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THE LAPLACIAN AND THE HEAT KERNEL ACTING ON DIFFERENTIAL FORMS ON SPHERES

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Abstract. We show that the Laplacian acting on differential forms on a sphere can be lifted to an operator on its rotation group which is intrinsically equivalent to the Laplacian acting on functions on the Lie group. Further, using the result and the Urakawa summation formula for the heat kernel of the latter Laplacian and the Weyl integration formula, we get a summation formula for the kernel of the former.

1. Introduction. Let us view the n-sphere S^n as a coset manifold $SO(n+1)/SO(n)$ and assume that the Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1)$ is endowed with an inner product $(\cdot,\cdot) = -B(\cdot,\cdot)$, where B is the Killing form, i.e., $B(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$, which is nondegenerate and negative definite. We will furnish S^n with a left $SO(n+1)$-invariant metric $(\cdot,\cdot)$ so that the canonical isomorphism from the orthogonal complement $\mathfrak{m}$ of the subalgebra $\mathfrak{k} = \mathfrak{so}(n)$ to $T_{[e]}S^n$ is isometric. The purpose of the paper is to study the horizontal lift of the Laplacian $\Delta^{S^n} = d^*d + dd^*$ to $SO(n+1)$, where d^* is the formal adjoint of the exterior derivative d, and, by using it, to give a summation formula for the heat kernel $e^{-t\Delta^{S^n}}$ at each diagonal point in $S^n \times S^n$.

On $T_eSO(n+1) = \mathfrak{g} = \mathfrak{m}+\mathfrak{k}$, we will take canonically a positively oriented orthonormal frame $e_\bullet(e) = (e_{\mathfrak{m}\bullet}(e), e_{\mathfrak{k}\bullet}(e)) = (e_1(e), \dots, e_{n(n+1)/2}(e)) = (e_{\mathfrak{m}1}(e), \dots, e_{\mathfrak{m}n}(e), e_{\mathfrak{k}1}(e), \dots, e_{\mathfrak{k}\,n(n-1)/2}(e)) = (e_{1,n+1}(e), \dots, e_{n,n+1}(e), e_{1,n}(e), \dots, e_{n-1,n}(e), e_{1,n-1}(e), \dots)$. That is, we set

$$e_{i,j}(e) = (2(n-1))^{-1/2}(E_{i,j} - E_{j,i}) \quad (i < j)\,, \tag{1.1}$$

where $E_{i,j}$ is the $(n+1)\times(n+1)$-matrix of which the (i,j)-entry is equal to 1 and all other entries are 0. By its left translation, we obtain a frame $e^L_\bullet = (e^L_{\mathfrak{m}\bullet}, e^L_{\mathfrak{k}\bullet})$ and its dual frame $e^\bullet_L = (e^{\mathfrak{m}\bullet}_L, e^{\mathfrak{k}\bullet}_L)$.

First, let us state an assertion concerning the horizontal lift of Δ^{S^n} by the Riemannian submersion

$$\pi : SO(n+1) \to SO(n+1)/SO(n) = S^n\,. \tag{1.2}$$

Consider the Laplacian $\Delta_0^{SO(n+1)} = -\sum e^L_i e^L_i$ acting on functions on $SO(n+1)$ (see (2.5)) and take a differential form ω on S^n. We define a differential operator acting on the horizontal

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lift $\bar{\omega}=\sum e_L^{\mathfrak{m}\alpha}\cdot\Omega_\alpha$ ($\alpha=(\alpha_1,\dots,\alpha_{|\alpha|})$, $e_L^{\mathfrak{m}\alpha}=e_L^{\mathfrak{m}\alpha_1}\wedge\cdots\wedge e_L^{\mathfrak{m}\alpha_{|\alpha|}}$) by

$$\mathfrak{D}^{SO(n+1)}\,\bar{\omega}=\sum e_L^{\mathfrak{m}\alpha}\cdot\Delta_0^{SO(n+1)}\Omega_\alpha\,. \tag{1.3}$$

Then we have the following theorem.

THEOREM 1.1. *The equality* $\overline{\Delta^{S^n}\omega}=\mathfrak{D}^{SO(n+1)}\,\bar{\omega}$ *holds.*

As is known ((2.20), [6, Chap. II, §4], [2, Lemma 2.1]), for a function f on S^n, we have $\overline{\Delta_0^{S^n}f}=\Delta_0^{SO(n+1)}\,\bar{f}$. In the case of forms, one may naturally expect (refer to (2.17)) that $\overline{\Delta^{S^n}\omega}$ is equal to the horizontal component $\mathcal{H}^*(\Delta^{SO(n+1)}\,\bar{\omega})$. However, it is the formula in the theorem that really holds, which is rather desirable because the study of heat kernel $e^{-t\Delta^{S^n}}$ acting on differential forms can be reduced to that of $e^{-t\Delta_0^{SO(n+1)}}$, which has been closely investigated by many people. Second, we want to describe the former kernel by using the latter. Let us consider the submersion $\mathfrak{m}\times SO(n)\to SO(n+1)$, $(X,k)\mapsto(\exp X)k$, and define the frames $f_\bullet^h=(f_{\mathfrak{m}\bullet}^h,f_{\mathfrak{k}\bullet}^h)$, $f_h^\bullet=(f_h^{\mathfrak{m}\bullet},f_h^{\mathfrak{k}\bullet})$ (with marked point $h=\exp X$) at $hk\in\pi^{-1}([h])$ by

$$f_i^h(hk)=R_{k*}(e_i^L(h))\,,\qquad f_h^i(hk)=R_{k*}(e_L^i(h))\,, \tag{1.4}$$

where R_k is the right translation by $k\in SO(n)$. Remark that the matrix $a^{\mathfrak{m}}(k)$ (see (2.6)) with (i,j)-entries

$$a_{ij}^{\mathfrak{m}}(k)=e^{\mathfrak{m}i}(e)(\mathrm{Ad}(k)e_{\mathfrak{m}j}(e)) \tag{1.5}$$

is the transition matrix between the frames, i.e.,

$$f_{\mathfrak{m}\bullet}^h(hk)\cdot a^{\mathfrak{m}}(k)=e_{\mathfrak{m}\bullet}(hk)\,,\qquad f_h^{\mathfrak{m}\bullet}(hk)\cdot a^{\mathfrak{m}}(k)=e^{\mathfrak{m}\bullet}(hk)\,, \tag{1.6}$$

and, in general, $\overline{\pi_*e_L^{\mathfrak{m}i}}$ is not equal to $e_L^{\mathfrak{m}i}$, while $\overline{\pi_*f_h^{\mathfrak{m}i}}$ is equal to $f_h^{\mathfrak{m}i}$.

Now, let us set

$$e^{-t\Delta^{S^n}}([h],[h'])=\sum_{I,I'}^{(|I|=|I'|)}(\pi_*f_h^{\mathfrak{m}I})([h])\boxtimes(\pi_*f_{h'}^{\mathfrak{m}I'})([h'])\cdot(e^{-t\Delta^{S^n}})_{II'}^{h,h'}([h],[h'])\,, \tag{1.7}$$

where we assume that $(e^{-t\Delta^{S^n}})_{\sigma(I)\tau(I')}^{h,h'}=\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)(e^{-t\Delta^{S^n}})_{II'}^{h,h'}$ ($\sigma,\tau\in\mathfrak{S}_{|I|}$), and denote by $\mu_{SO(n)}$ the Haar measure on $SO(n)$ given by the frame $e_{\mathfrak{k}\bullet}(e)$.

COROLLARY 1.2. *We have*

$$(e^{-t\Delta^{S^n}})_{II'}^{h,h'}([h],[h'])=\int_{SO(n)}d\mu_{SO(n)}(k)\,e^{-t\Delta_0^{SO(n+1)}}(h'^{-1}hk,e)\,A_{II'}^{\mathfrak{m}}(k) \tag{1.8}$$

with

$$A_{II'}^{\mathfrak{m}}(k)=\frac{1}{|I|!}\,\det(a^{\mathfrak{m}}(k))_{II'}=\frac{1}{|I|!}\,\det\begin{pmatrix}a_{i_1i'_1}^{\mathfrak{m}}(k)&\cdots&a_{i_1i'_{|I'|}}^{\mathfrak{m}}(k)\\\vdots&&\vdots\\a_{i_{|I|}i'_1}^{\mathfrak{m}}(k)&\cdots&a_{i_{|I|}i'_{|I'|}}^{\mathfrak{m}}(k)\end{pmatrix}, \tag{1.9}$$

which is interpreted as 1 *if* $|I|\,(=|I'|)=0$.

By applying the Poisson summation formula for heat kernel, proved by Urakawa ([15]), to the kernel $e^{-t\Delta_0^{SO(n+1)}}$ (see also [4], [2]) and using the Weyl integration formula (see [10] for example), we can describe the coefficients (1.8) for $h = h'$ explicitly in Corollary 1.3. For that, let us assume $n \geq 2$ and take $\mathbf{Z}^{[n/2]} \ni l = (l_1, \dots) \geq 0$ (i.e., $l_i \geq 0$ for all i). Then, for $t \in (0, \infty)$ and $\theta = (\theta_1, \dots) \in [-\pi, \pi]^{[n/2]}$, if $n = 2m$, we put

$$(1.10)\quad F_l(t,\theta) = \sum_{\varepsilon} e^{-2(n-1)\sum(\theta_i + 2\pi\varepsilon_i l_i)^2/4t} \frac{\prod_{i<j}\left\{\left(t^{-1/2}\sin\frac{\theta_i}{2}\right)^2 - \left(t^{-1/2}\sin\frac{\theta_j}{2}\right)^2\right\}}{\prod_i t^{-1/2}\sin\frac{\theta_i}{2}} \times (-1)^{\sum\varepsilon_i l_i} \prod_i \frac{\theta_i + 2\pi\varepsilon_i l_i}{(4t)^{1/2}} \prod_{i<j}\left\{\left(\frac{\theta_i + 2\pi\varepsilon_i l_i}{(4t)^{1/2}}\right)^2 - \left(\frac{\theta_j + 2\pi\varepsilon_j l_j}{(4t)^{1/2}}\right)^2\right\}$$

and, if $n = 2m - 1$, we put

$$(1.11)\quad F_l(t,\theta) = \sum_{\varepsilon} e^{-2(n-1)\sum(\theta_i + 2\pi\varepsilon_i l_i)^2/4t} \prod_{i<j}\left\{\left(t^{-1/2}\sin\frac{\theta_i}{2}\right)^2 - \left(t^{-1/2}\sin\frac{\theta_j}{2}\right)^2\right\} \times \prod_i \left(\frac{\theta_i + 2\pi\varepsilon_i l_i}{(4t)^{1/2}}\right)^2 \prod_{i<j}\left\{\left(\frac{\theta_i + 2\pi\varepsilon_i l_i}{(4t)^{1/2}}\right)^2 - \left(\frac{\theta_j + 2\pi\varepsilon_j l_j}{(4t)^{1/2}}\right)^2\right\},$$

where ε runs over the set of $(\varepsilon_1, \dots, \varepsilon_{[n/2]})$ satisfying

$$\varepsilon_i = \begin{cases} \pm 1 & (l_i > 0), \\ 0 & (l_i = 0), \end{cases}$$

and we understand $\prod_{i<j}\{\cdots\} = 1$ at (1.10) for $n = 2$ and at (1.11) for $n = 3$.

COROLLARY 1.3. *Each function $F_l(t,\theta)$ is smooth on $(0,\infty) \times [-\pi,\pi]^{[n/2]}$, and, given $t_0 > 0$, there exists $C > 0$ satisfying*

$$(1.12)\quad |F_l(t,\theta)| \leq Ce^{-2(n-1)(|\theta|^2 + \sum l_i)/5t} \ (\textit{for all } l), \quad \sum_{l \geq 0} |F_l(t,\theta)| \leq Ce^{-2(n-1)|\theta|^2/5t}$$

provided $(t,\theta) \in (0, t_0] \times [-\pi,\pi]^{[n/2]}$. Also, (1.8) *with $h = h'$ has an expression of termwise integration*

$$(1.13)\quad (e^{-t\Delta^{S^n}})^{h,h}_{II'}([h],[h]) = c_n \frac{e^{tn(n+1)/48}}{(4\pi t)^{n/2}} \sum_{l \geq 0} \frac{1}{(4\pi t)^{[n/2]/2}} \int_{[-\pi,\pi]^{[n/2]}} d\theta\, F_l(t,\theta) B^{\mathfrak{m}}_{II'}(\theta)$$

with

$$(1.14)\quad c_n = \frac{(2(n-1))^{[n/2](2[(n-1)/2]+1)/2}\, 2^{[(n-1)/2][(n+1)/2]}}{2!\,4!\cdots(2[(n-1)/2])!} \cdot \begin{cases} \frac{1}{m!} & (n = 2m), \\ 1 & (n = 2m-1), \end{cases}$$

$$(1.15)\qquad B^{\mathfrak{m}}_{II'}(\theta)=\int_{SO(n)/T} d\mu_{SO(n)/T}([k])\,A^{\mathfrak{m}}_{II'}(ku(\theta)k^{-1})$$
$$=\frac{1}{|I|!}\int_{SO(n)/T} d\mu_{SO(n)/T}([k])\,\det(a^{\mathfrak{m}}(k)D(\theta)\,a^{\mathfrak{m}}(k^{-1}))_{II'}\quad (n\geq 3)\,,$$

$$(1.16)\qquad B^{\mathfrak{m}}_{II'}(\theta)=\frac{1}{|I|!}\det D(\theta)_{II'}\ (=D(\theta)_{II'})\quad (n=2)\,.$$

Here we put

$$D(\theta_1)=\begin{pmatrix}\cos\theta_1 & \sin\theta_1\\ -\sin\theta_1 & \cos\theta_1\end{pmatrix},\quad D(\theta)=\begin{pmatrix}D(\theta_1) & & O\\ & \ddots & \\ O & & D(\theta_{[n/2]})\end{pmatrix}$$

and $T=T_{SO(n)}$ *is a maximal torus of* $SO(n)$ *consisting of* $u(\theta)=D(\theta)$ (*if* $n=2m$), $u(\theta)=\left(\begin{smallmatrix}D(\theta) & 0\\ 0 & 1\end{smallmatrix}\right)$ (*if* $n=2m-1$). (*Hence*, $A^{\mathfrak{m}}_{II'}(ku(\theta)k^{-1})$ *is independent of the choice of representative* k *of* $[k]$.) *Moreover*, *the canonical measure* $\mu_{SO(n)/T}$ *is normalized as* $\int_{SO(n)/T} d\mu_{SO(n)/T}([k])=1$.

The standard metric on $S^n=SO(n+1)/SO(n)$ will be $(2(n-1))^{-1}(\cdot,\cdot)$, so that the kernel for q-forms with respect to the standard one can be expressed as $e^{-t\Delta^{(S^n,\mathrm{st})}}=(2(n-1))^{n/2-q}e^{-2(n-1)t\Delta^{S^n}}$. Hence, if we express it using the orthonormal frame $(2(n-1))^{-1/2}(\pi_* f_h^{\mathfrak{m}\bullet})([h])$ in the same way as at (1.13), the coefficients can be written at diagonal points as

$$(1.17)\qquad (e^{-t\Delta^{(S^n,\mathrm{st})}})^{h,h}_{II'}([h],[h])$$
$$=\tilde{c}_n\frac{e^{tn(n^2-1)/24}}{(4\pi t)^{n/2}}\sum_{l\geq 0}\frac{1}{(4\pi t)^{[n/2]/2}}\int_{[-\pi,\pi]^{[n/2]}} d\theta\,\tilde{F}_l(t,\theta)B^{\mathfrak{m}}_{II'}(\theta)\,,$$

where $\tilde{c}_n$ is defined as c_n divided by $(2(n-1))^{[n/2](2[(n-1)/2]+1)/2}$ and $\tilde{F}_l(t,\theta)$ means $F_l(t,\theta)$ with $e^{-2(n-1)\sum(\theta_i+2\pi\varepsilon_i l_i)^2/4t}$ replaced by $e^{-\sum(\theta_i+2\pi\varepsilon_i l_i)^2/4t}$. As for $S^1=SO(2)/SO(1)=SO(2)$ with standard metric, as is known ([3, p. 306]), we have the Poisson summation formula

$$(1.18)\qquad e^{-t\Delta^{(S^1,\mathrm{st})}}([h],[h])=\frac{1}{(4\pi t)^{1/2}}\sum_{l\in\mathbf{Z}}e^{-l^2/4t}\,.$$

Thus (1.13) and (1.17) may be regarded as its generalizations.

The author has been studying heat kernel coefficients for a few years mainly by using the general adiabatic expansion theory ([11], [12]), and, in particular in the sphere case, the kernels acting on functions were closely investigated ([13]) by considering the duality relation between sphere and hyperbolic space. This paper comes from an attempt to generalize the results ([13]) to the differential form case. We note that, in some different appearance, on general compact symmetric spaces, another type of formula for the kernel acting on differential forms has already been obtained by Urakawa ([16]), and, in the function case, the formulas corresponding to (1.8) and (1.13) have already been obtained by Benabdallach ([2]).

2. The Laplacian on compact two-point homogeneous Riemannian manifolds. Let (G, K) be a symmetric pair with G compact, connected and semisimple, and assume that the Lie algebra $\mathfrak{g}$ of G possesses the inner product $(\cdot, \cdot) = -B(\cdot, \cdot)$, where B is its Killing form. We will furnish G/K with a left G-invariant metric $(\cdot, \cdot)$ so that the canonical isomorphism from the orthogonal complement $\mathfrak{m}$ of the subalgebra $\mathfrak{k}$ of K to $T_{[e]}(G/K)$ is isometric. Notice that there is an orthogonal decomposition ([14, Lemma 30 (p. 316)])

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{k} \quad \text{with } \mathrm{Ad}(k)\mathfrak{m} \subset \mathfrak{m}\ (k \in K)\,, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\,, \ [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}\,, \ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}\,, \tag{2.1}$$

and we have the Riemannian submersion

$$\pi : (G, (\cdot, \cdot)) \to (G/K, (\cdot, \cdot))\,. \tag{2.2}$$

The purpose of the section is to prove Propositions 2.2 and 2.3 which describe two kinds of differences $\mathcal{H}^*(\Delta^G \bar{\omega}) - \overline{\Delta^{G/K}\omega}$ and $\mathcal{H}^*(\Delta^G \bar{\omega}) - \mathfrak{D}^G \bar{\omega}$ minutely. The Riemannian connections ∇^G, $\nabla^{G/K}$, the horizontal lifts $\bar{X}$, $\bar{\omega}$ (of vector fields, differential forms) and the horizontal components $\mathcal{H}^{(*)}(\cdot)$ have the simple relations ([14, Lemma 45 (p. 212)])

$$\begin{aligned} &(\bar{X}, \bar{Y}) = (X, Y)\circ\pi\,, \quad \mathcal{H}([\bar{X}, \bar{Y}]) = \overline{[X, Y]}\,, \quad \mathcal{H}(\nabla^G_{\bar{X}}\bar{Y}) = \overline{\nabla^{G/K}_X Y}\,, \\ &(\bar{\omega}, \bar{\eta}) = (\omega, \eta)\circ\pi\,, \quad \mathcal{H}^*(\nabla^G_{\bar{X}}\bar{\omega}) = \overline{\nabla^{G/K}_X \omega}\,, \end{aligned} \tag{2.3}$$

but they are complicated if the Laplacians are implicated in them.

We will begin our discussion with reviewing the Weitzenböck formula. Let us take a positively oriented orthonormal local frame $e_\bullet$ of $(TG, (\cdot, \cdot))$ and its dual frame $e^\bullet$. We denote the exterior, interior products of a differential 1-form ξ by $\xi\wedge$, $\xi\vee$, respectively, and denote the curvature 2-form of ∇^G by $F(\nabla^G)$. Then we have the formula

$$\begin{aligned} \Delta^G &= d^G d^{G*} + d^{G*} d^G \\ &= -\sum(\nabla^G_{e_j}\nabla^G_{e_j} - \nabla^G_{\nabla^G_{e_j}e_j}) - \sum(F(\nabla^G)(e_{i_3}, e_{i_4})e_{i_2}, e_{i_1})\, e^{i_1}\wedge e^{i_2}\vee e^{i_3}\wedge e^{i_4}\vee\,. \end{aligned} \tag{2.4}$$

For the left invariant orthonormal frame $e^L_\bullet = (e^L_{\mathfrak{m}\bullet}, e^L_{\mathfrak{k}\bullet})$ given as in the introduction, we have $\nabla^G_{e^L_i} e^L_j = (1/2)[e^L_i, e^L_j]$ ([14, Proposition 9 (p. 304)]). Hence, in the function case, (2.4) can be simplified into

$$\Delta^G_0 = -\sum e^L_j e^L_j \quad (\text{acting on } C^\infty(G))\,. \tag{2.5}$$

Another frame we are interested in is a local frame $f_\bullet = (f_{\mathfrak{m}\bullet}, f_{\mathfrak{k}\bullet})$ at $hk = (\exp X)k \in \pi^{-1}([h])$ given as in the introduction but with X restricted to a neighborhood $V_{\mathfrak{m}}$ of 0. Since here the marked point $h = \exp X$ is given by a unique X, we omit the symbol h to simplify the notation. Set $V = \pi(\exp V_{\mathfrak{m}})$. Then the frames $e^L_{\mathfrak{m}\bullet}$, $f_{\mathfrak{m}\bullet}$ on $\pi^{-1}(V)$ have the relations (1.6), the transition matrix $a^{\mathfrak{m}}(k)$ of which has the properties

$$a^{\mathfrak{m}}(1) = \mathrm{id.}\,, \quad a^{\mathfrak{m}}(k^{-1}) = a^{\mathfrak{m}}(k)^{-1} = {}^t a^{\mathfrak{m}}(k)\,, \quad a^{\mathfrak{m}}(k_1)a^{\mathfrak{m}}(k_2) = a^{\mathfrak{m}}(k_1k_2)\,. \tag{2.6}$$

Similarly, we denote by $a^{\mathfrak{k}}(k)$ the transition matrix between $e^L_{\mathfrak{k}\bullet}$ and $f_{\mathfrak{k}\bullet}$, and let us consider the Maurer-Cartan structure constants given by

$$[e^L_{i_1}, e^L_{i_2}] = \sum e^L_{i_3} \cdot c^{i_3}_{i_1 i_2}, \quad [e^L_{\mathfrak{m}i_1}, e^L_{\mathfrak{m}i_2}] = \sum e^L_{\mathfrak{k}i_3} \cdot c^{(\mathfrak{k}i_3)}_{(\mathfrak{m}i_1)(\mathfrak{m}i_2)}, \tag{2.7}$$

etc. (see (2.1)). We will calculate the connection coefficients of ∇^G with respect to the frame $f_\bullet$. Henceforth in the proofs in the section, we adopt the Einstein convention on sums over repeated indices.

LEMMA 2.1. *At $g = hk = (\exp X)k \in \pi^{-1}(V)$, we have*

$$\nabla^G_{f_{\mathfrak{m}i}} f_{\mathfrak{m}j} = -\sum f_{\mathfrak{m}j'} \cdot e^{\mathfrak{m}j'}_L(h)\Big(\Big[\tanh\frac{\operatorname{ad} X}{2} e^L_{\mathfrak{m}i}, e^L_{\mathfrak{m}j}\Big]\Big) + \sum f_{\mathfrak{k}j'} \cdot \frac{1}{2} c^{(\mathfrak{k}j')}_{(\mathfrak{m}i)(\mathfrak{m}j)}, \tag{2.8}$$

$$\nabla^G_{f_{\mathfrak{m}i}} f_{\mathfrak{k}j} = \sum f_{\mathfrak{m}j'} \cdot \frac{1}{2} c^{(\mathfrak{m}j')}_{(\mathfrak{m}i)(\mathfrak{k}j)} - \sum f_{\mathfrak{k}j'} \cdot e^{\mathfrak{k}j'}_L(h)\Big(\Big[\tanh\frac{\operatorname{ad} X}{2} e^L_{\mathfrak{m}i}, e^L_{\mathfrak{k}j}\Big]\Big), \tag{2.9}$$

$$\nabla^G_{f_{\mathfrak{k}i}} f_{\mathfrak{m}j} = -\sum f_{\mathfrak{m}j'} \cdot \frac{1}{2} c^{(\mathfrak{m}j')}_{(\mathfrak{k}i)(\mathfrak{m}j)}, \quad \nabla^G_{f_{\mathfrak{k}i}} f_{\mathfrak{k}j} = -\sum f_{\mathfrak{k}j'} \cdot \frac{1}{2} c^{(\mathfrak{k}j')}_{(\mathfrak{k}i)(\mathfrak{k}j)}, \tag{2.10}$$

$$\nabla^G_{f_{\mathfrak{m}i}} f^{\mathfrak{m}j} = \sum f^{\mathfrak{m}j'} \cdot e^{\mathfrak{m}j}_L(h)\Big(\Big[\tanh\frac{\operatorname{ad} X}{2} e^L_{\mathfrak{m}i}, e^L_{\mathfrak{m}j'}\Big]\Big) - \sum f^{\mathfrak{k}j'} \cdot \frac{1}{2} c^{(\mathfrak{m}j)}_{(\mathfrak{m}i)(\mathfrak{k}j')}, \tag{2.11}$$

$$\nabla^G_{f_{\mathfrak{m}i}} f^{\mathfrak{k}j} = -\sum f^{\mathfrak{m}j'} \cdot \frac{1}{2} c^{(\mathfrak{k}j)}_{(\mathfrak{m}i)(\mathfrak{m}j')} + \sum f^{\mathfrak{k}j'} \cdot e^{\mathfrak{k}j}_L(h)\Big(\Big[\tanh\frac{\operatorname{ad} X}{2} e^L_{\mathfrak{m}i}, e^L_{\mathfrak{k}j'}\Big]\Big), \tag{2.12}$$

$$\nabla^G_{f_{\mathfrak{k}i}} f^{\mathfrak{m}j} = \sum f^{\mathfrak{m}j'} \cdot \frac{1}{2} c^{(\mathfrak{m}j)}_{(\mathfrak{k}i)(\mathfrak{m}j')}, \quad \nabla^G_{f_{\mathfrak{k}i}} f^{\mathfrak{k}j} = \sum f^{\mathfrak{k}j'} \cdot \frac{1}{2} c^{(\mathfrak{k}j)}_{(\mathfrak{k}i)(\mathfrak{k}j')}. \tag{2.13}$$

PROOF. We put $\exp(X)\exp(te_i(e)) = h \cdot k(\exp(X)\exp(te_i(e))) \in \exp\mathfrak{m} \cdot K$. Then we have the equalities

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} k(\exp(X)\exp(te_{\mathfrak{m}i}(e))) &= \tanh\frac{\operatorname{ad} X}{2} e_{\mathfrak{m}i}(e) \ \in \mathfrak{k}, \\ \frac{d}{dt}\Big|_{t=0} k(\exp(X)\exp(te_{\mathfrak{k}i}(e))) &= e_{\mathfrak{k}i}(e) \ \in \mathfrak{k}. \end{aligned} \tag{2.14}$$

The second equality is obvious. To show the first one, set $\exp(X)\exp(te_{\mathfrak{m}i}(e)) = \exp(X + Y(t)) \cdot k(t) \in \exp\mathfrak{m} \cdot K$. Then we have

$$\begin{aligned} e_{\mathfrak{m}i}(e) &= \frac{d}{dt}\Big|_{t=0} \exp(te_{\mathfrak{m}i}(e)) = \frac{d}{dt}\Big|_{t=0} \exp(-X)\exp(X + Y(t))\,k(t) \\ &= \frac{d}{dt}\Big|_{t=0} \exp(-X)\exp(X + Y(t)) + \dot{k}(0) = L^*_{\exp X}\frac{d}{dt}\Big|_{t=0} \exp(X + t\dot{Y}(0)) + \dot{k}(0) \end{aligned}$$

and, by using [5, Theorem 1.7 (p. 105)], we have

$$\begin{aligned} L^*_{\exp X}\frac{d}{dt}\Big|_{t=0} \exp(X + t\dot{Y}(0)) &= \frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X}(\dot{Y}(0)) \\ &= \frac{\sinh \operatorname{ad} X}{\operatorname{ad} X}(\dot{Y}(0)) + \Big(-\tanh\frac{\operatorname{ad} X}{2}\Big)\frac{\sinh \operatorname{ad} X}{\operatorname{ad} X}(\dot{Y}(0)) \ \in \mathfrak{m} + \mathfrak{k}. \end{aligned}$$

They imply

$$e_{\mathfrak{m}i}(e) = \frac{\sinh \operatorname{ad} X}{\operatorname{ad} X}(\dot{Y}(0)), \quad \tanh \frac{\operatorname{ad} X}{2} \frac{\sinh \operatorname{ad} X}{\operatorname{ad} X}(\dot{Y}(0)) = \dot{k}(0),$$

which yield the first equality at (2.14). Then by (2.14) we know

$$\begin{aligned}
&e^L_{\mathfrak{m}i'}(g)(a^{\mathfrak{m}}_{ij}(k(g)))|_{g=h} = -e^{\mathfrak{m}j}_L\Big(\Big[\tanh \frac{\operatorname{ad} X}{2}\, e^L_{\mathfrak{m}i'}, e^L_{\mathfrak{m}i}\Big]\Big), \\
&e^L_{\mathfrak{m}i'}(g)(a^{\mathfrak{k}}_{ij}(k(g)))|_{g=h} = -e^{\mathfrak{k}j}_L\Big(\Big[\tanh \frac{\operatorname{ad} X}{2}\, e^L_{\mathfrak{m}i'}, e^L_{\mathfrak{k}i}\Big]\Big), \qquad (2.15)\\
&e^L_{\mathfrak{k}i'}(g)(a^{\mathfrak{m}}_{ij}(k(g)))|_{g=h} = -c^{(\mathfrak{m}j)}_{(\mathfrak{k}i')(\mathfrak{m}i)}, \quad e^L_{\mathfrak{k}i'}(g)(a^{\mathfrak{k}}_{ij}(k(g)))|_{g=h} = -c^{(\mathfrak{k}j)}_{(\mathfrak{k}i')(\mathfrak{k}i)}.
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
e^L_{\mathfrak{m}i'}(g)(a^{\mathfrak{m}}_{ij}(k(g)))|_{g=h} &= \frac{d}{dt}\Big|_{t=0} a^{\mathfrak{m}}_{ij}(k(\exp(X)\exp(te_{\mathfrak{m}i'}(e)))) \\
&= \Big(e_{\mathfrak{m}i}(e), \Big[\tanh \frac{\operatorname{ad} X}{2} e_{\mathfrak{m}i'}(e), e_{\mathfrak{m}j}(e)\Big]\Big) = -\Big(\Big[\tanh \frac{\operatorname{ad} X}{2} e_{\mathfrak{m}i'}(e), e_{\mathfrak{m}i}(e)\Big], e_{\mathfrak{m}j}(e)\Big) \\
&= -e^{\mathfrak{m}j}(e)\Big(\Big[\tanh \frac{\operatorname{ad} X}{2}\, e_{\mathfrak{m}i'}(e), e_{\mathfrak{m}i}(e)\Big]\Big) = -e^{\mathfrak{m}j}_L\Big(\Big[\tanh \frac{\operatorname{ad} X}{2}\, e^L_{\mathfrak{m}i'}, e^L_{\mathfrak{m}i}\Big]\Big),
\end{aligned}$$

etc. Using (2.15), we can show the equalities of the lemma. For example, (2.8) is obtained as

$$\begin{aligned}
(\nabla^G_{f_{\mathfrak{m}i}} f_{\mathfrak{m}j})(h) &= \nabla^G_{e^L_{\mathfrak{m}i}(h)} f_{\mathfrak{m}j}(g) = \nabla^G_{e^L_{\mathfrak{m}i}(h)} a^{\mathfrak{m}}_{jj'}(k(g)) e^L_{\mathfrak{m}j'}(g) \\
&= e^L_{\mathfrak{m}i}(g)(a^{\mathfrak{m}}_{jj'}(k(g)))|_{g=h} e^L_{\mathfrak{m}j'}(h) + a^{\mathfrak{m}}_{jj'}(k(g))|_{g=h} \nabla^G_{e^L_{\mathfrak{m}i}(h)} e^L_{\mathfrak{m}j'}(g) \\
&= -e^L_{\mathfrak{m}j'}(h) \cdot e^{\mathfrak{m}j'}_L\Big(\Big[\tanh \frac{\operatorname{ad} X}{2}\, e^L_{\mathfrak{m}i}, e^L_{\mathfrak{m}j}\Big]\Big) + \frac{1}{2}[e^L_{\mathfrak{m}i}, e^L_{\mathfrak{m}j}](h) \\
&= -e^L_{\mathfrak{m}j'}(h) \cdot e^{\mathfrak{m}j'}_L\Big(\Big[\tanh \frac{\operatorname{ad} X}{2}\, e^L_{\mathfrak{m}i}, e^L_{\mathfrak{m}j}\Big]\Big) + e^L_{\mathfrak{k}j'}(h) \cdot \frac{1}{2} c^{(\mathfrak{k}j')}_{(\mathfrak{m}i)(\mathfrak{m}j)}, \\
(\nabla^G_{f_{\mathfrak{m}i}} f_{\mathfrak{m}j})(hk) &= R_{k*} \nabla^G_{e^L_{\mathfrak{m}i}(h)} f_{\mathfrak{m}j}(g) \\
&= -f_{\mathfrak{m}j'} \cdot e^{\mathfrak{m}j'}_L(h)\Big(\Big[\tanh \frac{\operatorname{ad} X}{2} e^L_{\mathfrak{m}i}, e^L_{\mathfrak{m}j}\Big]\Big) + f_{\mathfrak{k}j'} \cdot \frac{1}{2} c^{(\mathfrak{k}j')}_{(\mathfrak{m}i)(\mathfrak{m}j)}.
\end{aligned}$$

The equalities (2.11) through (2.13) follow from (2.8) through (2.10) together with the formula $\nabla^G_{f_i} f^j = -f^{j'} \otimes f^j(\nabla^G_{f_i} f_{j'})$. □

Now, let us generalize the formula (2.4) for $\Delta^{G/K}$ with replacing $e_\bullet$ etc. by $f_\bullet$ etc. Namely, we have

$$\begin{aligned}
\Delta^{G/K} = &-\sum \big(\nabla^{G/K}_{\pi_* f_{\mathfrak{m}i}} \nabla^{G/K}_{\pi_* f_{\mathfrak{m}i}} - \nabla^{G/K}_{\nabla^{G/K}_{\pi_* f_{\mathfrak{m}i}} f_{\pi_*\mathfrak{m}i}}\big) \\
&- \sum (F(\nabla^{G/K})(\pi_* f_{\mathfrak{m}i_3}, \pi_* f_{\mathfrak{m}i_4})\pi_* f_{\mathfrak{m}i_2}, \pi_* f_{\mathfrak{m}i_1}) \qquad (2.16)\\
&\cdot \pi_* f^{\mathfrak{m}i_1} \wedge \pi_* f^{\mathfrak{m}i_2} \vee \pi_* f^{\mathfrak{m}i_3} \wedge \pi_* f^{\mathfrak{m}i_4} \vee .
\end{aligned}$$

We assume that the symmetric space G/K is two-point homogeneous. We have the following proposition.

PROPOSITION 2.2. *Let ω be a differential form. On $\pi^{-1}(V)$, the difference between the horizontal part $\mathcal{H}^*(\Delta^G\bar{\omega})$ and the horizontal lift $\overline{\Delta^{G/K}\omega}$ can be written as*

$$\begin{aligned}\mathcal{H}^*(\Delta^G\bar{\omega})-\overline{\Delta^{G/K}\omega} &= -\frac{3}{4}\sum c^{(\mathfrak{m}j_2)}_{(\mathfrak{m}j)(\mathfrak{k}i)}c^{(\mathfrak{k}i)}_{(\mathfrak{m}j)(\mathfrak{m}j_1)}f^{\mathfrak{m}j_1}\wedge f^{\mathfrak{m}j_2}\vee\bar{\omega}\\ &\quad+\sum\frac{1}{4}\{3c^{(\mathfrak{m}j_1)}_{(\mathfrak{m}j_2)(\mathfrak{k}i)}c^{(\mathfrak{k}i)}_{(\mathfrak{m}j_3)(\mathfrak{m}j_4)}-c^{(\mathfrak{m}j_1)}_{(\mathfrak{m}j_4)(\mathfrak{k}i)}c^{(\mathfrak{k}i)}_{(\mathfrak{m}j_2)(\mathfrak{m}j_3)}\}\\ &\qquad\times f^{\mathfrak{m}j_1}\wedge f^{\mathfrak{m}j_2}\vee f^{\mathfrak{m}j_3}\wedge f^{\mathfrak{m}j_4}\vee\bar{\omega}\,.\end{aligned}\tag{2.17}$$

PROOF. Set $\omega=\pi_*f^{\mathfrak{m}J}\cdot\omega_J$. Then we have

$$\begin{aligned}\mathcal{H}^*(\Delta^G\bar{\omega})-\overline{\Delta^{G/K}\omega} &= \{\mathcal{H}^*(\Delta^G f^{\mathfrak{m}J})-\overline{\Delta^{G/K}\pi_*f^{\mathfrak{m}J}}\}\cdot\pi^*\omega_J\\ &\quad+f^{\mathfrak{m}J}\cdot\{\mathcal{H}^*(\Delta^G\pi^*\omega_J)-\overline{\Delta^{G/K}\omega_J}\}\\ &\quad+2\{-\mathcal{H}^*(\nabla^G_{f_{\mathfrak{m}i}}f^{\mathfrak{m}J})+\overline{\nabla^{G/K}_{\pi_*f_{\mathfrak{m}i}}\pi_*f^{\mathfrak{m}J}}\}\cdot f_{\mathfrak{m}i}(\pi^*\omega_J)\,.\end{aligned}\tag{2.18}$$

(2.3) and Lemma 2.1 imply

$$\begin{aligned}&-\mathcal{H}^*(\nabla^G_{f_{\mathfrak{m}i}}\nabla^G_{f_{\mathfrak{m}i}}f^{\mathfrak{m}J})+\overline{\nabla^{G/K}_{\pi_*f_{\mathfrak{m}i}}\nabla^{G/K}_{\pi_*f_{\mathfrak{m}i}}\pi_*f^{\mathfrak{m}J}}\\ &\quad=-\mathcal{H}^*(\nabla^G_{f_{\mathfrak{m}i}}\nabla^G_{f_{\mathfrak{m}i}}f^{\mathfrak{m}J})+\mathcal{H}^*\left(\nabla^G_{f_{\mathfrak{m}i}}\left\{\nabla^G_{f_{\mathfrak{m}i}}f^{\mathfrak{m}J}+f^{\mathfrak{k}j'}\wedge f^{\mathfrak{m}j}\vee f^{\mathfrak{m}J}\cdot\frac{1}{2}c^{(\mathfrak{m}j)}_{(\mathfrak{m}i)(\mathfrak{k}j')}\right\}\right)\\ &\quad=-f^{\mathfrak{m}j_1}\wedge f^{\mathfrak{m}j_2}\vee f^{\mathfrak{m}J}\cdot\frac{1}{4}c^{(\mathfrak{m}j_2)}_{(\mathfrak{m}i)(\mathfrak{k}j)}c^{(\mathfrak{k}j)}_{(\mathfrak{m}i)(\mathfrak{m}j_1)}\,,\\ &-\mathcal{H}^*(\nabla^G_{f_{\mathfrak{k}i}}\nabla^G_{f_{\mathfrak{k}i}}f^{\mathfrak{m}J})=-\mathcal{H}^*\left(\nabla^G_{f_{\mathfrak{k}i}}(f^{\mathfrak{m}j_1}\wedge f^{\mathfrak{m}j}\vee f^{\mathfrak{m}J})\cdot\frac{1}{2}c^{(\mathfrak{m}j)}_{(\mathfrak{k}i)(\mathfrak{m}j_1)}\right)\\ &\quad=-f^{\mathfrak{m}j_2}\wedge f^{\mathfrak{m}j}\vee f^{\mathfrak{m}J}\cdot\frac{1}{2}c^{(\mathfrak{m}j)}_{(\mathfrak{k}i)(\mathfrak{m}j_1)}\frac{1}{2}c^{(\mathfrak{m}j_1)}_{(\mathfrak{k}i)(\mathfrak{m}j_2)}[3pt]\\ &\qquad-f^{\mathfrak{m}j_1}\wedge f^{\mathfrak{m}j'}\wedge f^{\mathfrak{m}j_2}\vee f^{\mathfrak{m}j}\vee f^{\mathfrak{m}J}\cdot\frac{1}{2}c^{(\mathfrak{m}j)}_{(\mathfrak{k}i)(\mathfrak{m}j_1)}\frac{1}{2}c^{(\mathfrak{m}j_2)}_{(\mathfrak{k}i)(\mathfrak{m}j')}\\ &\quad=-f^{\mathfrak{m}j_1}\wedge f^{\mathfrak{m}j_2}\vee f^{\mathfrak{m}J}\cdot\frac{1}{4}c^{(\mathfrak{m}j_2)}_{(\mathfrak{k}i)(\mathfrak{m}j)}c^{(\mathfrak{m}j)}_{(\mathfrak{k}i)(\mathfrak{m}j_1)}-f^{\mathfrak{m}j_1}\wedge f^{\mathfrak{m}j_2}\vee f^{\mathfrak{m}J}\cdot\frac{1}{4}c^{(\mathfrak{m}j_2)}_{(\mathfrak{k}i)(\mathfrak{m}j_1)}c^{(\mathfrak{m}j)}_{(\mathfrak{k}i)(\mathfrak{m}j)}\\ &\qquad+f^{\mathfrak{m}j_1}\wedge f^{\mathfrak{m}j_2}\vee f^{\mathfrak{m}j_3}\wedge f^{\mathfrak{m}j_4}\vee f^{\mathfrak{m}J}\cdot\frac{1}{4}c^{(\mathfrak{m}j_4)}_{(\mathfrak{k}i)(\mathfrak{m}j_1)}c^{(\mathfrak{m}j_2)}_{(\mathfrak{k}i)(\mathfrak{m}j_3)}\,,\\ &\mathcal{H}^*\big(\nabla^G_{\nabla^G_{f_{\mathfrak{m}i}}f_{\mathfrak{m}i}}f^{\mathfrak{m}J}\big)-\overline{\nabla^{G/K}_{\nabla^{G/K}_{\pi_*f_{\mathfrak{m}i}}\pi_*f_{\mathfrak{m}i}}\pi_*f^{\mathfrak{m}J}}=0\,,\quad\mathcal{H}^*\big(\nabla^G_{\nabla^G_{f_{\mathfrak{k}i}}f_{\mathfrak{k}i}}f^{\mathfrak{m}J}\big)=0\,,\end{aligned}$$

$$
\begin{aligned}
&-\mathcal{H}^*((F(\nabla^G)(f_{i_3}, f_{i_4})f_{i_2}, f_{i_1}) \cdot (f^{i_1} \wedge f^{i_2} \vee f^{i_3} \wedge f^{i_4} \vee) f^{\mathfrak{m}J}) \\
&\quad = -(F(\nabla^G)(f_{\mathfrak{m}i_3}, f_{\mathfrak{m}i_4})f_{\mathfrak{m}i_2}, f_{\mathfrak{m}i_1}) \cdot f^{\mathfrak{m}i_1} \wedge f^{\mathfrak{m}i_2} \vee f^{\mathfrak{m}i_3} \wedge f^{\mathfrak{m}i_4} \vee f^{\mathfrak{m}J} \\
&\qquad - (F(\nabla^G)(f_{\mathfrak{k}i_2}, f_{\mathfrak{m}i_4})f_{\mathfrak{k}i_2}, f_{\mathfrak{m}i_1}) \cdot f^{\mathfrak{m}i_1} \wedge f^{\mathfrak{m}i_4} \vee f^{\mathfrak{m}J}, \\
&\overline{F(\nabla^{G/K})(\pi_* f_{\mathfrak{m}i_3}, \pi_* f_{\mathfrak{m}i_4})\pi_* f_{\mathfrak{m}i_2}} = \mathcal{H}\left(\nabla^G_{f_{\mathfrak{m}i_3}}\left\{\nabla^G_{f_{\mathfrak{m}i_4}} f_{\mathfrak{m}i_2} - f_{\mathfrak{k}j'} \cdot \frac{1}{2} c^{(\mathfrak{k}j')}_{(\mathfrak{m}i_4)(\mathfrak{m}i_2)}\right\}\right) \\
&\qquad - \mathcal{H}\left(\nabla^G_{f_{\mathfrak{m}i_4}}\left\{\nabla^G_{f_{\mathfrak{m}i_3}} f_{\mathfrak{m}i_2} - f_{\mathfrak{k}j'} \cdot \frac{1}{2} c^{(\mathfrak{k}j')}_{(\mathfrak{m}i_3)(\mathfrak{m}i_2)}\right\}\right) - \mathcal{H}\big(\nabla^G_{[f_{\mathfrak{m}i_3}, f_{\mathfrak{m}i_4}] - f_{\mathfrak{k}j'} \cdot c^{(\mathfrak{k}j')}_{(\mathfrak{m}i_3)(\mathfrak{m}i_4)}} f_{\mathfrak{m}i_2}\big) \\
&\quad = \mathcal{H}(F(\nabla^G)(f_{\mathfrak{m}i_3}, f_{\mathfrak{m}i_4})f_{\mathfrak{m}i_2}) - f_{\mathfrak{m}i'} \cdot \frac{1}{4} c^{(\mathfrak{m}i')}_{(\mathfrak{m}i_3)(\mathfrak{k}j)} c^{(\mathfrak{k}j)}_{(\mathfrak{m}i_4)(\mathfrak{m}i_2)} \\
&\qquad + f_{\mathfrak{m}i'} \cdot \frac{1}{4} c^{(\mathfrak{m}i')}_{(\mathfrak{m}i_4)(\mathfrak{k}j)} c^{(\mathfrak{k}j)}_{(\mathfrak{m}i_3)(\mathfrak{m}i_2)} - f_{\mathfrak{m}i'} \cdot \frac{1}{2} c^{(\mathfrak{m}i')}_{(\mathfrak{k}j)(\mathfrak{m}i_2)} c^{(\mathfrak{k}j)}_{(\mathfrak{m}i_3)(\mathfrak{m}i_4)}, \\
&(\overline{F(\nabla^{G/K})(\pi_* f_{\mathfrak{m}i_3}, \pi_* f_{\mathfrak{m}i_4})\pi_* f_{\mathfrak{m}i_2}}, \overline{\pi_* f_{\mathfrak{m}i_1}}) - (F(\nabla^G)(f_{\mathfrak{m}i_3}, f_{\mathfrak{m}i_4})f_{\mathfrak{m}i_2}, f_{\mathfrak{m}i_1}) \\
&\quad = -\frac{1}{4} c^{(\mathfrak{m}i_1)}_{(\mathfrak{m}i_3)(\mathfrak{k}j)} c^{(\mathfrak{k}j)}_{(\mathfrak{m}i_4)(\mathfrak{m}i_2)} + \frac{1}{4} c^{(\mathfrak{m}i_1)}_{(\mathfrak{m}i_4)(\mathfrak{k}j)} c^{(\mathfrak{k}j)}_{(\mathfrak{m}i_3)(\mathfrak{m}i_2)} - \frac{1}{2} c^{(\mathfrak{m}i_1)}_{(\mathfrak{k}j)(\mathfrak{m}i_2)} c^{(\mathfrak{k}j)}_{(\mathfrak{m}i_3)(\mathfrak{m}i_4)} \\
&\quad = \frac{3}{4} c^{(\mathfrak{m}i_1)}_{(\mathfrak{m}i_2)(\mathfrak{k}j)} c^{(\mathfrak{k}j)}_{(\mathfrak{m}i_3)(\mathfrak{m}i_4)}, \\
&(F(\nabla^G)(f_{\mathfrak{k}i_2}, f_{\mathfrak{m}i_4})f_{\mathfrak{k}i_2}, f_{\mathfrak{m}i_1}) = (\mathcal{H}(\nabla^G_{f_{\mathfrak{k}i_2}} \nabla^G_{f_{\mathfrak{m}i_4}} f_{\mathfrak{k}i_2}), f_{\mathfrak{m}i_1}) = -\frac{1}{4} c^{(\mathfrak{m}i_1)}_{(\mathfrak{k}i_2)(\mathfrak{m}j)} c^{(\mathfrak{m}j)}_{(\mathfrak{m}i_4)(\mathfrak{k}i_2)}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\mathcal{H}^*(\Delta^G f^{\mathfrak{m}J}) - \overline{\Delta^{G/K} \pi_* f^{\mathfrak{m}J}} \\
&\quad = -f^{\mathfrak{m}j_1} \wedge f^{\mathfrak{m}j_2} \vee f^{\mathfrak{m}J} \cdot \frac{3}{4} c^{(\mathfrak{m}j_2)}_{(\mathfrak{m}j)(\mathfrak{k}i)} c^{(\mathfrak{k}i)}_{(\mathfrak{m}j)(\mathfrak{m}j_1)} \\
&\qquad + f^{\mathfrak{m}j_1} \wedge f^{\mathfrak{m}j_2} \vee f^{\mathfrak{m}j_3} \wedge f^{\mathfrak{m}j_4} \vee f^{\mathfrak{m}J} \\
&\qquad \cdot \left(\frac{3}{4} c^{(\mathfrak{m}j_1)}_{(\mathfrak{m}j_2)(\mathfrak{k}i)} c^{(\mathfrak{k}i)}_{(\mathfrak{m}j_3)(\mathfrak{m}j_4)} - \frac{1}{4} c^{(\mathfrak{m}j_1)}_{(\mathfrak{m}j_4)(\mathfrak{k}i)} c^{(\mathfrak{k}i)}_{(\mathfrak{m}j_2)(\mathfrak{m}j_3)}\right).
\end{aligned}
\tag{2.19}
$$

The two-point homogeneity ([6, Proposition 4.11 (p. 288)]) implies

$$
\mathcal{H}^*(\Delta^G \pi^* \omega_J) - \overline{\Delta^{G/K} \omega_J} = \Delta^G(\omega_J \circ \pi) - (\Delta^{G/K} \omega_J) \circ \pi = 0. \tag{2.20}
$$

Then, by referring to (2.3), we have

$$
-\mathcal{H}^*(\nabla^G_{f_{\mathfrak{m}i}} f^{\mathfrak{m}J}) + \overline{\nabla^{G/K}_{\pi_* f_{\mathfrak{m}i}} \pi_* f^{\mathfrak{m}J}} = -\mathcal{H}^*(\nabla^G_{f_{\mathfrak{m}i}} f^{\mathfrak{m}J}) + \mathcal{H}^*(\nabla^G_{f_{\mathfrak{m}i}} f^{\mathfrak{m}J}) = 0. \tag{2.21}
$$

Thus we obtain (2.17). □

Next, let us define the operator $\mathfrak{D}^G$ in the same way as at (1.3). Then we have the following proposition.

PROPOSITION 2.3. *Take a differential form ω and set $\bar\omega = \sum e_L^{\mathfrak{m}\alpha} \cdot \Omega_\alpha$ with $\Omega_{\sigma(\alpha)} = \operatorname{sgn}(\sigma)\Omega_\alpha$ ($\sigma \in \mathfrak{S}_{|\alpha|}$). Then we have*

$$
\begin{aligned}
\mathcal{H}^*(\Delta^G \bar\omega) - \mathfrak{D}^G \bar\omega = {} & \sum e_L^{\mathfrak{m}i_1} \wedge e_L^{\mathfrak{m}i_2} \vee e_L^{\mathfrak{m}\alpha} \cdot \left\{ \sum \frac{c_{(\mathfrak{m}j)(\mathfrak{k}i)}^{(\mathfrak{m}i_1)} c_{(\mathfrak{m}i_2)(\mathfrak{m}j)}^{(\mathfrak{k}i)}}{2} \Omega_\alpha \right. \\
& \left. - \sum c_{(\mathfrak{m}\alpha_p)(\mathfrak{k}i)}^{(\mathfrak{m}\alpha'_p)} c_{(\mathfrak{m}i_1)(\mathfrak{m}i_2)}^{(\mathfrak{k}i)} \Omega_{\alpha\binom{\alpha'_p}{\alpha_p}} \right\} \\
& - \sum e_L^{\mathfrak{m}i_1} \wedge e_L^{\mathfrak{m}i_2} \vee e_L^{\mathfrak{m}i_3} \wedge e_L^{\mathfrak{m}i_4} \vee e_L^{\mathfrak{m}\alpha} \cdot \frac{c_{(\mathfrak{m}i_2)(\mathfrak{k}i)}^{(\mathfrak{m}i_1)} c_{(\mathfrak{m}i_3)(\mathfrak{m}i_4)}^{(\mathfrak{k}i)}}{2} \Omega_\alpha ,
\end{aligned}
\tag{2.22}
$$

where the expression $\alpha\binom{\alpha'_p}{\alpha_p}$ means α with α_p replaced by $\alpha'_p \in \{1, 2, \ldots, \dim \mathfrak{m}\}$.

PROOF. We will show it on V and $\pi^{-1}(V)$. The formula (2.4) with $e_\bullet$ etc. replaced by $e_\bullet^L$ etc. can be written as

$$
\Delta^G = -e_i^L e_i^L - c_{i_1 i}^{i_2} e_i^L \cdot e_L^{i_1} \wedge e_L^{i_2} \vee + \frac{(c_{i_2 i}^{i_1} - c_{i_1 i}^{i_2}) c_{i_3 i}^{i_4}}{4} \cdot e_L^{i_1} \wedge e_L^{i_2} \vee e_L^{i_3} \wedge e_L^{i_4} \vee . \tag{2.23}
$$

Hence the left hand side of (2.22) is equal to

$$
\begin{aligned}
& -e_L^{\mathfrak{m}\alpha} \cdot \Delta_0^G \Omega_\alpha - e_L^{\mathfrak{m}\alpha} \cdot e_i^L e_i^L(\Omega_\alpha) - e_L^{\mathfrak{m}i_1} \wedge e_L^{\mathfrak{m}i_2} \vee e_L^{\mathfrak{m}\alpha} \cdot e_i^L(\Omega_\alpha) c_{(\mathfrak{m}i_1)i}^{(\mathfrak{m}i_2)} \\
& \quad + e_L^{\mathfrak{m}i_1} \wedge e_L^{\mathfrak{m}i_2} \vee e_L^{\mathfrak{m}\alpha} \cdot \Omega_\alpha \frac{(c_{(\mathfrak{k}j)i}^{(\mathfrak{m}i_1)} - c_{(\mathfrak{m}i_1)i}^{(\mathfrak{k}j)}) c_{(\mathfrak{k}j)i}^{(\mathfrak{m}i_2)}}{4} \\
& \quad + e_L^{\mathfrak{m}i_1} \wedge e_L^{\mathfrak{m}i_2} \vee e_L^{\mathfrak{m}i_3} \wedge e_L^{\mathfrak{m}i_4} \vee e_L^{\mathfrak{m}\alpha} \cdot \Omega_\alpha \frac{(c_{(\mathfrak{m}i_2)i}^{(\mathfrak{m}i_1)} - c_{(\mathfrak{m}i_1)i}^{(\mathfrak{m}i_2)}) c_{(\mathfrak{m}i_3)i}^{(\mathfrak{m}i_4)}}{4} \\
& = -e_L^{\mathfrak{m}i_1} \wedge e_L^{\mathfrak{m}i_2} \vee e_L^{\mathfrak{m}\alpha} \cdot e_{\mathfrak{k}i}^L(\Omega_\alpha) c_{(\mathfrak{m}i_1)(\mathfrak{k}i)}^{(\mathfrak{m}i_2)} + e_L^{\mathfrak{m}i_1} \wedge e_L^{\mathfrak{m}i_2} \vee e_L^{\mathfrak{m}\alpha} \cdot \Omega_\alpha \frac{c_{(\mathfrak{m}j)(\mathfrak{k}i)}^{(\mathfrak{m}i_1)} c_{(\mathfrak{m}i_2)(\mathfrak{m}j)}^{(\mathfrak{k}i)}}{2} \\
& \quad - e_L^{\mathfrak{m}i_1} \wedge e_L^{\mathfrak{m}i_2} \vee e_L^{\mathfrak{m}i_3} \wedge e_L^{\mathfrak{m}i_4} \vee e_L^{\mathfrak{m}\alpha} \cdot \Omega_\alpha \frac{c_{(\mathfrak{m}i_2)(\mathfrak{k}i)}^{(\mathfrak{m}i_1)} c_{(\mathfrak{m}i_3)(\mathfrak{m}i_4)}^{(\mathfrak{k}i)}}{2} .
\end{aligned}
$$

If we put $\omega = \pi_* f^{\mathfrak{m}I} \cdot \omega_I$, then we have $\Omega_\alpha(g) = a_{I\alpha}^{\mathfrak{m}}(k(g))\, \omega_I([g])$, and by observing the proof of (2.15), we have

$$
\begin{aligned}
e_{\mathfrak{k}i}^L(g)(a_{i_p\alpha_p}^{\mathfrak{m}}(k(g))) &= \frac{d}{dt}\bigg|_{t=0} a_{i_p\alpha_p}^{\mathfrak{m}}(g c_{\mathfrak{k}i}(t)) = (e_{\mathfrak{m}i_p}(e), \operatorname{Ad}(k(g))[\dot c_{\mathfrak{k}i}(0), e_{\mathfrak{m}\alpha_p}(e)]) \\
&= (e_{\mathfrak{m}i_p}(e), \operatorname{Ad}(k(g)) e_{\mathfrak{m}\alpha'_p}(e))\, c_{(\mathfrak{k}i)(\mathfrak{m}\alpha_p)}^{(\mathfrak{m}\alpha'_p)} = a_{i_p\alpha'_p}^{\mathfrak{m}} c_{(\mathfrak{k}i)(\mathfrak{m}\alpha_p)}^{(\mathfrak{m}\alpha'_p)} .
\end{aligned}
$$

Hence we know

$$
e_{\mathfrak{k}i}^L(\Omega_\alpha) = e_{\mathfrak{k}i}^L(a_{I\alpha}^{\mathfrak{m}})\, \omega_I = a_{I,\alpha\binom{\alpha'_p}{\alpha_p}}^{\mathfrak{m}} c_{(\mathfrak{k}i)(\mathfrak{m}\alpha_p)}^{(\mathfrak{m}\alpha'_p)} \omega_I = \Omega_{\alpha\binom{\alpha'_p}{\alpha_p}} c_{(\mathfrak{k}i)(\mathfrak{m}\alpha_p)}^{(\mathfrak{m}\alpha'_p)} .
$$

Thus we obtain (2.22). □

3. Proofs of Theorem 1.1, Corollary 1.2. In the rest of the paper, unless otherwise specified, we set $G = SO(n+1) \supset K = SO(n)$ by the injection $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right) \hookleftarrow A$, and denote their Lie algebras by $\mathfrak{g} \supset \mathfrak{k}$ by $\left(\begin{smallmatrix} X_{\mathfrak{k}} & 0 \\ 0 & 0 \end{smallmatrix}\right) \hookleftarrow X_{\mathfrak{k}}$. By applying the results in §2 to the sphere $S^n = G/K = SO(n+1)/SO(n)$, which is certainly a two-point homogeneous compact symmetric Riemannian space, we will prove Theorem 1.1 and Corollary 1.2.

PROOF OF THEOREM 1.1. It is enough to prove it on a neighborhood $\pi^{-1}(V)$ of $e \in G$. On it, we have

$$\begin{aligned} &[e_{\mathfrak{m}i}(e), e_{\mathfrak{m}j}(e)] = \frac{-e_{i,j}(e)}{\sqrt{2(n-1)}}, \quad [e_{\mathfrak{m}i}(e), e_{j_1,j_2}(e)] = \frac{\delta_{ij_1}e_{\mathfrak{m}j_2}(e) - \delta_{ij_2}e_{\mathfrak{m}j_1}(e)}{\sqrt{2(n-1)}}, \\ &[e_{\mathfrak{m}i_2}(e), [e_{\mathfrak{m}i_3}(e), e_{\mathfrak{m}i_4}(e)]] = \frac{-\delta_{i_2i_3}e_{\mathfrak{m}i_4}(e) + \delta_{i_2i_4}e_{\mathfrak{m}i_3}(e)}{2(n-1)}. \end{aligned} \tag{3.1}$$

We assume that ω is a q-form. Let us investigate the right hand sides of (2.17) and (2.22). As for (2.17), since we have

$$\begin{aligned} &\sum c^{(\mathfrak{m}j_2)}_{(\mathfrak{m}j)(\mathfrak{k}i)} c^{(\mathfrak{k}i)}_{(\mathfrak{m}j)(\mathfrak{m}j_1)} = \sum e_L^{\mathfrak{m}j_2}([e^L_{\mathfrak{m}j}, [e^L_{\mathfrak{m}j}, e^L_{\mathfrak{m}j_1}]]) \\ &\quad = -\sum \frac{\delta_{jj}\delta_{j_2j_1} - \delta_{jj_1}\delta_{j_2j}}{2(n-1)} = -\frac{n\delta_{j_2j_1} - \delta_{j_2j_1}}{2(n-1)} = -\frac{1}{2}\delta_{j_2j_1}, \\ &\sum\{3c^{(\mathfrak{m}j_1)}_{(\mathfrak{m}j_2)(\mathfrak{k}i)} c^{(\mathfrak{k}i)}_{(\mathfrak{m}j_3)(\mathfrak{m}j_4)} - c^{(\mathfrak{m}j_1)}_{(\mathfrak{m}j_4)(\mathfrak{k}i)} c^{(\mathfrak{k}i)}_{(\mathfrak{m}j_2)(\mathfrak{m}j_3)}\} \\ &\quad = 3e_L^{\mathfrak{m}j_1}([e^L_{\mathfrak{m}j_2}, [e^L_{\mathfrak{m}j_3}, e^L_{\mathfrak{m}j_4}]]) - e^{\mathfrak{m}j_1}([e^L_{\mathfrak{m}j_4}, [e^L_{\mathfrak{m}j_2}, e^L_{\mathfrak{m}j_3}]]) \\ &\quad = -\frac{1}{2(n-1)}(3\delta_{j_2j_3}\delta_{j_1j_4} - 4\delta_{j_2j_4}\delta_{j_1j_3} + \delta_{j_4j_3}\delta_{j_1j_2}), \end{aligned}$$

we know

$$\begin{aligned} &\mathcal{H}^*(\Delta^G\bar{\omega}) - \overline{\Delta^{G/K}\omega} \\ &= \frac{3}{8}\sum f^{\mathfrak{m}j_1} \wedge f^{\mathfrak{m}j_1} \vee \bar{\omega} - \frac{1}{8(n-1)}\sum\{3\delta_{j_2j_3}\delta_{j_1j_4} f^{\mathfrak{m}j_1} \wedge f^{\mathfrak{m}j_2} \vee f^{\mathfrak{m}j_3} \wedge f^{\mathfrak{m}j_4} \vee \\ &\quad - 4\delta_{j_2j_4}\delta_{j_1j_3} f^{\mathfrak{m}j_1} \wedge f^{\mathfrak{m}j_2} \vee f^{\mathfrak{m}j_3} \wedge f^{\mathfrak{m}j_4} \vee \\ &\quad + \delta_{j_4j_3}\delta_{j_1j_2} f^{\mathfrak{m}j_1} \wedge f^{\mathfrak{m}j_2} \vee f^{\mathfrak{m}j_3} \wedge f^{\mathfrak{m}j_4} \vee\}\bar{\omega} \\ &= \frac{3q}{8}\bar{\omega} - \frac{1}{8(n-1)}\left\{3\sum_{j_1 \in I \not\ni j_2} f^{\mathfrak{m}j_1} \wedge f^{\mathfrak{m}j_2} \vee f^{\mathfrak{m}j_2} \wedge f^{\mathfrak{m}j_1} \vee f^{\mathfrak{m}I} \cdot \omega_I \right. \\ &\quad \left. + \sum_{\substack{j_1 \neq j_2 \\ j_1, j_2 \in I}} f^{\mathfrak{m}j_1} \wedge f^{\mathfrak{m}j_1} \vee f^{\mathfrak{m}j_2} \wedge f^{\mathfrak{m}j_2} \vee f^{\mathfrak{m}I} \cdot \omega_I \right\} \\ &= \frac{3q}{8}\bar{\omega} - \frac{3q(n-q) + q(q-1)}{8(n-1)}\bar{\omega} = \frac{q(q-1)}{4(n-1)}\bar{\omega}. \end{aligned} \tag{3.2}$$

As for (2.22), since we have

$$\sum e_L^{\mathfrak{m}i_1}\wedge e_L^{\mathfrak{m}i_2}\vee e_L^{\mathfrak{m}\alpha}\cdot\Omega_\alpha\frac{c_{(\mathfrak{m}j)(\mathfrak{k}i)}^{(\mathfrak{m}i_1)}c_{(\mathfrak{m}i_2)(\mathfrak{m}j)}^{(\mathfrak{k}i)}}{2}=\frac14\sum e_L^{\mathfrak{m}i_1}\wedge e_L^{\mathfrak{m}i_1}\vee e_L^{\mathfrak{m}\alpha}\cdot\Omega_\alpha=\frac{q}{4}\bar\omega\,,$$

$$-\sum e_L^{\mathfrak{m}i_1}\wedge e_L^{\mathfrak{m}i_2}\vee e_L^{\mathfrak{m}\alpha}\cdot\Omega_{\alpha\binom{\alpha'_p}{\alpha_p}}c_{(\mathfrak{m}\alpha_p)(\mathfrak{k}i)}^{(\mathfrak{m}\alpha'_p)}c_{(\mathfrak{m}i_1)(\mathfrak{m}i_2)}^{(\mathfrak{k}i)}$$

$$=-\sum_{i_1\notin\alpha\ni i_2}e_L^{\mathfrak{m}i_1}\wedge e_L^{\mathfrak{m}i_2}\vee e_L^{\mathfrak{m}\alpha}\cdot\Omega_{\alpha\binom{\alpha'_p}{\alpha_p}}\frac{\delta_{\alpha_p i_2}\delta_{\alpha'_p i_1}}{2(n-1)}$$

$$=-\frac{1}{2(n-1)}\sum_{i_1\notin\alpha\ni i_2}e_L^{\mathfrak{m}i_1}\wedge e_L^{\mathfrak{m}i_2}\vee e_L^{\mathfrak{m}\alpha}\cdot\Omega_{\alpha\binom{i_1}{i_2}}=-\frac{q(n-q)}{2(n-1)}\bar\omega\,,$$

$$-\sum e_L^{\mathfrak{m}i_1}\wedge e_L^{\mathfrak{m}i_2}\vee e_L^{\mathfrak{m}i_3}\wedge e_L^{\mathfrak{m}i_4}\vee e_L^{\mathfrak{m}\alpha}\cdot\Omega_\alpha\frac{c_{(\mathfrak{m}i_2)(\mathfrak{k}i)}^{(\mathfrak{m}i_1)}c_{(\mathfrak{m}i_3)(\mathfrak{m}i_4)}^{(\mathfrak{k}i)}}{2}$$

$$=\frac{1}{4(n-1)}\sum_{i_3\neq i_4}^{i_1\neq i_2}e_L^{\mathfrak{m}i_1}\wedge e_L^{\mathfrak{m}i_2}\vee e_L^{\mathfrak{m}i_3}\wedge e_L^{\mathfrak{m}i_4}\vee e_L^{\mathfrak{m}\alpha}\cdot\Omega_\alpha(\delta_{i_2i_3}\delta_{i_1i_4}-\delta_{i_2i_4}\delta_{i_1i_3})$$

$$=\frac{1}{4(n-1)}\sum_{i_1\in\alpha\not\ni i_2}e_L^{\mathfrak{m}i_1}\wedge e_L^{\mathfrak{m}i_2}\vee e_L^{\mathfrak{m}i_2}\wedge e_L^{\mathfrak{m}i_1}\vee e_L^{\mathfrak{m}\alpha}\cdot\Omega_\alpha=\frac{q(n-q)}{4(n-1)}\bar\omega\,,$$

we know

$$\mathcal{H}^*(\Delta^G\bar\omega)-\mathfrak{D}^G\bar\omega=\frac{q}{4}\bar\omega-\frac{q(n-q)}{2(n-1)}\bar\omega+\frac{q(n-q)}{4(n-1)}\bar\omega=\frac{q(q-1)}{4(n-1)}\bar\omega\,. \tag{3.3}$$

Then (3.2) and (3.3) imply the theorem. □

PROOF OF COROLLARY 1.2. Take a differential form ω on $S^n=G/K$. Theorem 1.1 yields that

$$(d/dt+\mathfrak{D}^G)\overline{e^{-t\Delta^{G/K}}\omega}=\overline{(d/dt+\Delta^{G/K})e^{-t\Delta^{G/K}}\omega}=0\,,\quad \lim_{t\to0}\overline{e^{-t\Delta^{G/K}}\omega}=\bar\omega\,.$$

Hence, by the uniqueness of the solution for the initial value problem associated to the heat equation (for $\mathfrak{D}^G$), we have $e^{-t\mathfrak{D}^G}\bar\omega=\overline{e^{-t\Delta^{G/K}}\omega}$.

Suppose now that $\omega([h'])=\sum\pi_*f_{h'}^{\mathfrak{m}I'}([h'])\cdot\omega_{I'}^{h'}([h'])$ has a support near a given point in G/K. Let μ_G, μ_K be the Haar measures on G, K given by the frames $e_\bullet^L$, $e_{\mathfrak{k}\bullet}^L$ and let $\mu_{G/K}$ be the left-invariant measure on G/K which is associated to $e_{\mathfrak{m}\bullet}^L$. Then we have

$$\overline{e^{-t\Delta^{G/K}}\omega}=\sum\overline{(\pi_*f_h^{\mathfrak{m}I})\cdot(e^{-t\Delta^{G/K}})_{II'}^{h,h'}\omega_{I'}^{h'}}=\sum f_h^{\mathfrak{m}I}\cdot\overline{(e^{-t\Delta^{G/K}})_{II'}^{h,h'}\omega_{I'}^{h'}}$$

$$=\sum f_h^{\mathfrak{m}I}(hk)\cdot\int_{G/K}d\mu_{G/K}([h'])(e^{-t\Delta^{G/K}})_{II'}^{h,h'}([h],[h'])\,\omega_{I'}^{h'}([h'])\,,$$

$$(e^{-t\mathfrak{D}^G}\bar\omega)(hk)=\sum e_L^{\mathfrak{m}J}(hk)\cdot\int_G d\mu_G(h'k')e^{-t\Delta_0^G}(hk,h'k')\,a_{I'J}^{\mathfrak{m}}(k')\,\omega_{I'}^{h'}([h'])$$

$$=\sum f_h^{\mathfrak{m}I}(hk)\cdot\int_{G/K}d\mu_{G/K}([h'])\left(\int_K d\mu_K(k')\,e^{-t\Delta_0^G}(hk,h'k')A_{II'}^{\mathfrak{m}}(kk'^{-1})\right)\omega_{I'}^{h'}([h'])\,.$$

By the equalities $e^{-t\Delta_0^G}(gg_1, gg_2) = e^{-t\Delta_0^G}(g_1, g_2) = e^{-t\Delta_0^G}(g_1 g, g_2 g)$, we have

$$\begin{aligned}(e^{-t\Delta^{G/K}})^{h,h'}_{II'}([h],[h']) &= \int_K d\mu_K(k')e^{-t\Delta_0^G}(h, h'k')\, A^{\mathfrak{m}}_{II'}(k'^{-1}) \\ &= \int_K d\mu_K(k')e^{-t\Delta_0^G}(h'^{-1}hk'^{-1}, e)\, A^{\mathfrak{m}}_{II'}(k'^{-1}) = \int_K d\mu_K(k)e^{-t\Delta_0^G}(h'^{-1}hk, e)\, A^{\mathfrak{m}}_{II'}(k)\,.\end{aligned}$$

Thus we obtain the formula (1.8). □

4. Proof of Corollary 1.3. Let G be a semisimple compact connected Lie group and let T_G be a maximal torus. The corresponding subalgebra $\mathfrak{t}_G$ of $\mathfrak{g}$ gives the Cartan subalgebra $\mathfrak{t}_G^{\boldsymbol{C}} = \sqrt{-1}\mathfrak{t}_G$ of the complexified Lie algebra $\mathfrak{g}^{\boldsymbol{C}} = \sqrt{-1}\mathfrak{g}$, which has the (positive) root systems $\Phi_G^{\boldsymbol{C}(+)} = \sqrt{-1}\Phi_G^{(+)}$ contained in $\sqrt{-1}\mathfrak{t}_G^*$. Let us take the lattice $\Gamma_G = \{\gamma \in \mathfrak{t}_G\,;\, \exp\gamma = e\}$. Then, if $u = \exp U \in T_G = \exp\mathfrak{t}_G$ satisfies $\alpha(U) \notin 2\pi\boldsymbol{Z}$ for all $\alpha \in \Phi_G^+$, that is, if u is a regular element of T_G^{reg}, the summation formula due to Urakawa ([15, Theorem 2], see also [4], [2]) implies that

$$e^{-t\Delta_0^G}(u, e) = \frac{e^{tn_0/24}}{(4\pi t)^{n_0/2}} \sum_{\gamma\in\Gamma_G} e^{-|U+\gamma|^2/4t} \prod_{\alpha\in\Phi_G^+} \frac{\alpha(U+\gamma)/2}{\sin\alpha(U+\gamma)/2} \tag{4.1}$$

for $n_0 = \dim G$, which converges absolutely.

For our $G = SO(n+1)$, let us take the maximal torus T_G as in the introduction, and take the frame $(U_1, \dots)$ of $\mathfrak{t}_G$ defined by $U_i = E_{2i-1,2i} - E_{2i,2i-1}$ (see (1.1)) and denote its dual frame by $(\lambda_1, \dots)$, i.e., $\lambda_i(U_j) = \delta_{ij}$. As is well-known ([10, pp. 684–685]), if G is equal to $SO(2m+1)$ (i.e., of type B_m), we have

$$\begin{aligned}&\mathfrak{t}_G = \{U_i\ ;\ 1 \le i \le m\}_{\boldsymbol{R}} \supset \Gamma_G = \{2\pi U_i\ ;\ 1 \le i \le m\}_{\boldsymbol{Z}}\,,\\ &\Phi_G^+ = \{\lambda_i - \lambda_j,\ \lambda_i + \lambda_j\ ;\ 1 \le i < j \le m\} \cup \{\lambda_i\ ;\ 1 \le i \le m\}\end{aligned} \tag{4.2}$$

and, if G is equal to $SO(2m)$ (i.e., of type D_m), we have

$$\begin{aligned}&\mathfrak{t}_G = \{U_i\ ;\ 1 \le i \le m\}_{\boldsymbol{R}} \supset \Gamma_G = \{2\pi U_i\ ;\ 1 \le i \le m\}_{\boldsymbol{Z}}\,,\\ &\Phi_G^+ = \{\lambda_i - \lambda_j, \lambda_i + \lambda_j\ ;\ 1 \le i < j \le m\}\,.\end{aligned} \tag{4.3}$$

Therefore, at (4.1) with $G = SO(n+1) = SO(2m+1)$ or $SO(2m)$, we can write $U = \sum_{i=1}^m \theta_i U_i$ ($\theta_i \in \boldsymbol{R}$) and $\gamma = 2\pi\sum_{i=1}^m l_i U_i$ ($l_i \in \boldsymbol{Z}$) so that, for $u \in T_G^{\mathrm{reg}}$, we have the absolutely convergent summation formula

$$\begin{aligned}e^{-t\Delta_0^G}(u, e) = {}& \frac{e^{tn_0/24}}{(4\pi t)^{n_0/2}} \prod_{\alpha\in\Phi_G^+} \frac{1}{\sin\alpha(U)/2} \\ &\times \sum_l \varepsilon(l)\, e^{-2(n-1)\sum(\theta_i+2\pi l_i)^2/4t} \prod_{\alpha\in\Phi_G^+} \alpha\Big(\sum(\theta_i + 2\pi l_i)U_i\Big)/2\,.\end{aligned} \tag{4.4}$$

Remark here that $l = (l_1, \dots)$ runs over $\boldsymbol{Z}^m$ and $\varepsilon(l)$ means 1 if $\sum l_i$ is even and $(-1)^{n+1}$ if $\sum l_i$ is odd. The Urakawa summation formula primarily deals with a simply connected group, that is, not $G = SO(n+1)$ but the universal covering group $\tilde{G} = \mathrm{Spin}(n+1)$. We have $\Gamma_G \supset \Gamma_{\tilde{G}} = \{2\pi \sum l_i U_i\,;\ l_i \in \boldsymbol{Z},\ \sum l_i \in 2\boldsymbol{Z}\}$. Also, for the nonzero element $[c]$ of $\pi_1(G) = \boldsymbol{Z}_2$, if we take such $\gamma_{[c]} \in \Gamma_G$ that the deck transformation of $\tilde{G}$ given by $[c]$ sends $e \in \tilde{G}$ to $\exp \gamma_{[c]} \in \tilde{G}$ (for example we set $\gamma_{[c]} = 2\pi U_1$), then we have $\exp((\sqrt{-1}/2)\sum_{\alpha\in\Phi_G^+}\alpha(\gamma_{[c]})) = -1$ if n is even and 1 if n is odd. It may be more faithful to his theory than above to say (4.4) is then obtained by applying his formula for $\tilde{G}$ to the right hand side of $e^{-t\Delta_0^G}(u,e) = \sum_{p(\tilde{u})=u} e^{-t\Delta_0^{\tilde{G}}}(\tilde{u},e)$, where $p : \tilde{G} \to G$ is the covering map.

Here, let us apply the Weyl integration formula ([10, Theorem 8.60]) to the right hand side of (1.8) ($n \geq 3$). We take the maximal torus T_K (as in the introduction) etc. of $K = SO(n)$. For $u = \exp U \in T_K$, we set

$$\Omega(u) = \prod_{\sqrt{-1}\alpha\in\Phi_K^C} \left(e^{\sqrt{-1}\alpha(U)/2} - e^{-\sqrt{-1}\alpha(U)/2}\right) = 2^{|\Phi_K|} \prod_{\alpha\in\Phi_K^+} \sin^2 \alpha(U)/2\,, \tag{4.5}$$

and normalize the measures μ_{T_K}, μ_{K/T_K} on T_K, K/T_K as $\int_{T_K} d\mu_{T_K}(u) = 1$, $\int_{K/T_K} d\mu_{K/T_K}([k]) = 1$. We denote by $\mathrm{vol}(K)$ the volume of K with respect to the metric of $G = SO(n+1)$, and by W_K the Weyl group of K. Then the Weyl formula says

$$\begin{aligned}(e^{-t\Delta^{G/K}})^{h,h'}_{II'}([h],[h']) &= \frac{\mathrm{vol}(K)}{|W_K|}\int_{K/T_K} d\mu_{K/T_K}([k]) \\ &\quad\times \int_{T_K} d\mu_{T_K}(u)\, \Omega(u) e^{-t\Delta_0^G}(h'^{-1}hkuk^{-1}, e) A^{\mathfrak{m}}_{II'}(kuk^{-1})\,,\end{aligned} \tag{4.6}$$

$$\begin{aligned}&(e^{-t\Delta^{G/K}})^{h,h}_{II'}([h],[h]) \\ &= \frac{\mathrm{vol}(K)}{|W_K|}\int_{T_K} d\mu_{T_K}(u)\, \Omega(u) e^{-t\Delta_0^G}(u,e) \int_{K/T_K} d\mu_{K/T_K}([k])\, A^{\mathfrak{m}}_{II'}(kuk^{-1})\,.\end{aligned} \tag{4.7}$$

We also have the following lemma.

LEMMA 4.1. *Assume $n \geq 3$. For $u = \exp U = \sum_{1\leq i\leq[n/2]} \theta_i U_i \in T_K \cap T_G^{\mathrm{reg}}$, we have the absolutely convergent summation formula*

$$\Omega(u) e^{-t\Delta_0^G}(u,e) = \frac{1}{\pi^{[n/2][(n-1)/2]}} \frac{e^{tn(n+1)/48}}{(4\pi t)^{n/2+[n/2]/2}} \sum_{l\geq 0} F_l(t,\theta)\,. \tag{4.8}$$

PROOF. We have $n_0 = n(n+1)/2$. As for the case $n = 2m$, by referring to (4.2) through (4.5), we have

$$\begin{aligned}\Omega(u)e^{-t\Delta_0^G}(u,e) &= \frac{e^{tn_0/24}\, 2^{2m(m-1)}}{(4\pi t)^{n_0/2}} \frac{\prod_{\alpha\in\Phi_K^+}\sin^2\alpha(U)/2}{\prod_{\alpha\in\Phi_G^+}\sin\alpha(U)/2} \\ &\quad\times \sum_l (-1)^{\sum l_i} e^{-2(n-1)\sum(\theta_i+2\pi l_i)^2/4t} \prod_{\alpha\in\Phi_G^+} \alpha\Big(\sum(\theta_i+2\pi l_i)U_i\Big)/2 \\ &= \frac{1}{\pi^{m[(n-1)/2]}} \frac{e^{tn(n+1)/48}}{(4\pi t)^{n/2+m/2}} \frac{1}{t^{m(m-1)}} \frac{\prod_{i<j}\{\sin^2\theta_i/2 - \sin^2\theta_j/2\}}{\prod_i \sin\theta_i/2} \sum_{l\in\mathbf{Z}^m}(-1)^{\sum l_i} \\ &\quad\cdot e^{-2(n-1)\sum(\theta_i+2\pi l_i)^2/4t} \prod_i \frac{\theta_i+2\pi l_i}{2} \prod_{i<j} \frac{(\theta_i+2\pi l_i)^2-(\theta_j+2\pi l_j)^2}{4}.\end{aligned} \tag{4.9}$$

Thus we obtain (4.8) with (1.10). Its absolute convergence is obvious because (4.4) converges absolutely. As for the case $n = 2m-1$, we have

$$\begin{aligned}\Omega(u)e^{-t\Delta_0^G}(u,e) &= \frac{e^{tn_0/24}\, 2^{2(m-1)^2}}{(4\pi t)^{n_0/2}} \frac{\prod_{\alpha\in\Phi_K^+}\sin^2\alpha(U)/2}{\prod_{\alpha\in\Phi_G^+}\sin\alpha(U)/2} \\ &\quad\times \sum_l e^{-2(n-1)\sum(\theta_i+2\pi l_i)^2/4t} \prod_{\alpha\in\Phi_G^+} \alpha\Big(\sum(\theta_i+2\pi l_i)U_i\Big)/2 \\ &= \frac{1}{\pi^{(m-1)[(n-1)/2]}} \frac{e^{tn(n+1)/48}}{(4\pi t)^{n/2+(m-1)/2}} \frac{1}{t^{(m-1)^2}} \prod_{i<j<m}\{\sin^2\theta_i/2 - \sin^2\theta_j/2\} \\ &\quad\times \sum_{l\in\mathbf{Z}^{m-1}} e^{-2(n-1)\sum(\theta_i+2\pi l_i)^2/4t} \prod_{i<m}\left(\frac{\theta_i+2\pi l_i}{2}\right)^2 \\ &\quad\times \prod_{i<j<m} \frac{(\theta_i+2\pi l_i)^2-(\theta_j+2\pi l_j)^2}{4}.\end{aligned} \tag{4.10}$$

Thus we obtain (4.8) with (1.11). Its absolute convergence is obvious. □

Now let us prove Corollary 1.3.

PROOF OF COROLLARY 1.3. As for the smoothness of $F_l(t,\theta)$, we have only to show it at $\theta = 0$. In the case $n = 2m-1$, it will be obvious. We will show it in the case $n = 2m$. Referring to (4.9), for a fixed $l \geq 0$, it suffices to show that

$$\sum_\varepsilon e^{-2(n-1)\sum(\theta_i+2\pi\varepsilon_i l_i)^2/4t} \prod_i \frac{\theta_i+2\pi\varepsilon_i l_i}{2} \prod_{i<j} \frac{(\theta_i+2\pi\varepsilon_i l_i)^2-(\theta_j+2\pi\varepsilon_j l_j)^2}{4} \tag{4.11}$$

is an odd function with respect to each θ_i. By using multi-indices $p = (p_1, \dots) \in \boldsymbol{Z}^m$ with $p \geq 0$ and certain constants a_p, (4.11) can be written as

$$\begin{aligned} &\sum_{\varepsilon, p}^{\text{(finite sum)}} a_p e^{-2(n-1)\sum(\theta_i + 2\pi\varepsilon_i l_i)^2/4t} (\theta + 2\pi\varepsilon l)^{2p+1} \\ &= \sum_p a_p e^{-2(n-1)\sum(\theta_i^2 + (2\pi l_i)^2)/4t} \prod_{l_i=0} \theta_i^{2p_i+1} \\ &\quad \times \prod_{l_i>0} \{ e^{-2\pi(n-1)\theta_i l_i/t} (\theta_i + 2\pi l_i)^{2p_i+1} + e^{2\pi(n-1)\theta_i l_i/t} (\theta_i - 2\pi l_i)^{2p_i+1} \}, \end{aligned} \tag{4.12}$$

which is certainly such an odd function.

Next, let us show the estimates (1.12). Assume $n = 2m$ and $|\theta_i| \leq \pi/4$. Referring to (4.12) and (1.10), there exist constants $C_1, C_2, \dots > 0$, independent of $l \geq 0$, such that

$$\begin{aligned} &\left| e^{-2\pi(n-1)\theta_i l_i/t} \left(\frac{\theta_i + 2\pi l_i}{t^{1/2}} \right)^{2p_i+1} + e^{2\pi(n-1)\theta_i l_i/t} \left(\frac{\theta_i - 2\pi l_i}{t^{1/2}} \right)^{2p_i+1} \right| \\ &\qquad \leq C_1 \{1 + (\theta_i/t^{1/2})^{2p_i+1}\}\{1 + (2\pi l_i/t^{1/2})^{2p_i+2}\} |\theta_i/t^{1/2}| e^{\pi^2(n-1)l_i/2t}, \end{aligned}$$

$$\begin{aligned} |F_l(t,\theta)| &\leq C_2 e^{-2(n-1)\sum\{\theta_i^2/4t + (2\pi l_i)^2/4t - \pi^2 l_i/2t\}} \\ &\qquad \times \prod_i \left| \frac{t^{-1/2}\theta_i}{t^{-1/2}\sin\theta_i/2} \right| \cdot \{1 + (\theta_i/t^{1/2})^{2p+1}\}\{1 + (2\pi l_i/t^{1/2})^{2p+2}\} \\ &\leq C_3 e^{-2(n-1)\sum\{\theta_i^2/5t + l_i/5t\}}, \end{aligned}$$

$$\sum_{l\geq 0} |F_l(t,\theta)| \leq C_3\, e^{-2(n-1)\sum\theta_i^2/5t} \sum_{l\geq 0} e^{-2(n-1)\sum l_i/5t} \leq C_4 e^{-2(n-1)\sum\theta_i^2/5t}.$$

If $\pi/4 \leq |\theta| \leq \pi$, then the estimate holds obviously. Next assume $n = 2m - 1$. Then we have

$$\begin{aligned} |F_l(t,\theta)| &\leq C_1 |\theta/t^{1/2}|^{\binom{m-1}{2}} \sum e^{-2(n-1)\sum(\theta_i + 2\pi\varepsilon_i l_i)^2/(9/2)t} \\ &\leq C_2\, |\theta/t^{1/2}|^{\binom{m-1}{2}} e^{2(n-1)\{-\sum\theta_i^2 - (2\pi)^2\sum l_i\}/(9/2)t} \leq C_3 e^{2(n-1)\{-\sum\theta_i^2 - \sum l_i\}/5t}, \end{aligned}$$

$$\sum_{l\geq 0} |F_l(t,\theta)| \leq C_4 e^{-2(n-1)\sum\theta_i^2/5t}.$$

Thus (1.12) were shown.

We will prove (1.13) through (1.15). The Haar measure μ_{T_K} is given by the volume element $(2\pi)^{-[n/2]} d\theta_{[n/2]} \wedge \cdots \wedge d\theta_1$. This, together with (1.12), (4.7) and (4.8) (hence, $n \geq 3$), yields the expression (1.13) of termwise integration with

$$c_n = \frac{\mathrm{vol}(K)}{|W_K|} \frac{1}{(2\pi^{[(n-1)/2]+1})^{[n/2]}} = \frac{\mathrm{vol}(K)}{|W_K|} \frac{1}{(2\pi^{[(n+1)/2]})^{[n/2]}}, \tag{4.13}$$

$$B^{\mathfrak{m}}_{II'}(\theta) = \int_{K/T_K} d\mu_{K/T_K}([k]) A^{\mathfrak{m}}_{II'}(ku(\theta)k^{-1}) . \tag{4.14}$$

The second equality at (1.15) certainly holds because

$$a^{\mathfrak{m}}_{jj'}(u(\theta)) = (e_{\mathfrak{m}j}(e), (\mathrm{Ad}(u(\theta))e_{\mathfrak{m}j'}(e)) = {}^t\boldsymbol{e}_j \cdot D(\theta)\boldsymbol{e}_{j'} = D(\theta)_{jj'} ,$$
$$|I|!\, A^{\mathfrak{m}}_{II'}(ku(\theta)k^{-1}) = \det(a^{\mathfrak{m}}_{ii'}(ku(\theta)k^{-1})) = \det(a^{\mathfrak{m}}(k)D(\theta)a^{\mathfrak{m}}(k^{-1}))_{II'} .$$

We need to show that (4.13) is equal to the right hand side of (1.14). Denote by $\mathrm{vol}(K, \mathfrak{k})$ and $\mathrm{vol}(K, \mathfrak{g})$ the volumes of $K = SO(n)$ with respect to the metrics $-B_{\mathfrak{k}}$ and $-B_{\mathfrak{g}}|\mathfrak{k}$, respectively. It follows from [1] that

$$\begin{aligned} \mathrm{vol}(K, \mathfrak{g}) &= \left(\frac{n-1}{n-2}\right)^{n(n-1)/4} \mathrm{vol}(K, \mathfrak{k}) \\ &= \begin{cases} \dfrac{2^{m^2-1}(4m-2)^{m(2m-1)/2}}{2!\,4!\cdots(2m-2)!}(2\pi^m)^m & (n = 2m) , \\ \dfrac{2^{m^2-1}(4m-4)^{(m-1)(2m-1)/2}(m-1)!}{2!\,4!\cdots(2m-2)!}(2\pi^m)^{m-1} & (n = 2m-1) . \end{cases} \end{aligned}$$

Since we have $\mathrm{vol}(K) = \mathrm{vol}(K, \mathfrak{g})$, $|W_{SO(2m)}| = m!\,2^{m-1}$ and $|W_{SO(2m-1)}| = (m-1)!\,2^{m-1}$ ([10, pp. 684–685]), (4.13) is certainly equal to (1.14). Thus the proof of the corollary with $n \geq 3$ is complete. As for the case $n = 2$, that is, $S^2 = G/K = SO(3)/SO(2)$, since $K = SO(2) = T_K \ni u = u(\theta_1)$, (4.4) says

$$e^{-t\Delta^G_0}(u, e) = \frac{e^{t/8}}{(4\pi t)^{2/2+1/2}} \sum_{l_1} e^{-2(\theta_1+2\pi l_1)^2/4t}(-1)^{l_1} \frac{(\theta_1 + 2\pi l_1)/2}{\sin\theta_1/2} ,$$

which coincides with (1.10) for $n = 2$. Hence (1.8) with $n = 2$ implies (1.13) with $n = 2$. □

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