Q-FACTORIAL GORENSTEIN TORIC FANO VARIETIES WITH LARGE PICARD NUMBER

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Abstract. In dimension d, Q-factorial Gorenstein toric Fano varieties with Picard number ρ_X correspond to simplicial reflexive polytopes with $\rho_X + d$ vertices. Casagrande showed that any d-dimensional simplicial reflexive polytope has at most $3d$ and $3d - 1$ vertices if d is even and odd, respectively. Moreover, for d even there is up to unimodular equivalence only one such polytope with $3d$ vertices, corresponding to the product of $d/2$ copies of a del Pezzo surface of degree six. In this paper we completely classify all d-dimensional simplicial reflexive polytopes having $3d - 1$ vertices, corresponding to d-dimensional Q-factorial Gorenstein toric Fano varieties with Picard number $2d - 1$. For d even, there exist three such varieties, with two being singular, while for $d > 1$ odd there exist precisely two, both being nonsingular toric fiber bundles over the projective line. This generalizes recent work of the second author.

1. Introduction. The goal of this paper is to finish the classification of simplicial reflexive polytopes with the maximal number of vertices, pursued in [Nil05, Cas06, Oeb08]. Before stating the main convex-geometric result Theorem 1.2, we recall necessary notions. The algebro-geometric version of Theorem 1.2 is given in Corollary 1.3.

1.1. **Lattice polytopes.** A *polytope* is the convex hull of finitely many points in a vector space. Given a lattice $N \cong \mathbf{Z}^d$, a polytope $P \subseteq N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}^d$ is called *lattice polytope*, if all vertices of P are lattice points. We denote the set of vertices of P by $\mathcal{V}(P)$. In other words, a lattice polytope is the convex hull of finitely many lattice points. We say two lattice polytopes are *isomorphic* or *unimodularly equivalent*, if there is a lattice automorphism mapping one vertex set onto the other. In what follows we always assume that P is a lattice polytope of full dimension d that contains the origin in its interior. In this case we can define the *dual polytope* P^*. For this, let us denote by $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ the dual lattice of N and by $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$ the dual vector space of $N_{\mathbf{R}}$. Then

$$P^* := \{x \in M_{\mathbf{R}} \,;\, \langle x, y \rangle \leq 1 \text{ for all } y \in P\},$$

is also a d-dimensional polytope containing the origin in its interior, however in general it is not a lattice polytope.

1.2. **Reflexive polytopes.** A d-dimensional lattice polytope $P \subseteq N_{\mathbf{R}}$ with the origin in its interior is called a *reflexive polytope* if P^* is also a lattice polytope. This definition was given by Batyrev [Bat94] in the context of mirror symmetry. It is known that there is only a finite number of isomorphism classes of reflexive polytopes in fixed dimension d, and

complete classification results exist for $d \leq 4$ (see [KS97, KS98, KS00]). The polytope P is called *simplicial*, if each facet (i.e., $(d-1)$-dimensional face) is a simplex. The most interesting case of a simplicial reflexive polytope is given by a lattice polytope containing the origin in its interior, where the vertices of each facet form a lattice basis. We call such a polytope a *smooth Fano polytope*. These special reflexive polytopes were studied quite intensely, and by now we have complete lists for $d \leq 8$ (see [Oeb07]).

1.3. Low dimensions. Let us look at simplicial reflexive polytopes with many vertices in low dimensions d. For $d = 1$ there is only one reflexive polytope, namely $[-1, 1] \subseteq \mathbf{R}$ (with respect to the lattice $\mathbf{Z}$). For $d = 2$ there are 16 isomorphism classes of reflexive polytopes (all necessarily simplicial; see [Nil05, Prop. 2.1] for a list and references). Only three of these (called $\tilde{V}_2$, E_1, E_2) have 5 vertices, and precisely one (called V_2) has 6 vertices (see Figure 1). $\tilde{V}_2$, V_2 are smooth Fano polytopes, while E_1, E_2 are not.

For $d = 3$ there are 4319 isomorphism classes of reflexive polytopes, of these are 194 simplicial. There are up to isomorphisms only two three-dimensional simplicial reflexive polytopes having the maximal number of 8 vertices (see Figure 2). Both are smooth Fano polytopes that are bipyramids over a hexagon, we denote them by Q_3 and Q'_3. While Q_3 is centrally symmetric, the two apexes v, v' of Q'_3 add up to a vertex w of the hexagon, i.e., $v + v' = w$.

1.4. The main theorem. To describe the general case, let us say a reflexive polytope $P \subseteq N_{\mathbf{R}}$ *splits* into $P_1, \ldots, P_n$ for $n \geq 2$, if P is the convex hull of lattice polytopes $P_1, \ldots, P_n$, and $N = N_1 \oplus_{\mathbf{Z}} \cdots \oplus_{\mathbf{Z}} N_n$, $P_1 \subseteq (N_1)_{\mathbf{R}}, \ldots, P_n \subseteq (N_n)_{\mathbf{R}}$. In this case, P_k is a reflexive polytope with respect to N_k for $k = 1, \ldots, n$. For instance, Q_3 splits into $[-1, 1]$ and V_2.

The following long-standing conjecture on the maximal number of vertices was finally proven by Casagrande [Cas06] in 2004 (here $| \cdot |$ denotes the cardinality):
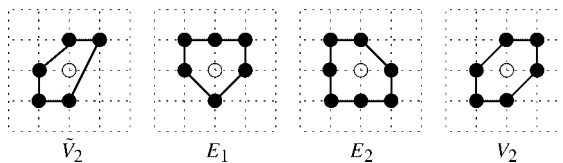

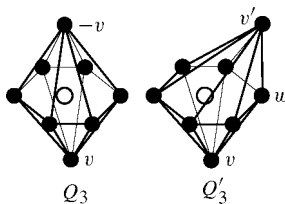FIGURE 1. Reflexive polytopes of dimension $d = 2$ with 5 or 6 vertices.

FIGURE 2. Simplicial reflexive polytopes of dimension $d = 3$ with 8 vertices.

THEOREM 1.1 (Casagrande 04). *Let $P \subseteq N_{\mathbf{R}}$ be a simplicial reflexive polytope of dimension d. Then*

$$|\mathcal{V}(P)| \leq \begin{cases} 3d & d \text{ even}, \\ 3d - 1 & d \text{ odd}. \end{cases}$$

If d is even and $|\mathcal{V}(P)| = 3d$, then P splits into $d/2$-copies of V_2.

Note that there are by now very short proofs of these upper bounds, cf. [KN09, Oeb08]. See also Subsection 2.3.

Here is our main result, the classification of simplicial reflexive polytopes of dimension d with $3d - 1$ vertices.

THEOREM 1.2. *Let $P \subseteq N_{\mathbf{R}}$ be a simplicial reflexive polytope of dimension $d \geq 3$ with $3d - 1$ vertices.*
If d is even, then P splits into $\tilde{V}_2$ (or E_1, or E_2) and $(d - 2)/2$ copies of V_2.
If d is odd, then P splits into Q_3 (or Q_3') and $(d - 3)/2$ copies of V_2.

This generalizes a recent result of the second author in [Oeb08], where this theorem was proven under the assumption that any lattice point on the boundary of P is a vertex (for instance, if P is a smooth Fano polytope). In this case E_1 and E_2 cannot occur, so there is only one type in Theorem 1.2 for d even.

1.5. Algebro-geometric interpretation. The algebro-geometric objects corresponding to reflexive polytopes P are *Gorenstein toric Fano varieties* X (i.e., normal complex projective varieties, where the anticanonical divisor is Cartier and ample). The relation is given via the toric dictionary (see [Ful93]): X is the toric variety associated to the fan spanned by the faces of P. For the *Picard number* ρ_X of X we have the equation $\rho_X = |\mathcal{V}(P)| - d$. For instance, V_2 corresponds to the del Pezzo surface S_6 with $\rho_{S_6} = 4$, which is $\mathbf{P}^2$ blown-up at three torus-invariant fixpoints. In the same way, $\tilde{V}_2$ corresponds to the del Pezzo surface S_7 with $\rho_{S_7} = 3$. Here, P is simplicial if and only if X is $\mathbf{Q}$-*factorial* (i.e., any Weil divisor has some multiple which is Cartier). Moreover, P is a smooth Fano polytope if and only if X is a *toric Fano manifold* (i.e., nonsingular). Since the splitting of reflexive polytopes corresponds to products of toric Fano varieties, we can reformulate Casagrande's result by saying that the Picard number of a $\mathbf{Q}$-factorial Gorenstein toric Fano variety X is at most $2d$, with equality only for d even and $X \cong (S_6)^{d/2}$. Here is the algebro-geometric version of our main result:

COROLLARY 1.3. *Let X be a $\mathbf{Q}$-factorial Gorenstein toric Fano variety of dimension $d \geq 3$ and with Picard number $\rho_X = 2d - 1$.*
If d is even, then X is a product of $(S_6)^{(d-2)/2}$ and a (possibly singular) del Pezzo surface S with $\rho_S = 3$, where there are three possibilities for S up to isomorphisms, only one of these, namely S_7, is nonsingular.
If d is odd, then X is a product of $(S_6)^{(d-3)/2}$ and a toric Fano 3-fold Y with $\rho_Y = 5$, where there are two possibilities for Y up to isomorphisms, namely $S_6 \times \mathbf{P}^1$ or a unique toric S_6-fiber bundle over $\mathbf{P}^1$.

1.6.   Organization of this article.   In the second section we recall preliminary results, namely properties of lattice points of reflexive polytopes, results about neighboring facets, and the notion of a special facet.

In the third section, we start the proof of the main result, which is then separated into Parts I–III, given in Sections 4–6. The proof is a combination of two different ideas. The first idea of the proof is the same that was successfully used in [Cas06, KN09, Oeb08], that is, having a large number of vertices implies that there is a special facet from which nearly all vertices have integral distance two or less. Then Parts I and II can be treated using the methods developed and applied by the second author in [Oeb08]. For Part III, we use as a second idea the essential property of reflexive polytopes, namely their duality, to get restrictions on the outer normals of their facets. Then we can apply the strong properties of pairs of vertices of simplicial reflexive polytopes proven by the first author in [Nil05].

2.   Preliminary results.   In this section we present basic results on simplicial reflexive polytopes.

2.1.   Lattice points in reflexive polytopes.   First let us recall an elementary property of reflexive polytopes (see [Bat94] or [Nil05, Prop. 1.12, Lemma 1.17]).

LEMMA 2.1.   *A reflexive polytope contains no interior lattice points different from the origin. In dimension two this property is equivalent to the reflexivity of the polytope.*

The following notation was introduced in [Nil05].

DEFINITION 2.2.   Let P be a polytope. We denote by ∂P its boundary.

For $x, y \in \partial P$, we write $x \sim y$, if x and y are contained in a common face (or equivalently, facet) of P.

Using this relation, we can describe a partial addition of lattice points in reflexive polytopes (see [Nil05, Prop. 4.1]).

LEMMA 2.3.   *Let $P \subseteq N_{\mathbf{R}}$ be a reflexive polytope, and $v, w \in \partial P \cap N$. Then $v + w \neq 0$ and $v \not\sim w$ if and only if $v + w \in \partial P \cap N$.*

Finally, in the simplicial case there is a strong restriction on pairs of vertices (see [Nil05, Lemma 5.11]).

LEMMA 2.4.   *Let $P \subseteq N_{\mathbf{R}}$ be a simplicial reflexive polytope. Let $v, w, w' \in \mathcal{V}(P)$ be pairwise different such that $w \neq -v \neq w'$, $v \not\sim w$ and $v \not\sim w'$. Then $P(v, w, w') := P \cap \mathrm{lin}(v, w, w')$ is a two-dimensional reflexive polytope with at least five vertices.*

2.2.   Neighboring facets.   *Throughout, let $P \subseteq N_{\mathbf{R}}$ be a simplicial reflexive polytope of dimension $d \geq 2$.*

Let us first fix our notation.

DEFINITION 2.5.   Let F be a facet of P.

• The vertices $\mathcal{V}(F)$ form a basis of $N_{\mathbf{R}}$. We denote by $\{u_F^v ; v \in \mathcal{V}(F)\}$ the *dual basis* in $M_{\mathbf{R}}$, i.e., $\langle u_F^v, w \rangle = \delta_{v,w}$ for $v, w \in \mathcal{V}(F)$.

• Let $v \in \mathcal{V}(F)$ be a vertex of F. Then there is a unique facet of P that contains all vertices of F except v. We call this facet the *neighboring facet* $N(F, v)$. The unique vertex of $N(F, v)$ that is not contained in F is called the *neighboring vertex* $n(F, v)$.

• There is a unique *outer normal* $u_F \in M_{\mathbf{R}}$ defined by $\langle u_F, F \rangle = 1$. The dual polytope P^* has as vertices precisely the outer normals of the facets of P. Since P is reflexive, the outer normal u_F is a lattice point. Hence, the lattice N is "sliced" into lattice hyperplanes

$$H(F, i) := \{ x \in N \; ; \; \langle u_F, x \rangle = i \} \, , \; i \in \mathbf{Z} \, .$$

Let us abbreviate $H_P(F, i) := H(F, i) \cap \mathcal{V}(P)$.

We are going to collect restrictions on neighboring vertices and facets. The first result is contained in [Oeb08, Lemmas 1 and 2]. The point (3) of the following lemma follows from (1).

LEMMA 2.6. *Let F be a facet of P and $v \in \mathcal{V}(F)$. Let F' be the neighboring facet $N(F, v)$ and v' the neighboring vertex $n(F, v)$. Then we have the following.*

(1) *For any point $x \in N_{\mathbf{R}}$,*

$$\langle u_{F'}, x \rangle = \langle u_F, x \rangle + (\langle u_{F'}, v \rangle - 1) \langle u_F^v, x \rangle \, ,$$

where $\langle u_{F'}, v \rangle - 1 \leq -1$.

(2) *For any $x \in P$,*

$$\langle u_F, x \rangle - 1 \leq \langle u_F^v, x \rangle \, .$$

In case of equality, x is on the facet $N(F, v)$.

(3) *If $n(F, v) \in H(F, 0)$ and $\mathcal{V}(F)$ is a lattice basis, then $\langle u_F^v, n(F, v) \rangle = -1$.*

Compare the next two results [Oeb08, Lemmas 3 and 4] with Remark 5(2) in Section 2.3 of [Deb03] and [Nil05, Lemma 5.5].

LEMMA 2.7. *Let F be a facet, and x a lattice point in $\partial P \cap H(F, 0)$. Then x lies on a neighboring facet of F. In particular, let $v \in H_P(F, 0)$. Then v is a neighboring vertex of F. Hence, there are at most d vertices of P in $H(F, 0)$. Moreover, it holds:*

(1) *For every $w \in \mathcal{V}(F)$, v is equal to $n(F, w)$ if and only if $\langle u_F^w, v \rangle < 0$. In particular, for every $w \in \mathcal{V}(F)$ there is at most one vertex $v \in H(F, 0)$ with $\langle u_F^w, v \rangle < 0$.*

(2) *If v is contained in precisely one neighboring facet $N(F, w)$ of F, then $v \not\succ w$, so $v + w \in F \cap N$.*

LEMMA 2.8. *Let F be a facet of P. Suppose there are at least $d - 1$ vertices $e_1, \ldots, e_{d-1}$ in $\mathcal{V}(F)$, such that $n(F, e_i) \in H(F, 0)$ and $\langle u_F^{e_i}, n(F, e_i) \rangle = -1$ for every $1 \leq i \leq d-1$. Then $\mathcal{V}(F)$ is a basis of the lattice N.*

The following lemma is due to the second author.

LEMMA 2.9. *Assume that, for any facet F of P, we have*

$$|\{ n(F, v) \in H(F, 0) \; ; \; v \in \mathcal{V}(F) \}| \geq d - 1 \, .$$

Then there exists a facet G such that $\mathcal{V}(G)$ is a $\mathbf{Z}$-basis of N.

PROOF.   By Lemma 2.8, we are done if there exists a facet G such that the set

$$\{v \in \mathcal{V}(G) \; ; \; n(G, v) \in H(G, 0) \text{ and } u_G^v(n(G, v)) = -1\}$$

is of size at least $d - 1$. So we suppose that no such facet exists.

Let $e_1, \ldots, e_d$ be a fixed basis of the lattice N, and write every vertex of P in this basis. For every facet F of P, we let $\det A_F$ denote the determinant of the matrix

$$A_F := \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix},$$

where $\mathcal{V}(F) = \{v_1, \ldots, v_d\}$. As $\det A_F$ is determined up to a sign, the number $r_F := |\det A_F|$ is well-defined.

Now, let F_0 be an arbitrary facet of P. By our assumptions, there must be at least one vertex v of F_0, such that $v' = n(F_0, v) \in H(F_0, 0)$ but $\langle u_{F_0}^v, v' \rangle \neq -1$. Then $-1 < \langle u_{F_0}^v, v' \rangle < 0$ by Lemma 2.6(2) and Lemma 2.7(1). Let F_1 denote the neighboring facet $N(F_0, v)$. Then $r_{F_0} > r_{F_1}$.

We can proceed in this way to produce an infinite sequence of facets

$$F_0, F_1, F_2, \ldots \quad \text{where} \quad r_{F_0} > r_{F_1} > r_{F_2} > \cdots .$$

However, there are only finitely many facets of P, a contradiction.   □

We also need [Oeb08, Lemma 5].

LEMMA 2.10. *Let F be a facet of P. Let $v_1, v_2 \in \mathcal{V}(F)$, $v_1 \neq v_2$, and set $y_1 = n(F, v_1)$ and $y_2 = n(F, v_2)$. Suppose $y_1 \neq y_2$, $y_1, y_2 \in H(F, 0)$ and $\langle u_F^{v_1}, y_1 \rangle = \langle u_F^{v_2}, y_2 \rangle = -1$. Then there is no vertex $x \in \mathcal{V}(P)$ in $H(F, -1)$ with $\langle u_F^{v_1}, x \rangle = \langle u_F^{v_2}, x \rangle = -1$.*

Finally, for convenience of the reader we cite [Oeb08, Lemmas 6 and 7] with a weaker assumption. However, one can check that the proofs are precisely the same, so they are omitted.

LEMMA 2.11. *Let F be a facet of P such that any lattice point in F is a vertex (for instance, $\mathcal{V}(F)$ is a lattice basis). If $|H_P(F, 0)| = d$, then the following holds:*

(1) $H_P(F, 0) = \{-y + z_y; y \in \mathcal{V}(F)\}$, *where $z_y \in \mathcal{V}(F)$ is determined by y. Moreover, $\mathcal{V}(F)$ is a lattice basis.*

(2)   *If $x \in H_P(F, -1)$, then $-x \in \mathcal{V}(F)$.*

2.3.   Special facets.   Here we recall the crucial notion of special facets introduced by the second author in [Oeb08], which in particular yields a short proof of the upper bound in Casagrande's theorem.

The goal is to show that knowing the number of vertices of a d-dimensional simplicial reflexive polytope P yields restrictions on the distribution of the vertices along the hyperplanes parallel to a special facet. For this, we define

$$v_P := \sum_{v \in \mathcal{V}(P)} v .$$

DEFINITION 2.12. A facet F of P with $v_P \in \mathbf{R}_{\geq 0} F$ is called *special facet*.

Obviously, P has a special facet, say F. Let us first deduce the following observation from the simpliciality of P and Lemma 2.7:

$$(1) \qquad\qquad |H_P(F, 1)| = d, \quad |H_P(F, 0)| \leq d.$$

Now, since F is a special facet, we get

$$(2) \qquad\qquad 0 \leq \langle u_F, v_P \rangle = \sum_{v \in \mathcal{V}(P)} \langle u_F, v \rangle = d + \sum_{i \leq -1} i \, |H_P(F, i)|.$$

In particular, there are at most d vertices lying in the union of hyperplanes $H(F, i)$ with $i \leq -1$. This yields together with Equation (1)

$$(3) \qquad\qquad |\mathcal{V}(P)| = d + |H_P(F, 0)| + \sum_{i \leq -1} |H_P(F, i)| \leq 3d,$$

which is the sharp upper bound in Theorem 1.1.

3. Outline of the proof of the main theorem. *For the remaining sections of this paper, let $P \subseteq N_{\mathbf{R}}$ be a simplicial reflexive polytope of dimension $d \geq 3$ having $3d - 1$ vertices.*

Let F be a special facet of P. Taking Equations (1)–(3) into account, we see that there are precisely three cases how the $3d - 1$ vertices of P can be distributed in the hyperplanes $H(F, i)$:

	Case A	Case B	Case C
$|H_P(F, 1)|$	d	d	d
$|H_P(F, 0)|$	d	$d - 1$	d
$|H_P(F, -1)|$	$d - 2$	d	$d - 1$
$|H_P(F, -2)|$	1	0	0

Now, let us look at the lattice point v_P, which is the sum of the vertices of P, in the three cases A, B, C:

	Case A	Case B	Case C
$\langle u_F, v_P \rangle$	0	0	1

Hence, the definition of a special facet implies: *In Cases* A *and* B *the sum of all the vertices of P equals the origin, while in Case* C *the sum is a lattice point on the facet F.*

Now, the proof falls into Parts I–III (Sections 4–6) depending on whether $v_P = 0$, v_P is a vertex, or otherwise. Then the main result, Theorem 1.2, follows directly from combining Propositions 4.1, 5.1 and 6.1.

4. Part I: $\nu_P = 0$. Here, we prove the following result:

PROPOSITION 4.1. *Let $\nu_P = 0$. Then either d is even and P splits into $(d-2)/2$ copies of V_2 and a single copy of the polytope E_2, or d is odd and P splits into $(d-3)/2$ copies of V_2 and a single copy of the polytope Q_3.*

PROOF. Since $\nu_P = 0$, every facet of P is special. Thus, for any facet F of P, we are in Cases A or B described above. In particular, there are at least $d-1$ vertices in $H(F, 0)$, hence by Lemma 2.7 the assumptions of Lemma 2.9 are satisfied, so we find a facet F whose vertex set $\mathcal{V}(F)$ is a lattice basis of N. Let us denote the vertices of $\mathcal{V}(F)$ by $e_1, \ldots, e_d$.

CLAIM. *We may assume we are in Case A.*

PROOF OF CLAIM. Suppose not. Then there are d vertices in $H(F, -1)$.

Let us first consider the case that P contains a centrally symmetric pair of facets. Then from [Nil07, Theorem 0.1] one easily derives that either d is even and P splits into $(d-2)/2$ copies of V_2 and a single copy of the polytope $\tilde{V}_2$, or d is odd and P splits into $(d-3)/2$ copies of V_2 and a single copy of the polytope Q_3. In the first case we have a contradiction to $\nu_P = 0$, while the second case is as desired.

Hence, we may assume that at least one of the vertices in $H(F, -1)$ is not equal to $-e_i$ for $i \in \{1, \ldots, d\}$, so this vertex has at least one positive e_j-coordinate for some j. Say, $w \in H_P(F, -1)$ and $\langle u_F^{e_1}, w \rangle > 0$. Then $\langle u_{N(F, e_1)}, w \rangle < -1$ by Lemma 2.6(1), which implies that the vertices of P are distributed in hyperplanes $H(N(F, e_1), \cdot)$ as in Case A. In particular, $\langle u_{N(F, e_1)}, w \rangle = -2$. Now, it remains to show that $\mathcal{V}(N(F, e_1))$ is a lattice basis.

If $n(F, e_1) \in H(F, 0)$, then $\langle u_F^{e_1}, n(F, e_1) \rangle = -1$ by Lemma 2.6(3), hence $\mathcal{V}(N(F, e_1))$ is a lattice basis, as desired. So suppose $n(F, e_1) \notin H(F, 0)$, thus $n(F, e_1) \in H_P(F, -1)$. Since $|H_P(F, 0)| \geq d-1$, we have $n(F, e_2), \ldots, n(F, e_d) \in H(F, 0)$ and they are all distinct. Furthermore, Lemma 2.6(3) yields

$$\langle u_F^{e_2}, n(F, e_2) \rangle = \cdots = \langle u_F^{e_d}, n(F, e_d) \rangle = -1 \,.$$

By Lemma 2.6(2) we get $\langle u_F^{e_i}, w \rangle \geq -2$, and moreover, if $\langle u_F^{e_i}, w \rangle = -2$ for some $i > 1$, then $w = n(F, e_i) \in H(F, 0)$, which is not possible. So $\langle u_F^{e_i}, w \rangle \geq -1$ for $i = 2, \ldots, d$. Now, since

$$\langle u_F^{e_1}, w \rangle > 0 \text{ and } \sum_{i=1}^{d} \langle u_F^{e_i}, w \rangle = -1 \,,$$

there are at least two indices $i \neq j$ in $\{2, \ldots, d\}$ such that $\langle u_F^{e_i}, w \rangle = \langle u_F^{e_j}, w \rangle = -1$. This is a contradiction to Lemma 2.10. □

So we may safely assume that $\mathcal{V}(F)$ is a lattice basis and there are d vertices of P in $H(F, 0)$, $d-2$ in $H(F, -1)$ and a single one, say v, in $H(F, -2)$. If $\langle u_F^{e_i}, v \rangle > 0$ for some i, then $\langle u_{N(F, e_i)}, v \rangle < -2$ by Lemma 2.6(1), which cannot happen. Furthermore, v cannot be equal to $-2e_i$ for some i. So (up to renumeration) $v = -e_1 - e_2$ since $\langle u_F, v \rangle = -2$. The
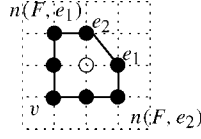
FIGURE 3. $\operatorname{conv}(e_1, e_2, v, n(F, e_1), n(F, e_2))$.

vertices of P in $H(F, 0)$ are by Lemma 2.11(1)

$$n(F, e_1) = -e_1 + e_{i_1}, \quad \ldots, \quad n(F, e_d) = -e_d + e_{i_d},$$

for $\{i_1, \ldots, i_d\} \subseteq \{1, \ldots, d\}$. By Lemma 2.11(2), the vertices in $H(F, -1)$ are

$$-e_{j_1}, \ldots, -e_{j_{d-2}}$$

for $\{j_1, \ldots, j_{d-2}\} \subseteq \{1, \ldots, d\}$. Suppose $i_1 \neq 2$. Then there are two cases:

$i_2 \neq 1$: This case leads to a contradiction. This is proven precisely as in the proof of Case 2 of the main result in [Oeb08] (starting from the line "Let $G = N(F, e_1)$", with $j = i_1$ and $i = i_2$).

$i_2 = 1$: Consider the facet $G = N(F, e_2)$. Since $v = -e_1 - e_2 = -2e_1 + n(F, e_2)$, we see $\langle u_G, v \rangle = -1$ and $\langle u_G^{e_1}, v \rangle = -2$. Then by Lemma 2.6(2), $v = n(G, e_1)$. However, $-e_1 + e_{i_1} \in H_P(G, 0)$ is also equal to $n(G, e_1)$ by Lemma 2.7(1), a contradiction.

So $i_1 = 2$. By symmetry, $i_2 = 1$. Now, as we see from Figure 3, $-e_1$ and $-e_2$ cannot be vertices (here, conv denotes the convex hull).

Hence,

$$H_P(F, -1) = \{-e_3, \ldots, -e_d\}.$$

Let us consider the facets $N(F, e_3), \ldots, N(F, e_d)$. Without loss of generality, it is enough to deal with $N(F, e_3)$. Since $n(F, e_3) = -e_3 + e_{i_3}$, the vertices of $N(F, e_3)$ form a lattice basis. Because we have $v \in H_P(N(F, e_3), -2)$, we are still in Case A. Consequently,

$$H_P(N(F, e_3), -1) = -\mathcal{V}(N(F, e_3)) \setminus \{-e_1, -e_2\} = \{-(-e_3 + e_{i_3}), -e_4, \ldots, -e_d\}.$$

In particular, $e_3 - e_{i_3}$ is also a vertex in $H(F, 0)$, so $e_3 - e_{i_3} = -e_k + e_{i_k}$ for some $k \in \{4, \ldots, d\}$. From this, we conclude that all the vertices of P in $H(F, 0)$ come in centrally symmetric pairs, so d is even and P splits into the claimed polytopes. □

5. Part II: v_P is a vertex of P. Here, we prove the following result:

PROPOSITION 5.1. *Let v_P be a vertex of P. Then either d is even and P splits into $(d-2)/2$ copies of V_2 and a single copy of the polytope $\tilde{V}_2$, or d is odd and P splits into $(d-3)/2$ copies of V_2 and a single copy of the polytope Q_3'.*

PROOF. A facet of P is special if and only if it contains v_P.

CLAIM. *Let F be a special facet. Then $\mathcal{V}(F)$ is a lattice basis.*

PROOF OF CLAIM.   Since we are in Case C, the vertices of P are distributed in hyperplanes $H(F, \cdot)$ as follows: d in $H(F, 1)$, d in $H(F, 0)$ and $d - 1$ in $H(F, -1)$. In particular, $n(F, w) \in H(F, 0)$ for every $w \in \mathcal{V}(F)$.

Consider the facet $N(F, w)$ for some $w \in \mathcal{V}(F)$, $w \neq v_P$. Since $N(F, w)$ is also special, there are d vertices in $H(N(F, w), 0)$. So $w \in H(N(F, w), 0)$ and $n(F, w) \in H(F, 0)$, and it follows from Lemma 2.6(1) that $\langle u_F^w, n(F, w) \rangle = -1$. This holds for all $w \in \mathcal{V}(F)$, $w \neq v_P$, and Lemma 2.8 yields that $\mathcal{V}(F)$ is a lattice basis.   □

Let $v_P = e_1$. Now, the remaining proof follows precisely as in Case 1 of the proof of the main result in [Oeb08] (starting from line "There are $d - 1$ vertices in $H(F, -1)$"). The only difference is that in our situation one refers to points (1) or (2) in Lemma 2.11 instead of refering to Lemmas 6 or 7 in [Oeb08].   □

6. Part III: $v_P \neq 0$ is not a vertex of P.   Here, we prove the following result, which finishes the proof of Theorem 1.2:

PROPOSITION 6.1.   *Let $v_P \neq 0$, and let v_P be not a vertex of P. Then d is even and P splits into $(d - 2)/2$ copies of V_2 and a single copy of the polytope E_1.*

PROOF.   Let F be a special facet of P. As described in Section 3, we are in Case C, and v_P is a lattice point of F but not a vertex.

Let $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$ and $H_P(F, 0) = \{v_1, \ldots, v_d\}$. By Lemma 2.7 we may assume that $v_i = n(F, e_i)$ for $i = 1, \ldots, d$. In particular, Lemma 2.7(2) implies:

FACT 1.   *For $i = 1, \ldots, d$, we have $v_i + e_i \in F$.*

Moreover, since any neighboring vertex of F is in $H(F, 0)$, Lemma 2.7 implies:

FACT 2.   *Any lattice point in $\partial P \cap H(F, 0)$ is a neighboring vertex of F.*

Let $G := \{x \in P \; ; \; \langle u_F, x \rangle = -1\}$. Since G contains $d - 1$ vertices, G is a $(d - 2)$-dimensional face of P. Let $\mathcal{V}(G) = \{b_1, \ldots, b_{d-1}\}$. There exist precisely two facets G_1, G_2 of P containing G. We have $G = G_1 \cap G_2$. Let $w_1 := u_{G_1}$, $w_2 := u_{G_2}$. The next observation is the crucial starting point of our proof.

CLAIM 1.   *By possibly interchanging w_1 and w_2, we have*
(1) $2w_1 + w_2 + 3u = 0$ *or*
(2) $w_1 + w_2 + 2u = 0$.

PROOF OF CLAIM 1.   By duality, w_1, w_2 are vertices of P^* joined by an edge that contains $-u$ in its relative interior. Let $T := \operatorname{conv}(w_1, w_2, u)$. Lemma 2.1 (applied to P^*) implies that T does not contain any lattice points different from the origin in its interior, thus it is a reflexive polygon. Figure 4 lists all reflexive triangles up to isomorphism (see, e.g., [Nil05, Prop. 2.1]).

Since T has a vertex (namely u) such that $-u$ is also a lattice point in T, we see that T cannot be isomorphic to the first or the last triangle in Figure 4. For the remaining three triangles we can check that the vertices of T, namely, w_1, w_2, u, satisfy either relation (1) or relation (2).   □
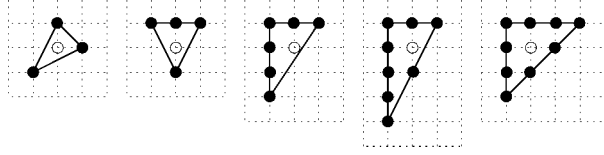
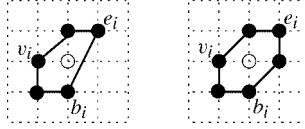FIGURE 4. Reflexive polytopes of dimension $d = 2$ with 3 vertices.

FIGURE 5. Two possibilities for $P(v_i, e_i, b_i)$.

Now, let $x_1 \in \mathcal{V}(G_1)$, $x_1 \notin G$ and $x_2 \in \mathcal{V}(G_2)$, $x_2 \notin G$, i.e., $x_1 = n(G_2, x_2)$ and $x_2 = n(G_1, x_1)$. By using $0 = \langle 2w_1 + w_2 + 3u, v_i \rangle$ in Case (1) of Claim 1, respectively $0 = \langle w_1 + w_2 + 2u, v_i \rangle$ in Case (2), we deduce:

FACT 3. *Let $i \in \{1, \ldots, d\}$ such that $v_i \notin \{x_1, x_2\}$. Then $v_i \in H(G_1, 0) \cap H(G_2, 0)$. In particular, by Lemma 2.7, v_i is a neighboring vertex of G_1, as well as of G_2.*

CLAIM 2. x_1 *and* x_2 *are in* $H(F, 0)$.

PROOF OF CLAIM 2. Assume not. First let us suppose that $x_1 \in F$ and $x_2 \in F$. Then by Fact 3 and Lemma 2.7 we have $H_P(G_1, 0) = \{v_1, \ldots, v_d\}$. Moreover, Lemma 2.7 implies $x_2 = n(G_1, x_1) \in \{v_1, \ldots, v_d\}$, a contradiction.

Since we are not going to distinguish between cases (1) and (2) for the proof of Claim 2, we may assume that x_1 is in $H(F, 0)$ and x_2 is in F. Let us suppose $x_1 = v_1$. Then by Fact 3, $x_2, v_2, \ldots, v_d$ are the d different neighboring vertices of G_1. By Lemma 2.7(2), we may permute $b_2, \ldots, b_d$ so that $v_2 + b_2, \ldots, v_d + b_d$ are in G_1. Now, by Fact 1, Lemma 2.4 implies, for $i \in \{2, \ldots, d\}$, that $P(v_i, e_i, b_i)$ is a reflexive polygon with at least five vertices. Looking at Figure 1 and Fact 2, we find that there are only two possibilities, which are shown in Figure 5.

In particular, $-b_i = v_i + e_i \in F$. Moreover, since, by Fact 3, $v_i \in H(G_2, 0)$, we note $\langle w_2, e_i \rangle = -1$. Hence, $e_i \notin G_2$ for $i = 2, \ldots, d$. Therefore, $x_2 = e_1$. This implies $\langle w_2, e_1 \rangle = \langle w_2, x_2 \rangle = 1$. Since $\langle w_2, -b_i \rangle = -1$, we get $-b_i \in \text{conv}(e_2, \ldots, e_d)$ for $i = 2, \ldots, d$. Hence,

$$-G = -\text{conv}(b_2, \ldots, b_d) \subseteq \text{conv}(e_2, \ldots, e_d).$$

Since, by Figure 5, $-e_2, \ldots, -e_d \in G$, this yields $G = -\text{conv}(e_2, \ldots, e_d)$ and $\{b_2, \ldots, b_d\} = \{-e_2, \ldots, -e_d\}$.

We conclude that, for $i = 2, \ldots, d$, each vertex v_i can be written as $v_i = -b_i - e_i = e_j - e_i$ for some $j \in \{2, \ldots, d\}$ with $j \neq i$. Hence, $\langle u_F^{e_i}, v_i \rangle = -1$ for $i = 2, \ldots, d$. Now, Lemma 2.8 yields that $\{e_1, \ldots, e_d\}$ is a lattice basis. Thus, any lattice point in F is a vertex, in particular, this holds for v_P, a contradiction. So Claim 2 is proven. $\square$

Assume we are in case (1) of Claim 1. Then $0 = \langle 2w_1 + w_2 + 3u, x_2 \rangle = 2\langle w_1, x_2 \rangle + 1$, thus $\langle w_1, x_2 \rangle = -1/2 \notin \mathbf{Z}$, a contradiction. Hence, we are in case (2). We may suppose $x_1 = v_1$ and $x_2 = v_2$. Now, Fact 3 implies

$$H_P(G_1, 1) = \{x_1, b_2, \ldots, b_d\}, \quad H_P(G_2, 1) = \{x_2, b_2, \ldots, b_d\},$$

$$H_P(G_1, 0) \supseteq \{v_3, \ldots, v_d\}, \quad H_P(G_2, 0) \supseteq \{v_3, \ldots, v_d\}.$$

Moreover, by $w_1 + w_2 + 2u = 0$, we get

$$x_2 \in H_P(G_1, -1), \quad x_1 \in H_P(G_2, -1).$$

In particular, since x_2 is a neighboring vertex of G_1 outside $H_P(G_1, 0)$, Lemma 2.7 implies the following observation (the same argument holds for G_2):

FACT 4. $|H_P(G_1, 0)| \leq d - 1$ and $|H_P(G_2, 0)| \leq d - 1$.

It is our next goal to determine on which slices with respect to G_1 and G_2 the vertices of F lie. For this, we need a preliminary result.

CLAIM 3. $\{y \in P \,;\, \langle w_1, y \rangle = -1\}$ is not a face of P. The same statement also holds for w_2.

PROOF OF CLAIM 3.   Assume the claim is wrong for w_1. Hence, $|H_P(G_1, -1)| \leq d$, and Fact 4 yields

$$(|H_P(G_1, 1)|, |H_P(G_1, 0)|, |H_P(G_1, -1)|) = (d, d - 1, d).$$

In this case $\langle w_1, v_P \rangle = 0$, thus $v_P \in F \cap H(G_1, 0)$. Now, since $v_3, \ldots, v_d \in H_P(G_1, 0)$, there exists $i \in \{1, \ldots, d\}$ such that $e_i \in H_P(G_1, 0)$, while $e_j \in H_P(G_1, -1)$ for $j \neq i$. Hence, $v_P \in F \cap H(G_1, 0)$ implies $v_P = e_i \in \mathcal{V}(P)$, a contradiction. $\square$

Since $w_1 + w_2 + 2u = 0$, we have $\langle w_1 + w_2, e_i \rangle = -2$ for $i = 1, \ldots, d$. Therefore, Claim 3 implies the existence of

$$r \in \{1, \ldots, d\} \,:\, \langle w_1, e_r \rangle = -2, \quad \langle w_2, e_r \rangle = 0,$$

$$s \in \{1, \ldots, d\} \,:\, \langle w_1, e_s \rangle = 0, \quad \langle w_2, e_s \rangle = -2.$$

Moreover, since $v_2 = x_2 = n(G_1, x_1)$, we get, by Fact 4 and Lemma 2.7,

$$\{e_s, v_3, \ldots, v_d\} = \{n(G_1, b_2), \ldots, n(G_1, b_d)\}.$$

We permute $b_2, \ldots, b_d$, and assume that $e_s = n(G_1, b_2)$ and $v_i = n(G_1, b_i)$ for $i = 3, \ldots, d$; moreover, by Lemma 2.7(2), we have $v_i \not\sim b_i$. Hence by Fact 1 we may apply, for $i = 3, \ldots, d$, Lemma 2.4 to v_i, e_i, b_i, and deduce as in the proof of Claim 2 the following result:

FACT 5. *For $i = 3, \ldots, d$, the polygon $P(v_i, e_i, b_i)$ looks as in Figure 5.*

In particular, $\langle w_1, e_i \rangle = \langle w_2, e_i \rangle = -1$ for $i = 3, \ldots, d$. Thus, $\{r, s\} = \{1, 2\}$. Since by Fact 1, $e_1 \not\sim v_1 = x_1$, however $e_s = n(G_1, b_2) \sim x_1 \in N(G_1, b_2)$, we get $s \neq 1$. Hence, $r = 1, s = 2$. Let us sum up what we just proved:

$$H_P(G_1, 0) = \{e_2, v_3, \ldots, v_d\}, \quad H_P(G_2, 0) = \{e_1, v_3, \ldots, v_d\}.$$

$$H_P(G_1, -1) = \{x_2, e_3, \ldots, e_d\}, \quad H_P(G_2, -1) = \{x_1, e_3, \ldots, e_d\},$$

$$H_P(G_1, -2) = \{e_1\}, \quad H_P(G_2, -2) = \{e_2\}.$$

Now, since $e_2 = n(G_1, b_2)$, we have by Lemma 2.7(2) that $e_2 + b_2 \in G_1 \cap H(F, 0)$. Thus, Fact 2 implies $e_2 + b_2 = x_1 = v_1 \in \mathcal{V}(P)$. By Fact 1, we may again apply Lemma 2.4 to e_2, v_2, b_2 to deduce that $P(v_2, e_2, b_2)$ is a reflexive polygon that has to look like one of the two reflexive polygons in Figure 6 (use $v_2 \sim b_2$).

CLAIM 4. *In Figure 6, only the right possibility occurs, moreover, $z = e_1$. In particular, $P(v_2, e_2, b_2) \cong E_1$.*

PROOF OF CLAIM 4. Assume $P(v_2, e_2, b_2)$ is given by the left reflexive polygon in Figure 6. Since $e_1 \in H(G_2, 0)$, Lemma 2.7(2) implies that $e_1 = n(G_2, b_j)$ for some $j \in \{2, \ldots, d\}$, even more, $e_1 \not\sim b_j$ and $e_1 \neq -b_j$. If $j \in \{3, \ldots, d\}$, then by Lemma 2.4, $e_1 \in P(b_j, v_j, e_1) = P(v_j, e_j, b_j)$, which implies by Figure 5 that $e_1 = -b_j$, a contradiction. Hence $j = 2$. Therefore, Lemma 2.4 implies $e_1 \in P(b_2, e_2, e_1) = P(v_2, e_2, b_2)$. Figure 6 yields $e_1 = z = -b_2$, a contradiction.

Finally, note that, in the right reflexive polygon, $z \in N(G_1, -2)$, and therefore $z = e_1$. □

By Fact 5 and Figure 5, we have $v_i + e_i = -b_i \in F$ for $i = 3, \ldots, d$.

CLAIM 5. *Let $i \in \{3, \ldots, d\}$. Then $-b_i \in \mathrm{conv}(e_3, \ldots, e_d)$.*

PROOF OF CLAIM 5. Assume not. For $j \in \{1, \ldots, d\}$ let $F_j := N(F, e_j)$ and $u_j := u_{F_j}$. For $j = 3, \ldots, d$, we deduce from Figure 5 that $\pm e_j \in P$ and, of course, $\pm e_j \notin F_j$. This implies

(4) $\langle u_j, e_j \rangle = 0 \quad \text{for} \quad j = 3, \ldots, d.$

Let $j \in \{3, \ldots, d\}$. Assume $-b_i \notin F_j$. Then, since also $b_i \notin F_j$, we get $\langle u_j, -b_i \rangle = 0$. Now, since $-b_i \in F$, Equation (4) yields $-b_i = e_j$, a contradiction to our assumption. Therefore,
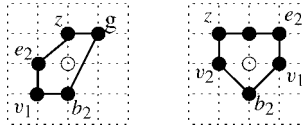

FIGURE 6. Two possibilities for $P(v_2, e_2, b_2)$.

$-b_i \in F_j$ for all $j \in \{3, \ldots, d\}$. Hence, $-b_i \in \mathrm{conv}(e_1, e_2)$. Now, looking at Figure 6 yields $b_i = b_2$, a contradiction.   □

By Figures 5 and 6, we have

$$-\mathrm{conv}\left(\frac{e_1 + e_2}{2}, e_3, \ldots, e_d\right) \subseteq G = \mathrm{conv}(b_2, b_3, \ldots, b_d).$$

Now, Claim 5 shows that equality holds. Moreover, we get

$$\{-e_3, \ldots, -e_d\} = \{b_3, \ldots, b_d\}.$$

Hence, there exists a permutation σ on $\{3, \ldots, d\}$ satisfying

$$e_{\sigma(i)} = -b_i \quad \text{and} \quad \sigma(i) \neq i \quad \text{for } i = 3, \ldots, d.$$

By Fact 1 and Figure 5, $v_{\sigma(i)} \not\succ e_{\sigma(i)} = -b_i \not\succ -e_i = b_{\sigma^{-1}(i)}$, thus by Lemma 2.4 we have $v_{\sigma(i)} \in P(e_{\sigma(i)}, v_{\sigma(i)}, b_{\sigma^{-1}(i)}) = P(v_i, e_i, b_i)$. Hence, we see that in Figure 5 the first possibility cannot occur, so $P(v_i, e_i, b_i) \cong V_2$, and we have

$$v_{\sigma(i)} = -v_i \quad \text{for } i = 3, \ldots, d.$$

Therefore, σ is a fix-point-free involution, thus, σ is a product of disjoint transpositions. In particular, d is even. It remains to show the following statement.

CLAIM 6. $\{e_1, b_2, e_3, \ldots, e_d\}$ *is a lattice basis.*

PROOF OF CLAIM 6.   These elements form a basis of $N_{\mathbf{R}}$ because of $e_2 = -e_1 - 2b_2$. Let $\{e_1^*, b_2^*, e_3^*, \ldots, e_d^*\}$ denote the dual basis of $M_{\mathbf{R}}$. By Figure 6 and Equation (4), we see

$$u_F = e_1^* - b_2^* + e_3^* + \cdots + e_d^*,$$

$$u_{N(F, e_2)} = u_F + b_2^*,$$

$$u_{N(F, e_i)} = u_F - e_i^* \quad \text{for } i = 3, \ldots, d.$$

Since, by reflexivity of P, the outer normals are lattice points in M, e_1^*, b_2^*, e_3^*, $\ldots, e_d^*$ are also lattice points in M. Thus $\{e_1, b_2, e_3, \ldots, e_d\}$ is a lattice basis of N.   □

This finishes the proof of Proposition 6.1, and hence of Theorem 1.2.   □

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