MULTIPLICITY OF SOLUTIONS FOR PARAMETRIC p-LAPLACIAN EQUATIONS WITH NONLINEARITY CONCAVE NEAR THE ORIGIN

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Abstract. We consider a nonlinear elliptic problem driven by the p-Laplacian and depending on a parameter. The right-hand side nonlinearity is concave, (i.e., p-sublinear) near the origin. For such problems we prove two multiplicity results, one when the right-hand side nonlinearity is p-linear near infinity and the other when it is p-superlinear. Both results show that there exists an open bounded interval such that the problem has five nontrivial solutions (two positive, two negative and one nodal), if the parameter is in that interval. We also consider the case when the parameter is in the right end of the interval.

Introduction. Let $Z \subseteq \mathbf{R}^N$ be a bounded domain with a C^2-boundary ∂Z. In this paper, we study the existence of multiple solutions of constant sign and of nodal (sign-changing) solutions for a class of parametric nonlinear elliptic problems, with right-hand side nonlinearity concave at the origin. Specifically, we are considering the following problem:

$$(1) \quad \begin{cases} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = \lambda|x(z)|^{q-2}x(z) + f(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0, \quad 1 < q < p < \infty, \quad \lambda > 0. \end{cases}$$

In problem (1), $|x|^{q-2}x$ is the concave term, i.e., a p-sublinear nonlinearity. Concerning the perturbation term f, we consider two cases. In the first case, we require that $f(z, \cdot)$ is p-linear near infinity; while in the second case we assume that $f(z, \cdot)$ is p-superlinear near infinity, i.e., problems with concave-convex nonlinearities.

Problems like (1) have been investigated primarily in the framework of semilinear equations, i.e., $p = 2$. The first case (with a p-linear perturbation of the concave term) was studied by Perera [20] as well as de Paiva and Massa [9]. The second case (with a p-superlinear perturbation) can be found in the paper of Ambrosetti, Brezis and Cerami [2]. Extensions to problems driven by the p-Laplacian differential operator were obtained by Ambrosetti, Garcia Azorero and Peral Alonso [3], Garcia Azorero, Manfredi and Peral Alonso [13] and Guo and Zhang [16]. However, all of them treat problems with a right-hand side nonlinearity of the form $\lambda|x|^{q-2}x + |x|^{r-2}x$ with $1 < q < p < r < p^*$. Here p^* is the Sobolev critical exponent, i.e.,

$$p^* = \begin{cases} Np/N - p & \text{if } p < N \\ \infty & \text{if } p \geq N. \end{cases}$$

They prove the existence of $\lambda^* > 0$ such that, for each $\lambda \in (0, \lambda^*)$, the problem has two positive solutions. In [3], the authors use the radial p-Laplacian and their method of proof is based on the Leray-Schauder degree theory. In [13], $Z \subseteq \boldsymbol{R}^N$ is an arbitrary bounded domain with a smooth boundary and the approach is variational. In [16], they assume $p > 2$ and, in addition to the case of a p-superlinear perturbation (of the special form $|x|^{r-2}x$, $p < r < p^*$), they also treat the case of a p-linear perturbation $f(x)$, which is assumed to be C^1 and monotone. Their approach is variational too. We should also mention the interesting work of Boccardo, Escobedo and Peral [6].

In [6] the authors consider a reaction term of the form $\lambda g(x) + |x|^{r-2}x$ with $\lambda > 0$, $g : \boldsymbol{R} \to \boldsymbol{R}$ continuous, $g(x) \le c_1 x^{q-1}$ for all $x \ge 0$, with $c_1 > 0$, $q \in (1, p)$ and $r > p$, and they also assume that $x \to \lambda g(x) + |x|^{r-1}$ is nondecreasing on $\boldsymbol{R}_+$. They prove the existence of a positive solution for λ taking values in a bounded interval. They do not produce a second positive solution or nodal solutions as we do here. Moreover, the monotonicity condition on the reaction $x \to \lambda g(x) + |x|^{r-1}$ makes the implementation of the subsolution-supersolution method easier, since it is possible to use the classical monotone iteration technique. Finally our derivation of the supersolution $\overline{u} \in \mathrm{int}C_+$ appears to be more straightforward (compare the proof of Proposition 2.1 in this paper with the proof of [6, Lemma 1]). The work here extends the aforementioned papers. Our approach is variational, combined with the method of subsolutions and supersolutions, and with suitable truncation techniques.

The rest of the paper is organized as follows. Section 2 deals with some background material, which will be used in the sequel. Section 3 produces multiple solutions of constant sign for the case of a p-linear perturbation f. In Section 4, we obtain an additional nodal solution. Finally, Section 5 treats the case of a p-superlinear perturbation f.

1. Mathematical background. In the analysis of problem (1), we will make use of the Sobolev space $W_0^{1,p}(Z)$ and of the space $C_0^1(\overline{Z}) = \{x \in C^1(\overline{Z}) \,; x|_{\partial Z} = 0\}$, which is dense in $W_0^{1,p}(Z)$. The space $C_0^1(\overline{Z})$ is an ordered Banach space with the order cone given by

$$C_+ = \{x \in C_0^1(\overline{Z}) \,; x(z) \ge 0 \text{ for all } z \in \overline{Z}\},$$

which has a nonempty interior given by

$$\mathrm{int}\, C_+ = \left\{x \in C_+ \,; x(z) > 0 \ \text{ for all } \ z \in Z \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\right\}.$$

Here $n(z)$ is the outward unit normal vector at $z \in \partial Z$. The following obvious lemma about ordered Banach spaces will be useful in our considerations.

LEMMA 1.1. *If X is an ordered Banach space, K is the order cone of X and x_0 is in* $\mathrm{int}\, K$, *then for any $y \in X$ there exists some $t = t(y) > 0$ such that $tx_0 - y$ is contained in K, i.e., $y \le tx_0$.*

Let us recall the following notion from nonlinear operator theory (see, e.g., Gasinski and Papageorgiou [14, p. 338]). So, let X be a Banach space and X^* its topological dual. Denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X, X^*).

DEFINITION 1.2. A map $A : X \to X^*$ is said to be of type $(S)_+$, if, for any sequence $\{x_n\} \subseteq X$ for which $x_n \overset{w}{\to} x$ in X and $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$, we have $x_n \to x$ in X.

In the sequel, $X = W_0^{1,p}(Z)$, $X^* = W^{-1,p'}(Z)$ with $1/p + 1/p' = 1$, and $\langle \cdot, \cdot \rangle$ will be the duality brackets for this dual pair. Let $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ be the nonlinear map defined by

$$(2) \qquad \langle A(x), y \rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbf{R}^N} dz \,,$$

for all $x, y \in W_0^{1,p}(Z)$.

LEMMA 1.3. $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$, *defined by* (2), *is of type* $(S)_+$.

PROOF. Let $\{x_n\} \subseteq W_0^{1,p}(Z)$ be a sequence such that $x_n \overset{w}{\to} x$ in $W_0^{1,p}(Z)$ and assume that

$$(3) \qquad \limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0 \,.$$

It is clear from (2) that A is continuous monotone, hence it is maximal monotone. Recall that a maximal monotone operator is generalized pseudomonotone (see [14, p. 330]). So, from (3) we have

$$\|Dx_n\|_p^p = \langle A(x_n), x_n \rangle \to \langle A(x), x \rangle = \|Dx\|_p^p \,.$$

Since $Dx_n \overset{w}{\to} Dx$ in $L^p(Z, \mathbf{R}^N)$ and $L^p(Z, \mathbf{R}^N)$ is uniformly convex, from the Kadec-Klee property, we have $Dx_n \to Dx$ in $L^p(Z, \mathbf{R}^N)$, hence $x_n \to x$ in $W_0^{1,p}(Z)$. □

Next we recall what we mean by supersolutions and subsolutions for problem (1).

DEFINITION 1.4. (a) A *supersolution* for problem (1), is a function $\overline{x} \in W^{1,p}(Z)$ such that $\overline{x}|_{\partial Z} \geq 0$ and

$$(4) \qquad \int_Z \|D\overline{x}\|^{p-2}(D\overline{x}, Dy)_{\mathbf{R}^N} dz \geq \lambda \int_Z |\overline{x}|^{q-2}\overline{x}y dz + \int_Z f(z, \overline{x})y dz$$

for all $y \in W_0^{1,p}(Z)$, $y(z) \geq 0$ a.e. on Z. We say that $\overline{x}$ is a *strict supersolution* for problem (1), if the inequality in (4) is strict for some $y \neq 0$.

(b) A *subsolution* for problem (1) is a function $\underline{x} \in W^{1,p}(Z)$ such that $\underline{x}|_{\partial Z} \leq 0$ and

$$(5) \qquad \int_Z \|D\underline{x}\|^{p-2}(D\underline{x}, Dy)_{\mathbf{R}^N} dz \leq \lambda \int_Z |\underline{x}|^{q-2}\underline{x}y dz + \int_Z f(z, \underline{x})y dz$$

for all $y \in W_0^{1,p}(Z)$, $y(z) \geq 0$ a.e. on Z. We say that $\underline{x}$ is a *strict subsolution* for problem (1), if the inequality in (5) is strict for some $y \neq 0$.

In the analysis of problem (1), we will use some basic facts about the spectrum of the negative Dirichlet p-Laplacian. In the sequel, we use the notation

$$\Delta_p u = \mathrm{div}(\|Du\|^{p-2}Du) \,.$$

For $0 \neq m \in L^\infty(Z)_+$, we consider the following nonlinear weighted (with weight m) eigenvalue problem:

(6)
$$\begin{cases} -\Delta_p x(z) = \widehat{\lambda} m(z)|x(z)|^{p-2}x(z) & \text{a.e. on } Z \\ x|_{\partial Z} = 0 \,. \end{cases}$$

The smallest number $\widehat{\lambda} \in R$ for which problem (6) has a nontrivial solution is the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(Z), m\right)$, and it is denoted by $\widehat{\lambda}_1(m)$. We know that $\widehat{\lambda}_1(m) > 0$, and it is isolated and simple, i.e., the corresponding eigenspace is one-dimensional. Moreover, $\widehat{\lambda}_1(m)$ admits the following variational characterization

(7)
$$\widehat{\lambda}_1(m) = \min\left\{ \|Dx\|_p^p \Big/ \left(\int_Z m|x|^p dz \right); x \in W_0^{1,p}(Z), x \neq 0 \right\}$$

(see Anane [4]).

In (7), the minimum is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1(m)$. Let $u_1 \in W_0^{1,p}(Z)$ be the eigenfunction such that $\int_Z m|u_1|^p dz = 1$. Evidently, $|u_1|$ also realizes the minimum in (7), and so, we may assume that $u_1(z) \geq 0$ a.e. on Z. In fact, using nonlinear regularity theory (see, e.g., [14, pp. 737–738]), we have $u_1 \in C_+$. We actually have $u_1 \in \text{int}\, C_+$ by an application of the nonlinear strong maximum principle of Vazquez [22]. The eigenvalue $\widehat{\lambda}_1(m)$ exhibits the following monotonicity property with respect to the weight function $m \in L^\infty(Z)$, which can be easily deduced from (7), namely,

$$\widehat{\lambda}_1(m_2) < \widehat{\lambda}(m_1) \text{ if } m_1(z) \leq m_2(z) \text{ a.e. on } Z \text{ and } m_1 \neq m_2 \,.$$

If $m \equiv 1$, then we write $\widehat{\lambda}_1(1) = \lambda_1$. For further details on the spectral properties of the negative Dirichlet p-Laplacian, we refer to Lê [17] and [14].

2. Solutions of constant sign for p-linear perturbations. In this section and the next, we deal with the case when the perturbation term $f(z, \cdot)$ is p-linear near infinity. So, the hypotheses on f are the following:

H(f)$_1$ $f : Z \times R \to R$ is a function such that $f(z, 0) = 0$ a.e. on Z and
    (i)   for all $x \in R, z \to f(z, x)$ is measurable,
    (ii)   for a.e. $z \in Z, x \to f(z, x)$ is continuous,
    (iii)   for every $r > 0$, there is some $a_r \in L^\infty(Z)_+$ such that

$$|f(z, x)| \leq a_r(z) \text{ a.e. on } Z, \text{ for all } |x| \leq r \,,$$

    (iv)   there exist $\eta, \widehat{\eta} \in R$ such that $\lambda_1 < \eta \leq \widehat{\eta}$ and

$$\eta \leq \liminf_{|x|\to\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{|x|\to\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \widehat{\eta}$$

uniformly for a.a. $z \in Z$,
    (v) $\lim_{x\to 0} f(z, x)\big/(|x|^{p-2}x) = 0$ uniformly for a.a. $z \in Z$,
    (vi) $f(z, x)x \geq 0$ for a.e. $z \in Z$ and all $x \in R$ (sign condition).

Let $\mathcal{L}_+ = \{\lambda > 0 \,;\ \text{problem (1) has a positive solution}\}$ and define

$$\widehat{\lambda}_+ = \sup \mathcal{L}_+ \,.$$

PROPOSITION 2.1.   *If hypotheses* $\mathbf{H}(f)_1$ *hold, then* $\mathcal{L}_+ \neq \emptyset$ *and* $\widehat{\lambda}_+ < \infty$.

PROOF.   First, we show that $\mathcal{L}_+ \neq \emptyset$. The hypotheses $\mathbf{H}(f)_1$(iii) through (v) imply that for any given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$(8) \qquad\qquad |f(z, x)| \le \varepsilon |x|^{p-1} + c_\varepsilon |x|^{r-1}$$

for a.e. $z \in Z$, all $x \in \mathbf{R}$, and $p < r < p^*$.

Let $e \in \operatorname{int} C_+$ be the unique solution of the Dirichlet problem

$$(9) \qquad\qquad -\Delta_p e(z) = 1 \quad \text{a.e. on} \quad Z \quad \text{and} \quad e|_{\partial Z} = 0 \,.$$

We claim that there exists some $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, we may choose $\xi_1 = \xi_1(\lambda) > 0$ satisfying

$$(10) \qquad\quad \lambda \, (\xi_1 \|e\|_\infty)^{q-1} + \varepsilon \, (\xi_1 \|e\|_\infty)^{p-1} + c_\varepsilon \, (\xi_1 \|e\|_\infty)^{r-1} < \xi_1^{p-1} \,.$$

To show this, we argue indirectly. So, suppose that there exist $\{\lambda_n\} \subseteq \mathbf{R}_+$ such that $\lambda_n \to 0^+$ and for every $\xi > 0$ we have

$$\xi^{p-1} \le \lambda_n (\xi \|e\|_\infty)^{q-1} + \varepsilon (\xi \|e\|_\infty)^{p-1} + c_\varepsilon (\xi \|e\|_\infty)^{r-1} \,.$$

Passing to the limit as $n \to \infty$, we obtain

$$\xi^{p-1} \le \varepsilon (\xi \|e\|_\infty)^{p-1} + c_\varepsilon (\xi \|e\|_\infty)^{r-1} \,,$$

and consequently for all $\xi > 0$ we have

$$(11) \qquad\qquad 1 \le \varepsilon \|e\|_\infty + c_\varepsilon \xi^{r-p} \|e\|_\infty^{r-1} \,.$$

Since $r > p$ and $\varepsilon > 0$ is arbitrary, we may choose $\varepsilon, \xi > 0$ so small that (11) is violated, which means that the claim is true.

Let $\xi_1 > 0$ be as above and define $\overline{x} = \xi_1 e \in \operatorname{int} C_+$. We have

$$\begin{aligned}
-\Delta_p \overline{x} &= -\xi_1^{p-1} \Delta_p e(z) \\
&= \xi_1^{p-1} \quad \text{(see (9))} \\
&> \lambda(\xi_1 \|e\|_\infty)^{q-1} + \varepsilon(\xi_1 \|e\|_\infty)^{p-1} \\
&\quad + c_\varepsilon (\xi_1 \|e\|_\infty)^{r-1} \quad \text{(see (10))} \\
&\ge \lambda \overline{x}(z)^{q-1} + f(z, \overline{x}(z)) \quad \text{(see (8))}.
\end{aligned}$$

Hence, $\overline{x} \in \operatorname{int} C_+$ is a strict supersolution for problem (1).

On the other hand, recall that $u_1 \in \operatorname{int} C_+$ is the L^p-normalized principal eigenfunction of $(-\Delta_p, W_0^{1,p}(Z))$. We can always choose small $\varepsilon > 0$ such that

$$(12) \qquad\qquad \lambda_1 \varepsilon^{p-1} u_1(z) < \lambda \varepsilon^{q-1} u_1(z)^{q-1}$$

for all $z \in \overline{Z}$ (recall that $r < q < p$). Set $\underline{x} = \varepsilon u_1 \in \text{int} \, C_+$. We have

$$
\begin{aligned}
-\Delta_p \underline{x}(z) &= -\Delta_p(\varepsilon u_1)(z) \\
&= \lambda_1 \varepsilon^{p-1} u_1(z)^{p-1} \\
&< \lambda \varepsilon^{q-1} u_1(z)^{q-1} \\
&\leq \lambda \underline{x}(z)^{q-1} + f(z, \underline{x}(z)) \quad (\text{see } \mathbf{H}(f)_1 \, (\text{vi})).
\end{aligned}
$$

So, $\underline{x} \in \text{int} \, C_+$ is a strict subsolution for problem (1). By choosing $\varepsilon > 0$ even smaller, we can also have $\underline{x} \leq \overline{x}$ (see Lemma 1.1).

Now that we have the ordered pair $\{\overline{x}, \underline{x}\}$ of supersolution and subsolution for problem (1), we consider the following truncation:

$$
\widehat{f_\lambda}(z, x) = \begin{cases} \lambda \underline{x}(z)^{q-1} + f(z, \underline{x}(z)) & \text{if } x < \underline{x}(z) \\ \lambda x^{q-1} + f(z, x) & \text{if } \underline{x}(z) \leq x \leq \overline{x}(z) \\ \lambda \overline{x}(z) + f(z, \overline{x}(z)) & \text{if } \overline{x}(z) < x. \end{cases}
$$

Evidently, $\widehat{f_\lambda}$ is a Carathéodory function, i.e., measurable in z and continuous in x. Let $\widehat{F_\lambda}(z, x) = \int_0^x \widehat{f_\lambda}(z, s) ds$, which is the primitive of $\widehat{f_\lambda}$. We consider the functional $\widehat{\varphi_\lambda} : W_0^{1,p}(Z) \to \mathbf{R}$, defined by

$$
\widehat{\varphi_\lambda}(x) = \frac{1}{p} \|Dx\|_p^p - \int_Z \widehat{F_\lambda}(z, x(z)) dz.
$$

Clearly, $\widehat{\varphi_\lambda}$ is in $C^1(W_0^{1,p}(Z))$ and for some $c_1 > 0$ and all $x \in W_0^{1,p}(Z)$,

$$
\widehat{\varphi_\lambda}(x) \geq \frac{1}{p} \|Dx\|_p^p - c_1.
$$

Hence, $\widehat{\varphi_\lambda}$ is coercive.

Moreover, exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we can easily verify that $\widehat{\varphi_\lambda}$ is sequentially weakly lower semicontinuous. So, invoking the theorem of Weierstrass, we can find some $x_0 \in W_0^{1,p}(Z)$ such that

$$
\widehat{\varphi_\lambda}(x_0) = \inf[\widehat{\varphi_\lambda}(x) ; x \in W_0^{1,p}(Z)].
$$

Hence, we have $\widehat{\varphi_\lambda}'(x_0) = 0$, and consequently

$$
(13) \qquad\qquad\qquad A(x_0) = \widehat{N_\lambda}(x_0),
$$

where $\widehat{N_\lambda}(x)(\cdot) = \widehat{f_\lambda}(\cdot, x(\cdot))$ for all $x \in L^p(Z)$ (the Nemytskii operator corresponding to $\widehat{f}$). On (13), we act with the test function $(x_0 - \overline{x})^+ \in W_0^{1,p}(Z)$. We obtain

$$
\begin{aligned}
\langle A(x_0), (x_0 - \overline{x})^+ \rangle &= \int_Z \widehat{f_\lambda}(z, x_0)(x_0 - \overline{x})^+ dz \\
&= \lambda \int_{\{x_0 > \overline{x}\}} \overline{x}(x_0 - \overline{x}) dz + \int_{\{x_0 > \overline{x}\}} f(z, \overline{x})(x_0 - \overline{x}) dz \\
&\leq \langle A(\overline{x}), (x_0 - \overline{x})^+ \rangle,
\end{aligned}
$$

where the last inequality is due to the fact that $\overline{x} \in \text{int}\, C_+$ is an upper solution. So, we have $\langle A(x_0) - A(\overline{x}), (x_0 - \overline{x})^+ \rangle \leq 0$ and

$$(14) \qquad \int_{\{x_0 > \overline{x}\}} (\|Dx_0\|^{p-2} Dx_0 - \|D\overline{x}\|^{p-2} D\overline{x}, Dx_0 - D\overline{x})_{\mathbf{R}^N} \leq 0 \,.$$

Since the map $\vartheta_p : \mathbf{R}^N \to \mathbf{R}^N$, defined by

$$\vartheta_p(y) \begin{cases} \|y\|^{p-2} y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

is a strictly monotone homeomorphism, from (14), it follows that

$$|\{x_0 > \overline{x}\}|_N = 0 \,.$$

Here, $|\cdot|_N$ denotes the Lebesgue measure on $\mathbf{R}^N$. So, we have

$$x_0 \leq \overline{x} \,.$$

In a similar fashion we can show that

$$\underline{x} \leq x_0 \,.$$

Then, we obtain $\widehat{\tau}(x_0) = x_0$ and $\widehat{N}(x_0) = N(x_0)$, where $N(x)(\cdot) = f(\cdot, x(\cdot))$ for all $x \in W_0^{1,p}(Z)$. So (13) becomes $A(x_0) = \lambda x_0^{q-1} + N(x_0)$, and we have

$$-\Delta_p x_0(z) = \lambda x_0(z)^{q-1} + f(z, x_0(z)) \quad \text{for a.e. } z \in Z, \text{ and } x_0|_{\partial Z} = 0 \,.$$

Nonlinear regularity theory implies that x_0 is in $\text{int}\, C_+$ (see, e.g., [14, pp. 737–738]). Therefore, $x_0 \in \mathcal{L}_+$ and so, $\mathcal{L}_+ \neq \emptyset$.

Next, we show that $\widehat{\lambda_+} = \sup \mathcal{L}_+ < \infty$. By $\mathbf{H}(f)$(iv), we can find some $\eta_1 > \lambda_1$ and $M_1 > 0$ such that

$$(15) \qquad f(z, x) \geq \eta_1 x^{p-1}$$

for a.e. $z \in Z$ and all $x \geq M_1$. In addition, $\mathbf{H}(f)_1$(v) implies that, for given $\varepsilon > 0$, we can find some $\delta = \delta(\varepsilon) \in (0, 1)$ such that

$$(16) \qquad f(z, x) \geq -\varepsilon x^{p-1}$$

for a.e. $z \in Z$ and all $x \in [0, \delta]$.

Choose $\overline{\lambda} > \max\{\lambda_1 + \varepsilon, \lambda_1 M_1^{p-1}/\delta^{q-1}\}$. Then we have

$$(17) \qquad \overline{\lambda} x^{q-1} + f(z, x) > \lambda_1 x^{p-1}$$

for a.e. $z \in Z$ and all $x > 0$. To see this, note that, for $x \geq M_1$, we have (see (15)) that

$$\overline{\lambda} x^{q-1} + f(z, x) > f(z, x) \geq \eta_1 x^{p-1} > \lambda_1 x^{p-1}$$

for a.e. $z \in Z$. For $\delta \leq x < M_1$, we have (see $\mathbf{H}(f)_1$(vi))

$$\lambda_1 x^{p-1} < \lambda_1 M_1^{p-1} < \overline{\lambda} \delta^{q-1} \leq \overline{\lambda} x^{q-1} \leq \overline{\lambda} x^{q-1} + f(z, x)$$

for a.e. $z \in Z$.

Finally, for $0 < x < \delta$, we have

$$
\begin{aligned}
\overline{\lambda} x^{q-1} + f(z, x) &\geq \overline{\lambda} x^{q-1} - \varepsilon x^{p-1} \quad \text{(see (16))} \\
&> (\overline{\lambda} - \varepsilon) x^{q-1} \quad \text{(since } \delta < 1 \text{ and } q < p) \\
&> (\overline{\lambda} - \varepsilon) x^{p-1} \\
&> \lambda_1 x^{p-1} \quad \text{(recall the choice of } \overline{\lambda}) \,.
\end{aligned}
$$

So, indeed (17) holds. Now suppose that $u \in C_+ \setminus \{\emptyset\}$ satisfies

(18) $-\Delta_p u(z) = \overline{\lambda} u(z)^{q-1} + f(z, u(z))$ a.e. on Z, and $u|_{\partial Z} = 0$.

Invoking the nonlinear strong maximum principle of Vazquez [22], we deduce that u is in $\operatorname{int} C_+$. So, by Lemma 1.1, we can find some $\xi > 0$ such that

$$
\xi u_1 \leq u \,.
$$

Consider the set $\mathcal{D} = \{\xi > 0 \,; \xi u_1 \leq u\}$ and set $\xi_0 = \sup \mathcal{D}$. We have just seen that $\mathcal{D} \neq \emptyset$. Also, note that $\xi_0 < \infty$. Indeed, otherwise we could find $\xi_n \to \infty$ such that $\xi_n u_1 \leq u$, hence $u_1 \leq \xi_n^{-1} u \to 0$, a contradiction to the fact that u_1 is in $\operatorname{int} C_+$. So, we have

(19) $\xi_0 u_1 \leq u$.

From (18) and (19), we have

(20)
$$
\begin{aligned}
-\Delta_p u(z) &= \overline{\lambda} u(z)^{q-1} + f(z, u(z)) \quad \text{(see (18))} \\
&> \lambda_1 u(z)^{p-1} \\
&\geq \lambda_1 (\xi_0 u_1(z))^{p-1} \quad \text{(see (19))} \\
&= -\Delta_p(\xi_0 u_1)(z) \quad \text{for a.e. } z \in Z \,.
\end{aligned}
$$

Then, from (20) and the result of Guedda and Veron [15, Proposition 2.2], we deduce that

$$
u - \xi_0 u_1 \in \operatorname{int} C_+ \,,
$$

which implies the existence of some small $\beta > 0$ such that

$$
u \geq (\xi_0 + \beta) u_1 \,,
$$

a contradiction to the fact that $\xi_0 = \sup \mathcal{D}$. So, (18) cannot have a positive solution, which in turn implies that

$$
\widehat{\lambda}_+ < \overline{\lambda} < \infty \,. \qquad \qquad \square
$$

Similarly, we define

$$
\mathcal{L}_- = \{\lambda > 0 \,; \text{ problem (1) has a negative solution}\} \,.
$$

Then, working as above, but this time on the negative semiaxis, we may produce a subsolution $\underline{v} \in -\operatorname{int} C_+$ and a supersolution $\overline{v} \in -\operatorname{int} C_+$ with $\underline{v} \leq \overline{v}$. Thus we obtain the following proposition.

PROPOSITION 2.2. *If hypotheses* $\mathbf{H}(f)_1$ *hold, then* $\mathcal{L}_- \neq \emptyset$ *and* $\widehat{\lambda}_- = \sup \mathcal{L}_- < \infty$.

Next, we show that problem (1) has a smallest positive solution and a biggest negative solution. To this end, we first prove a lattice-type property for the sets of supersolutions and subsolutions for problem (1).

We say that a nonempty set $S \subseteq W^{1,p}(Z)$ is downward (resp. upward) directed if, for every elements y_1, $y_2 \in S$, there exists $y \in S$ such that $y \leq y_1$ and $y \leq y_2$ (resp. $y \geq y_1$ and $y \geq y_2$).

Let us fix some $\lambda \in (0, \lambda^*)$, where λ^* is from the proof of Proposition 2.1. Then we have the following lemma.

LEMMA 2.3. *The set of supersolutions for problem* (1) *is downward directed. In fact, for any supersolutions* y_1, $y_2 \in W^{1,p}(Z)$ *for problem* (1), $y = \min\{y_1, y_2\} \in W^{1,p}(Z)$ *is a supersolution too.*

PROOF. Let y_1 and y_2 be two supersolutions for problem (1). Given $\varepsilon > 0$, we consider the truncation function $\xi_\varepsilon : \boldsymbol{R} \to \boldsymbol{R}$, defined by

$$\xi_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if } s < -\varepsilon \\ s & \text{if } s \in [-\varepsilon, \varepsilon] \\ \varepsilon & \text{if } \varepsilon < s. \end{cases}$$

Clearly, ξ_ε is Lipschitz continuous. So, from Marcus and Mizel [18], we have

$$\xi_\varepsilon((y_1 - y_2)^-) \in W^{1,p}(Z)$$

and

$$(21) \qquad D\xi_\varepsilon((y_1 - y_2)^-) = \xi'_\varepsilon((y_1 - y_2)^-)D(y_1 - y_2)^-.$$

Let $\psi \in C_c^1(Z)$ with $\psi \geq 0$. Then $\xi_\varepsilon((y_1 - y_2)^-)\psi$ is in $W^{1,p}(Z) \cap L^\infty(Z)$ and

$$(22) \qquad D\left(\xi_\varepsilon((y_1 - y_2)^-)\psi\right) = \psi D\xi_\varepsilon((y_1 - y_2)^-) + \xi_\varepsilon((y_1 - y_2)^-)D\psi.$$

Since y_1 and y_2 are supersolutions for problem (1), we have

$$\langle A(y_1), \xi_\varepsilon((y_1 - y_2)^-)\psi \rangle \geq \lambda \int_Z |y_1|^{q-2} y_1 \xi_\varepsilon((y_1 - y_2)^-)\psi\, dz$$
$$+ \int_Z f(z, y_1)\xi_\varepsilon((y_1 - y_2)^-)\psi\, dz,$$

and

$$\langle A(y_2), (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi \rangle \geq \lambda \int_Z |y_2|^{q-2} y_2 (\varepsilon - \xi_\varepsilon(y_1 - y_2)^-)\psi\, dz$$
$$+ \int_Z f(z, y_2)(\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi\, dz.$$

Adding the last two inequalities, we obtain

$$\langle A(y_1), \xi_\varepsilon((y_1 - y_2)^-)\psi \rangle + \langle A(y_2), (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi \rangle$$

(23)
$$\geq \lambda \int_Z |y_1|^{q-2} y_1 \xi_\varepsilon((y_1 - y_2)^-)\psi \, dz + \lambda \int_Z |y_2|^{q-2} y_2 (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi \, dz$$

$$+ \int_Z f(z, y_1)\xi_\varepsilon((y_1 - y_2)^-)\psi \, dz + \int_Z f(z, y_2)(\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi \, dz \,.$$

Using (21) and (22), we have

$$\langle A(y_1), \xi_\varepsilon((y_1 - y_2)^-)\psi \rangle$$

$$= \int_Z \|Dy_1\|^{p-2}(Dy_1, D(y_1 - y_2)^-)_{\mathbf{R}^N} \xi_\varepsilon'((y_1 - y_2)^-)\psi \, dz$$

(24)
$$+ \int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbf{R}^N} \xi_\varepsilon((y_1 - y_2)^-) \, dz$$

$$= - \int_{\{-\varepsilon \leq y_1 - y_2 \leq 0\}} \|Dy_1\|^{p-2}(Dy_1, D(y_1 - y_2))_{\mathbf{R}^N} \psi \, dz$$

$$+ \int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbf{R}^N} \xi_\varepsilon((y_1 - y_2)^-) \, dz \,.$$

In a similar way, we also have

$$\langle A(y_2), (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi \rangle$$

(25)
$$= \int_{\{-\varepsilon \leq y_1 - y_2 \leq 0\}} \|Dy_2\|^{p-2}(Dy_2, D(y_1 - y_2))_{\mathbf{R}^N} \psi \, dz$$

$$+ \int_Z \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbf{R}^N} (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-)) \, dz \,.$$

Using (24) and (25) and the fact that $\psi \geq 0$, we obtain

$$\langle A(y_1), \xi_\varepsilon((y_1 - y_2)^-)\psi \rangle + \langle A(y_2), (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi \rangle$$

$$= \int_{\{-\varepsilon \leq y_1 - y_2 \leq 0\}} (\|Dy_2\|^{p-2} Dy_2 - \|Dy_1\|^{p-2} Dy_1, D(y_1 - y_2)_{\mathbf{R}^N})\psi \, dz$$

$$+ \int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbf{R}^N} \xi_\varepsilon((y_1 - y_2)^-) \, dz$$

(26)
$$+ \int_Z \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbf{R}^N} (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-)) \, dz$$

$$\leq \int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbf{R}^N} \xi_\varepsilon((y_1 - y_2)^-) \, dz$$

$$+ \int_Z \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbf{R}^N} (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-)) \, dz \,.$$

We return to (23), use (26) and then divide by $\varepsilon > 0$. Thus

(27)
$$\int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbf{R}^N} \frac{1}{\varepsilon}\xi_\varepsilon((y_1 - y_2)^-) dz$$

$$+ \int_Z \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbf{R}^N} \left(1 - \frac{1}{\varepsilon}\xi_\varepsilon((y_1 - y_2)^-)\right) dz$$

$$\geq \lambda \int_Z |y_1|^{q-2}y_1 \frac{1}{\varepsilon}\xi_\varepsilon((y_1 - y_2)^-)\psi dz + \lambda \int_Z |y_2|^{q-2}y_2 \left(1 - \frac{1}{\varepsilon}\xi_\varepsilon((y_1 - y_2)^-)\right)\psi dz$$

$$+ \int_Z f(z, y_1) \frac{1}{\varepsilon}\xi_\varepsilon((y_1 - y_2)^-)\psi dz + \int_Z f(z, y_2)\left(1 - \frac{1}{\varepsilon}\xi_\varepsilon((y_1 - y_2)^-)\right)\psi dz .$$

Note that

$$\frac{1}{\varepsilon}\xi_\varepsilon((y_1 - y_2)^-(z)) \to \chi_{\{y_1 < y_2\}}(z) \text{ a.e. on } Z \quad \text{as} \quad \varepsilon \to 0,$$

and $\chi_{\{y_1 \geq y_2\}} = 1 - \chi_{\{y_1 < y_2\}}$.

Passing to the limit as $\varepsilon \to \infty$ in (27), we obtain

$$\int_{\{y_1 < y_2\}} \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbf{R}^N} dz + \int_{\{y_1 \geq y_2\}} \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbf{R}^N} dz$$

(28)
$$\geq \lambda \int_{\{y_1 < y_2\}} |y_1|^{q-2}y_1 \psi dz + \lambda \int_{\{y_1 \geq y_2\}} |y_2|^{q-2}y_2 \psi dz$$

$$+ \int_{\{y_1 < y_2\}} f(z, y_1)\psi dz + \int_{\{y_1 \geq y_2\}} f(z, y_2)\psi dz .$$

Since $y = \min\{y_1, y_2\}$ is in $W^{1,p}(Z)$, we have

$$Dy(z) = \begin{cases} Dy_1(z) & \text{for a.e. } z \in \{y_1 < y_2\} \\ Dy_2(z) & \text{for a.e. } z \in \{y_1 \geq y_2\}. \end{cases}$$

So, we can rewrite (28) as

(29)
$$\int_Z \|Dy\|^{p-2}(Dy, D\psi)_{\mathbf{R}^N} dz \geq \lambda \int_Z |y|^{p-2}y\psi dz + \int_Z f(z, y)\psi dz .$$

But $\psi \in C_c^1(Z)$ with $\psi \geq 0$ was arbitrary and $C_c^1(Z)_+$ is dense in $W_0^{1,p}(Z)$. So, we deduce that (29) is also true for all $\psi \in W_0^{1,p}(Z)$ with $\psi \geq 0$, which in turn implies that $y = \min\{y_1, y_2\}$ is a supersolution for problem (1).   □

In a similar fashion, we can also show the following lemma.

LEMMA 2.4.   *The set of subsolutions for problem* (1) *is upward directed. In fact, for any subsolutions* $v_1, v_2 \in W^{1,p}(Z)$ *for problem* (1), $v = \max\{v_1, v_2\} \in W^{1,p}(Z)$ *is a subsolution too.*

Recall that we already have an ordered pair $\{\overline{x}, \underline{x}\}$ of supersolution and subsolution for problem (1) with $\overline{x}, \underline{x} \in \text{int } C_+$, and an ordered pair $\{\overline{v}, \underline{v}\}$ of supersolution and subsolution

for problem (1) with $\overline{v}, \underline{v} \in - \operatorname{int} C_+$. Now we consider the following order intervals

$$I_+ = [\underline{x}, \overline{x}] = \{x \in W_0^{1,p}(Z)\,;\, \underline{x}(z) \le x(z) \le \overline{x}(z) \text{ a.e. on } Z\}$$

and

$$I_- = [\underline{v}, \overline{v}] = \{v \in W_0^{1,p}(Z)\,;\, \underline{v}(z) \le v(z) \le \overline{v}(z) \text{ a.e. on } Z\}\,.$$

PROPOSITION 2.5. *If hypotheses* $\mathbf{H}(f)_1$ *hold and* λ *is in* $(0, \widehat{\lambda}_+)$, *then problem* (1) *admits a smallest solution in* I_+.

PROOF. Let S_+ be the set of solutions of (1) belonging to I_+. From the proof of Proposition 2.1, we know that $S_+ \ne \emptyset$. We claim that the set S_+ is downward directed. To this end, let $x_1, x_2 \in S_+$. By Lemma 2.3, $\widehat{x} = \min\{x_1, x_2\} \in W_0^{1,p}(Z)$ is a supersolution too. We consider the order interval.

$$\widehat{I}_+ = [\underline{x}, \widehat{x}] = \{x \in W_0^{1,p}(Z)\,;\, \underline{x}(z) \le x(z) \le \widehat{x}(z) \text{ a.e. on } Z\}\,.$$

As before, truncating the nonlinearity f with respect to the pair $\{\underline{x}, \widehat{x}\}$ and reasoning similarly (see the proof of Proposition 5), we can get some $\widehat{x}_0 \in \widehat{I}_+$, a solution of problem (1). Nonlinear regularity theory implies $\widehat{x}_0 \in \operatorname{int} C_+$. Moreover, we can show that (see the proof of Proposition 2.1)

$$\underline{x} \le \widehat{x}_0 \le \widehat{x} = \min\{x_1, x_2\}\,,$$

hence S_+ is downward directed.

Consider a chain C of S_+, i.e., a totally ordered subset of S_+. From Dunford and Schwartz [11, Corollary 7, p. 336], we know that there exists a sequence $\{x_n\} \subseteq C$ such that

$$\inf_{n \ge 1} x_n = \inf C\,.$$

Because C is totally ordered, we may assume that $\{x_n\}$ is decreasing. As solutions of (1), $\{x_n\}$ satisfy

$$(30) \qquad\qquad A(x_n) = \lambda x_n^{q-1} + N(x_n)\,,$$

hence we have $\|Dx_n\|_p^p = \lambda\|x_n\|_q^q + \int_Z f(z, x_n)x_n dz$, and consequently,

$$(31) \qquad\qquad \|Dx_n\|_p^p \le c_2(\lambda\|Dx_n\|_p^q + \|Dx_n\|_p)$$

for some $c_2 > 0$ and all $n \ge 1$ (see $\mathbf{H}(f)_1$(iii)).

Recall that $q < p$. From (31), it follows that $\{x_n\}_{n \ge 1}$ is bounded in $W_0^{1,p}(Z)$. So, we may assume that

$$x_n \xrightarrow{w} \widehat{u} \text{ in } W_0^{1,p}(Z) \text{ and } x_n \to \widehat{u} \text{ in } L^p(Z)\,.$$

Acting on (30) with $x_n - \widehat{u} \in W_0^{1,p}(Z)$ and passing to the limit, we obtain

$$\lim_{n \to \infty} \langle A(x_n), x_n - \widehat{u} \rangle = 0\,.$$

This, by virtue of Lemma 1.3, implies that $x_n \to \widehat{u}$ in $W_0^{1,p}(Z)$ and $\overline{x} \leq \widehat{u}$. So, if in (30) we pass to the limit as $n \to \infty$, then

$$A(\widehat{u}) = \lambda \widehat{u}^{q-1} + N(\widehat{u}),$$

and hence

$$-\Delta \widehat{u}(z) = \lambda \widehat{u}(z)^{q-1} + f(z, \widehat{u}(z)) \text{ a.e. on } Z, \text{ and } \widehat{u}|_{\partial Z} = 0.$$

Nonlinear regularity theory implies that $\widehat{u}$ is in $\operatorname{int} C_+$ and of course $\widehat{u} = \inf C$. By Zorn's lemma we can find x_*, a minimal element of S_+. Because S_+ is downward directed, we conclude that $x_* \in \operatorname{int} C_+$ is the smallest solution of (1) in I_+.   □

In a similar fashion, we can also prove the following proposition.

PROPOSITION 2.6.   *If hypotheses* $\mathbf{H}(f)_1$ *hold and* λ *is in* $(0, \widehat{\lambda}_-)$, *then problem* (1) *admits a biggest solution in* $I_- = [\underline{v}, \overline{v}]$.

Using Propositions 2.5 and 2.6, we will produce a smallest positive solution and a biggest negative solution for problem (1).

PROPOSITION 2.7.   *If hypotheses* $\mathbf{H}(f)_1$ *hold and* λ *is in* $(0, \widehat{\lambda}_+)$ *(resp.* λ *is in* $(0, \widehat{\lambda}_-))$, *then problem* (1) *has a smallest positive solution* $x_+ \in \operatorname{int} C_+$ *(resp. a biggest negative solution* $v_- \in -\operatorname{int} C_+)$.

PROOF.   Let $\underline{x}_n = \varepsilon_n u_1$ with $\varepsilon_n \downarrow 0$ and set

$$I_+^n = [\underline{x}_n, \overline{x}] = \{x \in W_0^{1,p}(Z); \ \underline{x}_n(z) \leq x(z) \leq \overline{x}(z) \text{ a.e. on} Z\}.$$

From Proposition 2.5, problem (1) has a smallest solution $x_*^n \in I_+^n$. Moreover, from the proof of Proposition 2.5, we know that $\{x_*^n\}_{n \geq 1}$ is bounded in $W_0^{1,p}(Z)$. So, we may assume that

$$x_*^n \xrightarrow{w} x_+ \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad x_*^n \to x_+ \text{ in } L^p(Z).$$

We have

$$(32) \qquad\qquad A(x_*^n) = \lambda(x_*^n)^{q-1} + N(x_*^n) \quad \text{for } n \geq 1.$$

On (32), we act with $x_*^n - x_+$ and then pass to the limit as $n \to \infty$. So

$$\lim_{n \to \infty} \langle A(x_*^n), x_*^n - x_+ \rangle = 0.$$

Hence, $x_*^n \to x_+$ in $W_0^{1,p}(Z)$ (see Lemma 1.3).

We consider the following auxiliary Dirichlet problem:

$$(33) \qquad\qquad -\Delta_p u(z) = \lambda u(z)^{q-1} \text{ a.e. on } Z, \text{ and } u|_{\partial Z} = 0.$$

From Otani [19], we know that problem (33) has a solution u is in $\operatorname{int} C_+$. Because x_*^n is in $\operatorname{int} C_+$, invoking Lemma 1.1, we can find some $\vartheta_n > 0$ such that

$$(34) \qquad\qquad \vartheta_n u \leq x_*^n.$$

We can always take $\vartheta_n > 0$ to be the biggest positive real number for which (34) holds (see also the proof of Proposition 2.1). Suppose that $0 < \vartheta_n < 1$. Then

$$
\begin{aligned}
-\Delta_p x_*^n(z) &\geq \lambda x_*^n(z)^{q-1} \quad (\text{see } \mathbf{H}(f)_1(\text{vi})) \\
&\geq \lambda(\vartheta_n u(z))^{q-1} \\
&> \lambda \vartheta_n^{p-1} u(z)^{q-1} \quad (\text{since } 0 < \vartheta_n < 1 \text{ and } p > q) \\
&= -\Delta_p(\vartheta_n u)(z) \quad \text{a.e. on } Z \ (\text{see } (33)).
\end{aligned}
$$
(35)

From (33) and using Proposition 2.2 of [15], it follows that

$$
x_*^n - \vartheta_n u \in \text{int}\, C_+ \,,
$$

which contradicts the maximality of ϑ_n. Therefore, we must have $\vartheta_n \geq 1$. Hence, $u \leq x_*^n$ (see (34)). Thus,

$$
u \leq x_+, \quad \text{i.e., } x_+ \neq 0 \,.
$$

Also, if in (32) we pass to the limit as $n \to \infty$, we obtain

$$
A(x_+) = \lambda x_+^{q-1} + N(x_+) \,,
$$

thus

$$
-\Delta_p x_+(z) = \lambda x_+(z)^{q-1} + f(z, x_+(z)) \quad \text{a.e. on } Z, \quad \text{and } x_+|_{\partial Z} = 0 \,.
$$

Nonlinear regularity theory and the nonlinear strong maximum principle of Vazquez [22] imply that x_+ is in $\text{int}\, C_+$ (recall that $x_+ \neq 0$). We claim that x_+ is in $\text{int}\, C_+$ is the smallest positive solution of (1). Indeed, let $x \in W_0^{1,p}(Z)$, $x \geq 0$, $x \neq 0$, be such a solution. Automatically, we have x is in $\text{int}\, C_+$. Therefore, for n large, we have $\underline{x}_n = \varepsilon_n u_1 \leq x$, hence $x_*^n \leq x$. Thus

$$
x_+ \leq x \,.
$$

In a similar way, for $\lambda \in (0, \widehat{\lambda}_-)$, we can produce $v_- \in -\text{int}\, C_+$, the biggest negative solution of problem (1). $\qquad \square$

Now we are ready to state our main results concerning solutions of constant sign for problem (1), when the perturbation term f is asymptotically p-linear.

THEOREM 2.8.   *If hypotheses* $\mathbf{H}(f)_1$ *hold and* $\lambda = \widehat{\lambda}_+$ (*resp.* $\lambda = \widehat{\lambda}_-$), *then problem* (1) *has a solution* x *in* $\text{int}\, C_+$ (*resp.* v *in* $-\text{int}\, C_+$).

PROOF.   We shall give the proof for the case when $\lambda = \widehat{\lambda}_+$. The proof for $\lambda = \widehat{\lambda}_-$ is similar. Let $\{\lambda_n\} \subseteq (0, \widehat{\lambda}_+)$ and assume that $\lambda_n \uparrow \widehat{\lambda}_+$. From Proposition 2.7, we know that for each λ_n problem (1) has a smallest positive solution x_n in $\text{int}\, C_+$. Suppose that $\|x_n\| \to \infty$ and set $y_n = x_n / \|x_n\|$. Then, $\|y_n\| = 1$ for all $n \geq 1$ and so, we may assume that

$$
\begin{aligned}
y_n &\overset{w}{\to} y \text{ in } W_0^{1,p}(Z) \,, \\
y_n &\to y \text{ in } L^p(Z) \,, \\
y_n(z) &\to y(z) \text{ a.e. on } Z \,,
\end{aligned}
$$

$$|y_n(z)| \leq k(z) \text{ a.e. on } Z,$$

for all $n \geq 1$ with $k \in L^p(Z)_+$. We have $A(x_n) = \lambda_n x_n^{q-1} + N(x_n)$, hence

$$(36) \qquad A(y_n) = \frac{\lambda_n}{\|x_n\|^{p-q}} y_n^{q-1} + \frac{N(x_n)}{\|x_n\|^{p-1}}.$$

From $\mathbf{H}(f)_1$(iii) through (v), it follows that $|f(z, x)| \leq c_3 |x|^{p-1}$ for a.e. $z \in Z$, all $x \in \mathbf{R}$ and some $c_3 > 0$. Thus, we have

$$(37) \qquad \frac{|f(z, x_n(z))|}{\|x_n\|^{p-1}} \leq c_3 |y_n(z)|^{p-1} \text{ for a.e. } z \in Z, \text{ all } n \geq 1,$$

and hence $\{h_n = N(x_n)/\|x_n\|^{p-1}; n \geq 1\}$ is a bounded sequence in $L^{p'}(Z)$. So, we may assume that

$$(38) \qquad h_n \xrightarrow{w} h \text{ in } L^{p'}(Z) \text{ as } n \to \infty.$$

For every $\varepsilon > 0$ and $n \geq 1$, let

$$D_{\varepsilon,n} = \left\{ z \in Z ; x_n(z) > 0, \eta - \varepsilon \leq \frac{f(z, x_n(z))}{x_n(z)^{p-1}} \leq \widehat{\eta} + \varepsilon \right\}.$$

Note that $x_n(z) \to \infty$ for a.e. $z \in \{y > 0\}$. So, by virtue of $\mathbf{H}(f)_1$ (iv) we have

$$\chi_{D_{\varepsilon,n}}(z) \to 1 \text{ for a.e. } z \in \{y > 0\}.$$

From the dominated convergence theorem, we have $\|(1 - \chi_{D_{\varepsilon,n}}) h_n\|_{L^{p'}(\{y>0\})} \to 0$, and hence we obtain

$$(39) \qquad \chi_{D_{\varepsilon,n}} h_n \xrightarrow{w} h \text{ in } L^{p'}(\{y > 0\}) \quad (\text{see (38)}).$$

From the definition of $D_{\varepsilon,n}$, we have

$$\begin{aligned} \chi_{D_{\varepsilon,n}}(z)(\eta - \varepsilon) y_n(z)^{p-1} &= \chi_{D_{\varepsilon,n}}(z) h_n(z) \\ &= \chi_{D_{\varepsilon,n}}(z) \frac{f(z, x_n(z))}{x_n(z)^{p-1}} y_n(z)^{p-1} \\ &\leq \chi_{D_{\varepsilon,n}}(z)(\widehat{\eta} + \varepsilon) y_n(z)^{p-1}. \end{aligned}$$

We pass to the limit as $\varepsilon \downarrow 0$, using (39) together with Mazur's lemma. Then,

$$\eta y(z)^{p-1} \leq h(z) \leq \widehat{\eta} y(z)^{p-1}$$

a.e. on $\{y > 0\}$. On the other hand, from (37) it is clear that

$$h(z) = 0 \text{ a.e. on } \{y = 0\}.$$

Since $y \geq 0$, we have $Z = \{y > 0\} \cup \{y = 0\}$ and so it follows that

$$\eta y(z)^{p-1} \leq h(z) \leq \widehat{\eta} y(z)^{p-1}$$

a.e. on Z. Therefore, we have $h = gy^{p-1}$ with $g \in L^\infty(Z)_+$ satisfying $\eta \le g(z) \le \widehat{\eta}$ a.e. on Z. If on (36) we act with $y_n - y \in W_0^{1,p}(Z)$, pass to the limit as $n \to \infty$ and use Lemma 1.3, we obtain

$$y_n \to y \text{ in } W_0^{1,p}(Z) \text{ and so } \|y\| = 1.$$

Hence, from (36), in the limit as $n \to \infty$, we have $A(y) = gy^{p-1}$ for $y \ne 0$. Thus,

(40) $$-\Delta_p y(z) = g(z)|y(z)|^{p-2}y(z) \text{ a.e. on } Z, \quad y|_{\partial Z} = 0, \quad y \ne 0.$$

Exploiting the monotonicity of the principal eigenvalue on the weight function (see Section 2), we have

$$\widehat{\lambda}_1(g) < \widehat{\lambda}_1(\lambda_1) = 1.$$

Using this fact in (40), we deduce that $y \ge 0$, $y \ne 0$ is not a principal eigenfunction. Hence it must change sign, a contradiction. This proves that $\{x_n\}_{n\ge 1}$ is a bounded sequence in $W_0^{1,p}(Z)$ and so, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad x_n \to x \text{ in } L^p(Z).$$

We have

(41) $$A(x_n) = \lambda_n x_n^{q-1} + N(x_n).$$

As before, if on (41) we act with $x_n - x$ and pass to the limit, with the help of Lemma 1.3, we obtain $x_n \to x$ in $W_0^{1,p}(Z)$ as $n \to \infty$ and so, $\|x\| = 1$. Therefore, from (41) we have

$$A(x) = \widehat{\lambda}_+ x^{q-1} + N(x),$$

and hence,

$$-\Delta_p x(z) = \widehat{\lambda}_+ x(z)^{q-1} + f(z, x(z)) \text{ a.e. on } Z, \quad x|_{\partial Z} = 0, \quad x \ge 0, \quad x \ne 0.$$

Thus, we conclude that x is in $\operatorname{int} C_+$ (by nonlinear regularity theory and the nonlinear maximum principle) and that it is a solution of (1) when $\lambda = \widehat{\lambda}_+$.

Similarly, we obtain a solution v in $-\operatorname{int} C_+$ when $\lambda = \widehat{\lambda}_-$. ☐

Next we check the cases when $\lambda \in (0, \widehat{\lambda}_+)$ and $\lambda \in (0, \widehat{\lambda}_-)$. For these cases, we produce a second positive and negative solution, respectively, by using the mountain pass theorem on a functional resulting by truncating the reaction term (right-hand side of (1)) at the solution x_0 in $\operatorname{int} C_+$ obtained in Theorem 2.8 (see also [2]).

THEOREM 2.9. *If* $\lambda \in (0, \widehat{\lambda}_+)$ *(resp.* $\lambda \in (0, \widehat{\lambda}_-)$*), then problem* (1) *has at least two solutions* $x_0, \widehat{x}$ *in* $\operatorname{int} C_+$ *with* $x_0 < \widehat{x}$ *(resp. two solutions* $v_0, \widehat{v}$ *in* $-\operatorname{int} C_+$ *with* $\widehat{v} < v_0$*).*

PROOF. We shall give the proof for the pair of positive solutions. The proof for the other pair is similar.

From the proof of Proposition 2.1, we have a solution x_0 in $I_+ = [\underline{x}, \overline{x}]$. We may assume that this is the only solution of (1) in I_+. Then, we introduce the following truncation of the

concave term and of the perturbations f. Namely, let

$$\overline{f}_+^\lambda(z, x) = \begin{cases} \lambda x_0(z)^{q-1} + f(z, x_0(z)) & \text{if } x \le x_0(z) \\ \lambda x^{q-1} + f(z, x) & \text{if } x_0(z) < x. \end{cases}$$

Note that this is Carathéodory. We set $\overline{N}_+^\lambda(x)(\cdot) = \overline{f}_+^\lambda(\cdot, x(\cdot))$ for all $x \in W_0^{1,p}(Z)$. Set $\overline{F}_+^\lambda(z, x) = \int_0^x \overline{f}_+^\lambda(z, s)ds$ and consider the functional $\overline{\varphi}_\lambda^+ : W_0^{1,p}(Z) \to \boldsymbol{R}$ defined by

$$\overline{\varphi}_\lambda^+(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \overline{F}_+^\lambda(z, x(z))dz.$$

Clearly, $\overline{\varphi}_\lambda^+ \in C^1(W_0^{1,p}(Z))$.

We also consider the following auxiliary Dirichlet problem:

(42)
$$\begin{cases} -\Delta_p x(z) = \overline{f}_+^\lambda(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases}$$

Since $\underline{x} \le x_0$, we have

(43)
$$\overline{f}_+^\lambda(z, \underline{x}(z)) = \lambda x_0(z)^{q-1} + f(z, x_0(z)).$$

From the proof of Proposition 2.1, we know that

$$
\begin{aligned}
-\Delta_p \underline{x}(z) &= -\Delta_p(\varepsilon u_1)(z) \\
&= \lambda_1 \varepsilon^{p-1} u_1(z)^{p-1} \\
&< \lambda \varepsilon^{q-1} u_1(z)^{q-1} \quad \text{(see the proof of Proposition 2.1)} \\
&= \lambda \underline{x}(z)^{q-1} \\
&\le \lambda x_0(z)^{q-1} + f(z, x_0(z)) \\
&= \overline{f}_+^\lambda(z, \underline{z}(z)) \text{ a.e. on } Z \quad \text{(see (43))}.
\end{aligned}
$$

(44)

Thus, $\underline{x} = \varepsilon u_1 \in \text{int } C_+$ is a strict subsolution for problem (42).

In addition, it is clear from the definition of $\overline{f}_+^\lambda$ that $\overline{x} \in \text{int } C_+$ remains a strict supersolution for problem (42) too. Note that $\overline{\varphi}_\lambda^+|_{I_+}$ is coercive and it is easy to see that $\overline{\varphi}_\lambda^+$ is a sequentially weakly lower semicontinuous. So, by the theorem of Weierstrass, we can find $\overline{x}_0 \in I_+$ such that

$$\overline{\varphi}_\lambda^+(\overline{x}_0) = \inf_{I_+} \overline{\varphi}_\lambda^+.$$

Reasoning as in Filippakis and Papageorgiou [12, the proof of Theorem 4.2] (see also Struwe [21, Theorem 2.1, p. 14]), we obtain

(45) $-\Delta_p \overline{x}_0(z) = \lambda \overline{x}_0(z)^{q-1} + \overline{f}_+(z, \overline{x}_0(z))$ a.e. on Z, $\overline{x}_0|_{\partial Z} = 0$,

and $\overline{x}_0 \in \text{int } C_+, \overline{x}_0 \in I_+$. So, (45) becomes

$$-\Delta_p \overline{x}_0(z) = \lambda \overline{x}_0(z)^{q-1} + f_+(z, \overline{x}_0(z)) \text{ a.e. on } Z, \quad \overline{x}_0|_{\partial Z} = 0.$$

Hence, $\overline{x}_0 \in \text{int } C_+ \cap I_+$ is a solution of (1).

Since we assumed that x_0 is the unique solution of (1) in I_+, it follows that $\overline{x}_0 = x_0$. Now, from (44), we have

$$(46) \qquad\qquad -\Delta_p \underline{x}(z) < -\Delta_p x_0(z) \text{ a.e. on } Z,$$

while, from the proof of Proposition 2.1, we have

$$(47) \qquad\qquad -\Delta_p \overline{x}(z) > -\Delta_p x_0(z) \text{ a.e. on } Z.$$

From (46) and (47), and Proposition 2.2 of [15], we deduce that $\overline{x} - x_0 \in \operatorname{int} C_+$ and $x_0 - \underline{x} \in \operatorname{int} C_+$. So, x_0 is a local $C_0^1(\overline{Z})$-minimizer of $\overline{\varphi}_\lambda^+$ and from [13] (see also Brezis and Nirenberg [7], where the result was first proved for $p = 2$), it follows that x_0 is a local $W_0^{1,p}(Z)$-minimizer of $\overline{\varphi}_\lambda^+$. Then, as in Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 29], we can find $r > 0$ small such that

$$(48) \qquad\qquad \overline{\varphi}_\lambda^+(x_0) < \inf\left[\overline{\varphi}_\lambda^+(u); \|u - x_0\| = r\right] = \overline{c}_r^+.$$

Also, from $\mathbf{H}(f)_1$ (iii) and (iv), we see that there exist $c_4 > 0$ and $\eta_0 > \lambda_1$ such that

$$(49) \qquad\qquad f(z, x) \geq \eta_0 x^{p-1} - c_4$$

for a.e. $z \in Z$ and all $x \geq 0$. Since u_1 is in $\operatorname{int} C_+$, we can find some $t_0 > 0$ such that $t u_1 \geq x_0$ for all $t \geq t_0$. Thus, we have

$$\overline{f}_+(z, t u_1(z)) = f(z, t u_1(z)) \geq \eta_0 t^{p-1} u_1(z)^{p-1} - c_4 \text{ a.e. on } Z$$

(see (49)). Therefore,

$$(50) \qquad\qquad \overline{F}_+(z, t u_1(z)) \geq \frac{\eta_0}{p} t^p u_1(z)^p - c_4 t u_1(z) \text{ a.e. on } Z.$$

Thus, for some $c_5 > 0$ we have

$$
\begin{aligned}
(51) \qquad \overline{\varphi}_\lambda^+(t u_1) &= \frac{t^p}{p}\|Du_1\|_p^p - \frac{\lambda t^q}{q}\|u_1\|_q^q - \int_Z \overline{F}_+(z, t u_1)\, dz \\
&\leq \frac{t^p}{p}\|Du_1\|_p^p - \frac{\lambda t^q}{q}\|u_1\|_q^q - \frac{t^p \eta_0}{p}\|u_1\|_p^p + c_5 t \|u_1\|_p \\
&= \frac{t^p}{p}(\lambda_1 - \eta_0) - \frac{\lambda t^q}{q}\|u_1\|_q^q + c_5 t,
\end{aligned}
$$

where the last equality is due to the fact that $\|Du_1\|_p^p = \lambda_1 \|u_1\|_p^p$ and $\|u_1\|_p = 1$. Because $\lambda_1 < \eta_0$, from (51) it follows that

$$(52) \qquad\qquad \overline{\varphi}_\lambda^+(t u_1) \to -\infty \quad \text{as } t \to \infty.$$

Finally, we show that $\overline{\varphi}_\lambda^+$ satisfies the PS-condition. For this purpose, let $\{x_n\}$ be a sequence in $W_0^{1,p}(Z)$ such that, for some $M_1 > 0$,

$$(53) \qquad |\overline{\varphi}_\lambda^+(x_n)| \leq M_1 \text{ for all } n \geq 1, \quad \text{and} \quad (\overline{\varphi}_\lambda^+)'(x_n) \to 0 \text{ in } W^{-1,p'}(Z).$$

We have $|\langle (\overline{\varphi}_\lambda^+)'(x_n), v\rangle| \leq \varepsilon_n \|v\|$ for all $v \in W_0^{1,p}(Z)$ with $\varepsilon_n \downarrow 0$.

Let $v = -x_n^- \in W_0^{1,p}(Z)$. Then

$$\left| \|Dx_n^-\|_p^p + \lambda \int_Z x_0^{q-1} x_n^- dz + \int_Z f(z, x_0) x_n^- dz \right| \le \varepsilon_n \|x_n^-\|,$$

thus we have $\|Dx_n^-\|_p^p \le \varepsilon_n \|x_n^-\|$ (since $x_0, f(z, x_0) \ge 0$). Therefore, $\{x_n^-\}$ is bounded in $W_0^{1,p}(Z)$.

Suppose that $\|x_n\| \to \infty$. Then we must have $\|x_n^+\| \to \infty$. Set $y_n = x_n^+ / \|x_n^+\|$. We have $\|y_n\| = 1$, and so, we may assume that

$$y_n \overset{w}{\to} y \text{ in } W_0^{1,p}(Z), \quad y_n \to y \text{ in } L^p(Z), \quad y_n(z) \to y(z) \text{ a.e. on } Z,$$

and $|y_n(z)| \le k(z)$ for a.e. $z \in Z$, all $n \ge 1$ and with some $k \in L^p(Z)_+$.

From the choice of the sequence $\{x_n\}_{n \ge 1}$ from $W_0^{1,p}(Z)$, we have

$$|\langle A(x_n) - \lambda |\widehat{\tau}_0(x_n)|^{q-2} \widehat{\tau}_0(x_n) - \overline{N}_+(x_n), v \rangle| \le \varepsilon_n \|v\|$$

for all $v \in W_0^{1,p}(Z)$.

Since $\{x_n^-\}$ is bounded in $W_0^{1,p}(Z)$, $A(x_n) = A(x_n^+) - A(x_n^-)$ and $1 < q < p$, we have

$$\left\langle A(y_n) - \frac{\overline{N}_+(x_n)}{\|x_n^+\|^{p-1}}, v \right\rangle \le \varepsilon_n' \|v\|$$

for all $v \in W_0^{1,p}(Z)$ with $\varepsilon_n' \downarrow 0$.

Note that $x_n^+(z) \to \infty$ a.e. on $\{y > 0\}$ (recall $y \ge 0$). Then, arguing as in the proof of Theorem 2.8, we can show that

$$\frac{\overline{N}_+(x_n)}{\|x_n^+\|^{p-1}} \overset{w}{\longrightarrow} gy^{p-1} \quad \text{in } L^{p'}(Z)$$

for some $g \in L^\infty(Z)_+$ satisfying $\eta \le g(z) \le \widehat{\eta}$ a.e. on Z. Also, using Lemma 1.3, we can show that $y_n \to y$ in $W_0^{1,p}(Z)$, hence $\|y\| = 1$. Therefore, in the limit, we have $A(y) = gy^{p-1}$, $y \ge 0$ and $y \ne 0$. Thus,

$$(54) \qquad -\Delta_p y(z) = g(z) |y(z)|^{p-2} y(z) \text{ a.e. on } Z, \quad y|_{\partial Z} = 0, \quad y \ne 0.$$

However, $\widehat{\lambda}_1(g) < \widehat{\lambda}_1 = 1$ and so, y must change sign, a contradiction to the fact that $y \ge 0$. This proves that $\{x_n^+\}$ is bounded in $W_0^{1,p}(Z)$, hence $\{x_n\}$ is bounded in $W_0^{1,p}(Z)$. So, we may assume that

$$x_n \overset{w}{\to} x \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad x_n \to x \text{ in } L^p(Z).$$

We have

$$\varepsilon_n \|x_n - x\| \ge \left| \langle A(x_n), x_n - x \rangle - \lambda \int_Z |\widehat{\tau}_0(x_n)|^{q-2} \widehat{\tau}_0(x_n)(x_n - x) dz \right.$$

$$\left. - \int_Z \overline{f}_+(z, x_n)(x_n - x) dz \right|.$$

Thus, we arrive at $\lim \langle A(x_n), x_n - x \rangle = 0$.

Because of Lemma 1.3, we deduce that $x_n \to x$ in $W_0^{1,p}(Z)$ and so, we have proved that $\overline{\varphi}_\lambda^+$ satisfies the PS-condition.

This fact, together with (48) and (52), permits the application of the mountain pass theorem to yield some $\widehat{x} \in W_0^{1,p}(Z)$ such that

$$(\overline{\varphi}_\lambda^+)'(\widehat{x}) = 0 \quad \text{and} \quad \widehat{x} \neq x_0 \,.$$

Hence $A(\widehat{x}) = \lambda |\widehat{\tau}_0(\widehat{x})|^{q-2} \widehat{\tau}_0(\widehat{x}) + \overline{N}_+(\widehat{x})$, which in turns implies that

$$\langle A(\widehat{x}), (x_0 - \widehat{x})^+ \rangle = \lambda \int_{\{x_0 > \widehat{x}\}} |x_0|^{q-2} x_0 (x_0 - \widehat{x}) dz + \int_{\{x_0 > \widehat{x}\}} f(z, x_0)(x_0 - \widehat{x}) dz \,,$$

and then $\langle A(\widehat{x}) - A(x_0), (x_0 - \widehat{x})^+ \rangle = 0$. Therefore, we have

$$(55) \qquad \int_{\{x_0 > \widehat{x}\}} (\|D\widehat{x}\|^{P-2} D\widehat{x} - \|Dx_0\|^{P-2} Dx_0, Dx_0 - D\widehat{x})_{\mathbf{R}^N} dz = 0 \,.$$

By virtue of the strict monotonicity of the map $\vartheta_p : \mathbf{R}^N \to \mathbf{R}^N$ defined by

$$\vartheta_p(y) = \begin{cases} \|y\|^{p-2} y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \,, \end{cases}$$

from (55) it follows that $|\{x_0 > \widehat{x}\}|_N = 0$. Hence, $x_0 \leq \widehat{x}$. So,

$$\widehat{\tau}_0(\widehat{x}) = \widehat{x} \quad \text{and} \quad \overline{N}_+(\widehat{x}) = N(\widehat{x}) \,,$$

and $A(\widehat{x}) = \lambda \widehat{x}^{q-1} + N(\widehat{x})$. Therefore, we have

$$-\Delta_p \widehat{x}(z) = \lambda \widehat{x}(z)^{q-1} + f(z, \widehat{x}(z)) \text{ a.e. on } Z, \ \widehat{x}|_{\partial Z} = 0 \,.$$

By the nonlinear regularity theory, we have $\widehat{x} \in \text{int } C_+$ and so, $\widehat{x}$ is a second positive smooth solution of (1) distinct from $x_0 \in \text{int } C_+$ and $x_0 \leq \widehat{x}$.

Similarly, we obtain two negative solutions $\widehat{v}, v_0 \in -\text{int } C_+$ with $\widehat{v} \leq v_0$ and $\widehat{v} \neq v_0$, when $0 < \lambda < \widehat{\lambda}_-$. $\qquad\qquad \square$

As a consequence of Theorem 2.9, we have the following full multiplicity theorem concerning constant sign smooth solutions for problem (1).

COROLLARY 2.10. *If hypotheses* $\mathbf{H}(f)_1$ *hold and* $0 < \lambda < \min\{\widehat{\lambda}_+, \widehat{\lambda}_-\} = \widehat{\lambda}_0$, *then problem* (1) *has at least four nontrivial solutions of constant sign*:

$$x_0, \ \widehat{x} \in \text{int } C_+ \ \text{with} \ x_0 < \widehat{x}, \quad \text{and} \quad v_0, \widehat{v} \in -\text{int } C_+ \ \text{with} \ \widehat{v} < v_0 \,.$$

3. Nodal solutions for p-linear perturbations. In this section we shall produce a nodal solution and thus, we will have the full multiplicity result concerning problem (1) when $f(z, x)$ is p-linear near infinity.

THEOREM 3.1. *If hypotheses* $\mathbf{H}(f)_1$ *hold and* $\lambda \in (0, \widehat{\lambda}_0)$ *with* $\widehat{\lambda}_0 = \min\{\widehat{\lambda}_1, \widehat{\lambda}_-\}$, *then problem* (1) *has at least five nontrivial solutions, four of which are from Corollary* 2.10, *while the fifth one* $y_0 \in C_0^1(\overline{Z})$ *is nodal.*

PROOF. From Corollary 2.10, we already have four solutions of constant sign, namely,

$$x_0,\ \widehat{x} \in \operatorname{int} C_+ \ \text{with}\ x_0 < \widehat{x}, \quad \text{and} \quad v_0,\ \widehat{v} \in -\operatorname{int} C_+ \ \text{with}\ \widehat{v} < v_0\,.$$

It remains to establish the existence of a nodal solution.

From Proposition 2.7, we know that problem (1) has a smallest positive solution $x_+ \in \operatorname{int} C_+$ and a biggest negative solution $v_- \in -\operatorname{int} C_+$. We consider the following truncation

$$\widehat{f}_\lambda(z, x) = \begin{cases} \lambda|v_-(z)|^{q-2}v_-(z) + f(z, v_-(z)) & \text{if } x < v_-(z) \\ \lambda|x|^{q-2}x + f(z, x) & \text{if } v_-(z) \leq x \leq x_+(z) \\ \lambda|x_+(z)|^{q-2}x_+(z) + f(z, x_+(z)) & \text{if } x_+(z) < x\,. \end{cases}$$

This is Carathéodory. Let $\widehat{F}_\lambda(z, x) = \int_0^x \widehat{f}_\lambda(z, s)ds$. Now, consider the C^1-functional $\widehat{\varphi}_\lambda : W_0^{1,p}(Z) \to \mathbf{R}$, defined by

$$\widehat{\varphi}_\lambda(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{F}_\lambda(z, x(z))dz\,.$$

CLAIM 1. *The critical points of $\widehat{\varphi}_\lambda$ are in the order interval*

$$\widehat{T} = [v_-, x_+] = \{x \in W_0^{1,p}(Z)\,;\, v_-(z) \leq x(z) \leq x_+(z) \text{ a.e. on } Z\}\,.$$

Suppose that x is such a critical point. Then, $\widehat{\varphi}_\lambda'(x) = 0$ and so,

(56) $$A(x) = \widehat{N}_\lambda(x)$$

with $\widehat{N}_\lambda(x)(\cdot) = \widehat{f}_\lambda(\cdot, x(\cdot))$ for all $x \in W_0^{1,p}(Z)$. On (56), we act with the test function $(x - x_+)^+ \in W_0^{1,p}(Z)$ to obtain that

$$\langle A(x), (x - x_+)^+ \rangle = \lambda \int_{\{x>x_+\}} |x_+|^{q-2}x_+(x - x_+)dz$$
$$+ \int_{\{x>x_+\}} f(z, x_+)(x - x_+)dz$$
$$= \langle A(x_+), (x - x_+)^+ \rangle\,.$$

The last equality is due to the fact that $x_+ \in \operatorname{int} C_+$ is a solution of (1). Therefore, we have

(57) $$\int_{\{x>x_+\}} \left(\|Dx\|^{p-2}Dx - \|Dx_+\|^{p-2}Dx_+, Dx - Dx_+\right)_{\mathbf{R}^N} dz = 0\,.$$

By the strict monotonicity of the homeomorphism ϑ_p defined earlier and (57), we deduce that

$$|\{x > x_+\}|_N = 0\,,$$

hence $x \leq x_+$.

Similarly, we can show that $v_- \leq x$, hence $x \in \widehat{T}$. This proves Claim 1.

CLAIM 2. *The pair $\{v_-, x_+\}$ are local minimizers of the functional $\widehat{\varphi}_\lambda$.*

We consider the following additional truncations:

$$\widehat{f}_+^\lambda(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda x^{q-1} + f(z,x) & \text{if } 0 \le x \le x_+(z) \\ \lambda x_+(z)^{q-1} + f(z,x_+(z)) & \text{if } x_+(z) < x, \end{cases}$$

and

$$\widehat{f}_-^\lambda(z,x) = \begin{cases} \lambda |v_-(z)|^{q-2} v_-(z) + f(z,v_-(z)) & \text{if } x < v_-(z) \\ \lambda |x|^{q-2} x + f(z,x) & \text{if } v_-(z) \le x \le 0 \\ 0 & \text{if } 0 < x. \end{cases}$$

All are Carathéodory functions. Also, we set

$$\widehat{F}_\pm^\lambda(z,x) = \int_0^x \widehat{f}_\pm^\lambda(z,x)ds.$$

Finally, we introduce the C^1-functionals $(\widehat{\varphi}_\lambda)_\pm : W_0^{1,p}(Z) \to \mathbf{R}$ defined by

$$(\widehat{\varphi}_\lambda)_\pm(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{F}_\pm^\lambda(z,x(z))dz.$$

Arguing as in the proof of Claim 1 above, we can check that the critical points of $(\widehat{\varphi}_\lambda)_+$ are in $\widehat{T}_+ = [0,x_+] = \left\{ x \in W_0^{1,p}(Z) ; 0 \le x(z) \le x_+(z) \text{ a.e. on } Z \right\}$ and the critical points of $(\widehat{\varphi}_\lambda)_-$ are in

$$\widehat{T}_- = [v_-,0] = \left\{ v \in W_0^{1,p}(Z) ; v_-(z) \le v(z) \le 0 \text{ a.e. on } Z \right\}.$$

By the extremality of the solutions v_- and x_+, we deduce that

- the critical points of $(\widehat{\varphi}_\lambda)_+$ are $\{0, x_+\}$,
- the critical points of $(\widehat{\varphi}_\lambda)_-$ are $\{0, v_-\}$.

By virtue of $\mathbf{H}(f)_1$(vi), we have

(58) $$0 \le f(z,x)$$

for a.e. $z \in Z$ and all $x \ge 0$. Now we choose $\varepsilon > 0$ small enough so that

$$\varepsilon u_1(z) \le x_+(z)$$

for all $z \in \overline{Z}$ (recall that $x_+ \in \operatorname{int} C_+$ and use Lemma 1.1). Then, by (58), we have

$$\widehat{F}_+(z, \varepsilon u_1(z)) = \int_0^{\varepsilon u_1(z)} \widehat{f}_+(z,s)ds = \int_0^{\varepsilon u_1(z)} f(z,s)ds \ge 0.$$

Therefore,

$$\begin{aligned} (\widehat{\varphi}_\lambda)_+(\varepsilon u_1) &= \frac{\varepsilon^p}{p}\|Du_1\|_p^p - \int_Z \widehat{F}_+^\lambda(z,\varepsilon u_1)dz \\ &\le \frac{\varepsilon^p}{p}\lambda_1\|u_1\|_p^p - \frac{\varepsilon^q}{q}\|u_1\|_q^q \end{aligned}$$

$$= \frac{\lambda_1 \varepsilon^p}{p} - \frac{\varepsilon^q}{q} \|u_1\|_q^q \quad (\text{since } \|u_1\|_p = 1) .$$

Thus, $(\widehat{\varphi}_\lambda)_+ (\varepsilon u_1) < 0$ (since $q < p$) and hence

(59) $$\inf (\widehat{\varphi}_\lambda)_+ < 0 = (\widehat{\varphi}_\lambda)_+ (0) .$$

Evidently, $(\widehat{\varphi}_\lambda)_+$ is coercive and sequentially weakly lower semicontinuous. So, we can find some $y_+^0 \in W_0^{1,p}(Z)$, which is a minimizer of $(\widehat{\varphi}_\lambda)_+$ on $W_0^{1,p}(Z)$. From (59), it is clear that $y_+^0 \neq 0$ and so, we must have $y_+^0 = x_+$. However, recall that $x_+ \in \text{int} \, C_+$. We can find small $r > 0$ such that

$$\widehat{\varphi}_\lambda \big|_{\overline{B}_r^{C_0^1(\overline{Z})}(x_+)} = (\widehat{\varphi}_\lambda)_+ \big|_{\overline{B}_r^{C_0^1(\overline{Z})}(x_+)} ,$$

where $\overline{B}_r^{C_0^1(\overline{Z})}(x_+) = \{x \in C_0^1(\overline{Z}) ; \|x - x_+\|_{C_0^1(\overline{Z})} \leq r\}$. This implies that x_+ is a local $C_0^1(\overline{Z})$-minimizer of $(\widehat{\varphi}_\lambda)$. From [13], it follows that x_+ is a local $W_0^{1,p}(Z)$-minimizer of $\widehat{\varphi}_\lambda$.

Similarly, working with $(\widehat{\varphi}_\lambda)_-$ on $\widehat{T} = [v_-, 0]$, we conclude that $v_- \in -\text{int} \, C_+$ is a local minimizer of $\widehat{\varphi}_\lambda$. This proves Claim 2.

Using Claim 2 as in [1, the proof of Proposition 29] (see also the proof of Theorem 2.9), we can find small $r > 0$ such that

$$\widehat{\varphi}_\lambda(v_-) < \inf [\widehat{\varphi}_\lambda(x) ; \|x - v_-\| = r]$$

and

$$\widehat{\varphi}_\lambda(x_+) < \inf [\widehat{\varphi}_\lambda(x) ; \|x - x_+\| = r] .$$

Without loss of generality, we may assume that $\widehat{\varphi}_\lambda(v_-) \leq \widehat{\varphi}_\lambda(x_+)$. If we consider the sets $\widehat{T}_0 = \{v_-, x_+\}$ and $\widehat{T} = [v_-, x_+]$, and define

$$D = \partial B_r(x_+) = \{x \in W_0^{1,p}(Z) ; \|x - x_+\| = r\} ,$$

then we can easily check that the pair $\{\widehat{T}_0, \widehat{T}\}$ is linking with D in $W_0^{1,p}(Z)$ (see also [14, p. 642]). Moreover, the coercive functional $\widehat{\varphi}_\lambda$ satisfies the PS-condition. So, we may apply the linking theorem (see again [14, p. 644]) and obtain a critical point $y_0 \in W_0^{1,p}(Z)$ of $\widehat{\varphi}_\lambda$ of mountain pass type, which is different from v_- and x_+. From Claim 1, we know that $y_0 \in \widehat{T}$. So, we have

$$A(y_0) = \lambda |y_0|^{q-2} y_0 + N(y_0) ,$$

and hence

$$-\Delta_p y_0(z) = \lambda |y_0(z)|^{q-2} y_0(z) + f(z, y_0(z)) \text{ a.e. on } Z, \, y_0|_{\partial Z} = 0 .$$

Therefore, by the nonlinear regularity theory, we have $y_0 \in C_0^1(\overline{Z})$ and of course it solves problem (1).

Note that $\widehat{F} \geq 0$ (see the sign condition $\mathbf{H}(f)_1$(vi)). Hence, for $x \in W_0^{1,p}(Z)$, we have

$$\widehat{\varphi}_\lambda(x) \leq \frac{1}{p} \|Dx\|_p^p - \frac{\lambda}{q} \|\widehat{\tau}(x)\|_q^q .$$

Since $q < p$, it follows that the origin is not a critical point of the mountain pass type (see [21, p. 143]). Therefore, $y_0 \neq 0$ and hence, $y_0 \in C_0^1(\overline{Z})$ is a nodal solution. $\square$

4. p-Superlinear perturbations.   In a similar way, we can treat the case when the nonlinear perturbation $f(z, x)$ is p-superlinear near infinity. So, the new hypotheses on f are the following.

H$(f)_2$ $f : Z \times \boldsymbol{R} \to \boldsymbol{R}$ is a function such that $f(z, 0) = 0$ a.e. on Z and
  (i)   for all $x \in \boldsymbol{R}, z \to f(z, x)$ is measurable,
 (ii)   for a.e. $z \in Z, x \to f(z, x)$ is continuous,
(iii)   for a.e. $z \in Z$ and all $x \in \boldsymbol{R}$,

$$|f(z, x)| \leq a(z) + c|x|^{r-1}$$

with some $a \in L^\infty(Z)_+, c > 0$ and $p < r < p^*$,
 (iv)   there exist $\mu > p$ and $M > 0$ such that for a.e. $z \in Z$ and all $|x| \geq M$,

$$0 < \mu F(z, x) \leq f(z, x)x,$$

  (v) $\lim_{x \to 0} f(z, x)\big/(|x|^{p-2}x) = 0$ uniformly on Z,
 (vi) $f(z, x)x \geq 0$ for a.e. $z \in Z$ and all $x \in \boldsymbol{R}$ (sign condition).

The proofs are similar to those of the previous case (i.e. of a p-linear perturbation). In fact, in some occasions, the proofs are even simpler. So, they are omitted. We simply state the main theorem, summarizing the situation in the case of a p-superlinear perturbation $f(z, x)$.

THEOREM 4.1.   *If hypotheses* **H**$(f)_2$ *hold, then there exist* $\widehat{\lambda}_+, \widehat{\lambda}_- > 0$ *such that*
  (a)   *for* $\lambda = \widehat{\lambda}_+$ (*resp.* $\lambda = \widehat{\lambda}_-$)*, problem* (1) *has a solution* x *in* $\operatorname{int} C_+$ (*resp.* v *in* $-\operatorname{int} C_+$)*,*
  (b)   *for* $\lambda \in (0, \widehat{\lambda}_+)$ (*resp.* $\lambda \in (0, \widehat{\lambda}_-)$)*, problem* (1) *has at least two solutions*: $x_0, \widehat{x} \in \operatorname{int} C_+$ *with* $x_0 < \widehat{x}$ (*resp.* $v_0, \widehat{v} \in -\operatorname{int} C_+$ *with* $\widehat{v} < v_0$)*,*
  (c)   *for* $\lambda \in (0, \widehat{\lambda}_0)$ *with* $\widehat{\lambda}_0 = \min\{\widehat{\lambda}_+, \widehat{\lambda}_-\}$*, problem* (1) *has at least five solutions*: $x_0, \widehat{x} \in \operatorname{int} C_+$ *with* $x_0 < \widehat{x}, v_0, \widehat{v} \in -\operatorname{int} C_+$ *with* $\widehat{v} < v_0$*, and a fifth solution* $y_0 \in C_0^1(\overline{Z})$ *which is nodal.*

REMARK 4.2.   If $f(z, x) = f(x) = |x|^{\vartheta-2}x$ with $p < \vartheta < p^*$, then part (b) of Theorem 4.1 above recovers the result of [13] (see also [3]) where $Z = B_R = \{z \in \boldsymbol{R}^N; \|z\| < R\}$. In fact, even in this special case, our result is more general since it compares the two nontrivial positive solutions. Recently, there have been some works on the existence of nodal (sign-changing) solutions for certain p-Laplacian equations.

We mention the works of Bartsch and Liu [5], Carl and Perera [8], Zhang and Li [24] and Zhang, Chen and Li [23]. In [5], $\lambda = 0$ and the function $f(z, \cdot)$ is p-superlinear near infinity. They obtained three solutions (one positive, one negative and the third is a nodal) under hypotheses which exclude the presence of a concave term near zero. In [8], $\lambda = 0$, the nonlinearity $f(z, x)$ is p-linear near zero and near infinity, and the quotient $f(z, x)/(|x|^{p-2}x)$ admits finite limits as $x \to 0^\pm$ and $x \to \pm\infty$. Assuming the existence of super- and sub-solutions for their problem, they proved the existence of three solutions (one positive, one

negative and the third is nodal). In [24] and [23], again $\lambda = 0$ and the nonlinearity $f(z, x)$ is independent of z, locally Lipschitz in x and p-linear near zero and near infinity. Moreover, the quotient $f(x)\big/(|x|^{p-2}x)$ has finite limits as $x \to 0^{\pm}$ and as $x \to \pm\infty$, and this is crucial in their approach. In addition, [24] has treated the case $N < p$ (low dimensional problems). They produced three solutions (one positive, one negative and the third one is nodal). So, none of the aforementioned works can treat terms which are concave near the origin, and they do not produce five nontrivial smooth solutions with precise sign information. Finally we mention the recent work of de Paiva [7], where Morse Theory is used to obtain two nontrivial solutions but of unspecified sign.

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