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CARLESON INEQUALITIES ON PARABOLIC BERGMAN SPACES

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Abstract. We study Carleson inequalities on parabolic Bergman spaces on the upper half space of the Euclidean space. We say that a positive Borel measure satisfies a (p, q)-Carleson inequality if the Carleson inclusion mapping is bounded, that is, q-th order parabolic Bergman space is embedded in p-th order Lebesgue space with respect to the measure under considering. In a recent paper [6], we estimated the operator norm of the Carleson inclusion mapping for the case q is greater than or equal to p. In this paper we deal with the opposite case. When p is greater than q, then a measure satisfies a (p, q)-Carleson inequality if and only if its averaging function is σ-th integrable, where σ is the exponent conjugate to p/q. An application to Toeplitz operators is also included.

1. Introduction. Let H be the upper half space of the $(n+1)$-dimensional Euclidean space $\boldsymbol{R}^{n+1}$ $(n \geq 1)$, that is, $H = \{X = (x, t)\ ;\ x \in \boldsymbol{R}^n,\ t > 0\}$. For $0 < \alpha \leq 1$, let $L^{(\alpha)}$ be a parabolic operator

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta_x)^\alpha ,$$

where $\Delta_x := \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the Laplacian on the x-space $\boldsymbol{R}^n$. We say that a continuous function u on H is $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions (For the precise definition of $L^{(\alpha)}$-harmonic functions, see [4, §2]). For $1 \leq p < \infty$, the α-parabolic Bergman space $\boldsymbol{b}_\alpha^p$ is the set of all $L^{(\alpha)}$-harmonic functions on H with $\|u\|_{L^p(V)} < \infty$, where V is the Lebesgue volume measure on H and $\|\cdot\|_{L^p(V)}$ is the usual L^p norm. The α-parabolic Bergman space is a Banach space, and it was shown that $1/2$-parabolic Bergman spaces $\boldsymbol{b}_{1/2}^p$ coincide with usual harmonic Bergman spaces ([4, Corollary 4.4] and [5, §3]).

We have an interest in analysis of parabolic Bergman spaces. In this paper we discuss Carleson inequalities on them. Let $1 \leq p, q < \infty$ and μ be a positive Borel measure on H. We say that μ satisfies the (p, q)-Carleson inequality if a mapping $\iota_{\mu,p,q} : \boldsymbol{b}_\alpha^p \to L^q(\mu)$ defined by $\iota_{\mu,p,q}u = u$ is bounded, that is, there exists a constant $C > 0$ such that

$$\left(\int_H |u|^q d\mu\right)^{1/q} \leq C\left(\int_H |u|^p dV\right)^{1/p}$$

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holds for all $u \in \boldsymbol{b}_\alpha^p$. We call this mapping $\iota_{\mu,p,q}$ a Carleson inclusion.

In our paper [6], we studied the case $1 \leq p \leq q < \infty$ and gave a necessary and sufficient condition for μ to satisfy the (p, q)-Carleson inequality. Here we recall it. For $X = (x, t) \in H$, an α-parabolic Carleson box $Q^{(\alpha)}(X) = Q^{(\alpha)}(x, t)$ is defined by

$$Q^{(\alpha)}(x,t) := \{(z_1, \dots, z_n, r) \in H; t \leq r \leq 2t, |x_j - z_j| \leq 2^{-1}t^{1/2\alpha}, j = 1, \dots, n\}.$$

Clearly, $V(Q^{(\alpha)}(x, t)) = t^{n/2\alpha+1}$. For a positive Borel measure μ on H and a real number λ, we define a weighted averaging function $\widehat{Q}_\lambda\mu$ of μ by

$$\widehat{Q}_\lambda\mu(X) := \frac{\mu(Q^{(\alpha)}(X))}{t^{n/2\alpha+1+\lambda}}, \quad X = (x, t) \in H.$$

Our previous result is the following ([6, Theorem 1 and Remark 5]).

THEOREM A. *Let $1 \leq p \leq q < \infty$ and put $\lambda := (n/2\alpha + 1)(q/p - 1)$. Suppose that μ is a positive Borel measure on H. Then there exists a constant $C > 0$ independent of μ such that the inequalities*

$$C^{-1}\|\widehat{Q}_\lambda\mu\|_\infty^{1/q} \leq \|\iota_{\mu,p,q}\| \leq C\|\widehat{Q}_\lambda\mu\|_\infty^{1/q}$$

hold, where $\|\iota_{\mu,p,q}\|$ denotes the operator norm of $\iota_{\mu,p,q} : \boldsymbol{b}_\alpha^p \to L^q(\mu)$. Consequently, μ satisfies the (p, q)-Carleson inequality with $1 \leq p \leq q < \infty$ if and only if the function $\widehat{Q}_\lambda\mu$ is bounded.

Carleson inequalities on analytic or harmonic Bergman spaces were well investigated. The local submean inequality for analytic or harmonic functions is very useful for studying them. However, such a submean inequality is not available for our case. To overcome this difficulty, we used a Whitney type decomposition of the upper half space in the proof of Theorem A. In this paper we deal with the case $q < p$. Although the above argument is unsuitable for this case, an idea of Luecking [3] is effective (see also [2]). In his theory, interpolating sequences play an important role. By the aide of a $\boldsymbol{b}_\alpha^p$-interpolating theorem proved in [8], we have the following theorem. This is the main result of this paper (actually we will give a more general result in Theorem 5.3 below).

THEOREM 1.1. *Let $1 \leq q < p < \infty$ and $1/\sigma + 1/(p/q) = 1$. Suppose that μ is a positive Borel measure on H. Then there exists a constant $C > 0$ independent of μ such that the inequalities*

$$C^{-1}\|\widehat{Q}_0\mu\|_{L^\sigma(V)}^{1/q} \leq \|\iota_{\mu,p,q}\| \leq C\|\widehat{Q}_0\mu\|_{L^\sigma(V)}^{1/q} \tag{1.1}$$

hold. Consequently, μ satisfies the (p, q)-Carleson inequality with $1 \leq q < p < \infty$ if and only if the function $\widehat{Q}_0\mu$ is in $L^\sigma(V)$.

Note that, unlike in the case $p \leq q$, every bounded Carleson inclusion is compact when $q < p$. We discuss this fact in Theorem 5.4 below.

We display the plan of this paper. In Section 2, we describe some basic results on the fundamental solution of our parabolic operator. Since there need some results concerning $\boldsymbol{b}_\alpha^p$-interpolating sequences, we list them here, too. In Section 3, we give a Lipschitz type estimate of the reproducing kernel (the Bergman kernel) of $\boldsymbol{b}_\alpha^p$. This estimate brings a certain uniform continuity for functions in $\boldsymbol{b}_\alpha^p$. In Section 4, we give a lower estimate of the operator norm of Carleson inclusions by using some auxiliary functions. In Section 5, after discussing a relation between weighted averaging functions and our auxiliary functions, we complete the proof of the inequalities in the main result. Compactness of Carleson inclusions is also mentioned. In Section 6, we discuss a Toeplitz operator on parabolic Bergman spaces. Since there is a close relation between Carleson inclusions and the Toeplitz operators, our main result enables us to show the boundedness and compactness of Toeplitz operators.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Preliminaries. For $x \in \boldsymbol{R}^n$, let

$$W^{(\alpha)}(x,t) := \begin{cases} \dfrac{1}{(2\pi)^n}\displaystyle\int_{\boldsymbol{R}^n} \exp(-t|\xi|^{2\alpha} + i x\cdot\xi)d\xi & t>0 \\ 0 & t \le 0\,, \end{cases} \tag{2.1}$$

where $x\cdot\xi$ denotes the inner product on $\boldsymbol{R}^n$ and $|\xi| = (\xi\cdot\xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and it is $L^{(\alpha)}$-harmonic on H. Let $\gamma = (\gamma_1,\dots,\gamma_n) \in \boldsymbol{N}_0^n$ be a multi-index and $k \in \boldsymbol{N}_0$, where $\boldsymbol{N}_0 = \boldsymbol{N}\cup\{0\}$. We use the notation

$$\partial_x^\gamma \partial_t^k := \partial_{x_1}^{\gamma_1}\cdots\partial_{x_n}^{\gamma_n}\partial_t^k = \frac{\partial^{|\gamma|+k}}{\partial x_1^{\gamma_1}\cdots\partial x_n^{\gamma_n}\partial t^k}\,,$$

where $|\gamma| := \gamma_1+\cdots+\gamma_n$. The following estimate is [6, Lemma 1]: There exists a constant $C = C(n,\alpha,\gamma,k) > 0$ such that

$$|\partial_x^\gamma\partial_t^k W^{(\alpha)}(x,t)| \le \frac{C}{(t+|x|^{2\alpha})^{(n+|\gamma|)/2\alpha+k}} \tag{2.2}$$

for all $(x,t) \in H$. The α-parabolic Bergman kernel $R_\alpha(X,Y) = R_\alpha(x,t;y,s)$ is given by

$$R_\alpha(x,t;y,s) := -2\partial_t W^{(\alpha)}(x-y,t+s) = -2\partial_s W^{(\alpha)}(x-y,t+s)\,,$$

and it has the following reproducing property ([4, Theorem 6.3]): For any $1 \le p < \infty$,

$$u(X) = \int_H u(Y)R_\alpha(X,Y)dV(Y) \tag{2.3}$$

for all $X \in H$ and $u \in \boldsymbol{b}_\alpha^p$. We also use a kernel $R_\alpha^{\gamma,k}(X,Y) = R_\alpha^{\gamma,k}(x,t;y,s)$ defined by

$$R_\alpha^{\gamma,k}(x,t;y,s) := c_k s^{|\gamma|/2\alpha+k}\partial_x^\gamma\partial_t^k R_\alpha(x,t;y,s) = (-1)^{|\gamma|}c_k s^{|\gamma|/2\alpha+k}\partial_y^\gamma\partial_s^k R_\alpha(x,t;y,s)\,,$$

where $c_k = (-2)^k/k!$. Note that $R_\alpha(X,Y) = R_\alpha(Y,X)$, while $R_\alpha^{\gamma,k}(X,Y) \ne R_\alpha^{\gamma,k}(Y,X)$ in general.

We use the following lemma frequently in our later arguments.

LEMMA 2.1 ([6, Lemma 5]). *Let $\theta,\eta \in \boldsymbol{R}$. If $0 < 1+\theta < \eta - n/2\alpha$, then there exists a constant $C > 0$ such that*

$$\int_H \frac{t^\theta}{(t+s+|x-y|^{2\alpha})^\eta}dV(x,t) = Cs^{\theta-\eta+n/2\alpha+1}$$

for all $(y, s) \in H$.

Now we define an α-parabolic cylinder, which is used for the definition of separated sequences. For $X = (x, t) \in H$ and $0 < \eta < 1$, an α-parabolic cylinder $S_\eta^{(\alpha)}(X) = S_\eta^{(\alpha)}(x, t)$ is defined by

$$S_\eta^{(\alpha)}(x, t) := \left\{(z, r) \in H;\ |x - z| < \left(\frac{2\eta}{1-\eta^2}t\right)^{1/2\alpha},\ \frac{1-\eta}{1+\eta}t < r < \frac{1+\eta}{1-\eta}t\right\}.$$

Clearly $\lim_{\eta\to 1} S_\eta^{(\alpha)}(X) = H$ and $V(S_\eta^{(\alpha)}(x, t)) = 2B_n(2\eta(1-\eta^2)^{-1}t)^{n/2\alpha+1} = Ct^{n/2\alpha+1}$, where B_n is the volume of the unit ball in $\boldsymbol{R}^n$.

In order to study Carleson inequalities for $1 \leq q < p < \infty$, we need some lemmas concerning $\boldsymbol{b}_\alpha^p$-interpolating sequences. We begin with recalling the definition of separated sequences and interpolating sequences (for details, see [8]). Let $0 < \eta < 1$ and $\boldsymbol{X} = \{X_i\} = \{(x_i, t_i)\}$ be a sequence in H. Then, we say that $\{X_i\}$ is η-separated in the α-parabolic sense if α-parabolic cylinders $S_\eta^{(\alpha)}(X_i)$ are pairwise disjoint. Let $1 \leq p < \infty$. For given $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$ and $u \in \boldsymbol{b}_\alpha^p$, we denote a sequence of real numbers $T_p^{\gamma,k}u$ defined by

$$T_p^{\gamma,k}u := T_{p,\boldsymbol{X}}^{\gamma,k}u = \{t_i^{(n/2\alpha+1)/p+|\gamma|/2\alpha+k}\partial_x^\gamma\partial_t^k u(X_i)\}, \tag{2.4}$$

and we say that $\{X_i\}$ is a $\boldsymbol{b}_\alpha^p$-interpolating sequence of order (γ, k) if $T_p^{\gamma,k} : \boldsymbol{b}_\alpha^p \to \ell^p$ is bounded and onto. The following lemma is our $\boldsymbol{b}_\alpha^p$-interpolating theorem.

LEMMA 2.2 ([8, Theorem 2]). *Let $1 \leq p < \infty$, $\gamma \in \boldsymbol{N}_0^n$, and $k \in \boldsymbol{N}_0$. Then there exists a constant $0 < \eta_0 < 1$ with the following property: If $\{X_i\}$ is η-separated in the α-parabolic sense with $\eta \geq \eta_0$, then $\{X_i\}$ is a $\boldsymbol{b}_\alpha^p$-interpolating sequence of order (γ, k).*

REMARK 2.3. Let η_0 be the constant chosen in Lemma 2.2. Then, by the open mapping theorem and the proof of [8, Theorem 4.4], there exists a constant $M = M(n, \alpha, p, \gamma, k, \eta_0) > 0$ with the following property: For every η-separated sequence $\{X_i\}$ with $\eta \geq \eta_0$ and for each sequence $\{c_i\}$ of real numbers with $\sum |c_i|^p \leq 1$, there exists $u \in \boldsymbol{b}_\alpha^p$ such that $c_i = t_i^{(n/2\alpha+1)/p+|\gamma|/2\alpha+k}\partial_x^\gamma\partial_t^k u(X_i)$ and $\|u\|_{L^p(V)} \leq M$.

We also need the following result.

LEMMA 2.4 ([8, Corollary 2]). *If a sequence $\boldsymbol{X}$ is ε-separated in the α-parabolic sense for some $0 < \varepsilon < 1$, then for any $0 < \eta < 1$, $\boldsymbol{X}$ consists of a finite union of η-separated sequences in the α-parabolic sense.*

3. Lipschitz type estimates of parabolic Bergman kernels. We begin with the following elementary estimates.

LEMMA 3.1. *Let $0 < \alpha \leq 1$ and $\lambda \geq 1$. Suppose $0 < \delta \leq 1/3$. Then, there exists a constant $C = C(\alpha, \lambda) > 0$ independent of δ with the following properties*:

(1) *For every* $a > 0$ *and* $r, t > 0$ *with* $(1-\delta)(1+\delta)^{-1}t < r < (1+\delta)(1-\delta)^{-1}t$, *we have*

$$\left| \int_r^t \frac{1}{(\tau + a)^{\lambda+1}} d\tau \right| \le C \frac{\delta}{(t+a)^{\lambda}} . \tag{3.1}$$

(2) *For every* $a > 0$, $S > 0$ *and* $\xi, T > 0$ *with* $\xi < (2\delta(1-\delta^2)^{-1})^{1/2\alpha} T$, *we have*

$$\int_0^1 \frac{\xi}{(T + S + |a - \xi\tau|)^{\lambda+1}} d\tau \le C \frac{\delta^{1/2\alpha}}{(T+S+a)^{\lambda}} . \tag{3.2}$$

PROOF. (1) We may assume $t \ge r$. Then, since $t - r \le 2\delta t$ and $t + a \le 2(r+a)$, we have

$$\left| \int_r^t \frac{1}{(\tau + a)^{\lambda+1}} d\tau \right| \le \frac{t-r}{(r+a)^{\lambda+1}} \le \frac{C\delta t}{(t+a)^{\lambda+1}} \le \frac{C\delta}{(t+a)^{\lambda}} .$$

(2) Since $0 < \delta \le 1/3$ and $\xi < (2\delta(1-\delta^2)^{-1})^{1/2\alpha} T$, we have $\xi < (3/2)^{1/\alpha} \delta^{1/2\alpha} T$ and

$$T + S + |a - \xi\tau| > T + S + a - \xi > \{1 - (3/4)^{1/2\alpha}\}(T + S + a)$$

for every $0 < \tau < 1$. Hence

$$\int_0^1 \frac{\xi}{(T + S + |a - \xi\tau|)^{\lambda+1}} d\tau \le \frac{C\delta^{1/2\alpha} T}{(T+S+a)^{\lambda+1}} \le \frac{C\delta^{1/2\alpha}}{(T+S+a)^{\lambda}} . \qquad \square$$

Now we can state Lipschitz type estimates of parabolic Bergman kernels. In the sequel, we use the following notation frequently:

$$\varphi(\delta) := \delta + \delta^{1/2\alpha} .$$

PROPOSITION 3.2. *Let* $\gamma \in \boldsymbol{N}_0^n$ *and* $k \in \boldsymbol{N}_0$. *Then there exists a constant* $C = C(n, \alpha, k, \gamma) > 0$ *such that for all* (x, t), $(y, s) \in H$, $(z, r) \in S_\delta^{(\alpha)}(x, t)$, *and* $0 < \delta \le 1/3$,

$$|R_\alpha^{\gamma,k}(x, t; y, s) - R_\alpha^{\gamma,k}(z, r; y, s)| \le C\varphi(\delta) \frac{s^{|\gamma|/2\alpha + k}}{(t + s + |x - y|^{2\alpha})^{(n+|\gamma|)/2\alpha + k + 1}} \tag{3.3}$$

and

$$|R_\alpha^{\gamma,k}(x, t; y, s) - R_\alpha^{\gamma,k}(z, r; y, s)| \le C\varphi(\delta) \frac{s^{|\gamma|/2\alpha + k}}{(r + s + |z - y|^{2\alpha})^{(n+|\gamma|)/2\alpha + k + 1}} . \tag{3.4}$$

PROOF. To prove (3.3), it suffices to show the following inequalities:

$$|R_\alpha^{\gamma,k}(x, t; y, s) - R_\alpha^{\gamma,k}(x, r; y, s)| \le C\delta \frac{s^{|\gamma|/2\alpha + k}}{(t + s + |x - y|^{2\alpha})^{(n+|\gamma|)/2\alpha + k + 1}} \tag{3.5}$$

and

$$|R_\alpha^{\gamma,k}(x, r; y, s) - R_\alpha^{\gamma,k}(z, r; y, s)| \le C\delta^{1/2\alpha} \frac{s^{|\gamma|/2\alpha + k}}{(t + s + |x - y|^{2\alpha})^{(n+|\gamma|)/2\alpha + k + 1}} . \tag{3.6}$$

By the fundamental theorem of calculus and (2.2), we have

$$
\begin{aligned}
|R_\alpha^{\gamma,k}(x,t;y,s) - R_\alpha^{\gamma,k}(x,r;y,s)| &= \left| -2c_k s^{|\gamma|/2\alpha+k} \int_r^t \partial_x^\gamma \partial_s^{k+2} W^{(\alpha)}(x-y,\tau+s)d\tau \right| \\
&\le C s^{|\gamma|/2\alpha+k} \left| \int_r^t |\partial_x^\gamma \partial_s^{k+2} W^{(\alpha)}(x-y,\tau+s)| d\tau \right| \\
&\le C s^{|\gamma|/2\alpha+k} \left| \int_r^t \frac{1}{(\tau+s+|x-y|^{2\alpha})^{(n+|\gamma|)/2\alpha+k+2}} d\tau \right| .
\end{aligned}
$$

Since $(z,r) \in S_\delta^{(\alpha)}(x,t)$, (3.5) follows from (3.1) in Lemma 3.1. Also, since

$$
\begin{aligned}
&R_\alpha^{\gamma,k}(z,r;y,s) - R_\alpha^{\gamma,k}(x,r;y,s) \\
&\qquad = -2c_k s^{|\gamma|/2\alpha+k} \int_0^1 (z-x)\cdot \nabla_x \partial_x^\gamma \partial_t^{k+1} W^{(\alpha)}((\tau(z-x)-(y-x), r+s)) d\tau ,
\end{aligned}
$$

we have

$$
\begin{aligned}
&|R_\alpha^{\gamma,k}(z,r;y,s) - R_\alpha^{\gamma,k}(x,r;y,s)| \\
&\quad \le C s^{|\gamma|/2\alpha+k} \int_0^1 |z-x| \cdot |\nabla_x \partial_x^\gamma \partial_t^{k+1} W^{(\alpha)}((\tau(z-x)-(y-x), r+s))| d\tau \\
&\quad \le C s^{|\gamma|/2\alpha+k} \int_0^1 \frac{|z-x|}{(r+s+|\tau(z-x)-(y-x)|^{2\alpha})^{(n+|\gamma|+1)/2\alpha+k+1}} d\tau \\
&\quad \le C s^{|\gamma|/2\alpha+k} \\
&\qquad \times \int_0^1 \frac{|z-x|}{(((1-\delta)(1+\delta)^{-1}t)^{1/2\alpha} + s^{1/2\alpha} + |\tau|z-x| - |y-x||)^{n+|\gamma|+1+2\alpha(k+1)}} d\tau \\
&\quad \le 2^{(n+|\gamma|+1)/2\alpha+k+1} C s^{|\gamma|/2\alpha+k} \\
&\qquad \times \int_0^1 \frac{|z-x|}{(t^{1/2\alpha} + s^{1/2\alpha} + |\tau|z-x| - |y-x||)^{n+|\gamma|+1+2\alpha(k+1)}} d\tau .
\end{aligned}
$$

Hence, (3.2) in Lemma 3.1 implies

$$
\begin{aligned}
|R_\alpha^{\gamma,k}(z,r;y,s) - R_\alpha^{\gamma,k}(x,r;y,s)| &\le C\delta^{1/2\alpha} \frac{s^{|\gamma|/2\alpha+k}}{(t^{1/2\alpha}+s^{1/2\alpha}+|y-x|)^{n+|\gamma|+2\alpha(k+1)}} \\
&\le C\delta^{1/2\alpha} \frac{s^{|\gamma|/2\alpha+k}}{(t+s+|y-x|^{2\alpha})^{(n+|\gamma|)/2\alpha+k+1}} .
\end{aligned}
$$

To prove (3.4), we remark that $r \le 2t$ and $|y-z| < |y-x| + (3/4)^{1/2\alpha} t^{1/2\alpha}$ whenever $(z,r) \in S_\delta^{(\alpha)}(x,t)$, and hence

$$
r+s+|y-z|^{2\alpha} \le 2t+s+2^{2\alpha}\{|y-x|^{2\alpha} + (3/4)t\} \le 5(t+s+|y-x|^{2\alpha}) .
$$

The inequality (3.4) follows from (3.3) immediately. □

Let μ be a positive Borel measure on H and λ be a real number. We define an auxiliary function $\widehat{S}_{\delta,\lambda}\mu$ on H by

$$\widehat{S}_{\delta,\lambda}\mu(X) := \frac{\mu(S_\delta^{(\alpha)}(X))}{t^{n/2\alpha+1+\lambda}}, \quad X = (x, t) \in H . \tag{3.7}$$

We note that $\widehat{S}_{\delta,0}\mu(X)$ is a constant multiple of the average of μ on a cylinder $S_\delta^{(\alpha)}(X)$. The following theorem shows a certain kind of uniform continuity for derivatives of u in $\boldsymbol{b}_\alpha^p$, which is useful in Section 4.

THEOREM 3.3. *Let* $1 \le q < p < \infty$, $1/\sigma + 1/(p/q) = 1$, $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$, *and put* $\lambda := (|\gamma|/2\alpha + k)\,q$. *Suppose that* μ *is a positive Borel measure on* H *and* $\varphi(\delta) := \delta + \delta^{1/2\alpha}$. *Then, for* $0 < \eta < 1$, *there exists a constant* $L = L(n, \alpha, q, p, \gamma, k, \eta) > 0$ *independent of* u, δ, *and* μ *such that if* $\{X_i\} = \{(x_i, t_i)\}$ *is* η*-separated in the* α*-parabolic sense, then*

$$\begin{aligned}\sum_i \int_{S_\delta^{(\alpha)}(X_i)} &|\partial_x^\gamma \partial_t^k u(X_i) - \partial_x^\gamma \partial_t^k u(Z)|^q \,d\mu(Z) \\ &\le L\varphi(\delta)^q \|u\|_{L^p(V)}^q \left(\sum_i \widehat{S}_{\delta,\lambda}\mu(X_i)^\sigma t_i^{n/2\alpha+1} \right)^{1/\sigma}\end{aligned}$$

for all $u \in \boldsymbol{b}_\alpha^p$ *and* $0 < \delta \le \min\{\eta, 1/3\}$.

PROOF. Let $0 < \delta \le \min\{\eta, 1/3\}$ and $\{X_i\} = \{(x_i, t_i)\}$ be η-separated in the α-parabolic sense. Then, for any $Z \in S_\delta^{(\alpha)}(X_i)$ and $u \in \boldsymbol{b}_\alpha^p$, by (2.3) and (3.3), we have

$$\begin{aligned}|\partial_x^\gamma \partial_t^k u(X_i) - \partial_x^\gamma \partial_t^k u(Z)| &\le \int_H |u(Y)| \cdot |\partial_x^\gamma \partial_t^k R_\alpha(X_i, Y) - \partial_x^\gamma \partial_t^k R_\alpha(Z, Y)| dV(Y) \\ &\le C\varphi(\delta) \int_H |u(Y)| b_\alpha^{\gamma,k}(X_i, Y) dV(Y),\end{aligned}$$

where

$$b_\alpha^{\gamma,k}(X_i, Y) := \frac{1}{(t_i + s + |x_i - y|^{2\alpha})^{(n+|\gamma|)/2\alpha+k+1}} .$$

Let p' be the exponent conjugate to p. Then the Hölder inequality and Lemma 2.1 show

$$\begin{aligned}&|\partial_x^\gamma \partial_t^k u(X_i) - \partial_x^\gamma \partial_t^k u(Z)| \\ &\le C\varphi(\delta) \int_H |u(Y)| s^{1/pp'} \cdot s^{-1/pp'} b_\alpha^{\gamma,k}(X_i, Y) dV(Y) \\ &\le C\varphi(\delta) \left(\int_H |u(Y)|^p s^{1/p'} b_\alpha^{\gamma,k}(X_i, Y) dV(Y) \right)^{1/p} \left(\int_H s^{-1/p} b_\alpha^{\gamma,k}(X_i, Y) dV(Y) \right)^{1/p'} \\ &\le C\varphi(\delta) t_i^{-(|\gamma|/2\alpha+k+1/p)/p'} \left(\int_H |u(Y)|^p s^{1/p'} b_\alpha^{\gamma,k}(X_i, Y) dV(Y) \right)^{1/p} .\end{aligned}$$

Since $1/\sigma + 1/(p/q) = 1$, the Hölder inequality again yields

$$\sum_i \int_{S_\delta^{(\alpha)}(X_i)} |\partial_x^\gamma \partial_t^k u(X_i) - \partial_x^\gamma \partial_t^k u(Z)|^q \,d\mu(Z)$$

$$
\begin{aligned}
&\le C\varphi(\delta)^q \sum_i \mu(S_\delta^{(\alpha)}(X_i)) t_i^{-(|\gamma|/2\alpha+k+1/p)q/p'} \left(\int_H |u(Y)|^p s^{1/p'} b_\alpha^{\gamma,k}(X_i,Y)dV(Y)\right)^{q/p} \\
&= C\varphi(\delta)^q \sum_i t_i^{n/2\alpha+1} \widehat{S}_{\delta,\lambda}\mu(X_i) \left(t_i^{|\gamma|/2\alpha+k-1/p'} \int_H |u(Y)|^p s^{1/p'} b_\alpha^{\gamma,k}(X_i,Y)dV(Y)\right)^{q/p} \\
&\le C\varphi(\delta)^q \left(\sum_i \widehat{S}_{\delta,\lambda}\mu(X_i)^\sigma t_i^{n/2\alpha+1}\right)^{1/\sigma} \\
&\quad \times \left(\sum_i t_i^{(n+|\gamma|)/2\alpha+k+1-1/p'} \int_H |u(Y)|^p s^{1/p'} b_\alpha^{\gamma,k}(X_i,Y)dV(Y)\right)^{q/p}.
\end{aligned}
$$

Hence it suffices to show that there exists a constant $C > 0$ such that

$$
\sum_i t_i^{(n+|\gamma|)/2\alpha+k+1-1/p'} b_\alpha^{\gamma,k}(X_i,Y) \le C s^{-1/p'}
$$

for all $Y = (y,s) \in H$. If $X = (x,t) \in S_\eta^{(\alpha)}(X_i)$, then

$$
\begin{aligned}
t + s + |x-y|^{2\alpha} &\le t + s + \{|x-x_i| + |x_i - y|\}^{2\alpha} \\
&\le \frac{1+\eta}{1-\eta} t_i + s + \left\{ \left(\frac{2\eta}{1-\eta^2} t_i \right)^{1/2\alpha} + |x_i - y| \right\}^{2\alpha} \\
&\le \frac{12}{1-\eta^2} (t_i + s + |x_i - y|^{2\alpha}),
\end{aligned}
$$

so that, by Lemma 2.1, we have

$$
\begin{aligned}
&\sum_i t_i^{(n+|\gamma|)/2\alpha+k+1-1/p'} b_\alpha^{\gamma,k}(X_i,Y) \\
&\quad = \sum_i t_i^{(n+|\gamma|)/2\alpha+k+1-1/p'} b_\alpha^{\gamma,k}(X_i,Y) V(S_\eta^{(\alpha)}(X_i)) \left\{ 2B_n \left(\frac{2\eta}{1-\eta^2} t_i \right)^{n/2\alpha+1} \right\}^{-1} \\
&\quad = C_\eta \sum_i \frac{t_i^{|\gamma|/2\alpha+k-1/p'}}{(t_i + s + |x_i - y|^{2\alpha})^{(n+|\gamma|)/2\alpha+k+1}} \int_{S_\eta^{(\alpha)}(X_i)} dV(X) \\
&\quad \le C_\eta \sum_i \int_{S_\eta^{(\alpha)}(X_i)} \frac{t^{|\gamma|/2\alpha+k-1/p'}}{(t+s+|x-y|^{2\alpha})^{(n+|\gamma|)/2\alpha+k+1}} dV(X) \\
&\quad \le C_\eta \int_H \frac{t^{|\gamma|/2\alpha+k-1/p'}}{(t+s+|x-y|^{2\alpha})^{(n+|\gamma|)/2\alpha+k+1}} dV(X) \\
&\quad \le C_\eta s^{-1/p'},
\end{aligned}
$$

as required. □

4. The lower estimate of the operator norm of Carleson inclusions. Let $1 \le p < \infty$, $1 \le q < \infty$ and μ be a positive Borel measure on H. For $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$, and $u \in \boldsymbol{b}_\alpha^p$, we denote $\iota^{\gamma,k}_{\mu,p,q}u$ the function on H defined by

$$\iota^{\gamma,k}_{\mu,p,q}u(X) := \partial_x^{\gamma}\partial_t^{k}u(X)\,, \quad X \in H\,.$$

We call $\iota^{\gamma,k}_{\mu,p,q}$ a generalized Carleson inclusion mapping. When $(\gamma, k) = (0, 0)$, we write $\iota_{\mu,p,q} := \iota^{0,0}_{\mu,p,q}$. In this section, we give the lower estimate of the operator norm $\|\iota^{\gamma,k}_{\mu,p,q}\|$ using an auxiliary function $\widehat{S}_{\delta,\lambda}\mu$ defined in (3.7). We remark here that if $\|\iota^{\gamma,k}_{\mu,p,q}\| < \infty$, i.e., if there exists a constant $C > 0$ such that

$$\left(\int_H |\partial_x^{\gamma}\partial_t^{k}u|^q d\mu\right)^{1/q} \le C\left(\int_H |u|^p dV\right)^{1/p} \tag{4.1}$$

holds for all $u \in \boldsymbol{b}^p_\alpha$, then μ is a Radon measure (i.e., finite on each compact subset of H). In fact, by (2.2) and Lemma 2.1, the inequality (4.1) for the function $u(\,\cdot\,) = \partial_x^{\gamma} R_\alpha(X,\,\cdot\,) \in \boldsymbol{b}^p_\alpha$ implies

$$\int_H |\partial_x^{2\gamma}\partial_t^{k} R_\alpha(X, Y)|^q d\mu(Y) \le C^q t^{-((n+|\gamma|)/2\alpha+1)q+(n/2\alpha+1)q/p}$$

for all $X = (x, t) \in H$. Hence [10, Corollary 1] implies $\mu(Q^{(\alpha)}(X)) < \infty$ for all $X \in H$. Since every compact set is covered by a finite union of $Q^{(\alpha)}(X)$'s, μ is a Radon measure.

Given a sequence $\{t_i\}$ of nonnegative numbers and $1 \le p < \infty$, we consider the weighted sequence space

$$\ell^p_\alpha(\{t_i\}) := \left\{\{\xi_i\}\,;\ \sum_i |\xi_i|^p t_i^{n/2\alpha+1} < \infty\right\}.$$

We begin with showing the following two lemmas.

LEMMA 4.1. *Let* $1 \le q < p < \infty$, $1/\sigma + 1/(p/q) = 1$, $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$, *and put* $\lambda := (|\gamma|/2\alpha + k)\,q$. *Suppose that* μ *is a positive Radon measure on* H *and* η_0 *is the constant chosen in Lemma* 2.2. *Then, for an* η*-separated sequence* $\boldsymbol{X} = \{X_i\} = \{(x_i, t_i)\}$ *with* $\eta \ge \eta_0$, $0 < \delta < 1$, *and a compact set* $K \subset H$, *there exists* $u = u_{\boldsymbol{X},\delta,K} \in \boldsymbol{b}^p_\alpha$ *such that*

$$\left(\sum_i \widehat{S}_{\delta,\lambda}\mu_K(X_i)^{\sigma} t_i^{n/2\alpha+1}\right)^{1/\sigma} \le \sum_i |\partial_x^{\gamma}\partial_t^{k}u(X_i)|^q \mu_K(S^{(\alpha)}_\delta(X_i))$$

and $\|u\|_{L^p(V)} \le M$, *where* $d\mu_K := \chi_K d\mu$, χ_K *is the characteristic function of* K *and* $M = M(n, \alpha, p, \gamma, k, \eta_0)$ *is a constant chosen in Remark* 2.3.

PROOF. Since $\{X_i\}$ is η-separated in the α-parabolic sense and K is compact, $\{\widehat{S}_{\delta,\lambda}\mu_K(X_i)\}_i$ is a finite sequence, so that $\{\widehat{S}_{\delta,\lambda}\mu_K(X_i)\}_i \in \ell^{\sigma}_\alpha(\{t_i\})$. By the Hahn-Banach theorem, there exists $\{\xi_i\} \in \ell^{p/q}_\alpha(\{t_i\})$ such that

$$\left(\sum_i |\xi_i|^{p/q} t_i^{n/2\alpha+1}\right)^{q/p} = 1 \tag{4.2}$$

and

$$\left(\sum_i \widehat{S}_{\delta,\lambda}\mu_K(X_i)^\sigma t_i^{n/2\alpha+1}\right)^{1/\sigma} = \sum_i \xi_i \cdot \widehat{S}_{\delta,\lambda}\mu_K(X_i)t_i^{n/2\alpha+1}. \tag{4.3}$$

Since

$$1 = \sum_i |\xi_i|^{p/q} t_i^{n/2\alpha+1} = \sum_i \{|\xi_i|^{1/q} t_i^{(n/2\alpha+1)/p}\}^p ,$$

by Remark 2.3, there exists $u \in \boldsymbol{b}_\alpha^p$ such that $\|u\|_{L^p(V)} \le M$ and

$$|\xi_i|^{1/q} t_i^{(n/2\alpha+1)/p} = t_i^{(n/2\alpha+1)/p+|\gamma|/2\alpha+k} \partial_x^\gamma \partial_t^k u(X_i) ,$$

so that $|\xi_i|^{1/q} = t_i^{|\gamma|/2\alpha+k} \partial_x^\gamma \partial_t^k u(X_i)$. Hence, by (4.3), we obtain

$$\begin{aligned}\left(\sum_i \widehat{S}_{\delta,\lambda}\mu_K(X_i)^\sigma t_i^{n/2\alpha+1}\right)^{1/\sigma} &= \sum_i \xi_i \cdot \widehat{S}_{\delta,\lambda}\mu_K(X_i)t_i^{n/2\alpha+1} \\ &\le \sum_i |\xi_i| \cdot \widehat{S}_{\delta,\lambda}\mu_K(X_i)t_i^{n/2\alpha+1} = \sum_i |\partial_x^\gamma \partial_t^k u(X_i)|^q \mu_K(S_\delta^{(\alpha)}(X_i)) . \qquad \square\end{aligned}$$

LEMMA 4.2. *Let $1 \le \sigma < \infty$, $0 < \delta < 1$, and λ be a real number. Suppose that μ is a positive Borel measure on H. Then there exist a positive integer $m = m(\alpha, \delta)$ and a constant $C = (n, \alpha, \sigma, \delta, \lambda, m) > 0$ with the following property: If a sequence $\{X_j\} = \{(x_j, t_j)\} \subset H$ satisfies the condition $H = \bigcup_j S_{\delta/m}^{(\alpha)}(X_j)$, then*

$$\int_H \widehat{S}_{\delta/m,\lambda}\mu(X)^\sigma dV(X) \le C \sum_j \widehat{S}_{\delta,\lambda}\mu(X_j)^\sigma t_j^{n/2\alpha+1} .$$

PROOF. For fixed $0 < \delta < 1$, take an integer m sufficiently large such that

$$\left(\frac{1+\delta/m}{1-\delta/m}\right)^2 < \frac{1+\delta}{1-\delta}$$

and

$$\left(\frac{2(\delta/m)}{1-(\delta/m)^2}\frac{1+\delta/m}{1-\delta/m}\right)^{1/2\alpha} + \left(\frac{2(\delta/m)}{1-(\delta/m)^2}\right)^{1/2\alpha} < \left(\frac{2\delta}{1-\delta^2}\right)^{1/2\alpha} .$$

Then, for each j, we have $S_{\delta/m}^{(\alpha)}(X) \subset S_\delta^{(\alpha)}(X_j)$ for $X \in S_{\delta/m}^{(\alpha)}(X_j)$. Hence it follows that

$$\begin{aligned}\int_H \widehat{S}_{\delta/m,\lambda}\mu(X)^\sigma dV(X) &\le \sum_j \int_{S_{\delta/m}^{(\alpha)}(X_j)} \widehat{S}_{\delta/m,\lambda}\mu(X)^\sigma dV(X) \\ &\le C \sum_j \widehat{S}_{\delta,\lambda}\mu(X_j)^\sigma V(S_{\delta/m}^{(\alpha)}(X_j)) \le C \sum_j \widehat{S}_{\delta,\lambda}\mu(X_j)^\sigma t_j^{n/2\alpha+1} . \qquad \square\end{aligned}$$

We introduce the notion of a δ-lattice. Given $0 < \delta < 1$, we say that a sequence $\{X_j\} \subset H$ is a δ-lattice in the α-parabolic sense if $H = \bigcup_j S_\delta^{(\alpha)}(X_j)$ and $\{X_j\}$ is ε-separated in the α-parabolic sense for some $0 < \varepsilon < \delta$.

REMARK 4.3. There is a δ-lattice in the α-parabolic sense for every $0 < \delta < 1$. We will give a concrete example. For each fixed $0 < \delta < 1$, take a real number ε_1 with $0 < \varepsilon_1 < \delta$. Put $t_j := \big((1+\varepsilon_1)(1-\varepsilon_1)^{-1}\big)^{2j}$ $(j \in \boldsymbol{Z})$ and let $T_j(\delta)$ be an open interval such that

$$T_j(\delta) := \left(\frac{1-\delta}{1+\delta}t_j, \frac{1+\delta}{1-\delta}t_j\right).$$

Clearly $\bigcup_j T_j(\delta) = (0,\infty)$, and $T_j(\varepsilon)$ are pairwise disjoint whenever $0 < \varepsilon \le \varepsilon_1$. Now, for each fixed $j \in \boldsymbol{Z}$, we choose a sequence $\{x_{j,i}\}_i$ in $\boldsymbol{R}^n$ as follows: Let $x_{j,0}$ be the origin in $\boldsymbol{R}^n$. Pick $x_{j,1}$ in $\boldsymbol{R}^n$ with $B(x_{j,1};\rho/2) \cap B(x_{j,0};\rho/2) = \emptyset$ and which minimizes $|x_{j,1}|$, where $B(x;\rho) := \{y \in \boldsymbol{R}^n; |x-y| < \rho\}$ with $\rho = (2\delta(1-\delta^2)^{-1}t_j)^{1/2\alpha}$. Continuing inductively, we can pick $x_{j,i}$ in $\boldsymbol{R}^n$ such that

$$|x_{j,i}| = \min\{\,|x|\,;\ B(x;\rho/2) \cap B(x_{j,m};\rho/2) = \emptyset \text{ for } 0 \le m \le i-1\}\,.$$

Then the balls $B(x_{j,i};\rho/2)$ are pairwise disjoint. Further $\boldsymbol{R}^n = \bigcup_i B(x_{j,i};\rho)$. In fact, if there exists $x' \in \boldsymbol{R}^n$ such that $x' \notin \bigcup_i B(x_{j,i};\rho)$, then $B(x';\rho/2) \cap B(x_{j,i};\rho/2) = \emptyset$ for all $i \ge 0$. Hence the choice of $x_{j,i}$ implies that $|x_{j,i}| \le |x'|$ for all $i \ge 0$. However, this is a contradiction because each $B(x_{j,i};\rho/2)$ has the same volume. Finally, we take $0 < \varepsilon_2 < \delta$ such that

$$\left(\frac{2\varepsilon_2}{1-\varepsilon_2^2}\right)^{1/2\alpha} = \frac{1}{2}\left(\frac{2\delta}{1-\delta^2}\right)^{1/2\alpha}$$

and put $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$. Then $H = \bigcup_{i,j} S_\delta^{(\alpha)}(x_{j,i}, t_j)$ and $\{(x_{j,i}, t_j)\}_{i,j}$ is ε-separated in the α-parabolic sense whenever $0 < \varepsilon \le \varepsilon_0$.

We now give a lower estimate of the operator norm $\|\iota_{\mu,p,q}^{\gamma,k}\|$.

PROPOSITION 4.4. *Let* $1 \le q < p < \infty$, $1/\sigma + 1/(p/q) = 1$, $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$, *and put* $\lambda := (|\gamma|/2\alpha + k)\,q$. *Suppose that* μ *is a positive Borel measure on* H. *Then there exist constants* $0 < \delta < 1$ *and* $C > 0$ *independent of* μ *such that*

$$\|\widehat{S}_{\delta,\lambda}\mu\|_{L^\sigma(V)}^{1/q} \le C\|\iota_{\mu,p,q}^{\gamma,k}\|\,. \tag{4.4}$$

Consequently, if μ *satisfies the inequality* (4.1), *then* $\widehat{S}_{\delta,\lambda}\mu \in L^\sigma(V)$ *for some* $\delta > 0$.

PROOF. Let η_0 be a constant chosen in Lemma 2.2. Also, let $M = M(n,\alpha,p,\gamma,k,\eta_0)$ and $L = L(n,\alpha,q,p,\gamma,k,\eta_0)$ be constants chosen in Remark 2.3 and Theorem 3.3, respectively. Further we take $0 < \delta_0 \le \min\{\eta_0, 1/3\}$ such that $2^{1-q} - L\varphi(\delta_0)^q M^q > 0$, where $\varphi(\delta) = \delta + \delta^{1/2\alpha}$. Finally let $m = m(\alpha,\delta_0)$ be a positive integer chosen in Lemma 4.2.

Now we take any δ_0/m-lattice $\{X_i\} = \{(x_i,t_i)\}$ in the α-parabolic sense. Then, by Lemma 4.2, in order to show the inequality (4.4) for $\delta = \delta_0/m$, it suffices to show

$$\left(\sum_i \widehat{S}_{\delta_0,\lambda}\mu(X_i)^\sigma t_i^{n/2\alpha+1}\right)^{1/\sigma} \le C\|\iota_{\mu,p,q}^{\gamma,k}\|^q\,. \tag{4.5}$$

Moreover, since $\{X_i\}$ is ε-separated in the α-parabolic sense for some $\varepsilon < \delta_0/m$, by Lemma 2.4, we can decompose $\{X_i\}$ into a finite many, say N, union of η_0-separated sequences in the α-parabolic sense. Let $\{Y_i\}$ be one of them. Since N is independent of μ, it is enough to prove the inequality (4.5) for the sequence $\boldsymbol{Y} = \{Y_i\} = \{(y_i, s_i)\}$.

Let K be a compact subset of H. Then the boundedness of $\iota^{\gamma,k}_{\mu,p,q}$ and Theorem 3.3 imply that (since $\{Y_i\}$ is η_0-separated)

$$
\begin{aligned}
&\|\iota^{\gamma,k}_{\mu,p,q}\|^q \left(\int_H |u|^p dV \right)^{q/p} \\
&\geq \int_H |\partial_x^\gamma \partial_t^k u|^q d\mu \geq \sum_i \int_{S^{(\alpha)}_{\delta_0}(Y_i)} |\partial_x^\gamma \partial_t^k u|^q d\mu_K \\
&\geq 2^{1-q} \sum_i |\partial_x^\gamma \partial_t^k u(Y_i)|^q \mu_K(S^{(\alpha)}_{\delta_0}(Y_i)) - \sum_i \int_{S^{(\alpha)}_{\delta_0}(Y_i)} |\partial_x^\gamma \partial_t^k u(Y_i) - \partial_x^\gamma \partial_t^k u(Z)|^q d\mu_K(Z) \\
&\geq 2^{1-q} \sum_i |\partial_x^\gamma \partial_t^k u(Y_i)|^q \mu_K(S^{(\alpha)}_{\delta_0}(Y_i)) - L\varphi(\delta_0)^q \|u\|^q_{L^p(V)} \left(\sum_i \widehat{S}_{\delta_0,\lambda}\mu_K(Y_i)^\sigma s_i^{n/2\alpha+1} \right)^{1/\sigma}
\end{aligned}
$$

for all $u \in \boldsymbol{b}^p_\alpha$. In particular, if we take $u = u_{\boldsymbol{Y},\delta_0,K} \in \boldsymbol{b}^p_\alpha$ in Lemma 4.1, the above inequalities give us

$$
\|\iota^{\gamma,k}_{\mu,p,q}\|^q M^q \geq \left(2^{1-q} - L\varphi(\delta_0)^q M^q\right) \left(\sum_i \widehat{S}_{\delta_0,\lambda}\mu_K(Y_i)^\sigma s_i^{n/2\alpha+1} \right)^{1/\sigma},
$$

so that

$$
\left(\sum_i \widehat{S}_{\delta_0,\lambda}\mu_K(Y_i)^\sigma s_i^{n/2\alpha+1} \right)^{1/\sigma} \leq M^q \left(2^{1-q} - L\varphi(\delta_0)^q M^q\right)^{-1} \|\iota^{\gamma,k}_{\mu,p,q}\|^q . \tag{4.6}
$$

Since the right-hand side of (4.6) is independent of K, increasing K to H, we have the desired inequality. □

5. The proof of the main result. In this section, we complete the proof of the main result. It is important to examine the relation between weighted averaging functions $\widehat{Q}_\lambda\mu$ and our auxiliary functions $\widehat{S}_{\delta,\lambda}\mu$. For this purpose, we define generalized averaging functions. Let S be a Borel set in H of finite V-volume. For a positive Radon measure μ on H, we define a generalized averaging function $\widehat{A}_S\mu$ of μ by

$$
\widehat{A}_S\mu(X) := \frac{1}{V(\Phi_X(S))} \int_{\Phi_X(S)} d\mu , \quad X \in H , \tag{5.1}
$$

where $\Phi_X : H \mapsto H$ $(X = (x,t) \in H)$ is the α-parabolic similarity defined by

$$
\Phi_X(Z) := (t^{1/2\alpha} z + x, tr) , \quad Z = (z,r) \in H
$$

(see [9]). Remark that $S^{(\alpha)}_\eta(X) = \Phi(S^{(\alpha)}_\eta(0,1))$. Moreover, $\widehat{Q}_0\mu(X) = \widehat{A}_Q\mu(X)$ and $\widehat{S}_{\delta,0}\mu(X) = C_\delta \widehat{A}_{S_\delta}\mu(X)$ for some $C_\delta > 0$, where $Q := Q^{(\alpha)}(0,1)$ and $S_\delta := S^{(\alpha)}_\delta(0,1)$, respectively. We also use the following function: For $1 \le \rho < \infty$, $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$, and a real number c, we put

$$\text{(5.2)} \qquad B_{\gamma,k,\rho,c}\mu(Y) := s^{(\rho-1)(n/2\alpha+1)-c} \int_H t^c |R_\alpha^{\gamma,k}(X,Y)|^\rho d\mu(X)\,, \quad Y = (y,s) \in H\,.$$

This function acts as an intermediary between $\widehat{Q}_\lambda\mu$ and $\widehat{S}_{\delta,\lambda}\mu$.

PROPOSITION 5.1. *Let $1 \le \sigma < \infty$, $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$, and λ a real number. Suppose that μ is a positive Radon measure on H. For a real number θ satisfying $-1/\sigma < \theta < 1 - 1/\sigma$, we put $c := |\gamma|/2\alpha + k - \lambda - \theta$. Then the following statements hold*:

(1) *There exists a constant $C_1 > 0$ independent of μ such that*

$$C_1^{-1} \int_H \widehat{Q}_\lambda\mu(Y)^\sigma dV(Y) \le \int_H \{B_{\gamma,k,1,c}\mu(Y)s^{-\lambda}\}^\sigma dV(Y) \le C_1 \int_H \widehat{Q}_\lambda\mu(Y)^\sigma dV(Y)\,.$$

(2) *For each $0 < \delta < 1$, there exists a constant $C_2 > 0$ independent of μ such that*

$$C_2^{-1} \int_H \widehat{S}_{\delta,\lambda}\mu(Y)^\sigma dV(Y) \le \int_H \{B_{\gamma,k,1,c}\mu(Y)s^{-\lambda}\}^\sigma dV(Y) \le C_2 \int_H \widehat{S}_{\delta,\lambda}\mu(Y)^\sigma dV(Y)\,.$$

The proof relies on the following result.

LEMMA 5.2 ([9, Proposition 3]). *Let $1 \le \rho < \infty$, $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$, and c a real number. Then, for a positive Radon measure μ on H, we have the following assertions.*

(1) *Let $1 \le \sigma < \infty$ and $\tau \in \boldsymbol{R}$. For a compact set K in H, there exists a constant $C_1 > 0$ independent of μ such that*

$$\text{(5.3)} \qquad \int_H \widehat{A}_K\mu(Y)^\sigma s^\tau dV(Y) \le C_1 \int_H B_{\gamma,k,\rho,c}\mu(Y)^\sigma s^\tau dV(Y)\,.$$

(2) *If $1 \le \sigma < \infty$ and $\tau \in \boldsymbol{R}$ satisfy*

$$\text{(5.4)} \qquad (1-\rho)\left(\frac{n}{2\alpha}+1\right) + c - \rho\left(\frac{|\gamma|}{2\alpha}+k\right) < \frac{\tau+1}{\sigma} < c+1\,,$$

then, for a relatively compact open set $U \neq \emptyset$ in H, there exists a constant $C_2 > 0$ independent of μ such that

$$\text{(5.5)} \qquad \int_H B_{\gamma,k,\rho,c}\mu(Y)^\sigma s^\tau dV(Y) \le C_2 \int_H \widehat{A}_U\mu(Y)^\sigma s^\tau dV(Y)\,.$$

PROOF OF PROPOSITION 5.1. We only prove the assertion (1), because the proof of (2) is similar. The first inequality of (1) immediately follows from (1) of Lemma 5.2 for $K = Q^{(\alpha)}(0,1)$ and $\tau = -\lambda\sigma$. Accordingly, we show the second inequality of (1).

Since $-1/\sigma < \theta < 1 - 1/\sigma$ and $c = |\gamma|/2\alpha + k - \lambda - \theta$, if we define $\rho := 1$ and $\tau := -\lambda\sigma$, then we have

$$c + 1 - \frac{\tau+1}{\sigma} = \frac{|\gamma|}{2\alpha} + k - \theta + 1 - \frac{1}{\sigma} > 0$$

and

$$(1-\rho)\left(\frac{n}{2\alpha}+1\right) + c - \rho\left(\frac{|\gamma|}{2\alpha}+k\right) - \frac{\tau+1}{\sigma} = -\theta - \frac{1}{\sigma} < 0\,.$$

This is (5.4), so that, by (2) of Lemma 5.2 for $U = Q = Q^{(\alpha)}(0, 1)$, we obtain

$$\begin{aligned}\int_H \{B_{\gamma,k,1,c}\mu(Y)s^{-\lambda}\}^\sigma dV(Y) &= \int_H B_{\gamma,k,1,c}\mu(Y)^\sigma s^\tau dV(Y)\\ &\le C\int_H \widehat{A}_Q\mu(Y)^\sigma s^\tau dV(Y) = C\int_H \widehat{Q}_\lambda\mu(Y)^\sigma dV(Y). \qquad \square\end{aligned}$$

Now we state the main result of this paper. This contains Theorem 1 as a special case $(\gamma, k) = (0, 0)$ and $\lambda = 0$.

THEOREM 5.3. *Let* $1 \le q < p < \infty$, $1/\sigma + 1/(p/q) = 1$, $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$, *and put* $\lambda := (|\gamma|/2\alpha + k)\,q$. *Then there exists a constant* $C > 0$ *such that, for every positive Borel measure* μ *on* H, *we have*

$$C^{-1}\|\widehat{Q}_\lambda\mu\|_{L^\sigma(V)}^{1/q} \le \|\iota_{\mu,p,q}^{\gamma,k}\| \le C\|\widehat{Q}_\lambda\mu\|_{L^\sigma(V)}^{1/q}. \tag{5.6}$$

PROOF. To show the first inequality, we may assume $\|\iota_{\mu,p,q}^{\gamma,k}\| < \infty$. Then, as we mentioned in the beginning of Section 4, μ is a Radon measure. Hence the first inequality follows from Propositions 4.4 and 5.1.

To show the second inequality, we may also assume that $\|\widehat{Q}_\lambda\mu\|_{L^\sigma(V)}^{1/q} < \infty$. Then $Q_\lambda\mu(X) < \infty$ (V-a.e.). By the same reason as above, μ is a Radon measure. Hence by (1) of Proposition 5.1, it suffices to show that there exists a real number θ with $-1/\sigma < \theta < 1-1/\sigma$ such that

$$\|\iota_{\mu,p,q}^{\gamma,k}\|^{q\sigma} \le C\int_H \{B_{\gamma,k,1,c}\mu(Y)s^{-\lambda}\}^\sigma dV(Y),$$

where $c = |\gamma|/2\alpha + k - \lambda - \theta$. Now, by (2.3), we have

$$\partial_x^\gamma \partial_t^k u(X) = \int_H u(Y)\partial_x^\gamma \partial_t^k R_\alpha(X, Y)dV(Y) \tag{5.7}$$

for all $u \in \boldsymbol{b}_\alpha^p$ and $X \in H$.

Suppose $1 < q < \infty$ and let q' be the exponent conjugate to q. Since $\sigma > 1$, we can take a real number η such that $0 < \eta < q$ and $\eta < (1 - 1/\sigma)q'$. Then, as in the proof of Theorem 3.3, by the Hölder inequality, (2.3) and Lemma 2.1, we have

$$\begin{aligned}|\partial_x^\gamma \partial_t^k u(X)| &= \int_H |u(Y)|s^{\eta/qq'} \cdot s^{-\eta/qq'}|\partial_x^\gamma \partial_t^k R_\alpha(X, Y)|dV(Y)\\ &\le \left(\int_H |u(Y)|^q s^{\eta/q'}|\partial_x^\gamma \partial_t^k R_\alpha(X, Y)|dV(Y)\right)^{1/q}\\ &\quad \times\left(\int_H s^{-\eta/q}|\partial_x^\gamma \partial_t^k R_\alpha(X, Y)|dV(Y)\right)^{1/q'}\\ &\le Ct^{-(|\gamma|/2\alpha+k+\eta/q)/q'}\left(\int_H |u(Y)|^q s^{\eta/q'}|\partial_x^\gamma \partial_t^k R_\alpha(X, Y)|dV(Y)\right)^{1/q},\end{aligned}$$

where the constant C is independent of μ. Hence the Fubini theorem yields

$$\int_H |\partial_x^\gamma \partial_t^k u(X)|^q d\mu(X) \\ \leq C \int_H |u(Y)|^q s^{\eta/q'} \int_H t^{-(|\gamma|/2\alpha+k)q/q'-\eta/q'} |\partial_x^\gamma \partial_t^k R_\alpha(X,Y)| d\mu(X) dV(Y)\,.$$

Now put $\theta := \eta/q'$. Then $-1/\sigma < \theta < 1 - 1/\sigma$. Since

$$c = \frac{|\gamma|}{2\alpha} + k - \left(\frac{|\gamma|}{2\alpha} + k\right) q - \frac{\eta}{q'} = -\left(\frac{|\gamma|}{2\alpha} + k\right) \frac{q}{q'} - \frac{\eta}{q'}$$

and

$$-c - \lambda = -\left(\frac{|\gamma|}{2\alpha} + k\right) + \frac{\eta}{q'}\,,$$

we have

$$s^{\eta/q'} \int_H t^{-(|\gamma|/2\alpha+k)q/q'-\eta/q'} |\partial_x^\gamma \partial_t^k R_\alpha(X,Y)| d\mu(X) \\ = \frac{1}{(-1)^{|\gamma|} c_k} s^{-(|\gamma|/2\alpha+k)+\eta/q'} \int_H t^{-(|\gamma|/2\alpha+k)q/q'-\eta/q'} |R_\alpha^{\gamma,k}(X,Y)| d\mu(X) \\ = \frac{1}{(-1)^{|\gamma|} c_k} s^{-\lambda} B_{\gamma,k,1,c}\mu(Y)\,.$$

Hence the Hölder inequality shows that

$$\int_H |\partial_x^\gamma \partial_t^k u(X)|^q d\mu(X) \\ \leq C \int_H |u(Y)|^q s^{-\lambda} B_{\gamma,k,1,c}\mu(Y) dV(Y) \\ \leq C\left(\int_H |u(Y)|^p dV(Y)\right)^{q/p} \left(\int_H \{s^{-\lambda} B_{\gamma,k,1,c}\mu(Y)\}^\sigma dV(Y)\right)^{1/\sigma}.$$

This implies

$$\|\iota_{\mu,p,q}^{\gamma,k}\|^{q\sigma} \leq C \int_H \{s^{-\lambda} B_{\gamma,k,1,c}\mu(Y)\}^\sigma dV(Y)\,,$$

where the constant C is independent of μ.

The case of $q = 1$ is easily proved from (5.7). □

We remark the compactness of Carleson inclusions. Unlike the case $p \leq q$ (see [7]), if $1 < q < p < \infty$, then $\iota_{\mu,p,q}$ is always compact whenever $\|\iota_{\mu,p,q}\| < \infty$. Moreover the following theorem is established.

THEOREM 5.4. *Let $1 \leq q < p < \infty$, $\gamma \in \boldsymbol{N}_0^n$, $k \in \boldsymbol{N}_0$ and let μ be a positive Radon measure on H. Then the generalized Carleson inclusion $\iota_{\mu,p,q}^{\gamma,k}$ is compact if it is bounded.*

PROOF. As in [7, Proposition 1], if μ has compact support, then $\iota^{\gamma,k}_{\mu,p,q}$ is compact for every $1 \le p < \infty$ and $1 \le q < \infty$. To study the compactness for general measure μ, we take an exhaustion $(\omega_j)_j$ of H and set

$$\mu_j := \chi_{\omega_j}\mu \quad \text{and} \quad \nu_j := \mu - \mu_j .$$

If $\|\iota^{\gamma,k}_{\mu,p,q}\| < \infty$, then by Theorem 5.3, $\widehat{Q}_\lambda\mu \in L^\sigma(V)$ and

$$\|\iota^{\gamma,k}_{\mu,p,q} - \iota^{\gamma,k}_{\mu_j,p,q}\| = \|\iota^{\gamma,k}_{\nu_j,p,q}\| \le C\|\widehat{Q}_\lambda\nu_j\|^{1/q}_{L^\sigma(V)} ,$$

where $1/\sigma + 1/(p/q) = 1$. Since $\widehat{Q}_\lambda\mu \ge \widehat{Q}_\lambda\nu_j$ and $\widehat{Q}_\lambda\nu_j$ decreases to 0 pointwisely as $j \to \infty$, $\|\iota^{\gamma,k}_{\mu,p,q} - \iota^{\gamma,k}_{\mu_j,p,q}\| \to 0$ as $j \to \infty$. This brings the compactness of $\iota^{\gamma,k}_{\mu,p,q}$. □

6. An application to Toeplitz operators. In this section we will give an observation on Toeplitz operators. For a positive Radon measure μ on H, the Toeplitz operator with symbol μ is defined by

$$(T_\mu u)(X) := \int_H R_\alpha(X,Y)u(Y)d\mu(Y) .$$

Our concern is the boundedness of the operator $T_\mu = T_{\mu,p,q} : \boldsymbol{b}^p_\alpha \mapsto \boldsymbol{b}^q_\alpha$ for the case $1 \le q < p < \infty$, because we have already studied the case $p \le q$ in [6]. We assume that there is an integer $m \ge 1$ such that

$$\int_H |R^{0,m}_\alpha(X,Y)|d\mu(X) < \infty \quad \text{for every } Y \in H . \tag{6.1}$$

Remark that if $\int_H (1 + t + |x|^{1/2\alpha})^{-c} d\mu(x,t) < \infty$ with $m > n/2\alpha - c$, then (6.1) follows from (2.2).

The following theorem is a consequence of Theorem 1.1.

THEOREM 6.1. *For $1 < q < p < \infty$, put $\tau := pq/(p-q)$. Let μ be a positive Radon measure on H satisfying* (6.1) *for some integer $m \ge 1$. If $\widehat{Q}_0\mu \in L^\tau(V)$, then the Toeplitz operator $T_{\mu,p,q} : \boldsymbol{b}^p_\alpha \to \boldsymbol{b}^q_\alpha$ is well-defined and*

$$\|T_{\mu,p,q}\| \le C\|\widehat{Q}_0\mu\|_{L^\tau(V)} , \tag{6.2}$$

where the constant C is independent of μ.

PROOF. Let τ' and q' be the exponents conjugate to τ and q, respectively. Then, since $\widehat{Q}_0\mu \in L^\tau(V)$, we see from Theorem 1.1 that both Carleson inclusions $\iota_{\mu,p,p/\tau'}$ and $\iota_{\mu,q',q'/\tau'}$ are bounded. Take any $u \in \boldsymbol{b}^p_\alpha$ and any $X \in H$. Then $v := R_\alpha(X,\ \cdot\ )$ belongs to $\boldsymbol{b}^q_\alpha$. Hence $u \in L^{p/\tau'}(\mu)$ and $v \in L^{q'/\tau'}(\mu)$. Since p/τ' is the exponent conjugate to q'/τ', the integral

$$T_\mu u(X) := \int_H R_\alpha(X,Y)\,u(Y)\,d\mu(Y)$$

is well-defined. Moreover, because of $\boldsymbol{b}^q_\alpha = (\boldsymbol{b}^{q'}_\alpha)^*$, the adjoint operator $\iota^*_{\mu,q',q'/\tau'}$ of $\iota_{\mu,q',q'/\tau'}$ is a bounded operator from $L^{p/\tau'}(V)(= (L^{q'/\tau'}(V))^*)$ to $\boldsymbol{b}^q_\alpha$, so that $w := \iota^*_{\mu,q',q'/\tau'} \cdot \iota_{\mu,p,p/\tau'} u$

belongs to $\boldsymbol{b}_\alpha^q$. By the reproducing property for w, we have

$$\begin{aligned} w(X) &= \int_H R_\alpha(X, Y)\, w(Y)\, dV(Y) \\ &= \langle v, w\rangle_{(\boldsymbol{b}_\alpha^{q'}, \boldsymbol{b}_\alpha^q)} = \langle \iota_{\mu,q',q'/\tau'} v, \iota_{\mu,p,p/\tau'} u\rangle_{(L^{q'/\tau'}(\mu), L^{p/\tau'}(\mu))} \\ &= \int_H R_\alpha(X, Y)\, u(Y)\, d\mu(Y) = T_\mu u(X)\,. \end{aligned}$$

This implies that the integral operator $T_\mu = T_{\mu,p,q} : \boldsymbol{b}_\alpha^p \to \boldsymbol{b}_\alpha^q$ is well-defined and

$$T_{\mu,p,q} = \iota^*_{\mu,q',q'/\tau'} \cdot \iota_{\mu,p,p/\tau'}\,.$$

Hence by Theorem 1.1, we have

$$\|T_{\mu,p,q}\| \le \|\iota_{\mu,q',q'/\tau'}\| \cdot \|\iota_{\mu,p,p/\tau'}\| \le C\|\widehat{Q}_0\mu\|^{\tau'/q'}_{L^\tau(V)} \cdot \|\widehat{Q}_0\mu\|^{\tau'/p}_{L^\tau(V)} = C\|\widehat{Q}_0\mu\|_{L^\tau(V)}\,,$$

which shows Theorem 6.1. □

As for an opposite inequality to (6.2), we have the following result for the case $q = p'$.

THEOREM 6.2. *For $p > 2$, let $\tau := p/(p-2)$ and p' be the exponent conjugate to p. Let μ be a positive Radon measure on H satisfying* (6.1) *for some $m \ge 1$. If the Toeplitz operator $T_{\mu,p,p'} : \boldsymbol{b}_\alpha^p \mapsto \boldsymbol{b}_\alpha^{p'}$ is bounded, then there exists a constant $C > 0$ independent of μ such that*

$$\|\widehat{Q}_0\mu\|_{L^\tau(V)} \le C\|T_{\mu,p,p'}\|\,. \tag{6.3}$$

PROOF. We first remark that $2 < p$ and $1/\tau + 1/(p/2) = 1$. Hence $\|\widehat{Q}_0\mu\|^{1/2}_{L^\tau(V)} \le C\|\iota_{\mu,p,2}\|$ holds by Theorem 1.1. As in the proof of [7, Proposition 6],

$$\int_H |u(X)|^2 d\mu(X) = \int_H u(X) T_{\mu,p,p'} u(X) dV(X)$$

for every u in $\mathcal{E}$, where $\mathcal{E}$ is a dense subset of $\boldsymbol{b}_\alpha^p$. Then

$$\|u\|^2_{L^2(\mu)} \le \|u\|_{L^p(V)} \|T_{\mu,p,p'} u\|_{L^{p'}(V)} \le \|T_{\mu,p,p'}\| \|u\|^2_{L^p(V)}\,,$$

which implies $\|\iota_{\mu,p,2}\| \le \|T_{\mu,p,p'}\|^{1/2}$ and hence (6.3) follows. □

By the parallel argument as in Theorem 5.4, we also see the compactness of Toeplitz operators.

THEOREM 6.3. *Let $1 < q < p < \infty$ and let μ be a positive Radon measure on H satisfying* (6.1) *for some $m \ge 1$. Then the Toeplitz operator $T_{\mu,p,q}$ is compact whenever $\widehat{Q}_0\mu \in L^\tau(V)$, where $\tau = pq/(p-q)$.*

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