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BOUNDEDNESS OF THE MARCINKIEWICZ INTEGRALS WITH ROUGH KERNEL ASSOCIATED TO SURFACES

YONG DING, QINGYING XUE AND KÔZÔ YABUTA

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Abstract. In this paper, the authors discuss the weighted L^p boundedness for the rough Marcinkiewicz integrals associated to surfaces. More precisely, the kernel of our operator lacks smoothness not only on the unit sphere, but also in the radial directions. Moreover, the surface is defined by using a differentiable function with monotonicity and some properties on the positive real line. The results given in this paper improve and extend some known results.

1. Introduction.

Let $\boldsymbol{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\boldsymbol{R}^n$ with the area element $d\sigma(x')$. Let Ω be a homogeneous function of degree zero with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0\,, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$. Suppose that Φ is a nonnegative monotone C^1 function on $\boldsymbol{R}_+ := (0, \infty)$ such that

$$\varphi(t) := \frac{\Phi(t)}{t\Phi'(t)} \quad \text{and} \quad |\varphi(t)| \leq C \quad \text{for all } t \in \boldsymbol{R}_+\,. \tag{1.2}$$

For $1 \leq \gamma < \infty$, we define the function set Δ_γ on $\boldsymbol{R}_+$ by

$$\Delta_\gamma = \left\{ b\,;\ \|b\|_{\Delta_\gamma} := \sup_{R>0} \left(\frac{1}{R} \int_0^R |b(t)|^\gamma\, dt \right)^{1/\gamma} < \infty \right\} \tag{1.3}$$

and $\Delta_\infty = L^\infty(\boldsymbol{R}_+)$. Obviously, for $1 < \gamma_1 < \gamma_2 < \infty$,

$$\Delta_\infty \subset \Delta_{\gamma_2} \subset \Delta_{\gamma_1} \subset \Delta_1 \quad \text{and} \quad \|b\|_{\Delta_1} \leq \|b\|_{\Delta_{\gamma_1}} \leq \|b\|_{\Delta_{\gamma_2}} \leq \|b\|_{\Delta_\infty}\,. \tag{1.4}$$

For $\rho > 0$, we define the parametrized Marcinkiewicz integral $\mu_{\Omega,\rho,\Phi,b}$ associated to Ω, Φ and b by

$$\mu_{\Omega,\rho,\Phi,b}(f)(x) := \left(\int_0^\infty |F_{t,\Omega,\rho,\Phi,b}(x)|^2 \frac{dt}{t} \right)^{1/2}, \tag{1.5}$$

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where

$$F_{t,\Omega,\rho,\Phi,b}(x) := \frac{1}{t^\rho}\int_{|y|<t} \frac{b(|y|)\Omega(y')f(x-\Phi(|y|)y')}{|y|^{n-\rho}}dy\,. \tag{1.6}$$

If $b \equiv 1$, $\rho = 1$ and $\Phi(t) = t$ in the above definition, we simply denote $\mu_{\Omega,\rho,\Phi,b}$ by μ_Ω. It is well known that the operator μ_Ω was first defined by Stein in [21]. Stein proved that if Ω is continuous and satisfies a Lip_α $(0 < \alpha \le 1)$ condition on S^{n-1}, then μ_Ω is the operator of type (p, p) for $1 < p \le 2$ and of weak type (1.1). In [6], Benedek, Calderón and Panzone proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is of type (p, p) for $1 < p < \infty$. In 2000, Ding, Fan and Pan [10] improved all the results mentioned above. They proved that if $\Omega \in H^1(S^{n-1})$, where $H^1(S^{n-1})$ denotes the Hardy spaces on S^{n-1} (see [7] or [8] for the definition of $H^1(S^{n-1})$), then μ_Ω is bounded on $L^p(\boldsymbol{R}^n)$ for $1 < p < \infty$. In 2002, Al-Salman, Al-Qassem, Cheng and Pan [5] gave the L^p boundedness of μ_Ω for $\Omega \in L(\log L)^{1/2}(S^{n-1})$. If $b \equiv 1$ and $\Phi(t) = t$, we simply denote $\mu_{\Omega,\rho,\Phi,b}$ by $\mu_{\Omega,\rho}$. The L^p $(1 < p < \infty)$ boundedness of $\mu_{\Omega,\rho}$ was first studied by Hörmander [17] for real ρ in 1960, and later studied by Sakamoto and Yabuta [20] for complex number ρ in 1999 when the kernel Ω is in $\mathrm{Lip}_\alpha(S^{n-1})$.

On the other hand, motivated by Stein's work on singular integrals [22], in 2002, Ding, Fan and Pan [11] discussed the L^p boundedness of the Marcinkiewicz integral $\mu_{\Omega,\Phi}$, where $\Omega \in H^1(S^{n-1})$ and Φ is the mapping of polynomials, mappings of finite type, homogeneous mappings and surface of revolution, respectively. An important fact is that

$$H^1(S^{n-1}) \nsubseteq L(\log L)^{1/2}(S^{n-1}) \quad \text{and} \quad L(\log L)^{1/2}(S^{n-1}) \nsubseteq H^1(S^{n-1})\,.$$

In this paper, we will consider the boundedness of $\mu_{\Omega,\rho,\Phi,b}$ on the $L^p(\boldsymbol{R}^n)$ and the weighted $L^p(\boldsymbol{R}^n)$ for Ω belonging to different function spaces on S^{n-1}, such as the Hardy space $H^1(S^{n-1})$ and the Orlicz space $L(\log L)^{1/2}(S^{n-1})$. As a consequence of the above results, we also get the boundedness of $\mu_{\Omega,\rho,\Phi,b}$ on the $L^p(\boldsymbol{R}^n)$ and the weighted $L^p(\boldsymbol{R}^n)$ when Ω is in the block space $B_q^{(0,-1/2)}(S^{n-1})$. Before stating our results, let us recall the definitions of the weights.

Suppose that a nonnegative function ω is in $L^1_{\mathrm{loc}}(\boldsymbol{R}_+)$. For $1 < p < \infty$, we say that ω is in $A_p(\boldsymbol{R}_+)$ if there is a constant $C > 0$ such that for any interval $I \subset \boldsymbol{R}_+$,

$$\left(\frac{1}{|I|}\int_I \omega(r)dr\right)\left(\frac{1}{|I|}\int_I \omega(r)^{-1/(p-1)}dr\right)^{p-1} \le C < \infty\,. \tag{1.7}$$

If there is a constant $C > 0$ such that

$$\omega^*(r) \le C\omega(r) \quad \text{for a.e. } r \in \boldsymbol{R}_+\,, \tag{1.8}$$

where ω^* denotes the standard Hardy-Littlewood maximal function of ω on $\boldsymbol{R}_+$, then we say ω is in $A_1(\boldsymbol{R}_+)$. For $1 < p < \infty$, we define the weight classes as follows:

$$\tilde{A}_p(\boldsymbol{R}_+) := \{\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p}\,;\ \nu_1, \nu_2 \in A_1(\boldsymbol{R}_+) \text{ are decreasing or } \nu_1^2, \nu_2^2 \in A_1(\boldsymbol{R}_+)\}\,,$$

and

$$\bar{A}_p(\boldsymbol{R}_+) = \{\omega(x) = \sqrt{\nu(|x|)}\,;\ \nu \in A_p(\boldsymbol{R}_+)\}\,.$$

We know by [12] that $\bar{A}_p(\boldsymbol{R}_+) \subseteq \tilde{A}_p(\boldsymbol{R}_+)$, and by [13] that the Hardy-Littlewood maximal operator M is bounded on $L^p(\boldsymbol{R}^n, \omega)$ for $\omega \in \tilde{A}_p(\boldsymbol{R}_+)$, and thus $\tilde{A}_p(\boldsymbol{R}_+) \subset A_p(\boldsymbol{R}^n)$, the latter is the usual Muckenhoupt weight class on $\boldsymbol{R}^n$ and $L^p(\boldsymbol{R}^n, \omega)$ denotes the weighted L^p spaces associated to the weight ω defined by

$$L^p(\boldsymbol{R}^n, \omega) = \left\{ f;\ \|f\|_{L^p(\omega)} := \left(\int_{\boldsymbol{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty \right\}.$$

Let $\tilde{A}^I_p(\boldsymbol{R}_+) = \tilde{A}_p(\boldsymbol{R}_+) \cap A^I_p(\boldsymbol{R}^n)$ and $\bar{A}^I_p(\boldsymbol{R}_+) = \bar{A}_p(\boldsymbol{R}_+) \cap A^I_p(\boldsymbol{R}^n)$, where $A^I_p(\boldsymbol{R}^n)$ is defined as follows: For $1 < p < \infty$, we say that ω is in $A^I_p(\boldsymbol{R}^n)$ if there is a constant $C > 0$ such that for any n-dimensional intervals J with sides parallel to coordinate axes

$$\left(\frac{1}{|J|} \int_J \omega(x) dx \right) \left(\frac{1}{|J|} \int_J \omega(x)^{-1/(p-1)} dx \right)^{p-1} \leq C < \infty . \tag{1.9}$$

Now let us state our results obtained in this paper. Note that in the conditions of Theorems 1.1, 1.3, 1.5 and 1.7, we always assume that Ω satisfies the cancellation condition (1.1).

THEOREM 1.1. *Suppose* $\Omega \in H^1(S^{n-1})$ *and* $b \in \Delta_\gamma$ *for some* $\gamma > 1$. *Suppose* Φ *satisfies one of the following conditions*:

(i) Φ *is increasing, and* $\Phi(t) \leq c_1 \Phi(t/2)$.
(ii) Φ *is increasing, and* $t\Phi'(t)$ *is increasing.*
(iii) Φ *is decreasing, and* $\Phi(t/2) \leq c_2 \Phi(t)$.
(iv) Φ *is decreasing and convex.*

Then $\mu_{\Omega,\rho,\Phi,b}$ *is bounded on* $L^p(\boldsymbol{R}^n)$ *for* p *satisfying* $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$.

Furthermore, if $\gamma \geq 2$, *then* $\mu_{\Omega,\rho,\Phi,b}$ *is bounded on* $L^p(\omega)$ *for* $p \in (\gamma', \infty)$ *and* $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$.

REMARK 1.2. (1) If Φ is positive, increasing, and $\Phi(t)/(t\Phi'(t))$ is decreasing, then $t\Phi'(t)$ is increasing on $(0, \infty)$. If Φ is positive, increasing and convex, then $t\Phi'(t)$ is increasing on $(0, \infty)$. We mention three examples giving some including relations between (i) and (ii) and the monotonicity of $\Phi(t)/(t\Phi'(t))$. (a) $\Phi(t) = t^{1/2}e^t$ is nonconvex, positive, increasing, and $\Phi(t)/(t\Phi'(t))$ is decreasing and bounded. $t\Phi'(t)$ is increasing. But there is no $C > 0$ such that $\Phi(2t) \leq C\Phi(t)$. (b) $\Phi(t) = (t^2 - \sin^2 t)e^{t/2}$ is convex, positive, increasing, and $\Phi(t)/(t\Phi'(t))$ is bounded but nonmonotonic and $t\Phi'(t)$ is increasing. But there is no $C > 0$ such that $\Phi(2t) \leq C\Phi(t)$. (c) Let $\psi(t) = 1$ for $0 \leq t \leq \pi/2$, $\psi(t) = \sin t$ for $t \geq \pi/2$, and $\Phi(t) = 2t^2 + t\psi(t)$. Then $\Phi(t)$ is positive and increasing on $(0, \infty)$ and satisfies $\Phi(2t) \leq 7\Phi(t)$, but $\Phi(t)$ is not convex and $t\Phi'(t)$ is not monotone. Moreover, $|\Phi(t)/(t\Phi'(t))| < 1$ and $\Phi(t)/(t\Phi'(t))$ is not monotone.

(2) If Φ is positive, decreasing, and $-t\Phi'(t)$ is decreasing on $(0, \infty)$, then $\Phi(t)$ is convex. If Φ is positive, decreasing, and $-\Phi(t)/(t\Phi'(t))$ is increasing, then $-t\Phi'(t)$ is decreasing, and hence $\Phi(t)$ is convex. We mention two examples giving some including relations between (iii) and (iv). (d) Let $\psi(t)$ be as above and $\Phi(t) = 3/t + (1/t^2)\psi(t)$. Then $\Phi(t)$ is positive and decreasing on $(0, \infty)$ and satisfies $|\Phi(t)/(t\Phi'(t))| < 2$, $\Phi(2t) \geq (1/7)\Phi(t)$, but

$\Phi(t)$ is not convex. (e) Let $\Phi(t) = t^{-\alpha}e^{1/t}$, $\alpha > 0$. Then $\Phi(t)$ is positive, decreasing and convex on $(0, \infty)$, and $|\Phi(t)/(t\Phi'(t))| < 1$, but $\lim_{t\to 0} \Phi(t)/\Phi(2t) = +\infty$.

THEOREM 1.3. *Suppose $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and $b \in \Delta_\gamma$ for some $\gamma > 1$. If Φ satisfies the condition in Theorem* 1.1, *then $\mu_{\Omega,\rho,\Phi,b}$ is bounded on $L^p(\boldsymbol{R}^n)$ for p satisfying $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$. Furthermore, if $\gamma \geq 2$, then $\mu_{\Omega,\rho,\Phi,b}$ is bounded on $L^p(\omega)$ for $p \in (\gamma', \infty)$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$.*

REMARK 1.4. Al-Qassem [3] showed the above theorem for $\rho = 1$ under the condition that Φ is a C^2, convex, increasing function with $\Phi(0) = 0$. His condition automatically implies our condition $\Phi(t)/(t\Phi'(t)) \in L^\infty(0, \infty)$ and (ii).

It is worthwhile to note that (1.2) implies that $\Phi(2t) \geq 2^{1/\|\varphi\|_\infty}\Phi(t)$ $(t > 0)$ in the case Φ is increasing, and $\Phi(2t) \leq 2^{-1/\|\varphi\|_\infty}\Phi(t)$ $(t > 0)$ in the case Φ is decreasing. These conditions combined with (1.2) and our (i) and (iii) are used to prove weighted norm inequalities for rough singular integrals in Fan, Pan and Yang [16]. Hence, in the case $\gamma \geq 2$, our Theorem 1.1 is just the counterpart in Marcinkiewicz integrals to their Theorems 1 and 2 in singular integrals.

The following two facts are useful to check the conditions (i) and (iii) in the above theorems, which can be seen easily:

(v) The case where Φ is positive and increasing. If $\Phi(t)\,t^{-\delta}$ is non-increasing for some $\delta > 0$, then $\Phi(2t) \leq 2^\delta\Phi(t)$ $(t > 0)$.

(vi) The case where Φ is positive and decreasing. If $\Phi(t)\,t^{\delta}$ is non-decreasing for some $\delta > 0$, then $\Phi(2t) \geq 2^{-\delta}\Phi(t)$ $(t > 0)$.

Recently, Al-Qassem gave the L^p boundedness and the weighted L^p boundedness of $\mu_{\Omega,\rho,\Phi,b}$ when Ω belongs to some block spaces $B_q^{(0,-1/2)}(S^{n-1})$ in [1] and [2], respectively. In 2006, Ye and Zhu gave the following including relationship in [25]: For $q > 1$ and $v > -1$

$$B_q^{(0,v)}(S^{n-1}) \subset H^1(S^{n-1}) + L(\log L)^{1+v}(S^{n-1})\,. \tag{1.10}$$

Therefore, applying (1.10) and the conclusions of Theorems 1.1 and 1.3, we get immediately the following result:

THEOREM 1.5. *Suppose $\Omega \in B_q^{(0,-1/2)}(S^{n-1})$ for some $q > 1$ and $b \in \Delta_\gamma$ for some $\gamma > 1$. If Φ satisfies the condition in Theorem* 1.1, *then $\mu_{\Omega,\rho,\Phi,b}$ is bounded on $L^p(\boldsymbol{R}^n)$ for p satisfying $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$. Furthermore, if $\gamma \geq 2$, then $\mu_{\Omega,\rho,\Phi,b}$ is bounded on $L^p(\omega)$ for $p \in (\gamma', \infty)$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$.*

REMARK 1.6. Al-Qassem [2] gave the same result under the following two conditions on Φ (for the sake of simplicity, we only state in the case where $\Phi \in C^1(\boldsymbol{R}_+)$ is nonnegative and increasing): (a) $\Phi(2t) \geq \eta\Phi(t)$ for some fixed $\eta > 1$ and $\Phi(2t) \leq c\Phi(t)$ for some constant $c \geq \eta$. (b) $\Phi'(t) \geq \alpha\Phi(t)/t$ on $\boldsymbol{R}_+$ for some fixed $0 < \alpha \leq \log_2 c$ and $\Phi'(t)$ is monotone on $\boldsymbol{R}_+$. Obviously, Al-Qassem's assumption implies automatically our condition. However, there is a function Φ which satisfies our condition but does not satisfy Al-Qassem's condition. $\Phi(t) = \sqrt{t}\log(1+t)$ is such an example. Hence, Theorem 1.5 improves Al-Qassem's result.

Finally, if $\Omega \in L^q(S^{n-1})$ and $\gamma \geq 2$, we can obtain weighted norm estimates for the usual Muckenhoupt's A_p weights. We formulate it as follows.

THEOREM 1.7. *Suppose $\Omega \in L^q(S^{n-1})$ for some $q > 1$ and $b \in \Delta_\gamma$ for some $\gamma \geq 2$. If Φ satisfies the condition in Theorem* 1.1, *then $\mu_{\Omega,\rho,\Phi,b}$ is bounded on $L^p(\omega)$ provided p, q, ω satisfy one of the following conditions*:

(a) $\gamma' \leq q' < p < \infty$ *and* $\omega \in A_{p/q'}(\boldsymbol{R}^n)$.

(b) $\gamma' \leq p < q$ *and* $\omega^{1-p'} \in A_{p'/q'}(\boldsymbol{R}^n)$.

(c) $\gamma' \leq p < \infty$ *and* $\omega^{q'} \in A_p(\boldsymbol{R}^n)$.

Now, we would like to explain the reason why we discuss the parametrized Marcinkiewicz integrals here.

We first note the following facts: If $g(t) \in C^1(\boldsymbol{R}_+)$ is nonnegative and increasing (resp. decreasing) on $\boldsymbol{R}_+$ and $g(t)/(tg'(t))$ is bounded on $\boldsymbol{R}_+$, then $\lim_{t\to 0} g(t) = 0$ (resp. $\lim_{t\to 0} g(t) = +\infty$) and $\lim_{t\to+\infty} g(t) = +\infty$ (resp. $\lim_{t\to+\infty} g(t) = 0$). See [24, Remark 2] for the proof.

EXAMPLE 1.8. In the case where $\Phi(t) = t^a$ for some $a > 0$, we see that

$$\Phi'(t) = at^{a-1}, \quad \Phi^{-1}(t) = t^{1/a}, \quad \varphi(t) = \frac{\Phi(t)}{t\Phi'(t)} = \frac{t^a}{t\cdot at^{a-1}} = \frac{1}{a},$$

and hence

$$\mu_{\Omega,\rho,\Phi,b}(f)(x) = \frac{1}{a^{3/2}}\left(\int_0^\infty \left|\frac{1}{s^{\rho/a}}\int_{|y|<s} b(|y|^{1/a})\frac{\Omega(y')}{|y|^{n-\rho/a}} f(x-y)dy\right|^2 \frac{ds}{s}\right)^{1/2}.$$

This shows that it is natural to consider parametrized Marcinkiewicz integrals when we study Marcinkiewicz integrals associated to surfaces.

In the case where $\Phi(t) = t^a$ for some $a < 0$, we see that

$$\mu_{\Omega,\rho,\Phi,b}(f)(x) = \frac{1}{|a|^{3/2}}\left(\int_0^\infty \left|\frac{1}{s^{\rho/a}}\int_{|y|>s} b(|y|^{1/a})\frac{\Omega(y')}{|y|^{n-\rho/a}} f(x-y)dy\right|^2 \frac{ds}{s}\right)^{1/2}.$$

To prove Theorem 1.1, we borrowed many ideas from the proofs of the corresponding theorems by Ding, Fan and Pan [10], by Fan, Pan and Yang [16] and by Al-Qassem [1], [2]. To prove Theorem 1.3, we used the ideas from the proof of the corresponding theorem by Al-Salman, Al-Qassem, Cheng and Pan [5]. This work is a revision of our former one, and is stimulated by a recent paper by Al-Qassem and Pan [4].

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Preliminary lemmas. We prepare two lemmas, whose proofs can be found in Fan and Pan [15]. We present some notations. Let Ω be a regular ∞-atom in $H^1(S^{n-1})$ whose support satisfies

$$\operatorname{supp}\Omega \subset S^{n-1}\cap B(\xi',\tau)\,,\quad (\xi'\in S^{n-1})\,.$$

When $n \geq 3$, set

$$E_\Omega(s,\xi') = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}}\Omega(s,\ \sqrt{1-s^2}\,\tilde{y})d\sigma(\tilde{y})\,, \tag{2.1}$$

and when $n = 2$, set

$$e_\Omega(s,\xi') = \frac{1}{\sqrt{1-s^2}}\chi_{(-1,1)}(s)[\Omega(s,\ \sqrt{1-s^2}) + \Omega(s,\ -\sqrt{1-s^2})]\,. \tag{2.2}$$

LEMMA 2.1. *There exists a constant $c > 0$, independent of Ω, such that $cE_\Omega(s,\xi')$ is an ∞-atom in $H^1(\boldsymbol{R})$. That is, $cE_\Omega(s,\xi')$ satisfies*

$$\begin{aligned} &\|cE_\Omega\|_{L^\infty} \leq \frac{1}{4r(\xi')}\,,\quad \operatorname{supp}E_\Omega \subset (\xi_1' - 2r(\xi'),\ \xi_1' + 2r(\xi'))\,,\\ &\qquad\qquad \textit{and}\quad \int_{\boldsymbol{R}} E_\Omega(s,\xi')ds = 0\,, \end{aligned} \tag{2.3}$$

where $r(\xi') = |\xi|^{-1}|A_\tau\xi|$ and $A_\tau(\xi) = (\tau^2\xi_1, \tau\xi_2, \dots, \tau\xi_n)$.

LEMMA 2.2. *For $1 < q < 2$, there exists a constant $c > 0$, independent of Ω, such that $ce_\Omega(s,\xi')$ is a q-atom in $H^1(\boldsymbol{R})$, the center of whose support is ξ_1' and the radius $r(\xi') = |\xi|^{-1}(\tau^4\xi_1^2 + \tau^2\xi_2^2)^{1/2}$.*

We prepare the following facts about directional maximal function.

LEMMA 2.3. *Suppose Φ is a positive function on $(0,\infty)$ with $|\Phi(t)/(t\Phi'(t))| \leq b$ and satisfies one of the following conditions*:

(i) *Φ is increasing, and $\Phi(2t) \leq c_1\Phi(t)$.*
(ii) *Φ is increasing, and $t\Phi'(t)$ is increasing.*
(iii) *Φ is decreasing, and $\Phi(t) \leq c_2\Phi(2t)$.*
(iv) *Φ is decreasing and convex.*

Then
(a)

$$\left|\int_{t/2<|y|<t}\frac{f(x-\Phi(|y|)y')}{|y|^n}dy\right| \leq C(1+b)Mf(x)\,,$$

where $Mf(x)$ is the usual Hardy-Littlewood maximal function of f, and
(b) *if $\Omega \in L^1(S^{n-1})$,*

$$\left|\int_{t/2<|y|<t}\frac{\Omega(y')f(x-\Phi(|y|)y')}{|y|^n}dy\right| \leq C(1+b)\int_{S^{n-1}}|\Omega(y')|M_{y'}f(x)d\sigma(y')\,,$$

where $M_{y'}f(x)$ is the directional Hardy-Littlewood maximal function of f defined by

$$M_{y'}f(x) = \sup_{r>0}\frac{1}{2r}\int_{|t|<r}|f(x-ty')|dt\,.$$

PROOF. (a)

(i) The case where Φ is increasing and $\Phi(2t) \le c_1\Phi(t)$. By a change of variable $r = \Phi(s)$, we have

$$\begin{aligned}\left|\int_{t/2<|y|<t} \frac{f(x-\Phi(|y|)y')}{|y|^n}dy\right| &= \left|\int_{S^{n-1}}\int_{t/2}^{t} \frac{f(x-\Phi(s)y')}{s}dsd\sigma(y')\right| \\ &= \int_{S^{n-1}}\int_{\Phi(t/2)}^{\Phi(t)} |f(x-ry')|\frac{r}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\frac{dr}{r}d\sigma(y') \\ &\le b\int_{S^{n-1}}\int_{\Phi(t/2)}^{\Phi(t)} |f(x-ry')|\frac{dr}{r}d\sigma(y') \\ &\le \frac{b}{\Phi(t/2)^n}\int_{S^{n-1}}\int_{\Phi(t/2)}^{\Phi(t)} |f(x-ry')|r^{n-1}drd\sigma(y') \\ &\le \frac{bc_1^n}{\Phi(t)^n}\int_{S^{n-1}}\int_{0}^{\Phi(t)} |f(x-ry')|r^{n-1}drd\sigma(y') \\ &\le c_1^n v_n b\, Mg(x)\,,\end{aligned}$$

where v_n denotes the volume of the unit ball in $\boldsymbol{R}^n$.

(ii) The case where Φ is increasing and $t\Phi'(t)$ is increasing. By a change of variable $r = \Phi(s)$, we have

$$\begin{aligned}&\left|\int_{t/2<|y|<t} \frac{f(x-\Phi(|y|)y')}{|y|^n}dy\right| \\ &= \int_{S^{n-1}}\int_{\Phi(t/2)}^{\Phi(t)} |f(x-ry')|\frac{1}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}drd\sigma(y')\,.\end{aligned} \tag{2.4}$$

We set

$$a_t(r) = \begin{cases} ((t/2)\Phi'(t/2))^{-1} & 0 < r < \Phi(t/2)\,, \\ (\Phi^{-1}(r)\Phi'(\Phi^{-1}(r)))^{-1} & \Phi(t/2) \le r < \Phi(t)\,, \\ 0 & r \ge \Phi(t)\,. \end{cases}$$

Then

$$\begin{aligned}\int_0^\infty a_t(r)dr &= \frac{1}{(t/2)\Phi'(t/2)} \times \Phi(t/2) + \int_{\Phi(t/2)}^{\Phi(t)} \frac{1}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}dr \\ &\le b + \int_{t/2}^{t}\frac{ds}{s} = b + \log 2\,.\end{aligned}$$

Since $1/(t\Phi'(t))$ is decreasing, it follows that $a_t(r)$ is nonnegative, decreasing and integrable on $(0,\infty)$. So, $a_t(r)/t^{n-1}$ is nonnegative, decreasing, and $\int_{S^{n-1}} a_t(|y|)/|y|^{n-1}dy = b+\log 2$. Hence, we have by (2.4)

$$\begin{aligned}\left|\int_{t/2<|y|<t} \frac{f(x-\Phi(|y|)y')}{|y|^n}dy\right| &\le \int_{S^{n-1}}\int_0^\infty |f(x-ry')|a_t(r)drd\sigma(y') \\ &= \int_{\boldsymbol{R}^n} |f(x-y)|\frac{a_t(|y|)}{|y|^{n-1}}dy \le (b+\log 2)Mf(x)\,.\end{aligned}$$

(iii) The case where Φ is decreasing and $\Phi(t) \geq c_2\Phi(2t)$. In the same way as in the case (i), we get

$$\left|\int_{t/2<|y|<t} \frac{f(x-\Phi(|y|)y')}{|y|^n}dy\right| \leq c_2^n v_n b\, Mg(x)\,.$$

(iv) The case where Φ is decreasing and convex. Since $\Phi(t)$ is positive and decreasing, we see that $\Phi^{-1}(t)$ is also decreasing, and hence $1/\Phi^{-1}(t)$ is increasing. Hence we get

$$\begin{aligned}&\left|\int_{t/2<|y|<t} \frac{f(x-\Phi(|y|)y')}{|y|^n}dy\right| \\ &\leq \int_{S^{n-1}}\int_{\Phi(t)}^{\Phi(t/2)} |f(x-ry')|\frac{1}{-\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}drd\sigma(y') \qquad (2.5)\\ &\leq \frac{2}{t}\int_{S^{n-1}}\int_{\Phi(t)}^{\Phi(t/2)} |f(x-ry')|\frac{1}{-\Phi'(\Phi^{-1}(r))}drd\sigma(y')\,.\end{aligned}$$

We set

$$a_t(r) = \begin{cases} -2(t\Phi'(t))^{-1} & 0 < r < \Phi(t)\,, \\ -2(t\Phi'(\Phi^{-1}(r)))^{-1} & \Phi(t) \leq r < \Phi(t/2)\,, \\ 0 & r \geq \Phi(t/2)\,. \end{cases}$$

Then

$$\begin{aligned}\int_0^\infty a_t(r)dr &= -\frac{2}{t\Phi'(t)} \times \Phi(t) + \int_{\Phi(t)}^{\Phi(t/2)} \frac{-2}{t\Phi'(\Phi^{-1}(r))}dr \\ &\leq 2b + \frac{2}{t}\int_{t/2}^t ds = 2b+1\,.\end{aligned}$$

Furthermore, because of the convexity of $\Phi(t)$, it follows that $-\Phi'(\Phi^{-1}(t))$ is increasing. So, we see that $a_t(r)$ is nonnegative, decreasing and integrable on $(0,\infty)$. Hence, as in (ii), we obtain

$$\left|\int_{t/2<|y|<t} \frac{f(x-\Phi(|y|)y')}{|y|^n}dy\right| = \int_{\boldsymbol{R}^n} |f(x-y)|\frac{a_t(|y|)}{|y|^{n-1}}dy \leq (2b+1)Mf(x)\,.$$

(b) Using (a) with $n=1$, we can easily deduce the conclusions in (b). □

Next, we prepare the following estimates about Fourier transforms of some measures on $\boldsymbol{R}^n$. In the case where Φ is positive and increasing, we have the following.

LEMMA 2.4. *Let* $1 < q \leq \infty$, $\Omega \in L^q(S^{n-1})$. *If* Φ *is positive, increasing,* $\Phi(2t) \leq c_0\Phi(t)$, *and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$, *then it holds for any* $0 < \alpha \leq 1/q'$

$$\int_{t/2}^t \left|\int_{S^{n-1}} \Omega(x)e^{-i(\Phi(s)\xi\cdot x')}d\sigma(x')\right|^2 \frac{ds}{s} \leq \frac{C_\alpha 2^\alpha(\log c_0)^{1-\alpha}\|\varphi\|_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t/2)\xi|^\alpha}\,.$$

PROOF. We have

$$\begin{aligned}
&\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(x')e^{-i(\Phi(s)\xi\cdot x')}d\sigma(x')\right|^2\frac{ds}{s}\\
&=\int_{\Phi(t/2)}^{\Phi(t)}\left|\int_{S^{n-1}}\Omega(x')e^{-i(r\xi\cdot x')}d\sigma(x')\right|^2\frac{r}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\frac{dr}{r}\\
&\le\left\|\frac{\Phi(t)}{t\Phi'(t)}\right\|_\infty\int_{\Phi(t/2)}^{\Phi(t)}\left|\int_{S^{n-1}}\Omega(x')e^{-i(r\xi\cdot x')}d\sigma(x')\right|^2\frac{dr}{r}\\
&=\left\|\frac{\Phi(t)}{t\Phi'(t)}\right\|_\infty\int_{S^{n-1}\times S^{n-1}}\Omega(x')\overline{\Omega(y')}\left(\int_{\Phi(t/2)}^{\Phi(t)}e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\right)d\sigma(x')d\sigma(y')\,.
\end{aligned}\tag{2.6}$$

In the second equation, we used the change of variable $r=\Phi(s)$. Clearly we have

$$\left|\int_{\Phi(t/2)}^{\Phi(t)}e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\right|\le\log\frac{\Phi(t)}{\Phi(t/2)}\le\log c_0$$

and

$$\left|\int_{\Phi(t/2)}^{\Phi(t)}e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\right|\le\frac{2}{\Phi(t/2)|\xi||\xi'\cdot(x'-y')|}\,,$$

and so we have for any $0<\alpha\le 1$

$$\left|\int_{\Phi(t/2)}^{\Phi(t)}e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\right|\le\frac{(\log c_0)^{1-\alpha}2^\alpha}{|\Phi(t/2)\xi|^\alpha|\xi'\cdot(x'-y')|^\alpha}\,.$$

This combined with (2.6) yields the desired estimate. □

LEMMA 2.5. *Let $1<q\le\infty$, $\Omega\in L^q(S^{n-1})$. If Φ is positive, increasing, $t\Phi'(t)$ is increasing, and $\varphi(t):=\Phi(t)/(t\Phi'(t))\in L^\infty(0,\infty)$, then it holds for any $0<\alpha<1/q'$*

$$\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(x)e^{-i(\Phi(s)\xi\cdot x')}d\sigma(x')\right|^2\frac{ds}{s}\le\frac{C_\alpha 4^\alpha(\log 2)^{1-\alpha}\|\varphi\|_\infty^\alpha\|\Omega\|_{L^q(S^{n-1})}^2}{|\Phi(t/2)\xi|^\alpha}\,.$$

PROOF. We have

$$\begin{aligned}
&\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(x')e^{-i(\Phi(s)\xi\cdot x')}d\sigma(x')\right|^2\frac{ds}{s}\\
&=\int_{S^{n-1}\times S^{n-1}}\Omega(x')\overline{\Omega(y')}\left(\int_{t/2}^{t}e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right)d\sigma(x')d\sigma(y')\,.
\end{aligned}\tag{2.7}$$

Clearly we have

$$\left|\int_{t/2}^{t}e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right|\le\log 2\,.\tag{2.8}$$

Applying the change of variable $r=\Phi(s)$, we have

$$\int_{t/2}^{t}e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}=\int_{\Phi(t/2)}^{\Phi(t)}e^{-ir\xi\cdot(x'-y')}\frac{1}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}dr\,.$$

Since Φ is positive and increasing, and $t\Phi'(t)$ is increasing, we see that $\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))$ is increasing. Hence we obtain

$$\left|\int_{\Phi(t/2)}^{\Phi(t)} \cos(-r\xi\cdot(x'-y'))\frac{dr}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\right| \le \frac{1}{t/2\Phi'(t/2)}\frac{2}{|\xi\cdot(x'-y')|} \le \frac{\Phi(t/2)}{t/2\Phi'(t/2)}\frac{2}{\Phi(t/2)|\xi\cdot(x'-y')|}\,.$$

We get a similar estimate for sin part, and hence we obtain

$$\left|\int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right| \le \left\|\frac{\Phi(s)}{s\Phi'(s)}\right\|_\infty \frac{4}{\Phi(t/2)|\xi\cdot(x'-y')|}\,.$$

Thus, combining this with (2.8) we have for any $0<\alpha\le 1$

$$\left|\int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right| \le \left\|\frac{\Phi(t)}{t\Phi'(t)}\right\|_\infty^\alpha \frac{(\log 2)^{1-\alpha}4^\alpha}{|\Phi(t/2)\xi|^\alpha|\xi'\cdot(x'-y')|^\alpha}\,.$$

This combined with (2.7) yields the desired estimate. □

In the case where Φ is positive and decreasing, we have the following

LEMMA 2.6. *Let* $1<q\le\infty$, $\Omega\in L^q(S^{n-1})$. *If* Φ *is positive, decreasing,* $\Phi(2t)\ge c_1\Phi(t)$, *and* $\varphi(t):=\Phi(t)/(t\Phi'(t))\in L^\infty(0,\infty)$, *then it holds for any* $0<\alpha<1/q'$

$$\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(x)e^{-i(\Phi(s)\xi\cdot x')}d\sigma(x')\right|^2\frac{ds}{s} \le \frac{C_\alpha 2^\alpha(\log 1/c_1)^{1-\alpha}\|\varphi\|_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t)\xi|^\alpha}\,.$$

PROOF. We have

$$\begin{aligned}
&\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(x')e^{-i(\Phi(s)\xi\cdot x')}d\sigma(x')\right|^2\frac{ds}{s}\\
&= \int_{\Phi(t)}^{\Phi(t/2)}\left|\int_{S^{n-1}}\Omega(x')e^{-i(r\xi\cdot x')}d\sigma(x')\right|^2\frac{r}{-\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\frac{dr}{r}\\
&\le \left\|\frac{\Phi(t)}{t\Phi'(t)}\right\|_\infty\int_{\Phi(t)}^{\Phi(t/2)}\left|\int_{S^{n-1}}\Omega(x')e^{-i(r\xi\cdot x')}d\sigma(x')\right|^2\frac{dr}{r}\\
&= \left\|\frac{\Phi(t)}{t\Phi'(t)}\right\|_\infty\int_{S^{n-1}\times S^{n-1}}\Omega(x')\overline{\Omega(y')}\left(\int_{\Phi(t)}^{\Phi(t/2)}e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\right)d\sigma(x')d\sigma(y')\,.
\end{aligned} \tag{2.9}$$

In the second equation, we used the change of variable $r=\Phi(s)$. Clearly we have

$$\left|\int_{\Phi(t)}^{\Phi(t/2)}e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\right| \le \log\frac{\Phi(t/2)}{\Phi(t)} \le \log\frac{1}{c_1}$$

and

$$\left|\int_{\Phi(t)}^{\Phi(t/2)}e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\right| \le \frac{2}{\Phi(t)|\xi||\xi'\cdot(x'-y')|}\,,$$

and so we have for any $0 < \alpha \le 1$

$$\left|\int_{\Phi(t)}^{\Phi(t/2)} e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\right| \le \frac{(\log 1/c_1)^{1-\alpha}2^\alpha}{|\Phi(t)\xi|^\alpha|\xi'\cdot(x'-y')|^\alpha}\,.$$

This combined with (2.9) yields the desired estimate. □

LEMMA 2.7. *Let* $1 < q \le \infty$, $\Omega \in L^q(S^{n-1})$. *If* Φ *is positive, decreasing and convex, and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$, *then it holds for any* $0 < \alpha < 1/q'$

$$\int_{t/2}^t \left|\int_{S^{n-1}} \Omega(x)e^{-i(\Phi(s)\xi\cdot x')}d\sigma(x')\right|^2 \frac{ds}{s} \le \frac{C_\alpha 8^\alpha(\log 2)^{1-\alpha}\|\varphi\|_\infty^\alpha\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t)\xi|^\alpha}\,.$$

PROOF. We have

$$\begin{aligned}(2.10)\qquad &\int_{t/2}^t \left|\int_{S^{n-1}} \Omega(x')e^{-i(\Phi(s)\xi\cdot x')}d\sigma(x')\right|^2\frac{ds}{s}\\ &= \int_{S^{n-1}\times S^{n-1}} \Omega(x')\overline{\Omega(y')}\left(\int_{t/2}^t e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right)d\sigma(x')d\sigma(y')\,.\end{aligned}$$

Clearly we have

$$(2.11)\qquad \left|\int_{t/2}^t e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right| \le \log 2\,.$$

Applying the change of variable $r = \Phi(s)$, we have

$$\int_{t/2}^t e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s} = \int_{\Phi(t)}^{\Phi(t/2)} e^{-ir\xi\cdot(x'-y')}\frac{1}{-\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\,dr\,.$$

Since Φ is positive, increasing and convex, we see that $-\Phi'(t)$ is decreasing, and hence $-\Phi'(\Phi^{-1}(r))$ is positive and increasing. Hence we see by the second mean value theorem that there exists c with $\Phi(t) \le c \le \Phi(t/2)$ such that

$$\begin{aligned}&\int_{\Phi(t)}^{\Phi(t/2)} \cos(-r\xi\cdot(x'-y'))\frac{dr}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\\ &\qquad = \frac{1}{-\Phi'(t)}\int_{\Phi(t)}^{c} \cos(-r\xi\cdot(x'-y'))\frac{dr}{\Phi^{-1}(r)}\,.\end{aligned}$$

Since Φ is positive and decreasing, we see that $\Phi^{-1}(r)$ is also positive and decreasing. Hence we have

$$\begin{aligned}&\left|\int_{\Phi(t)}^{\Phi(t/2)} \cos(-r\xi\cdot(x'-y'))\frac{dr}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\right|\\ &\le \frac{1}{-\Phi'(t)}\frac{1}{\Phi^{-1}(c)}\frac{2}{|\xi\cdot(x'-y')|} \le \frac{1}{-\Phi'(t)}\frac{1}{t/2}\frac{2}{|\xi||\xi'\cdot(x'-y')|}\\ &\le \left\|\frac{\Phi(s)}{s\Phi'(s)}\right\|_\infty \frac{4}{\Phi(t)|\xi\cdot(x'-y')|}\,.\end{aligned}$$

We get a similar estimate for sin part, and hence we obtain

$$\left|\int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right| \le \left\|\frac{\Phi(s)}{s\Phi'(s)}\right\|_\infty \frac{8}{\Phi(t)|\xi\cdot(x'-y')|}\,.$$

Thus, combining this with (2.11), we have for any $0<\alpha\le 1$

$$\left|\int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right| \le \left\|\frac{\Phi(s)}{s\Phi'(s)}\right\|_\infty^\alpha \frac{(\log 2)^{1-\alpha}8^\alpha}{|\Phi(t)\xi|^\alpha|\xi'\cdot(x'-y')|^\alpha}\,.$$

Combining this with (2.10) yields the desired estimate. □

Finally in this section, we will note the lacunarity of the sequence $\{\Phi(a^k)\}_{k\in\mathbf{Z}}$.

LEMMA 2.8. *Suppose Φ is positive, increasing, and $\Phi(t)/(t\Phi'(t)) \le b$. Then, if $a>1$, $\Phi(a^{k+1})/\Phi(a^k)\ge a^{1/b}$ for $k\in\mathbf{Z}$. Hence $\{\Phi(a^k)\}_{k\in\mathbf{Z}}$ is a lacunary sequence.*

PROOF. From the assumption we get

$$\frac{1}{bt}\le\frac{\Phi'(t)}{\Phi(t)}=(\log\Phi(t))'\,,$$

and it implies

$$\frac{1}{b}\log a=\int_{a^k}^{a^{k+1}}\frac{dt}{bt}\le\int_{a^k}^{a^{k+1}}(\log\Phi(t))'dt=\log\frac{\Phi(a^{k+1})}{\Phi(a^k)}\,,$$

i.e.,

$$\Phi(a^{k+1})/\Phi(a^k)\ge a^{1/b}\,.$$ □

LEMMA 2.9. *Suppose Φ is positive, decreasing, and $-\Phi(t)/(t\Phi'(t)) \le b$. Then, if $a>1$, $\Phi(a^{-(k+1)})/\Phi(a^{-k})\ge a^{1/b}$ for $k\in\mathbf{Z}$. Hence $\{\Phi(a^{-k})\}_{k\in\mathbf{Z}}$ is a lacunary sequence.*

PROOF. From the assumption we get

$$\frac{1}{bt}\le-\frac{\Phi'(t)}{\Phi(t)}=-(\log\Phi(t))'\,,$$

and it implies

$$\frac{1}{b}\log a=\int_{a^{-(k+1)}}^{a^{-k}}\frac{dt}{bt}\le-\int_{a^{-(k+1)}}^{a^{-k}}(\log\Phi(t))'dt=\log\frac{\Phi(a^{-(k+1)})}{\Phi(a^{-k})}\,,$$

i.e.,

$$\Phi(a^{-(k+1)})/\Phi(a^{-k})\ge a^{1/b}\,.$$ □

3. Proof of Theorem 1.1. Using the definition of $\mu_{\Omega,\rho,\Phi,b}(f)(x)$ and Cauchy-Schwarz's inequality, we obtain via change of variables

$$\begin{aligned}
&\mu_{\Omega,\rho,\Phi,b}(f)(x)\\
&= \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|y|<t}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t}\right)^{1/2}\\
&= \left(\int_0^\infty \left|\sum_{k=0}^\infty\frac{1}{t^\rho}\int_{2^{-k-1}t\le|y|<2^{-k}t}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t}\right)^{1/2}\\
&\le \sum_{k=0}^\infty\left(\int_0^\infty \left|\sum_{k=0}^\infty\frac{1}{t^\rho}\int_{2^{-k-1}t\le|y|<2^{-k}t}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t}\right)^{1/2}\\
&= \sum_{k=0}^\infty\frac{1}{2^{\rho k}}\left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{2^{-k-1}t\le|y|<2^{-k}t}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t}\right)^{1/2}\\
&= \frac{1}{1-2^{-\rho}}\left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{t/2<|y|<t}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t}\right)^{1/2}.
\end{aligned}$$

Hence, it is sufficient to estimate the modified operator

$$\tilde{\mu}_{\Omega,\rho,\Phi,b}(f)(x) = \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{t/2<|y|<t}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t}\right)^{1/2}. \tag{3.1}$$

For a homogeneous kernel Ω and $\rho > 0$, we define the family $\{\sigma_t;\ t \in \boldsymbol{R}_+\}$ of measures and the maximal operator σ^* on $\boldsymbol{R}^n$ by

$$\int_{\boldsymbol{R}^n} f(x)d\sigma_t(x) = \frac{1}{t^\rho}\int_{2/t<|x|<t} f(\Phi(|x|)x')\frac{b(|x|)\Omega(x')}{|x|^{n-\rho}}dx\,, \tag{3.2}$$

$$\sigma^* f(x) = \sup_{t>0} ||\sigma_t| * f(x)|\,, \tag{3.3}$$

where $|\sigma_t|$ is defined in the same way as σ_t, but with Ω replaced by $|\Omega|$ and b replaced by $|b|$. Thus,

$$\tilde{\mu}_{\Omega,\rho,\Phi,b}(f)(x) = \left(\int_0^\infty |\sigma_t * f(x)|^2\frac{dt}{t}\right)^{1/2}. \tag{3.4}$$

We first check the following.

LEMMA 3.1. *Let $\Omega \in L^1(S^{n-1})$ and $b \in \Delta_\gamma$ for some $\gamma \ge 1$. Suppose Φ is a nonnegative and monotonic C^1 function on $\boldsymbol{R}_+$ satisfying the condition* (1.2). *Then the total measure of σ_t denoted by $\|\sigma_t\|$ is estimated as follows*:

$$\|\sigma_t\| := \int_{\boldsymbol{R}^n} |d\sigma_t(x)| \le C\|b\|_{\Delta_\gamma}\|\Omega\|_{L^1(S^{n-1})}\,. \tag{3.5}$$

PROOF. For any nonnegative $f \in C(\boldsymbol{R}^n)$, we obtain

$$\begin{aligned}\int_{\boldsymbol{R}^n} f(y)|d\sigma_t(y)| &= \frac{1}{t^\rho}\int_{t/2<|y|<t} f(\Phi(|y|)y')\frac{|b(|y|)\Omega(y')|}{|y|^{n-\rho}}dy \\ &\le \|f\|_\infty \frac{1}{t^\rho}\int_{t/2}^{t}|b(r)|\frac{dr}{r^{1-\rho}}\int_{S^{n-1}}|\Omega(y')|d\sigma(y') \\ &\le 2\|f\|_\infty\|b\|_{\Delta_1}\|\Omega\|_{L^1(S^{n-1})},\end{aligned}$$

which shows (3.5). □

LEMMA 3.2. *Let $\Omega \in L^1(S^{n-1})$ and $b \in \Delta_\gamma$ for some $\gamma \ge 1$. Suppose Φ is a nonnegative and monotonic C^1 function on $\boldsymbol{R}_+$ satisfying the condition in Theorem* 1.1. *Then there exists $C > 0$ such that*

$$\begin{aligned}\sigma^*(f)(x) &:= \sup_{0<t<\infty} ||\sigma_t| * f(x)| \\ &\le C\|b\|_{\Delta_\gamma}\|\Omega\|^{1/\gamma}_{L^1(S^{n-1})}\left(\int_{S^{n-1}}|\Omega(y')|M_{y'}(|f|^{\gamma'})(x)d\sigma(y')\right)^{1/\gamma'}.\end{aligned} \tag{3.6}$$

As a consequence, for $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}^n)$ there exists $C > 0$ such that

$$\|\sigma^*(f)\|_{L^p(\omega)} \le C\|b\|_{\Delta_\gamma}\|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^p(\omega)}. \tag{3.7}$$

PROOF. By Hölder's inequality and Lemma 2.3, we see that

$$\begin{aligned}||\sigma_t| * f(x)| &= \left|\int_{t/2<|y|<t}\frac{1}{t^\rho}\frac{|b(|y|)\Omega(y')|}{|y|^{n-\rho}}f(x-\Phi(|y|)y')dy\right| \\ &\le \frac{1}{t^\rho}\left(\int_{t/2<|y|<t}\frac{|b(|y|)|^\gamma|\Omega(y')|}{|y|^{n-\gamma\rho}}dy\right)^{1/\gamma} \\ &\quad\times\left(\int_{t/2<|y|<t}\frac{|\Omega(y')|}{|y|^n}|f(x-\Phi(|y|)y')|^{\gamma'}dy\right)^{1/\gamma'} \\ &\le C\left(\int_{t/2<|y|<t}\frac{|b(r)|^\gamma}{r}dr\int_{S^{n-1}}|\Omega(y')|d\sigma(y')\right)^{1/\gamma} \\ &\quad\times\left(\int_{S^{n-1}}|\Omega(y')|M_{y'}(|f|^{\gamma'})(x)d\sigma(y')\right)^{1/\gamma'} \\ &\le C\|b\|_{\Delta_\gamma}\|\Omega\|^{1/\gamma}_{L^1(S^{n-1})}\left(\int_{S^{n-1}}|\Omega(y')|M_{y'}(|f|^{\gamma'})(x)d\sigma(y')\right)^{1/\gamma'},\end{aligned} \tag{3.8}$$

which shows (3.6). It is known that, for $1 < r < \infty$ and $\omega \in \tilde{A}_r(\boldsymbol{R}^n)$, $\|M_{y'}\|_{L^r(\omega)} \le C_r\|f\|_{L^r(\omega)}$ uniformly in y' (see for example [13, Theorem 7, p. 875]). From this, for $p > \gamma'$ it follows by Minkowski's inequality

$$\left(\int_{\boldsymbol{R}^n}\left(\int_{S^{n-1}}|\Omega(y')|M_{y'}(|f|^{\gamma'})(x)d\sigma(y')\right)^{p/\gamma'}\omega(x)dx\right)^{1/p} \le \|\Omega\|^{1/\gamma'}_{L^1(S^{n-1})}\|f\|_{L^p(\omega)},$$

which shows (3.7). □

LEMMA 3.3. *Let Ω be a regular $H^1(S^{n-1})$-atom whose support is contained in $S^{n-1}\cap B(\boldsymbol{e}_1,\tau)$, where $\boldsymbol{e}_1=(1,0,\dots,0)$. Suppose Φ is a nonnegative and monotonic C^1 function on $\boldsymbol{R}_+$ satisfying the condition* (1.2). *Then, if Φ is increasing,*

$$|\widehat{\sigma_t}(\xi)|\le C\|b\|_{\Delta_1}\Phi(t)|A_\tau\xi|\,, \tag{3.9}$$

and if Φ is decreasing,

$$|\widehat{\sigma_t}(\xi)|\le C\|b\|_{\Delta_1}\Phi(t/2)|A_\tau\xi|\,, \tag{3.10}$$

where $A_\tau(\xi)=(\tau^2\xi_1,\tau\xi_2,\dots,\tau\xi_n)$.

PROOF. We only prove the case $n\ge 3$, since one can prove the case $n=2$ with a slight modification. Let Ω be a regular $H^1(S^{n-1})$-atom whose support is contained in $S^{n-1}\cap B(\boldsymbol{e}_1,\tau)$. For $0\ne\xi\in\boldsymbol{R}^n$, we choose a rotation $\mathcal{O}$ such that $\mathcal{O}(\xi)=|\xi|\boldsymbol{e}_1$. Then $\mathcal{O}^2(\xi')=(\xi_1',\eta_2',\dots,\eta_n')$ by virtue of $\mathcal{O}^{-1}={}^t\mathcal{O}$. Set Q_{n-1} be a rotation in $\boldsymbol{R}^{n-1}$ such that $Q_{n-1}(\xi_2',\dots,\xi_n')=(\eta_2',\dots,\eta_n')$ and $R=\begin{pmatrix}1&0\\0&Q_{n-1}\end{pmatrix}$. Then, for any $y'=(u,y_2',\dots,y_n')\in S^{n-1}$, we have $\boldsymbol{e}_1\cdot Ry'=\boldsymbol{e}_1\cdot y'=u$ and $\Omega(\mathcal{O}^{-1}Ry')$ is a regular $H^1(S^{n-1})$-atom supported in $S^{n-1}\cap B(\xi',\tau)$. Thus we have

$$\begin{aligned}
\widehat{\sigma_t}(\xi)&=\frac{1}{t^\rho}\int_{t/2<|y|<t}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}e^{-i\Phi(|y|)y'\cdot\xi}dy\\
&=\frac{1}{t^\rho}\int_{t/2}^{t}\frac{b(r)}{r^{1-\rho}}\left(\int_{S^{n-1}}\Omega(y')e^{-i\Phi(r)y'\cdot\xi}d\sigma(y')\right)dr\\
&=\frac{1}{t^\rho}\int_{t/2}^{t}\frac{b(r)}{r^{1-\rho}}\left(\int_{S^{n-1}}\Omega(\mathcal{O}^{-1}Ry')e^{-i\Phi(r)\mathcal{O}^{-1}Ry'\cdot\xi}d\sigma(y')\right)dr\\
&=\frac{1}{t^\rho}\int_{t/2}^{t}\frac{b(r)}{r^{1-\rho}}\left(\int_{S^{n-1}}\Omega(\mathcal{O}^{-1}Ry')e^{-i\Phi(r)u|\xi|}d\sigma(y')\right)dr\\
&=\frac{1}{t^\rho}\int_{t/2}^{t}\frac{b(r)}{r^{1-\rho}}\left(\int_{\boldsymbol{R}}E_\Omega(u,\xi')e^{-i\Phi(r)u|\xi|}du\right)dr\\
&=\frac{1}{t^\rho}\int_{t/2}^{t}\frac{b(r)}{r^{1-\rho}}\left(\int_{\boldsymbol{R}}E_\Omega(u,\xi')(e^{-i\Phi(r)u|\xi|}-e^{-i\Phi(r)\xi_1'|\xi|})du\right)dr\,.
\end{aligned}$$

In the last equality, we used the cancellation property of E_Ω, guaranteed by Lemma 2.1. Using the last expression of $\widehat{\sigma_t}(\xi)$ in the above, we get

$$|\widehat{\sigma_t}(\xi)|\le|\xi|\Phi(t)\int_{t/2}^{t}\frac{|b(r)|}{r}dr\left(\int_{\boldsymbol{R}}|E_\Omega(u,\xi')|\,|u-\xi_1'|du\right).$$

So, by Lemma 2.1 we obtain

$$|\widehat{\sigma_t}(\xi)|\le C\|b\|_{\Delta_1}\Phi(t)|A_\tau(\xi)|\,,$$

which shows (3.9). Similarly, we can show (3.10). □

LEMMA 3.4. *Suppose Φ is a nonnegative and monotonic C^1 function on $(0,\infty)$ such that $\varphi(t):=\Phi(t)/(t\Phi'(t))$ is bounded. Suppose $b\in\Delta_{\gamma_0}$ for some γ_0 $(1<\gamma_0\le\infty)$. Let*

Ω be a regular $H^1(S^{n-1})$-atom whose support is contained in $S^{n-1}\cap B(\boldsymbol{e}_1,\tau)$. Then, for γ satisfying $1<\gamma\le\min(2,\gamma_0)$, there exists $C=C_{\gamma,\rho}>0$ such that, if Φ is increasing,

$$|\widehat{\sigma_t}(\xi)|\le C\|b\|_{\Delta_\gamma}\|\varphi\|_\infty^{1-1/\gamma}\frac{1}{(\Phi(t/2)|A_\tau\xi|)^{1-1/\gamma}}\,, \tag{3.11}$$

and if Φ is decreasing,

$$|\widehat{\sigma_t}(\xi)|\le C\|b\|_{\Delta_\gamma}\|\varphi\|_\infty^{1-1/\gamma}\frac{1}{(\Phi(t)|A_\tau\xi|)^{1-1/\gamma}}\,, \tag{3.12}$$

where $A_\tau(\xi)=(\tau^2\xi_1,\tau\xi_2,\dots,\tau\xi_n)$.

PROOF. Write

$$\widehat{E_\Omega}(r)=\int_{\boldsymbol{R}}E_\Omega(u,\xi')e^{-iru}du\,.$$

Using the change of variable $|\xi|r=t$, Hausdorff-Young's inequality and Lemma 2.1, we have

$$\begin{aligned}\Big(\int_{\boldsymbol{R}}|\widehat{E_\Omega}(|\xi|r)|^{\gamma'}dr\Big)^{1/\gamma'}&=|\xi|^{-1/\gamma'}\Big(\int_{\boldsymbol{R}}|\widehat{E_\Omega}(t)|^{\gamma'}dt\Big)^{1/\gamma'}\\&\le C|\xi|^{-1/\gamma'}\Big(\int_{\boldsymbol{R}}|E_\Omega(s,\xi')|^{\gamma}ds\Big)^{1/\gamma}\\&\le\frac{C}{|\xi|^{1/\gamma'}}\Big(\frac{|A_\tau\xi|}{|\xi|}\Big(\frac{|\xi|}{|A_\tau\xi|}\Big)^{\gamma}\Big)^{1/\gamma}=\frac{C}{|A_\tau\xi|^{1-1/\gamma}}\,.\end{aligned}\tag{3.13}$$

Hence, using the fifth expression of $\widehat{\sigma_t}(\xi)$ in the proof of Lemma 3.3, Hölder's inequality and the increasingness of Φ, and applying the change of variable $\Phi(r)=s$, we get

$$\begin{aligned}|\widehat{\sigma_t}(\xi)|&\le\int_{t/2}^{t}|b(r)||\widehat{E_\Omega}(\Phi(r)|\xi|)|\frac{dr}{r}\\&\le\Big(\int_{t/2}^{t}|b(r)|^{\gamma}\frac{dr}{r}\Big)^{1/\gamma}\Big(\int_{t/2}^{t}|\widehat{E_\Omega}(\Phi(r)|\xi|)|^{\gamma'}dr\Big)^{1/\gamma'}\\&\le 2^{1/\gamma}\|b\|_{\Delta_\gamma}\Big(\int_{\Phi(t/2)}^{\Phi(t)}|\widehat{E_\Omega}(s|\xi|)|^{\gamma'}\frac{s}{\Phi^{-1}(s)\Phi'(\Phi^{-1}(s))}\frac{ds}{s}\Big)^{1/\gamma'}\\&\le\frac{2^{1/\gamma}\|b\|_{\Delta_\gamma}\|\varphi\|_\infty^{1/\gamma'}}{(\Phi(t/2))^{1/\gamma'}}\Big(\int_{\Phi(t/2)}^{\Phi(t)}|\widehat{E_\Omega}(s|\xi|)|^{\gamma'}ds\Big)^{1/\gamma'}\\&\le\frac{C\|b\|_{\Delta_\gamma}\|\varphi\|_\infty^{1-1/\gamma}}{(\Phi(t/2)|A_\tau\xi|)^{1-1/\gamma}}\,,\end{aligned}$$

which shows (3.11). In a similar way, we can show (3.12). □

LEMMA 3.5. *Let $\Omega\in L^1(S^{n-1})$, and $b\in\Delta_\gamma$ for some $1<\gamma\le 2$. Suppose Φ is a nonnegative and monotonic C^1 function on $\boldsymbol{R}_+$ satisfying the condition in Theorem* 1.1. *Then,* *for any p satisfying $|1/p - 1/2| < 1/\gamma'$, there exists $C > 0$ such that*

$$\begin{aligned}(3.14)\quad &\left\| \left(\sum_{k\in \boldsymbol{Z}} \int_{2^k}^{2^{k+1}} |(\sigma_t * g_k(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\boldsymbol{R}^n)} \\ &\le C\|\Omega\|_{L^1(S^{n-1})} \left\| \left(\sum_{k\in \boldsymbol{Z}} |g_k(\cdot)|^2 \right)^{1/2} \right\|_{L^p(\boldsymbol{R}^n)} .\end{aligned}$$

PROOF. It suffices to show

$$\begin{aligned}(3.15)\quad &\left\| \left(\sum_{k\in \boldsymbol{Z}} \int_{2^k}^{2^{k+1}} |(\sigma_t * g_k(\cdot, t))(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\boldsymbol{R}^n)} \\ &\le C\|\Omega\|_{L^1(S^{n-1})} \left\| \left(\sum_{k\in \boldsymbol{Z}} \int_{2^k}^{2^{k+1}} |g_k(\cdot, t)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\boldsymbol{R}^n)} .\end{aligned}$$

To show (3.15), we may assume $p > 2$ by duality. So, we assume $2 \le p < 2\gamma/(2-\gamma) = 2\gamma'/(\gamma'-2)$. We use a similar argument as in the proof of Fan and Pan [15, Theorem 7.5]. By duality, there exists a nonnegative function $h \in L^{(p/2)'}(\boldsymbol{R}^n)$ with unit norm such that

$$\begin{aligned}(3.16)\quad I :&= \left\| \left(\sum_{k\in \boldsymbol{Z}} \int_{2^k}^{2^{k+1}} |(\sigma_t * g_k(\cdot, t))(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\boldsymbol{R}^n)}^2 \\ &= \int_{\boldsymbol{R}^n} \sum_{k\in \boldsymbol{Z}} \int_{2^k}^{2^{k+1}} |\sigma_t * g_k(\cdot, t)(x)|^2 \frac{dt}{t} h(x)dx .\end{aligned}$$

By Cauchy-Schwarz's inequality, we get

$$\begin{aligned}&|\sigma_t * g_k(\cdot, t)(x)|^2 \\ &\le \left| \frac{1}{t^\rho} \int_{t/2<|y|<t} \frac{b(|y|)\Omega(y')}{|y|^{n-\rho}} g_k(x - \Phi(|y|)y', t)dy \right|^2 \\ &\le \left(\int_{t/2<|y|<t} \frac{|b(|y|)|^\gamma |\Omega(y')|}{|y|^n} dy \right) \\ &\quad \times \frac{1}{t^\rho} \int_{t/2<|y|<t} \frac{|b(|y|)|^{2-\gamma} |\Omega(y')|}{|y|^{n-\rho}} |g_k(x - \Phi(|y|)y', t)|^2 dy \\ &\le C\|b\|_{\Delta_\gamma}^\gamma \|\Omega\|_{L^1(S^{n-1})} \frac{1}{t^\rho} \int_{t/2<|y|<t} \frac{|b(|y|)|^{2-\gamma} |\Omega(y')|}{|y|^{n-\rho}} |g_k(x - \Phi(|y|)y', t)|^2 dy .\end{aligned}$$

We set

$$\sigma_{\gamma,t} * f(x) = \frac{1}{t^\rho} \int_{\boldsymbol{R}^n} \frac{|b(|y|)|^{2-\gamma} |\Omega(-y')|}{|y|^{n-\rho}} f(x - \Phi(|y|)y')dy .$$

Then we have, using the change of variable $u = x - \Phi(|y|)y'$,

$$\begin{aligned} I &\le C\|b\|_{\Delta_\gamma}^{\gamma}\|\Omega\|_{L^1(S^{n-1})} \\ &\quad\times \sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} h(x)\frac{|b(|y|)|^{2-\gamma}|\Omega(y')|}{t^\rho|y|^{n-\rho}}|g_k(x-\Phi(|y|)y',t)|^2dxdy\right)\frac{dt}{t} \\ &\le C\|b\|_{\Delta_\gamma}^{\gamma}\|\Omega\|_{L^1(S^{n-1})} \\ &\quad\times \sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} h(u+\Phi(|y|)y')\frac{|b(|y|)|^{2-\gamma}|\Omega(y')|}{t^\rho|y|^{n-\rho}}|g_k(u,t)|^2dudy\right)\frac{dt}{t} \\ &\le C\|b\|_{\Delta_\gamma}^{\gamma}\|\Omega\|_{L^1(S^{n-1})} \\ &\quad\times \sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n}\frac{|b(|y|)|^{2-\gamma}|\Omega(y')|}{t^\rho|y|^{n-\rho}}h(u-\Phi(|y|)y')dy\right)|g_k(u,t)|^2du\,\frac{dt}{t} \\ &= C\|b\|_{\Delta_\gamma}^{\gamma}\|\Omega\|_{L^1(S^{n-1})}\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}\int_{\mathbf{R}^n}(\sigma_{\gamma,t}*h(x))\,|g_k(y,t)|^2dy\,\frac{dt}{t} \\ &\le C\|b\|_{\Delta_\gamma}^{\gamma}\|\Omega\|_{L^1(S^{n-1})}\int_{\mathbf{R}^n}\left(\sup_{t>0}|\sigma_{\gamma,t}*h(x)|\right)\left(\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}|g_k(y,t)|^2\,\frac{dt}{t}\right)dy\,. \end{aligned}$$

From $b\in\Delta_\gamma$ and $1<\gamma\le 2$, it follows that $|b|^{2-\gamma}\in\Delta_{\gamma/(2-\gamma)}$ and $\gamma/(2-\gamma)>1$. So, using Hölder's inequality and Lemma 3.2, we have

$$I \le C\|b\|_{\Delta_\gamma}^2\|\Omega\|_{L^1(S^{n-1})}^2\|h\|_{L^{(p/2)'}}\left\|\left(\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}|g_k(\cdot,t)|^2\,\frac{dt}{t}\right)^{1/2}\right\|_{L^p}^2,$$

which implies the desired estimate (3.15). □

LEMMA 3.6. *Let $\Omega\in L^1(S^{n-1})$, $b\in\Delta_\gamma$ for some $\gamma\ge 2$. Let $p>\gamma'$ and $\omega\in\tilde{A}_{p/\gamma'}(\mathbf{R}_+)$. Suppose Φ is a nonnegative and monotonic C^1 function on $\mathbf{R}_+$ satisfying the condition in Theorem* 1.1. *Then there exists $C>0$ such that*

$$\left\|\left(\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}|\sigma_t*g_k(\cdot)|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p(\omega)} \le C\|\Omega\|_{L^1(S^{n-1})}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k(\cdot)|^2\right)^{1/2}\right\|_{L^p(\omega)}. \tag{3.17}$$

PROOF. By change of variables we get

$$\left(\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}|\sigma_t*g_k(x)|^2\frac{dt}{t}\right)^{1/2} = \left(\sum_{k\in\mathbf{Z}}\int_1^2|\sigma_{2^kt}*g_k(x)|^2\frac{dt}{t}\right)^{1/2}. \tag{3.18}$$

Now, using Hölder's inequality, we obtain

$$
\begin{aligned}
&|\sigma_t * g_k(x)| \\
&\le \frac{1}{t^\rho}\left(\int_{t/2<|y|<t} \frac{|b(|y|)|^\gamma |\Omega(y')|}{|y|^{n-\gamma\rho}}\right)^{1/\gamma} \\
&\quad \times \left(\int_{t/2<|y|<t} \frac{|g_k(x-\Phi(|y|)y')|^{\gamma'}|\Omega(y')|}{|y|} dy\right)^{1/\gamma'} \\
&\le 2^{1/\gamma}\|b\|_{\Delta_\gamma}\|\Omega\|^{1/\gamma}_{L^1(S^{n-1})}\left(\int_{t/2<|y|<t} \frac{|g_k(x-\Phi(|y|)y')|^{\gamma'}|\Omega(y')|}{|y|} dy\right)^{1/\gamma'} .
\end{aligned}
\tag{3.19}
$$

Let $d = p/\gamma'$. By duality, there is a nonnegative function $f \in L^{d'}(\omega^{1-d'}, \boldsymbol{R}^n)$ with unit norm such that

$$
\left\|\left(\sum_{k\in\boldsymbol{Z}}\int_1^2 |\sigma_{2^k t} * g_k(\cdot)|^{\gamma'}\frac{dt}{t}\right)^{1/\gamma'}\right\|^{\gamma'}_{L^p(\omega)} = \int_{\boldsymbol{R}^n}\sum_{k\in\boldsymbol{Z}}\int_1^2 |\sigma_{2^k t} * g_k(x)|^{\gamma'}\frac{dt}{t}\, f(x)dx . \tag{3.20}
$$

Combining (3.20) with (3.19) yields

$$
\begin{aligned}
&\left\|\left(\sum_{k\in\boldsymbol{Z}}\int_1^2 |\sigma_{2^k t} * g_k(\cdot)|^{\gamma'}\frac{dt}{t}\right)^{1/\gamma'}\right\|^{\gamma'}_{L^p(\omega)} \\
&\le C\sum_{k\in\boldsymbol{Z}}\int_{\boldsymbol{R}^n}\int_1^2\left(\int_{2^{k-1}t<|y|<2^k t}\frac{|\Omega(y')||g_k(x-\Phi(|y|)y')|^{\gamma'}}{|y|^n}dy\right)\frac{dt}{t}\, f(x)dx \\
&= C\sum_{k\in\boldsymbol{Z}}\int_1^2\left(\int_{2^{k-1}t<|y|<2^k t}\int_{\boldsymbol{R}^n}|g_k(x-\Phi(|y|)y')|^{\gamma'} f(x)dx\frac{|\Omega(y')|}{|y|^n}dy\right)\frac{dt}{t} \\
&= C\sum_{k\in\boldsymbol{Z}}\int_1^2\left(\int_{2^{k-1}t<|y|<2^k t}\int_{\boldsymbol{R}^n}|g_k(u)|^{\gamma'} f(u+\Phi(|y|)y')du\frac{|\Omega(y')|}{|y|^n}dy\right)\frac{dt}{t} \\
&= C\sum_{k\in\boldsymbol{Z}}\int_1^2\int_{\boldsymbol{R}^n}\left(\int_{2^{k-1}t<|y|<2^k t}\frac{|\Omega(y')|}{|y|^n} f(u+\Phi(|y|)y')dy\right)|g_k(u)|^{\gamma'}du\,\frac{dt}{t} \\
&\le C\int_{\boldsymbol{R}^n}\sum_{k\in\boldsymbol{Z}}|g_k(u)|^{\gamma'}\left(\int_{S^{n-1}}|\Omega(y')|M_{y'}\tilde{f}(-u)d\sigma(y')\right)\omega^{1/d}\omega^{-1/d}du \\
&\le \left(\int_{\boldsymbol{R}^n}\left(\sum_{k\in\boldsymbol{Z}}|g_k(u)|^{\gamma'}\right)^d\omega du\right)^{1/d} \\
&\quad \times\left(\int_{\boldsymbol{R}^n}\left(\int_{S^{n-1}}|\Omega(y')|M_{y'}\tilde{f}(-u)d\sigma(y')\right)^{d'}\omega^{-d'/d}du\right)^{1/d'} .
\end{aligned}
$$

It is known that $\omega \in \tilde{A}_d(\boldsymbol{R}_+)$ if and only if $\omega^{1-d'} \in \tilde{A}_{d'}(\boldsymbol{R}_+)$. Hence, we have

$$
\left\|\left(\int_{S^{n-1}}|\Omega(y')|M_{y'}\tilde{f}(-u)d\sigma(y')\right)\right\|_{L^{d'}(\omega^{1-d'})} \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^{d'}(\omega^{1-d'})} .
$$

Thus, noting $-d'/d = 1 - d'$, we get

$$\begin{aligned}(3.21)\quad &\left\| \left(\sum_{k\in\mathbf{Z}} \int_1^2 |\sigma_{2^k t} * g_k(\cdot)|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \right\|_{L^p(\omega)} \\ &\leq C\|\Omega\|_{L^1(S^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k(\cdot)|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)}.\end{aligned}$$

On the other hand, using Lemma 3.2 and noting that $\omega \in \tilde{A}_{p/\gamma'}$, we get

$$\begin{aligned}(3.22)\quad \left\| \sup_{k\in\mathbf{Z}} \sup_{1<t<2} |\sigma_{2^k t} * g_k| \right\|_{L^p(\omega)} &\leq \left\| \sigma^*\big(\sup_{k\in\mathbf{Z}} |g_k| \big) \right\|_{L^p(\omega)} \\ &\leq C\|\Omega\|_{L^1(S^{n-1})} \left\| \sup_{k\in\mathbf{Z}} |g_k| \right\|_{L^p(\omega)}.\end{aligned}$$

Since $\gamma > 2$, $\gamma' < 2$, and hence interpolating between (3.21) and (3.22), we obtain

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_1^2 |\sigma_{2^k t} * g_k(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\omega)} \leq C\|\Omega\|_{L^1(S^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k(\cdot)|^2 \right)^{1/2} \right\|_{L^p(\omega)}.$$

Hence, via (3.18) we have the desired inequality (3.17). □

Now, using Lemmas 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6, we can prove our Theorem 1.1 in a way quite similar to the proof of Al-Qassem [2, Theorem 1.4]. We first prove Theorem 1.1 (i) in the case $\gamma \geq 2$. We may assume Ω is a regular atom whose support is contained in $S^{n-1} \cap B(\boldsymbol{e}_1, \tau)$ $(0 < \tau < 2)$. Note that we have by Lemma 2.8

$$(3.23)\qquad \frac{\Phi(2^{k+1})}{\Phi(2^k)} \geq 2^{1/\|\varphi\|_\infty} k \in \mathbf{Z}.$$

Let $\eta \in C^\infty(\mathbf{R})$ such that $0 \leq \eta(t) \leq 1$, $\eta(t) = 1$ $(-1 \leq t \leq 1)$ and $\eta(t) = 0$ $(|t| \geq a)$, where $a = 2^{1/\|\varphi\|_\infty}$. We set

$$\psi_j(\xi) = \eta(\Phi(2^{j-1})|\xi|) - \eta(\Phi(2^j)|\xi|).$$

Then we have

$$\sum_{j\in\mathbf{Z}} \psi_j(\xi) = 1, \quad \xi \neq 0, \quad \text{and} \quad \psi_j(\xi) = \begin{cases} 0 & |\xi| \leq 1/\Phi(2^j),\ |\xi| \geq a/\Phi(2^{j-1}), \\ 1 & a/\Phi(2^j) \leq |\xi| \leq 1/\Phi(2^{j-1}). \end{cases}$$

Define the multiplier operators S_j in $\mathbf{R}^n$ by $(S_j f)\hat{}(\xi) = \hat{f}(\xi)\psi_j(A_\tau \xi)$, i.e., $S_j f = \Psi_j * f$ where $\Psi_j(x) = \mathcal{F}^{-1}(\psi_j(A_\tau \cdot))(x)$. Then

$$\begin{aligned}\sigma_t * f(x) &= \sum_{k\in\mathbf{Z}} \sum_{j\in\mathbf{Z}} \Psi_{j+k} * \sigma_t * f(x) \chi_{[2^k, 2^{k+1})}(t) \\ &= \sum_{j\in\mathbf{Z}} \left(\sum_{k\in\mathbf{Z}} \Psi_{j+k} * \sigma_t * f(x) \chi_{[2^k, 2^{k+1})}(t) \right) \\ &=: \sum_{j\in\mathbf{Z}} G_j(x, t).\end{aligned}$$

Define

$$T_j f(x) = \left(\int_0^\infty |G_j(x,t)|^2 \frac{dt}{t} \right)^{1/2} .$$

Then

$$\tilde{\mu}_{\Omega,\rho,\Phi,b}(x) \le C \sum_{j \in \boldsymbol{Z}} T_j f(x) . \tag{3.24}$$

First, we estimate L^2 bound of $T_j f$. By the Plancherel theorem, we get

$$\begin{aligned}
\|T_j f\|_{L^2}^2 &= \int_{\boldsymbol{R}^n} \int_0^\infty \left| \sum_{k \in \boldsymbol{Z}} \Psi_{j+k} * \sigma_t * f(x) \chi_{[2^k, 2^{k+1})}(t) \right|^2 \frac{dt}{t} dx \\
&\le \sum_{k \in \boldsymbol{Z}} \int_{\boldsymbol{R}^n} \int_{2^k}^{2^{k+1}} |\Psi_{j+k} * \sigma_t * f(x)|^2 \frac{dt}{t} dx \\
&= \sum_{k \in \boldsymbol{Z}} \int_{\boldsymbol{R}^n} \int_{2^k}^{2^{k+1}} |\psi_{j+k}(A_\tau \xi) \widehat{\sigma_t}(\xi) \hat{f}(\xi)|^2 \frac{dt}{t} d\xi \\
&\le \sum_{k \in \boldsymbol{Z}} \int_{1/\Phi(2^{(j+k)}) < |A_\tau \xi| < a/\Phi(2^{(j+k-1)})} \int_{2^k}^{2^{k+1}} |\widehat{\sigma_t}(\xi)|^2 \frac{dt}{t} |\hat{f}(\xi)|^2 d\xi .
\end{aligned}$$

Using Lemma 3.3 for $j \ge 2$, Lemma 3.4 for $j < 0$, and Lemma 3.1 for $j = 0, 1$ we get

$$\|T_j f\|_{L^2} \le C a^{-(1-1/\gamma)|j|} \|f\|_{L^2} . \tag{3.25}$$

Next, for $p > \gamma'$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$, we have, via Lemma 3.6 and Littlewood-Paley theory (cf. Lee and Lin [18, p. 216]),

$$\|T_j f\|_{L^p(\omega)} \le C \left\| \left(\sum_{k \in \boldsymbol{Z}} |\Psi_{j+k} * f(\cdot)|^2 \right)^{1/2} \right\|_{L^p(\omega)} \le C \|f\|_{L^p(\omega)} . \tag{3.26}$$

Interpolating between (3.25) and modified (3.26) (taking $\omega = 1$), we can find a number $0 < \theta < 1$ such that

$$\|T_j f\|_{L^p} \le C a^{-\theta(1-1/\gamma)|j|} \|f\|_{L^p} . \tag{3.27}$$

On the other hand, there exists $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$ (see for example [16, p. 151]). Hence by (3.26), we have

$$\|T_j f\|_{L^p(\omega^{1+\varepsilon})} \le C \|f\|_{L^p(\omega^{1+\varepsilon})} . \tag{3.28}$$

Therefore, using Stein and Weiss' interpolation theorem with change of measures, we can interpolate between (3.27) and (3.28) to obtain a positive number ν such that

$$\|T_j f\|_{L^p(\omega)} \le C a^{-\nu|j|} \|f\|_{L^p(\omega)} . \tag{3.29}$$

This combined with (3.24) yields the desired estimate

$$\|\mu_{\Omega,\rho,\Phi,b}(f)\|_{L^p(\omega)} \le C \|f\|_{L^p(\omega)} ,$$

which completes the proof of Theorem 1.1 in the case $\gamma \geq 2$. In the case $1 < \gamma < 2$, we can prove in the same way as above, using Lemma 3.5 in place of Lemma 3.6. □

4. Proof of Theorem 1.3. We shall prove Theorem 1.3 in the case Φ is increasing, since the other case can be proved in a quite similar way as in the proof of Theorem 1.1.

For $k \in \boldsymbol{N}$, set

$$E_k = \{y' \in S^{n-1}\,;\, 2^{k-1} \leq |\Omega(y')| < 2^k\}$$

and

$$\Omega_k(y') = \Omega(y')\chi_{E_k}(y') - \frac{1}{|S^{n-1}|}\int_{E_k} \Omega(x')d\sigma(x')\,.$$

Then

$$\int_{S^{n-1}} \Omega_k(x')d\sigma(x') = 0 \tag{4.1}$$

for all $k \in \boldsymbol{N}$. Let

$$\Lambda = \{k \in \boldsymbol{N}\,;\, |E_k| > 2^{-4k}\}\,,$$

where $|E_k|$ denotes the measure of E_k on S^{n-1} induced by Lebesgue measure. Set

$$\Omega_0 = \Omega - \sum_{k\in\Lambda} \Omega_k\,.$$

Then it is easy to verify that $\Omega_0 \in L^2(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega_0(x')d\sigma(x') = 0\,. \tag{4.2}$$

For $k \in \Lambda$, define a family of measures $\sigma^{(k)} = \{\sigma_{k,t}\,;\, t \in \boldsymbol{R}\}$ on $\boldsymbol{R}^n$, and the maximal operator $(\sigma^{(k)})^*$ in the same way as in the proof of Theorem 1.1, by

$$\int_{\boldsymbol{R}^n} f(x)d\sigma_{k,t}(x) = \frac{1}{t^\rho}\int_{2/t<|x|<t} f(\Phi(|x|)x')\frac{b(|x|)\Omega_k(x')}{|x|^{n-\rho}}dx\,, \tag{4.3}$$

$$(\sigma^{(k)})^* f(x) = \sup_{t>0} ||\sigma_{k,t}| * f(x)|\,. \tag{4.4}$$

Then, as is easily checked like in the proof of Theorem 1.1, we have only to estimate

$$\tilde{\mu}_{\Omega_k,\rho,\Phi,b}(f)(x) = \left(\int_0^\infty |\sigma_{k,t} * f(x)|^2\frac{dt}{t}\right)^{1/2}. \tag{4.5}$$

We notice that we can apply Lemmas 3.1, 3.2, 3.5 and 3.6. Hence, we have

$$\|\sigma_{k,t}\| \leq C\|b\|_{\Delta_1}\int_{E_k} |\Omega(y')|d\sigma(y')\,, \tag{4.6}$$

and

$$(\sigma^{(k)})^* f(x) \leq C\|b\|_{\Delta_\gamma}\|\Omega_k\|_{L^1(S^{n-1})}^{1/\gamma}\left(\int_{S^{n-1}} |\Omega_k(y')|M_{y'}(|f|^{\gamma'})(x)d\sigma(y')\right)^{1/\gamma'}. \tag{4.7}$$

(4.6) implies

$$|\widehat{\sigma_{k,t}}(\xi)| \leq C\|b\|_{\Delta_1}\int_{E_k} |\Omega(y')|d\sigma(y')\,. \tag{4.8}$$

Using (4.1), we see that

$$\widehat{\sigma_{k,t}}(\xi) = \frac{1}{t^\rho}\int_{2/t<|y|<t} \frac{b(|y|)\Omega_k(y')}{|y|^{n-\rho}} e^{-i\Phi(|y|)y'\cdot\xi}dy$$
$$= \frac{1}{t^\rho}\int_{2/t<|y|<t} \frac{b(|y|)\Omega_k(y')}{|y|^{n-\rho}} (e^{-i\Phi(|y|)y'\cdot\xi}-1)dy\,.$$

From this we have

$$|\widehat{\sigma_{k,t}}(\xi)| \le C\|b\|_{\Delta_1}\Phi(t)|\xi|\int_{E_k}|\Omega(y')|d\sigma(y')\,. \tag{4.9}$$

Combining (4.8) with (4.9), we get for any $0<\alpha\le 2$

$$|\widehat{\sigma_{k,t}}(\xi)| \le C(\Phi(t)|\xi|)^{\alpha/(2k)}\int_{E_k}|\Omega(y')|d\sigma(y')\,, \tag{4.10}$$

where C is independent of $k\in\Lambda$.

Next, we assume $\gamma\ge 2$. (We treat the case $1<\gamma<2$ later.) Using the Cauchy-Schwarz inequality, we get

$$|\widehat{\sigma_{k,t}}(\xi)|^2 = \frac{1}{t^{2\rho}}\left|\int_{t/2}^t b(r)r^\rho \int_{S^{n-1}}\Omega_k(y')e^{-i\Phi(r)y'\cdot\xi}d\sigma(y')\,\frac{dr}{r}\right|^2$$
$$\le \int_{t/2}^t \frac{|b(r)|^2}{r}\,dr\int_{t/2}^t\left|\int_{S^{n-1}}\Omega_k(y')e^{-i\Phi(r)y'\cdot\xi}d\sigma(y')\right|^2\frac{dr}{r}\,.$$

So, by Lemma 2.4 in the case of (i) and Lemma 2.5 in the case of (ii), we have for $0<\alpha<1/2$

$$|\widehat{\sigma_{k,t}}(\xi)| \le \frac{C\|b\|_{\Delta_2}\|\Omega_k\|_{L^2(S^{n-1})}}{(\Phi(t/2)|\xi|)^{\alpha/2}} \le \frac{C2^{2k+2}}{(\Phi(t/2)|\xi|)^{\alpha/2}}\int_{E_k}|\Omega(y')|d\sigma(y')\,. \tag{4.11}$$

Combining (4.8) with (4.11) yields

$$|\widehat{\sigma_{k,t}}(\xi)| \le CA_k^{(k-1)/k}\left(\frac{2^{2k+2}A_k}{(\Phi(t/2)|\xi|)^{\alpha/2}}\right)^{1/k} \le CA_k(\Phi(t/2)|\xi|)^{-\alpha/(2k)}\,, \tag{4.12}$$

where $A_k=\int_{E_k}|\Omega(y')|d\sigma(y')$.

As in the proof of Theorem 1.1, for each $k\in\boldsymbol{N}$, let $\eta\in C^\infty(\boldsymbol{R})$ such that $0\le\eta(t)\le 1$, $\eta(t)=1$ $(-1\le t\le 1)$ and $\eta(t)=0$ $(|t|\ge a)$, where $a=2^{k/\|\varphi\|_\infty}$. Note that we have by Lemma 2.8

$$\Phi(2^{k(j+1)})/\Phi(2^{kj}) \ge 2^{k/\|\varphi\|_\infty}\,,\ \ j\in\boldsymbol{Z}\,. \tag{4.13}$$

We set

$$\psi_j^{(k)}(\xi) = \eta(\Phi(2^{k(j-1)})|\xi|) - \eta(\Phi(2^{kj})|\xi|)\,.$$

Then we have

$$\sum_{j\in\boldsymbol{Z}}\psi_j^{(k)}(\xi)=1\,,\quad \xi\ne 0\,,\quad \text{and } \psi_j^{(k)}(\xi)=\begin{cases}0 & |\xi|\le 1/\Phi(2^{kj})\,,\ |\xi|\ge a/\Phi(2^{k(j-1)})\,,\\ 1 & a/\Phi(2^{kj})\le|\xi|\le 1/\Phi(2^{k(j-1)})\,.\end{cases}$$

Define the multiplier operators $S_j^{(k)}$ in $\boldsymbol{R}^n$ by $(S_j^{(k)} f)\hat{}(\xi) = \hat{f}(\xi)\psi_j^{(k)}(\xi)$. Then

$$\begin{aligned}\sigma_{k,t} * f(x) &= \sum_{l\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} \Psi_{j+l}^{(k)} * \sigma_{k,t} * f(x)\chi_{[2^{kl},2^{k(l+1)})}(t)\\ &= \sum_{j\in\mathbf{Z}}\bigg(\sum_{l\in\mathbf{Z}} \Psi_{j+l}^{(k)} * \sigma_{k,t} * f(x)\chi_{[2^{kl},2^{k(l+1)})}(t)\bigg)\\ &=: \sum_{j\in\mathbf{Z}} H_j^{(k)}(x,t)\,.\end{aligned}$$

Define

$$U_j^{(k)} f(x) = \bigg(\int_0^\infty |H_j^{(k)}(x,t)|^2 \frac{dt}{t}\bigg)^{1/2}.$$

Then

$$\tilde{\mu}_{\Omega_k,\rho,\Phi,b}(x) \le C\sum_{j\in\mathbf{Z}} U_j^{(k)} f(x)\,. \tag{4.14}$$

First, we compute L^2 norm of $U_j^{(k)} f$. By Plancherel's theorem we get

$$\begin{aligned}\|U_j^{(k)} f\|_{L^2}^2 &= \int_{\boldsymbol{R}^n}\int_0^\infty\bigg|\sum_{l\in\mathbf{Z}} \Psi_{j+l}^{(k)} * \sigma_{k,t} * f(x)\chi_{[2^{kl},2^{k(l+1)})}(t)\bigg|^2 \frac{dt}{t}dx\\ &\le \sum_{l\in\mathbf{Z}}\int_{\boldsymbol{R}^n}\int_{2^{kl}}^{2^{k(l+1)}} \big|\Psi_{j+l}^{(k)} * \sigma_{k,t} * f(x)\big|^2 \frac{dt}{t}dx\\ &= \sum_{l\in\mathbf{Z}}\int_{\boldsymbol{R}^n}\int_{2^{kl}}^{2^{k(l+1)}} \big|\psi_{j+l}^{(k)}(\xi)\widehat{\sigma_{k,t}}(\xi)\hat{f}(\xi)\big|^2 \frac{dt}{t}\,d\xi\\ &\le \sum_{l\in\mathbf{Z}}\int_{1/\Phi(2^{k(j+l)})\le|\xi|\le a/\Phi(2^{k(j+l-1)})}\int_{2^{kl}}^{2^{k(l+1)}} |\widehat{\sigma_{k,t}}(\xi)|^2\frac{dt}{t}\,|\hat{f}(\xi)|^2\,d\xi\,.\end{aligned}$$

For $j \ge 2$ and $1/\Phi(2^{k(j+l)}) \le |\xi| \le a/\Phi(2^{k(j+l-1)})$, we get, using (4.10) and (4.13),

$$\int_{2^{kl}}^{2^{k(l+1)}} |\widehat{\sigma_{k,t}}(\xi)|^2\frac{dt}{t} \le CA_k^2(\log 2^k)\bigg(\frac{a\Phi(2^{k(l+1)})}{\Phi(2^{k(j+l-1)})}\bigg)^{\alpha/k} \le CA_k^2(\log 2^k)2^{-(j-3)\alpha/\|\varphi\|_\infty}\,.$$

Also, for $j \le -1$ and $1/\Phi(2^{k(j+l)}) \le |\xi| \le a/\Phi(2^{k(j+l-1)})$, we get, using (4.12) and (4.13),

$$\int_{2^{kl}}^{2^{k(l+1)}} |\widehat{\sigma_{k,t}}(\xi)|^2\frac{dt}{t} \le CA_k^2(\log 2^k)\bigg(\frac{\Phi(2^{k(l+1)})}{\Phi(2^{k(l-1)})}\bigg)^{\alpha/k} \le CA_k^2(\log 2^k)2^{(j+1)\alpha/\|\varphi\|_\infty}\,.$$

Hence, combining these with (4.8) for $0 \le j \le 1$, we obtain

$$\|U_j^{(k)} f\|_{L^2} \le CA_k\sqrt{k}\,2^{-|j|\alpha/(2\|\varphi\|_\infty)}\|f\|_{L^2}\,. \tag{4.15}$$

Next, for $p > \gamma'$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$, we apply Lemma 3.6 to the set of functions $\{h_k\}_{k\in\boldsymbol{Z}}$, where $g_{lk+m} = h_l$ ($l \in \boldsymbol{Z}$ and $m = 0, 1, \dots, k-1$). Then we have

$$\left\|\left(\sum_{l\in\boldsymbol{Z}}\int_{2^{kl}}^{2^{k(l+1)}} |\sigma_{k,t} * h_l(\cdot)|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^p(\omega)} \le C\sqrt{k}\, A_k \left\|\left(\sum_{l\in\boldsymbol{Z}} |h_l(\cdot)|^2\right)^{1/2}\right\|_{L^p(\omega)}, \tag{4.16}$$

where C is independent of k. Hence, using (4.16) and Littlewood-Paley theory for $\{\Psi^{(k)}\}$, we obtain

$$\|U_j^{(k)} f\|_{L^p(\omega)} \le CA_k\sqrt{k}\left\|\left(\sum_{l\in\boldsymbol{Z}} |\Psi_{j+l}^{(k)} * f(\cdot)|^2\right)^{1/2}\right\|_{L^p(\omega)} \le CA_k\sqrt{k}\,\|f\|_{L^p(\omega)}. \tag{4.17}$$

Interpolating between (4.15) and modified (4.17) (taking $\omega = 1$), we can find a number $0 < \theta < 1$ such that

$$\|U_j^{(k)} f\|_{L^p} \le CA_k\sqrt{k}\, 2^{-|j|\theta\alpha/2}\|f\|_{L^p}. \tag{4.18}$$

On the other hand, there exists $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$ (see for example [16, p. 151]). Hence by (4.17), we have

$$\|U_j^{(k)} f\|_{L^p(\omega^{1+\varepsilon})} \le CA_k\sqrt{k}\,\|f\|_{L^p(\omega^{1+\varepsilon})}. \tag{4.19}$$

Therefore, using Stein and Weiss' interpolation theorem with change of measures, we can interpolate between (4.18) and (4.19) to obtain a positive number ν such that

$$\|U_j^{(k)} f\|_{L^p(\omega)} \le CA_k\sqrt{k}\, 2^{-\nu|j|}\|f\|_{L^p(\omega)}. \tag{4.20}$$

This combined with (4.14) yields the desired estimate

$$\|\tilde{\mu}_{\Omega_k,\rho,\Phi,b}\|_{L^p(\omega)} \le CA_k\sqrt{k}\,\|f\|_{L^p(\omega)}. \tag{4.21}$$

By Minkowski's inequality, we have

$$\tilde{\mu}_{\Omega,\rho,\Phi,b}(f)(x) \le \tilde{\mu}_{\Omega_0,\rho,\Phi,b}(f)(x) + \sum_{k\in\Lambda}\tilde{\mu}_{\Omega_k,\rho,\Phi,b}(f)(x). \tag{4.22}$$

Using Theorem 1.1, (4.21) and (4.22), we obtain

$$\begin{aligned}\|\tilde{\mu}_{\Omega,\rho,\Phi,b}(f)\|_{L^p(\omega)} &\le C_p\left(1 + \sum_{k\in\Lambda}\sqrt{k}\int_{E_k}|\Omega(y')|d\sigma(y')\right)\|f\|_{L^p} \\ &\le C_p(1 + \|\Omega\|_{L(\log^+ L)^{1/2}})\|f\|_{L^p}.\end{aligned}$$

The proof of Theorem 1.3 ($\gamma \ge 2$) is now complete.

In the case where $1 < \gamma < 2$, we proceed as in Al-Qassem [2, p. 7]. Since

$$\left|\int_{S^{n-1}} \Omega_k(y') e^{-i\Phi(r)y'\cdot\xi} d\sigma(y')\right| \le \|\Omega_k\|_{L^1(S^{n-1})} =: A_k,$$

using Hölder's inequality, we get

$$|\widehat{\sigma_{k,t}}(\xi)| \le \frac{1}{t^\rho}\left(\int_{t/2}^t \frac{|b(r)|^\gamma r^{\gamma\rho}}{r}\,dr\right)^{1/\gamma}\left(\int_{t/2}^t \frac{1}{r}\left|\int_{S^{n-1}} \Omega_k(y')e^{-i\Phi(r)y'\cdot\xi}d\sigma(y')\right|^{\gamma'} dr\right)^{1/\gamma'}$$
$$\le C\|b\|_{\Delta_\gamma}\|\Omega_k\|_{L^1(S^{n-1})}^{1-2/\gamma'}\left(\int_{t/2}^t\left|\int_{S^{n-1}} \Omega_k(y')e^{-i\Phi(r)y'\cdot\xi}d\sigma(y')\right|^2\frac{dr}{r}\right)^{1/\gamma'}.$$

So, by Lemma 2.4 in the case of (i) and Lemma 2.5 in the case of (ii), we have for $0 < \alpha < 1/2$

$$\int_{t/2}^t\left|\int_{S^{n-1}} \Omega_k(y')e^{-i\Phi(r)y'\cdot\xi}d\sigma(y')\right|^2\frac{dr}{r} \le \frac{C\|\Omega_k\|_{L^2(S^{n-1})}^2}{(\Phi(t/2)|\xi|)^\alpha}.$$

Hence, we get

$$|\widehat{\sigma_{k,t}}(\xi)| \le \frac{C\|b\|_{\Delta_\gamma}\|\Omega_k\|_{L^1(S^{n-1})}^{1-2/\gamma'}\|\Omega_k\|_{L^2(S^{n-1})}^{2/\gamma'}}{(\Phi(t/2)|\xi|)^{\alpha/\gamma'}} \le \frac{C2^{(4k+4)/\gamma'}}{(\Phi(t/2)|\xi|)^{\alpha/\gamma'}}A_k. \tag{4.23}$$

Combining (4.23) with (4.8) yields

$$|\widehat{\sigma_{k,t}}(\xi)| \le CA_k^{(k-1)/k}\left(\frac{2^{(4k+4)/\gamma'}A_k}{(\Phi(t/2)|\xi|)^{\alpha/\gamma'}}\right)^{1/k} \le CA_k(\Phi(t/2)|\xi|)^{-\alpha/(\gamma' k)}. \tag{4.24}$$

Now, using Lemma 3.5 in place of Lemma 3.6, we can prove Theorem 1.3 for $1 < \gamma < 2$ in the same way as in the case $\gamma \ge 2$. This completes the proof of Theorem 1.3. □

5. Proof of Theorem 1.7. As before we shall prove Theorem 1.7 only in the case Φ is increasing. We shall modify Lemmas 3.3, 3.4, 3.2 and 3.6.

As for Lemmas 3.3 and 3.4, we have, in the same way as in the proof of Theorem 1.3,

$$|\widehat{\sigma_t}(\xi)| \le C\int_{S^{n-1}}|\Omega(y')|d\sigma(y') \le C, \tag{5.1}$$

$$|\widehat{\sigma_t}(\xi)| \le C(\Phi(t)|\xi|)^{\alpha/2}\int_{S^{n-1}}|\Omega(y')|d\sigma(y') \le C(\Phi(t)|\xi|)^{\alpha/2} \tag{5.2}$$

for any $0 < \alpha \le 2$, and

$$|\widehat{\sigma_t}(\xi)| \le C\|\Omega\|_{L^q(S^{n-1})}(\Phi(t/2)|\xi|)^{-\alpha/2} \tag{5.3}$$

for any $0 < \alpha < 1/q'$.

As for Lemmas 3.2 and 3.6, we have the following.

LEMMA 5.1. *Let $\Omega \in L^q(S^{n-1})$ for some $q > 1$, and $b \in \Delta_\gamma$ for some $\gamma \ge q$. Suppose Φ is a nonnegative and monotonic C^1 function on* $(0,\infty)$ *such that $\varphi(t) := \Phi(t)/(t\Phi'(t))$ is bounded, and Φ satisfies the condition* (A-2). *Then there exists $C > 0$ such that*

$$\sigma^*(f)(x) := \sup_{0<t<\infty}||\sigma_t| * f(x)| \le C\|b\|_{\Delta_\gamma}\|\Omega\|_{L^q(S^{n-1})}(M(|f|^{q'})(x))^{1/q'}. \tag{5.4}$$

As a consequence, for $p > q'$ and $\omega \in A_{p/q'}(\boldsymbol{R}^n)$, there exists $C > 0$ such that

$$\|\sigma^*(f)\|_{L^p(\omega)} \le C\|b\|_{\Delta_\gamma}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\omega)}. \tag{5.5}$$

PROOF. By Hölder's inequality and Lemma 2.3 (a), we see that

$$\begin{aligned}||\sigma_t| * f(x)| &= \left|\frac{1}{t^\rho}\int_{t/2<|y|<t}\frac{|b(|y|)||\Omega(y')|}{|y|^{n-\rho}}f(x-\Phi(|y|)y')dy\right| \\ &\le \left(\frac{1}{t^{q\rho}}\int_{t/2<|y|<t}\frac{|b(|y|)|^q|y|^{q\rho}|\Omega(y')|^q}{|y|^n}dy\right)^{1/q}\left(\int_{t/2<|y|<t}\frac{|f(x-\Phi(|y|)y')|^{q'}}{|y|^n}dy\right)^{1/q'} \\ &\le C\|b\|_{\Delta_q}\|\Omega\|_{L^q(S^{n-1})}(M(|f|^{q'})(x))^{1/q'},\end{aligned}$$

which shows (5.4) because of $\Delta_\gamma \subset \Delta_q$ and $\|b\|_{\Delta_q} \le \|b\|_{\Delta_\gamma}$ for $\gamma \ge q$. It is well-known that, for $1 < r < \infty$ and $\omega \in A_r(\boldsymbol{R}^n)$, $\|M(f)\|_{L^r(\omega)} \le C_r\|f\|_{L^r(\omega)}$. From this, it follows (5.5) for $p > q'$. □

LEMMA 5.2. *Let $\Omega \in L^q(S^{n-1})$ for some $q > 1$, $b \in \Delta_\gamma$ for some $\gamma \ge \max\{2, q\}$. Suppose p, q, ω satisfy the conditions* (a), (b) *and* (c) *of Theorem* 1.7. *Suppose Φ is a non-negative and monotonic C^1 function on* $(0,\infty)$ *such that $\varphi(t) := \Phi(t)/(t\Phi'(t))$ is bounded, and Φ satisfies the condition* (A-2).

Then there exists $C > 0$ such that

$$\left\|\left(\sum_{k\in\boldsymbol{Z}}\int_{2^k}^{2^{k+1}}|\sigma_t * g_k(\cdot)|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p(\omega)} \le C\|\Omega\|_{L^q(S^{n-1})}\left\|\left(\sum_{k\in\boldsymbol{Z}}|g_k(\cdot)|^2\right)^{1/2}\right\|_{L^p(\omega)}. \tag{5.6}$$

PROOF. We only prove (5.6) in the case Φ is increasing, since the decreasing case can be proved similarly. We shall prove (5.6) only in the case where $q \ge 2$, since the other cases can be proved in similar ways to the proof of Lemma 1 in Ding, Fan and Pan [9].

Now, by changing variables, we get

$$\left(\sum_{k\in\boldsymbol{Z}}\int_{2^k}^{2^{k+1}}|\sigma_t * g_k(x)|^2\frac{dt}{t}\right)^{1/2} = \left(\sum_{k\in\boldsymbol{Z}}\int_1^2|\sigma_{2^kt} * g_k(x)|^2\frac{dt}{t}\right)^{1/2}. \tag{5.7}$$

Using Hölder's inequality and noting $\|b\|_{\Delta_q} \le \|b\|_{\Delta_\gamma}$ $(\gamma \ge q)$, we obtain

$$\begin{aligned}|\sigma_t * g_k(x)|^{q'} &= \left|\frac{1}{t^\rho}\int_{t/2<|y|<t}\frac{b(|y|)|y|^\rho\Omega(y')}{|y|^n}g_k(x-\Phi(|y|)y')dy\right|^{q'} \\ &\le C\|b\|_{\Delta_\gamma}^{q'}\|\Omega\|_{L^q(S^{n-1})}^{q'}\left(\int_{t/2<|y|<t}\frac{|g_k(x-\Phi(|y|)y')|^{q'}}{|y|^n}dy\right).\end{aligned} \tag{5.8}$$

We set $B_0 = \|b\|_{\Delta_\gamma}\|\Omega\|_{L^q(S^{n-1})}$. Let $d = p/q'$. By duality, there is a nonnegative function $f \in L^{d'}(\omega^{1-d'}, \boldsymbol{R}^n)$ with unit norm such that

$$\left\|\left(\sum_{k\in\boldsymbol{Z}}\int_1^2|\sigma_{2^kt} * g_k(\cdot)|^{q'}\frac{dt}{t}\right)^{1/q'}\right\|_{L^p(\omega)}^{q'} = \int_{\boldsymbol{R}^n}\sum_{k\in\boldsymbol{Z}}\int_1^2|\sigma_{2^kt} * g_k(x)|^{q'}\frac{dt}{t}f(x)dx. \tag{5.9}$$

Combining (5.9) with (5.8) yields via Lemma 2.3

$$\begin{aligned}
&\left\|\left(\sum_{k\in\boldsymbol{Z}}\int_1^2|\sigma_{2^kt}*g_k(\cdot)|^{q'}\frac{dt}{t}\right)^{1/q'}\right\|_{L^p(\omega)}^{q'}\\
&\le CB_0^{q'}\sum_{k\in\boldsymbol{Z}}\int_{\boldsymbol{R}^n}\int_1^2\left(\int_{2^{k-1}t<|y|<2^kt}\frac{|g_k(x-\Phi(|y|)y')|^{q'}}{|y|^n}dy\right)\frac{dt}{t}\,f(x)dx\\
&=CB_0^{q'}\sum_{k\in\boldsymbol{Z}}\int_1^2\int_{\boldsymbol{R}^n}\left(\int_{2^{k-1}t<|y|<2^kt}\frac{1}{|y|^n}f(\Phi(|y|)y'+u)dy\right)|g_k(u)|^{q'}du\,\frac{dt}{t}\\
&\le CB_0^{q'}\int_{\boldsymbol{R}^n}\sum_{k\in\boldsymbol{Z}}|g_k(u)|^{q'}M(f)(u)\omega^{1/d}(u)\omega^{-1/d}(u)du\\
&\le CB_0^{q'}\left(\int_{\boldsymbol{R}^n}\left(\sum_{k\in\boldsymbol{Z}}|g_k(u)|^{q'}\right)^d\omega(u)du\right)^{1/d}\left(\int_{\boldsymbol{R}^n}(M(f)(u))^{d'}\omega^{-d'/d}(u)du\right)^{1/d'}.
\end{aligned}$$

It is known that $\omega\in A_d(\boldsymbol{R}^n)$ if and only if $\omega^{1-d'}\in A_{d'}(\boldsymbol{R}^n)$. Hence, we have

$$\|M(f)\|_{L^{d'}(\omega^{1-d'})}\le C\|f\|_{L^{d'}(\omega^{1-d'})}.$$

Thus, noting $-d'/d=1-d'$, we get

$$\left\|\left(\sum_{k\in\boldsymbol{Z}}\int_1^2|\sigma_{2^kt}*g_k(\cdot)|^{q'}\frac{dt}{t}\right)^{1/q'}\right\|_{L^p(\omega)}\le CB_0\left\|\left(\sum_{k\in\boldsymbol{Z}}|g_k(\cdot)|^{q'}\right)^{1/q'}\right\|_{L^p(\omega)}. \tag{5.10}$$

On the other hand, using Lemma 5.1 and noting that $\omega\in A_{p/q'}$, we get

$$\left\|\sup_{k\in\boldsymbol{Z}}\sup_{1<t<2}|\sigma_{2^kt}*g_k|\right\|_{L^p(\omega)}\le\left\|\sigma^*\left(\sup_{k\in\boldsymbol{Z}}|g_k|\right)\right\|_{L^p(\omega)}\le CB_0\left\|\sup_{k\in\boldsymbol{Z}}|g_k|\right\|_{L^p(\omega)}. \tag{5.11}$$

Now, if $q>2$, $q'<2$, and hence interpolating between (5.10) and (5.11), we obtain

$$\left\|\left(\sum_{k\in\boldsymbol{Z}}\int_1^2|\sigma_{2^kt}*g_k(\cdot)|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p(\omega)}\le CB_0\left\|\left(\sum_{k\in\boldsymbol{Z}}|g_k(\cdot)|^2\right)^{1/2}\right\|_{L^p(\omega)}. \tag{5.12}$$

If $q=2$, (5.11) is just (5.12). Hence, via (5.7) we have the desired inequality (5.6). □

Now, preparing (5.2), (5.3) and Lemma 5.2, we can prove Theorem 1.7 in the same way as before. We leave the rest of the proof to the reader. □

ADDED IN PROOF. One of us recently showed that $\tilde{A}_p(\boldsymbol{R}_+)\subset A_p^I(\boldsymbol{R}^n)$ $(1\le p<\infty)$. So, we can replace $\tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$ by $\tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$ in Theorems 1.1, 1.3 and 1.5.

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School of Mathematical Sciences
Laboratory of Mathematics and
Complex Systems (BNU)
Ministry of Education, China
Beijing Normal University
Beijing 100875
China

E-mail address: dingy@bnu.edu.cn
qyxue@bnu.edu.cn

Research Center for Mathematical
Sciences
Kwansei Gakuin University
Gakuen 2–1, Sanda 669–1337
Japan