it suffices to check that

$$a^{(N-2M+4)\alpha} \ge M\,. \tag{15}$$

(15) follows immediately from the choice of α, so that this part of the proof is complete. We are now able to assume that $l > M - 2$ and, by symmetry, that $k > M - 2$. By a further appeal to symmetry we may assume

$$x_{k-M+2}/p_{k-M+2} \le y_{l-M+2}/q_{l-M+2} \tag{16}$$

We will choose the same products as before and define r, u in the same way. The change arises in the estimation of the terms $((1 - x_j)/(1 - p_j))^\alpha$ for $j \ge k + M - 1$. These are of the form $(y_j/q_j)^\alpha$ for $j \le l - M + 2$. (Recall (10).) By assumption (16) none of these is less than u^α, and so we have an additional factor of $u^{(l-M+2)\alpha}$ compared with (13). Because

$$k - M + 1 + l - M + 2 = N - 2M + 3\,,$$

the lower bound is

$$u^{(N-2M+3)\alpha}\left(u^\alpha + \sum_{t=1}^{M-1}((1 - p_r u)/(1 - p_r))^{t\alpha}\right),$$

and our task is identical to (14). This completes the proof of the theorem. □

The proof of Theorem 1 is now also complete. That theorem extends all the results of [3] but does not capture the main result of [4]. We will give a simple proof of the latter in the next section.

4. Binary coin-tossing. The next theorem is a mild extension of the main result of [4] but the proof is much simpler than before. Once more we take $0 < p_{j,n} < 1$, $j = 1, 2$, and

$$\mu_1 = \mathop{*}_{n=1}^{\infty} (p_{1,n}\delta(0) + (1 - p_{1,n})\delta(2^{-n}))\,, \quad \mu_2 = \mathop{*}_{n=1}^{\infty} (p_{2,n}\delta(0) + (1 - p_{2,n})\delta(2^{-n}))\,.$$

THEOREM 4.1. *Suppose that*

$$1 > a^{-1} \geq \sup_n \max(p_{1,n}(1 - p_{1,n})) , \quad 1 > b^{-1} \geq \sup_n \max(p_{2,n}, (1 - p_{2,n})) .$$

Then for any Borel sets E_1, E_2, *of* $[0, 1)$ *we have*

$$\lambda(E_1 + E_2) \geq \mu_1(E_1)^\alpha \mu_2(E_2)^\beta$$

provided $\alpha \log a + \beta \log b \geq 1$, *where* $0 < \alpha, \beta \leq 1$.

Either by simple modification of the arguments in Section 2 of this paper or those of [4, Section 3] we can reduce the task of proving Theorem 3 to that of establishing the following combinatorial result.

THEOREM 4.2. *Let* $a^{-1} = \max(p, (1 - p)), b^{-1} = \max(q, (1 - q))$, *where* $0 < p, q < 1$, *let* $0 \leq x, y, \leq 1$ *and let* $0 < \alpha, \beta \leq 1$ *with* $\alpha \log a + \beta \log b \geq \log 2$. *Then*

$$\begin{aligned} &\max\{(x/p)^\alpha (y/q)^\beta, ((1 - x)/(1 - p))^\alpha ((1 - y)/(1 - q))^\beta\} \\ &\quad + \max\{(x/p)^\alpha ((1 - y)/(1 - q))^\beta, ((1 - x)/(1 - p))^\alpha (y/q)^\beta\} \geq 2 . \end{aligned}$$

PROOF. By interchanging $x, (1 - x)$; $p, (1 - p)$ and/or $y, (1 - y)$; $q, (1 - q)$ we can and do assume that $x < p, y < q$. Now we can divide both sides of the required inequality by $((1 - x)^\alpha/(1 - p)^\alpha)((1 - y)^\beta/(1 - q)^\beta)$ and observe that it suffices to prove

$$\begin{aligned} &1 + \max\{(x/p)^\alpha/((1 - x)/(1 - p))^\alpha, (y/q)^\beta/((1 - y)/(1 - q))^\beta\} \\ &\quad \geq 2((1 - x)/(1 - p))^{-\alpha}((1 - y)/(1 - q))^{-\beta} . \end{aligned} \tag{17}$$

Write $u = (x/p)^\alpha((1 - x)/(1 - p))^{-\alpha}$, $v = (y/q)^\beta((1 - y)/(1 - q))^{-\beta}$ and note that (17) transforms to

$$1 + \max(u, v) \geq 2(1 - p + pu^{1/\alpha})^\alpha (1 - q + qv^{1/\beta})^\beta ,$$

where $0 \leq u, v \leq 1$.

We may take $u \geq v$ and note that it will suffice to prove

$$1 + u - 2(1 - p + pu^{1/\alpha})^\alpha (1 - q + qu^{1/\beta})^\beta \geq 0 . \tag{18}$$

Now for positive A, B and $0 < \alpha \leq 1$, the function $(A + Bu^{1/\alpha})^\alpha$ is convex non-decreasing and non-negative, as is the product of two such functions. Accordingly, the left side of (18) is a concave function of u and we need to check only the end points, $u = 0, 1$. At 1, $2 - 2 \geq 0$ and at 0 we need to check that

$$1 \geq 2(1 - p)^\alpha (1 - q)^\beta . \tag{19}$$

The logarithmic condition in the hypothesis gives $1 \geq 2a^{-\alpha}b^{-\beta}$, and, by definition, $a^{-1} \geq (1 - p)$, $b^{-1} \geq (1 - q)$. This shows that (19) holds and completes the proof $\square$

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