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ISOMETRIC IMMERSIONS OF EUCLIDEAN PLANE INTO EUCLIDEAN 4-SPACE WITH VANISHING NORMAL CURVATURE

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Abstract. Every isometric immersion of $\mathbf{R}^2$ into $\mathbf{R}^4$ with vanishing normal curvature is assosiated with a pair of real-valued functions satisfying a system of second order partial differential equations of hyperbolic type, and vice versa. An isometric immersion with vanishing normal curvature is revealed to be multiple-valued in general as is shown by some concrete examples.

Introduction. Hartman [12] showed that, for each pair of integers (n, p) with $1 \leq p < n$, an isometric immersion f of $\boldsymbol{R}^n$ into $\boldsymbol{R}^{n+p}$ is written as

$$f = B \circ (1 \times h) \circ A\,,$$

where A is an isometry of $\boldsymbol{R}^n$, 1 is the identity mapping in $\boldsymbol{R}^{n-p}$, h is an isometric immersion of $\boldsymbol{R}^p$ into $\boldsymbol{R}^{2p}$ and B is an isometry of $\boldsymbol{R}^{n+p}$. In view of this, the problem of describing all isometric immersions f of $\boldsymbol{R}^n$ into $\boldsymbol{R}^{n+p}$ is reduced to that of describing all isometric immersions h of $\boldsymbol{R}^p$ into $\boldsymbol{R}^{2p}$. For $p = 1$, every h is completely characterized by a real-valued function of single variable (see Chern-Kuiper [5] and Dajczer *et al.* [6] for more detailed informations). For $p \geq 2$, the problem of describing all h remains elusive, even for $p = 2$.

In a paper [2], do Carmo and Dajczer have constructed all local flat immersions of $\boldsymbol{R}^2$ into $\boldsymbol{R}^4$ which are nowhere composition and whose first normal spaces have dimension 2 (see [7] for related works). They constructed an immersion f of a small open subset U of $\boldsymbol{R}^2$ into $\boldsymbol{R}^4$ by means of a quartet $\{\xi, u, w, \gamma\}$, where ξ is an immersion of U into S^3, u and w are linearly independent unit vector fields on U with respect to the pull-back g of the standard metric on S^3 through ξ, and γ is a function on U. They also showed that the normal curvature of the immersion f is zero or non-zero according as the integral curves of w are geodesic or not, respectively, in the Riemannian manifold (U, g). Their construction depends on a geometric argument, however it is local one. Based on the results of [2], Dajczer and Tojeiro [8] have classified all local flat surfaces in $\boldsymbol{R}^2$ with flat normal bundle, that is to say, with vanishing normal curvature.

Recently, Gálvez and Mira [9] announced that they generated new complete flat cylinders in $\boldsymbol{R}^4$ with vanishing normal curvature and regular Gauss maps as small perturbations of Hopf cylinders.

In this article, we restrict ourselves to isometric immerions of $\boldsymbol{R}^2$ into $\boldsymbol{R}^4$ with vanishing normal curvature and start from the general theory by É. Cartan [3]. The structure equations

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of $\boldsymbol{R}^4$ involve ten exterior differential 1-forms (see (3)). These forms are to be represented by means of a standard coordinate system (x, y) of $\boldsymbol{R}^2$. Five 1-forms among them are known owing to [3, no. 50]. We can omit another 1-form by assuming that the normal curvature vanishes identically. Then, remaining four 1-forms contain only two real-valued functions. This is a reduction of our problem.

Our first result is the following. Every isometric immersion f of $\boldsymbol{R}^2$ into $\boldsymbol{R}^4$ with vanishing normal curvature is associated with a pair of real-valued functions (u_1, u_2) satisfying a system of partial differential equations

$$(1)\qquad \begin{aligned} &({\partial_x}^2 u_1 - {\partial_y}^2 u_1)\partial_x\partial_y u_2 = ({\partial_x}^2 u_2 - {\partial_y}^2 u_2)\partial_x\partial_y u_1\,,\\ &(\partial_x\partial_y u_1)^2 + (\partial_x\partial_y u_2)^2 = ({\partial_x}^2 u_1)({\partial_y}^2 u_1) + ({\partial_x}^2 u_2)({\partial_y}^2 u_2)\,, \end{aligned}$$

and vice versa (see Proposition 1.1 in §1). f is global if and only if (u_1, u_2) is global. We can leave the structure equations aside. The question is how to solve (1).

A key is the invariance of (1) by isometries of the xy-plane and by isometries of the u_1u_2-plane. This leads us to the definition of polar coordinates (ρ, a, b, c) of (u_1, u_2) (see Lemma 2.1 in §2). Above all, ρ and a are invariant and the rank of a matrix

$$P = \begin{pmatrix} {\partial_x}^2 u_1 & \sqrt{2}\,\partial_x\partial_y u_1 & {\partial_y}^2 u_1 \\ {\partial_x}^2 u_2 & \sqrt{2}\,\partial_x\partial_y u_2 & {\partial_y}^2 u_2 \end{pmatrix}$$

is also invariant (see (21)). In this article, we study two cases where the rank of P is identically equal to 1 or identically equal to 2. In the first case, an equation

$$(24)\qquad \partial_x \sin\theta = \partial_y \cos\theta$$

is fundamental (see Proposition 3.1 in §3). In the second case, an important part of (1) is reduced to a semi-linear equation

$$(36)\qquad \frac{\partial^2 \hat{A}}{\partial {s_1}^2} - \frac{\partial^2 \hat{A}}{\partial {s_2}^2} + \hat{B} = 0$$

after a change of independent variables from (x, y) to (s_1, s_2) (see Proposition 4.2 in §4).

We can conclude that isometric immersions and the solutions of (1) are multiple-valued in general. Solutions θ of (24) are in fact multiple-valued in the first case (see Lemma 3.3 in §3 and Examples 3, 4 in §5), and the transformations from (x, y) to (s_1, s_2) are multiple-valued in the second case. For the isometric immersion in Example 6, a classical asymptotic analysis guarantees an infinite number of function elements (see Lemmas 5.2 and 5.3 in §5) and the image of $\boldsymbol{R}^2$ is a Riemann surface realized in $\boldsymbol{R}^4$.

1. Reduction of the structure equations. The Euclidean space of dimension n is the set $\boldsymbol{R}^n$ with the coordinate system $t = {}^t(t_1, \ldots, t_n)$ and endowed with the scalar product $\langle t, t'\rangle = t_1t_1' + \cdots + t_nt_n'$. A cartesian coordinate system $s = {}^t(s_1, \ldots, s_n)$ of $\boldsymbol{R}^n$ is said to be standard if the scalar product is equal to $\langle s, s'\rangle = s_1s_1' + \cdots + s_ns_n'$. The tangent space at every point of $\boldsymbol{R}^n$ is identified with $\boldsymbol{R}^n$ itself. A mapping $s \mapsto t$ of $\boldsymbol{R}^n$ into itself is an isometry if

$t = As + b$ with a constant orthogonal matrix A with $\det A = 1$ and a constant vector $b \in \boldsymbol{R}^n$ (see Cartan [4, no. 27]).

Let (x, y) be a standard coordinate system of the Euclidean space $\boldsymbol{R}^2$. The distance of two points (x, y), (x', y') is defined to be $\sqrt{(x'-x)^2 + (y'-y)^2}$. Let (t_1, t_2, t_3, t_4) be a standard coordinate system of the Euclidean space $\boldsymbol{R}^4$. The distance of two points t, t' is defined to be $|t' - t| = \sqrt{(t_1' - t_1)^2 + \cdots + (t_4' - t_4)^2}$. A function f of x, y of class C^1 defined on $\boldsymbol{R}^2$ (or on an non-empty domain of $\boldsymbol{R}^2$) with values in $\boldsymbol{R}^4$ is said to be an isometric immersion of $\boldsymbol{R}^2$ (or of the domain of $\boldsymbol{R}^2$) into $\boldsymbol{R}^4$ if $|df|^2 = (dx)^2 + (dy)^2$ in the domain of definition of f.

PROPOSITION 1.1. (i) *Given any isometric immersion f of $\boldsymbol{R}^2$ into $\boldsymbol{R}^4$ with vanishing normal curvature, we can find a pair of real-valued functions (u_1, u_2) of class C^2 satisfying a system of partial differential equations*

$$\begin{aligned} &(\partial_x{}^2 u_1 - \partial_y{}^2 u_1)\partial_x\partial_y u_2 = (\partial_x{}^2 u_2 - \partial_y{}^2 u_2)\partial_x\partial_y u_1 \,, \\ &(\partial_x\partial_y u_1)^2 + (\partial_x\partial_y u_2)^2 = (\partial_x{}^2 u_1)(\partial_y{}^2 u_1) + (\partial_x{}^2 u_2)(\partial_y{}^2 u_2) \,, \end{aligned} \tag{1}$$

where (x, y) is a standard coordinate system of $\boldsymbol{R}^2$ and $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$.

(ii) *Given any pair of real-valued functions (u_1, u_2) of class C^2 satisfying* (1), *we can find an isometric immersion f of $\boldsymbol{R}^2$ into $\boldsymbol{R}^4$ with vanishing normal curvature.*

PROOF. (i) We formulate the problem following Cartan [3, no. 50]. A system $\{M, e_1, e_2, e_3, e_4\}$ is said to be an orthonormal moving frame of $\boldsymbol{R}^4$ if M is a point of $\boldsymbol{R}^4$ and if $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of the tangent space of $\boldsymbol{R}^4$ at M. In this article, a moving frame is always assumed to be *direct*, that is to say, endowing $\boldsymbol{R}^4$ with the same orientation as a standard coordinate system does. The infinitesimal variation of an orthonormal moving frame is written by a system of equations

$$dM = \sum_{j=1}^{4} \omega_j e_j \,, \quad de_j = \sum_{k=1}^{4} \omega_{jk} e_k \;\; (1 \le j \le 4) \,, \tag{2}$$

where ω_j and ω_{jk} are exterior differential 1-forms which satisfy the structure equations

$$d\omega_j = \sum_{l=1}^{4} \omega_l \wedge \omega_{lj} \,, \quad d\omega_{jk} = \sum_{l=1}^{4} \omega_{jl} \wedge \omega_{lk} \,, \quad \omega_{jk} + \omega_{kj} = 0 \quad (1 \le j, k \le 4) \,. \tag{3}$$

Given an isometric immersion f with vanishing normal curvature of $\boldsymbol{R}^2$ or of a non-empty domain of $\boldsymbol{R}^2$ into $\boldsymbol{R}^4$, we define

$$e_1 = \partial_x f \,, \quad e_2 = \partial_y f \,. \tag{4}$$

Then, $\{e_1, e_2\}$ is a standard orthonormal basis of the tangent space of the image of $\boldsymbol{R}^2$ by f at $f(x, y)$. Take any orthonormal basis $\{e_3, e_4\}$ of the normal space to the image at $f(x, y)$ such that $\{e_1, e_2, e_3, e_4\}$ be direct. Let (2) be the equation of infinitesimal variation of the frame $\{M, e_1, e_2, e_3, e_4\}$ of $\boldsymbol{R}^4$ with $M = f(x, y)$ defined on the image by f and let (3) be the integrability condition for (2). Then, $\omega_j = \langle df, e_j\rangle$ and $\omega_{jk} = \langle de_j, e_k\rangle$ are linear combinations of dx, dy whose coefficients are real-valued functions of x, y. We have at first $\omega_1 = dx, \omega_2 = dy, \omega_3 = 0, \omega_4 = 0$.

We set $\omega_{jk} = a_{jk}dx + b_{jk}dy$, where a_{jk} and b_{jk} are real-valued functions of x and y. Then, $d\omega_1 = -dy \wedge \omega_{12} = 0$ and $d\omega_2 = dx \wedge \omega_{12} = 0$ imply $\omega_{12} = 0$. Next, $d\omega_3 = dx \wedge \omega_{13} + dy \wedge \omega_{23} = 0$ and $d\omega_4 = dx \wedge \omega_{14} + dy \wedge \omega_{24} = 0$ imply $b_{13} = a_{23}$ and $b_{14} = a_{24}$. So, we have only eight functions $a_{13}, b_{13} = a_{23}, a_{14}, b_{14} = a_{24}, a_{34}, b_{23}, b_{24}, b_{34}$ to be distinguished among a_{jk}'s and b_{jk}'s. Changing the notation, we set

$$\text{(a)}\qquad \begin{aligned} &\omega_{13} = h_2dx + g_2dy\,, \quad \omega_{14} = -h_1dx - g_1dy\,, \quad \omega_{34} = sdx + tdy\,,\\ &\omega_{23} = g_2dx + k_2dy\,, \quad \omega_{24} = -g_1dx - k_1dy\,, \quad \omega_{12} = 0\,. \end{aligned}$$

Then, the second structure equation $d\omega_{jk} = \sum \omega_{jl} \wedge \omega_{lk}$ in (3) is written as follows.

$$\text{(b)}\qquad \begin{aligned} &\partial_x g_1 - sg_2 = \partial_y h_1 - th_2\,, \quad \partial_y g_1 - tg_2 = \partial_x k_1 - sk_2\,,\\ &\partial_x g_2 + sg_1 = \partial_y h_2 + th_1\,, \quad \partial_y g_2 + tg_1 = \partial_x k_2 + sk_1\,,\\ &\partial_x t - \partial_y s = (h_2 - k_2)g_1 - (h_1 - k_1)g_2\,, \quad h_1k_1 + h_2k_2 = {g_1}^2 + {g_2}^2\,. \end{aligned}$$

The exterior differential of $\omega_{34} = \langle de_3, e_4\rangle$ is independent of the choice of orthonormal basis $\{e_3, e_4\}$ of the normal space. The normal curvature R_n is defined as follows.

$$d\omega_{34} = R_n dx \wedge dy\,, \quad \text{or} \quad R_n = \partial_x t - \partial_y s\,.$$

R_n is independent of the choice of standard coordinate system (x, y). In the present work, we assume that $R_n = 0$ identically in the domain under consideration. So, there exists a real-valued function χ such that $s = \partial_x \chi$, $t = \partial_y \chi$ if the domain under consideration is simply connected. Once a χ chosen, we set

$$\begin{aligned} &g\ = g_1 \cos\chi - g_2 \sin\chi\,, \quad h\ = h_1 \cos\chi - h_2 \sin\chi\,, \quad k\ = k_1 \cos\chi - k_2 \sin\chi\,,\\ &g' = g_1 \sin\chi + g_2 \cos\chi\,, \quad h' = h_1 \sin\chi + h_2 \cos\chi\,, \quad k' = k_1 \sin\chi + k_2 \cos\chi\,. \end{aligned}$$

The first four equations of (b) yield $\partial_x g = \partial_y h$, $\partial_y g = \partial_x k$, $\partial_x g' = \partial_y h'$, $\partial_y g' = \partial_x k'$. So, there exists a pair of real-valued functions (u_1, u_2) such that

$$g = \partial_x\partial_y u_1\,, \quad h = {\partial_x}^2 u_1\,, \quad k = {\partial_y}^2 u_1\,, \quad g' = \partial_x\partial_y u_2\,, \quad h' = {\partial_x}^2 u_2\,, \quad k' = {\partial_y}^2 u_2\,.$$

In fact,

$$\begin{aligned} u_1(x, y) = &\int_0^x dx' \int_0^y g(x', y')dy' + \int_0^x (x - x')h(x', 0)dx'\\ &+ \int_0^y (y - y')k(0, y')dy' + px + qy + r\,, \end{aligned}$$

where p, q, r are real constants. u_2 is given similarly. g_j, h_j, k_j are represented as

$$\text{(c)}\qquad \begin{aligned} &g_1 = (\cos\chi)\partial_x\partial_y u_1 + (\sin\chi)\partial_x\partial_y u_2\,, \quad g_2 = -(\sin\chi)\partial_x\partial_y u_1 + (\cos\chi)\partial_x\partial_y u_2\,,\\ &h_1 = (\cos\chi){\partial_x}^2 u_1 + (\sin\chi){\partial_x}^2 u_2\,, \qquad h_2 = -(\sin\chi){\partial_x}^2 u_1 + (\cos\chi){\partial_x}^2 u_2\,,\\ &k_1 = (\cos\chi){\partial_y}^2 u_1 + (\sin\chi){\partial_y}^2 u_2\,, \qquad k_2 = -(\sin\chi){\partial_y}^2 u_1 + (\cos\chi){\partial_y}^2 u_2\,. \end{aligned}$$

And then, the last two equations of (b) imply (1).

(ii) Conversely, let (u_1, u_2) be a pair of real-valued functions satisfying (1). Then, the equation (3) holds if ω_j, ω_{jk} are defined to be

$$\begin{aligned}
&\omega_{13} = (\cos\chi)d(\partial_x u_2) - (\sin\chi)d(\partial_x u_1)\,, \quad \omega_{41} = (\cos\chi)d(\partial_x u_1) + (\sin\chi)d(\partial_x u_2)\,,\\
&\omega_{23} = (\cos\chi)d(\partial_y u_2) - (\sin\chi)d(\partial_y u_1)\,, \quad \omega_{42} = (\cos\chi)d(\partial_y u_1) + (\sin\chi)d(\partial_y u_2)\,,\\
&\omega_{34} = d\chi\,, \quad \omega_{12} = 0\,, \quad \omega_1 = dx\,, \quad \omega_2 = dy\,, \quad \omega_3 = \omega_4 = 0\,,
\end{aligned} \tag{5}$$

where χ is an arbitrary real-valued function. We eliminate χ by setting

$$\hat{e}_1 = e_1\,, \quad \hat{e}_2 = e_2\,, \quad \hat{e}_3 = (\cos\chi)e_3 - (\sin\chi)e_4\,, \quad \hat{e}_4 = (\sin\chi)e_3 + (\cos\chi)e_4\,. \tag{6}$$

Then, $de_j = \sum \omega_{jk} e_k$ are rewritten as

$$\begin{aligned}
d\hat{e}_1 &= (d\partial_x u_2)\hat{e}_3 - (d\partial_x u_1)\hat{e}_4\,, \quad & d\hat{e}_3 &= -(d\partial_x u_2)\hat{e}_1 - (d\partial_y u_2)\hat{e}_2\,,\\
d\hat{e}_2 &= (d\partial_y u_2)\hat{e}_3 - (d\partial_y u_1)\hat{e}_4\,, \quad & d\hat{e}_4 &= (d\partial_x u_1)\hat{e}_1 + (d\partial_y u_1)\hat{e}_2\,.
\end{aligned} \tag{7}$$

To integrate (7), we fix throughout this article a direct orthonormal basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of $\boldsymbol{R}^4$ independent of x, y and identify it with a basis of quaternions over the real number field $\boldsymbol{R}$. Let $\mathbf{1}$ be the unit of multiplication and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}\,, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}\,, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}\,, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}\,.$$

This endows $\boldsymbol{R}^4$ with the structure of an algebra over $\boldsymbol{R}$. Every point of $\boldsymbol{R}^4$ is identified with a quaternion $\mathbf{q} = q_1\mathbf{1} + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$. Given a $\mathbf{q}$, we define $\exp\mathbf{q}$ to be

$$\exp\mathbf{q} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!}\mathbf{q}^n\,.$$

In particular, $\exp(t\mathbf{q}) = (\cos t)\mathbf{1} + (\sin t)\mathbf{q}$ if $\mathbf{q}^2 = -\mathbf{1}$ and if t is a real number.

Remark that $\langle \mathbf{q}, \mathbf{q}' \rangle = \Re(\bar{\mathbf{q}}\mathbf{q}')$, that is to say, the scalar product of two vectors $\mathbf{q}$ and $\mathbf{q}'$ in $\boldsymbol{R}^4$ is equal to the real part of the product of two quaternions $\bar{\mathbf{q}}$ and $\mathbf{q}'$, where $\bar{\mathbf{q}} = q_1\mathbf{1} - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}$. Let S^3 be the unit sphere of $\boldsymbol{R}^4$. Every orthogonal matrix A of order 4 with $\det A = 1$ is associated with two points μ, ν of S^3 such that

$$A\mathbf{q} = \mu\mathbf{q}\bar{\nu}\,.$$

Two pairs (μ_1, ν_1), (μ_2, ν_2) represent the same A if and only if either $(\mu_2, \nu_2) = (\mu_1, \nu_1)$ or $(\mu_2, \nu_2) = (-\mu_1, -\nu_1)$ (see Cartan [4, no. 281, 282] and Yokota [17, p. 100]). Every direct orhonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\boldsymbol{R}^4$ is therefore represented as

$$e_1 = \mu\mathbf{1}\bar{\nu}\,, \quad e_2 = \mu\mathbf{i}\bar{\nu}\,, \quad e_3 = \mu\mathbf{j}\bar{\nu}\,, \quad e_4 = \mu\mathbf{k}\bar{\nu}\,. \tag{8}$$

Let μ, ν be functions of x, y with values in S^3 satisfying

$$d\mu = \frac{1}{2}\mu d(g_{12}\mathbf{j} - g_{11}\mathbf{k})\,, \quad d\nu = \frac{1}{2}\nu d(g_{22}\mathbf{j} + g_{21}\mathbf{k})\,, \tag{9}$$

where

$$\begin{aligned}
g_{11} &= \partial_x u_1 - \partial_y u_2\,, \quad & g_{12} &= \partial_y u_1 + \partial_x u_2\,,\\
g_{21} &= \partial_x u_1 + \partial_y u_2\,, \quad & g_{22} &= \partial_y u_1 - \partial_x u_2\,.
\end{aligned} \tag{10}$$

(7) is interpreted as (9) combined with (10). Also, (1) is equivalent to

$$dg_{11} \wedge dg_{12} = 0\,, \quad dg_{21} \wedge dg_{22} = 0\,. \tag{11}$$

Hence, g_{11} and g_{12} are functionally dependent, and so are g_{21} and g_{22}. So, (9) consists of two independent systems of ordinary differential equations, one for μ and the other for ν. (9) is integrable as is verified by exterior differentiation of both sides. We determine μ and ν from (9) by prescribing arbitrary values of them at a fixed point. Then, we have a following solution of (7).

$$\begin{aligned} e_1 = \hat{e}_1 = \mu\bar{\nu}\,, \quad e_2 = \hat{e}_2 = \mu\mathbf{i}\bar{\nu}\,, \quad & e_3 = \mu\exp(\chi\mathbf{i})\mathbf{j}\bar{\nu}\,, \quad & e_4 = \mu\exp(\chi\mathbf{i})\mathbf{k}\bar{\nu}\,, \\ & \hat{e}_3 = \mu\mathbf{j}\bar{\nu}\,, & \hat{e}_4 = \mu\mathbf{k}\bar{\nu}\,. \end{aligned}$$

Finally, by integrating $df = (dx)e_1 + (dy)e_2$ (see (4)) or equivalently

$$df = \mu\{(dx)\mathbf{1} + (dy)\mathbf{i}\}\bar{\nu}\,, \tag{12}$$

we obtain an isometric immersion f with vanishing normal curvature. □

COROLLARY 1.2. *Let f and $\tilde{f}$ be isometric immersions with vanishing normal curvature, and (u_1, u_2) and $(\tilde{u}_1, \tilde{u}_2)$ be solutions of* (1) *associated with f and $\tilde{f}$ in Proposition* 1.1, *respectively. Then,*

(i) *$\tilde{f} = f$ if and only if there exist real constants p_j, q_j, r_j $(j = 1, 2)$ such that*

$$\tilde{u}_j(x, y) - u_j(x, y) = p_j x + q_j y + r_j \qquad (j = 1, 2)\,. \tag{13}$$

(ii) *There exists an isometry S of $\boldsymbol{R}^4$ with $\tilde{f} = Sf$ if and only if there exist real constants l, p_j, q_j, r_j $(j = 1, 2)$ such that*

$$\begin{aligned} \tilde{u}_1(x, y) &= u_1(x, y)\cos l - u_2(x, y)\sin l + p_1 x + q_1 y + r_1\,, \\ \tilde{u}_2(x, y) &= u_1(x, y)\sin l + u_2(x, y)\cos l + p_2 x + q_2 y + r_2\,. \end{aligned} \tag{14}$$

PROOF. (i) If two solutions (u_1, u_2), $(\tilde{u}_1, \tilde{u}_2)$ represent the same g_j, h_j, k_j as in (c), we have $\partial_x{}^2(\tilde{u}_j - u_j) = 0$, $\partial_x\partial_y(\tilde{u}_j - u_j) = 0$ and $\partial_y{}^2(\tilde{u}_j - u_j) = 0$, which imply (13).

(ii) Denote $df = (dx)e_1 + (dy)e_2$ and $d\tilde{f} = (dx)\tilde{e}_1 + (dy)\tilde{e}_2$ by means of moving frames $\{f, e_1, e_2, e_3, e_4\}$ and $\{\tilde{f}, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ with $\omega_{34} = \tilde{\omega}_{34} = 0$. Then, from (7),

$$\begin{aligned} de_1 &= (d\partial_x u_2)e_3 - (d\partial_x u_1)e_4\,, \quad & de_3 &= -(d\partial_x u_2)e_1 - (d\partial_y u_2)e_2\,, \\ de_2 &= (d\partial_y u_2)e_3 - (d\partial_y u_1)e_4\,, \quad & de_4 &= (d\partial_x u_1)e_1 + (d\partial_y u_1)e_2\,; \end{aligned} \tag{d}$$

$$\begin{aligned} d\tilde{e}_1 &= (d\partial_x \tilde{u}_2)\tilde{e}_3 - (d\partial_x \tilde{u}_1)\tilde{e}_4\,, \quad & d\tilde{e}_3 &= -(d\partial_x \tilde{u}_2)\tilde{e}_1 - (d\partial_y \tilde{u}_2)\tilde{e}_2\,, \\ d\tilde{e}_2 &= (d\partial_y \tilde{u}_2)\tilde{e}_3 - (d\partial_y \tilde{u}_1)\tilde{e}_4\,, \quad & d\tilde{e}_4 &= (d\partial_x \tilde{u}_1)\tilde{e}_1 + (d\partial_y \tilde{u}_1)\tilde{e}_2\,. \end{aligned} \tag{e}$$

Set $\tilde{f} = Sf = Af + f^0$ with a fixed orthogonal matrix A ($\det A = 1$) and a fixed point f^0 of $\boldsymbol{R}^4$. Then, $d\tilde{f} = Adf$ implies $\tilde{e}_1 = Ae_1$ and $\tilde{e}_2 = Ae_2$. So, $\tilde{e}_3$ and $\tilde{e}_4$ are spanned by Ae_3 and Ae_4. Since $\{Ae_1, Ae_2, Ae_3, Ae_4\}$ and $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ are direct, we set

$$\tilde{e}_3 = (\cos l)Ae_3 - (\sin l)Ae_4\,, \quad \tilde{e}_4 = (\sin l)Ae_3 + (\cos l)Ae_4\,. \tag{f}$$

We have $\langle d\tilde{e}_j, Ae_k\rangle = 0$ for $3 \le j, k \le 4$ because $\langle d\tilde{e}_j, \tilde{e}_k\rangle = 0$ for $3 \le j, k \le 4$. So, l is a constant. By (f) combined with (d), (e) and by $\tilde{e}_j = Ae_j$ for $j = 1, 2$, we have

$$d\partial_x(\tilde{u}_1 - u_1 \cos l + u_2 \sin l) = 0\,, \quad d\partial_y(\tilde{u}_1 - u_1 \cos l + u_2 \sin l) = 0\,,$$
$$d\partial_x(\tilde{u}_2 - u_1 \sin l - u_2 \cos l) = 0\,, \quad d\partial_y(\tilde{u}_2 - u_1 \sin l - u_2 \cos l) = 0\,.$$

So, (14) holds.

Suppose conversely that (14) holds. Let $\{e_1, e_2, e_3, e_4\}$ and $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ be two orthonormal moving frames of $\boldsymbol{R}^4$ which satisfy (d), (e) and which are equal to $\{e_1^0, e_2^0, e_3^0, e_4^0\}$ and $\{\tilde{e}_1^0, \tilde{e}_2^0, \tilde{e}_3^0, \tilde{e}_4^0\}$, respectively, at a fixed point (x^0, y^0). Then, there exists a constant orthogonal matrix A ($\det A = 1$) such that

$$\tilde{e}_1^0 = Ae_1^0\,, \quad \tilde{e}_2^0 = Ae_2^0\,, \quad (\cos l)\tilde{e}_3^0 + (\sin l)\tilde{e}_4^0 = Ae_3^0\,, \quad -(\sin l)\tilde{e}_3^0 + (\cos l)\tilde{e}_4^0 = Ae_4^0\,.$$

$\{\tilde{e}_1, \tilde{e}_2, (\cos l)\tilde{e}_3 + (\sin l)\tilde{e}_4, -(\sin l)\tilde{e}_3 + (\cos l)\tilde{e}_4\}$ and $\{Ae_1, Ae_2, Ae_3, Ae_4\}$ satisfy the same system of equations (d) and they are equal at (x^0, y^0). So, they are identically equal. And hence, $d\tilde{f} = Adf$ and $\tilde{f} = Af + f^0 = Sf$, where f^0 is a fixed point. □

2. Invariance of (1) by isometries. Given a solution (u_1, u_2) of (1), we set

$$p_{j1} = \partial_x{}^2 u_j\,, \quad p_{j2} = \partial_x\partial_y u_j\,, \quad p_{j3} = \partial_y{}^2 u_j \quad (j = 1, 2)\,. \tag{15}$$

Then, p_{jk} satisfy quadratic equations and integrability condition

$$(p_{11} - p_{13})p_{22} = (p_{21} - p_{23})p_{12}\,, \quad p_{11}p_{13} + p_{21}p_{23} = p_{12}^2 + p_{22}^2\,, \tag{16}$$

$$\partial_y p_{11} = \partial_x p_{12}\,, \quad \partial_y p_{12} = \partial_x p_{13}\,, \quad \partial_y p_{21} = \partial_x p_{22}\,, \quad \partial_y p_{22} = \partial_x p_{23}\,. \tag{17}$$

Conversely, if a set of six real-valued functions p_{jk} satisfies (16) and (17) in a simply-connected domain, it is associated with a solution of (1). In fact, we set

$$v_j = \int_{(0,0)}^{(x,y)} (p_{j1}dx + p_{j2}dy)\,, \quad w_j = \int_{(0,0)}^{(x,y)} (p_{j2}dx + p_{j3}dy) \quad (j = 1, 2)\,.$$

Then curvi-linear integrals are well-defined in view of (17),

$$\partial_x v_j = p_{j1}\,, \quad \partial_y v_j = \partial_x w_j = p_{j2} \quad \text{and} \quad \partial_y w_j = p_{j3}$$

for $j = 1, 2$. So, we set further

$$u_j = \int_{(0,0)}^{(x,y)} (v_j dx + w_j dy) \quad (j = 1, 2)\,.$$

Then curvi-linear integrals are well-defined, (15) holds and (u_1, u_2) satisfies (1).

In this way, every solution (u_1, u_2) of (1) is associated with a 2×3 matrix

$$P = \begin{pmatrix} p_{11} & \sqrt{2}\,p_{12} & p_{13} \\ p_{21} & \sqrt{2}\,p_{22} & p_{23} \end{pmatrix} = \begin{pmatrix} \partial_x{}^2 u_1 & \sqrt{2}\,\partial_x\partial_y u_1 & \partial_y{}^2 u_1 \\ \partial_x{}^2 u_2 & \sqrt{2}\,\partial_x\partial_y u_2 & \partial_y{}^2 u_2 \end{pmatrix} \tag{18}$$

whose entries satisfy (16) and (17). Observe that

$$\begin{aligned}\operatorname{trace}(P\,{}^tP) &= p_{11}^2 + 2p_{12}^2 + p_{13}^2 + p_{21}^2 + 2p_{22}^2 + p_{23}^2 = (p_{11}+p_{13})^2 + (p_{21}+p_{23})^2 \\ &= (p_{11}-p_{13}-2p_{22})^2 + (p_{21}-p_{23}+2p_{12})^2 \\ &= (p_{11}-p_{13}+2p_{22})^2 + (p_{21}-p_{23}-2p_{12})^2 .\end{aligned}$$

Let us define the polar coordinates ρ, a, b, c of $P(\neq O)$ in the following way.

$$\begin{aligned}&\rho = \sqrt{\operatorname{trace}(P\,{}^tP)} = (p_{11}^2 + 2p_{12}^2 + p_{13}^2 + p_{21}^2 + 2p_{22}^2 + p_{23}^2)^{1/2} \quad (\geq 0)\,, \\ &p_{11}+p_{13} = \rho\cos b\,, \quad p_{21}+p_{23} = \rho\sin b\,, \\ &p_{11}-p_{13}-2p_{22} = \rho\cos(b+a+c)\,, \quad p_{21}-p_{23}+2p_{12} = \rho\sin(b+a+c)\,, \\ &p_{11}-p_{13}+2p_{22} = \rho\cos(b+a-c)\,, \quad p_{21}-p_{23}-2p_{12} = \rho\sin(b+a-c)\,.\end{aligned} \tag{19}$$

We do not define a, b, c at $P = O$. Real numbers a, b, c can be defined independently as follows.

$$\tan a = \frac{(p_{11}+p_{13})(p_{21}-p_{23}+2p_{22}\,i) - (p_{21}+p_{23})(p_{11}-p_{13}+2p_{12}\,i)}{(p_{11}+p_{13})(p_{11}-p_{13}+2p_{12}\,i) + (p_{21}+p_{23})(p_{21}-p_{23}+2p_{22}\,i)}\,,$$

$$(p_{11}+p_{13}) + (p_{21}+p_{23})i = \rho\, e^{bi}, \quad \tan c = \frac{2p_{12}+2p_{22}\,i}{p_{11}-p_{13}+(p_{21}-p_{23})i} \quad (i = \sqrt{-1})\,.$$

Although we will not use these equalities in what follows, we can verify that the fractions are real-valued and never reduced to $0/0$ as far as $\rho > 0$. Two points (ρ, a, b, c), (ρ', a', b', c') of the coordinate space represent the same $P(\neq O)$ if $\rho' = \rho > 0$ and if $(a', b', c') = (a + j\pi, b + 2k\pi, c + (j + 2l)\pi)$ for some integers j, k, l.

If a 2×3 matrix P is a function of x and y satisfying (16), the polar coordinates ρ, a, b, c defined by (19) are also functions of x and y, and vice versa. If P satisfies (17), the entries p_{jk} are the second order derivatives of some functions u_1, u_2 of x and y whose couple satisfies the system of equations (1). Therefore, the system (1) will be rewritten as a certain system of partial differential equations whose unknown is a quadruple of functions (ρ, a, b, c).

LEMMA 2.1. *ρ, a, b and dc are invariant by isometries of the xy-plane. ρ, a, c and db are invariant by those of the u_1u_2-plane. The system* (1) *and the rank of P are invariant by isometries of the xy-plane and the u_1u_2-plane.*

PROOF. We proceed by linear algebra introducing seven matrices

$$Q_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

$$Q_5 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad Q_6 = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By an isometry of the xy-plane

$$\hat{x} = x\cos\lambda - y\sin\lambda + x^0\,, \quad \hat{y} = x\sin\lambda + y\cos\lambda + y^0$$

and by an isometry of the u_1u_2-plane (see Corollary 1.2, (ii))

$$\check{u}_1 = u_1\cos\lambda' - u_2\sin\lambda' + u_1^0\,, \quad \check{u}_2 = u_1\sin\lambda' + u_2\cos\lambda' + u_2^0\,,$$

P is subjected to transformations $\hat{P} = P\hat{L}$ and $\check{P} = \check{L}P$, respectively, where

$$\hat{L} = \begin{pmatrix} \cos^2\lambda & \frac{1}{\sqrt{2}}\sin 2\lambda & \sin^2\lambda \\ \frac{-1}{\sqrt{2}}\sin 2\lambda & \cos 2\lambda & \frac{1}{\sqrt{2}}\sin 2\lambda \\ \sin^2\lambda & \frac{-1}{\sqrt{2}}\sin 2\lambda & \cos^2\lambda \end{pmatrix}, \quad \check{L} = \begin{pmatrix} \cos\lambda' & -\sin\lambda' \\ \sin\lambda' & \cos\lambda' \end{pmatrix}.$$

$\hat{L}$, $\check{L}$ are orthogonal, $\hat{L}H = H\hat{L}$ and $\hat{P}\,{}^t\hat{P} = {}^t\check{L}\check{P}\,{}^t\check{P}\check{L} = P\,{}^tP$, $\hat{P}H{}^t\hat{P} = {}^t\check{L}\check{P}H\,{}^t\check{P}\check{L} = PH\,{}^tP$. A 3×3 orthogonal matrix L is equal to $\hat{L}$ for some λ if $\det L = 1$ and if $LH = HL$. Any polynomial of p_{jk} which is invariant by these two kinds of isometries is a polynomial of four variables

(a) $\quad h_1 = \mathrm{trace}(P\,{}^tP)\,, \quad h_2 = \mathrm{trace}(PH\,{}^tP)\,, \quad h_3 = \det(P\,{}^tP)\,, \quad h_4 = \det(PH\,{}^tP)$

due to the Hamilton-Cayley theorem. Now, (1) implies $h_2 = h_3 + h_4 = 0$ because

$$h_2 = 2(p_{11}p_{13} + p_{21}p_{23} - p_{12}^2 - p_{22}^2)\,, \quad h_3 + h_4 = 2\{(p_{11} - p_{13})p_{22} - (p_{21} - p_{23})p_{12}\}^2\,.$$

We switch over to the polar coordinate system. Then, $h_1 = \rho^2$ and $h_3 = (\rho^4/4)\sin^2 a$. So, any function of p_{jk}'s satisfying (16) which is invariant by all these isometries is a function of ρ and a. We see that

(20) $\quad$ Rank $P = 0$ if $\rho = 0$, $\quad$ rank $P = 1$ if $\rho > \sin a = 0$, $\quad$ rank $P = 2$ if $\rho \sin a \neq 0$.

So, the system (1) and the rank of P are invariant by all these isometries.

Next, we rewrite (19) in the following way.

(b)
$$\begin{aligned}(4/\rho)P = {} & 2\cos b\, Q_1 + 2\sin b\, Q_2 + \cos(b + a + c)\, Q_3 + \sin(b + a + c)\, Q_4 \\ & + \cos(b + a - c)\, Q_5 + \sin(b + a - c)\, Q_6\,.\end{aligned}$$

This and

(c)
$$\begin{aligned} & Q_1\hat{L} = Q_1\,, \quad Q_3\hat{L} = Q_3\cos 2\lambda + Q_4\sin 2\lambda\,, \quad Q_5\hat{L} = Q_5\cos 2\lambda - Q_6\sin 2\lambda\,, \\ & Q_2\hat{L} = Q_2\,, \quad Q_4\hat{L} = -Q_3\sin 2\lambda + Q_4\cos 2\lambda\,, \quad Q_6\hat{L} = Q_5\sin 2\lambda + Q_6\cos 2\lambda\,, \\ & \check{L}Q_j = Q_j\cos\lambda' + Q_{j+1}\sin\lambda'\,, \quad \check{L}Q_{j+1} = -Q_j\sin\lambda' + Q_{j+1}\cos\lambda' \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 1, 3, 5)\end{aligned}$$

combined with $\hat{P} = P\hat{L}$, $\check{P} = \check{L}P$ imply

$$\begin{aligned} & \hat{b} = b\,, && \hat{b} + \hat{a} + \hat{c} = b + a + c + 2\lambda\,, && \hat{b} + \hat{a} - \hat{c} = b + a - c - 2\lambda\,, \\ & \check{b} = b + \lambda'\,, && \check{b} + \check{a} + \check{c} = b + a + c + \lambda'\,, && \check{b} + \check{a} - \check{c} = b + a - c + \lambda'\,.\end{aligned}$$

We have finally the following law of transformation.

(21)
$$\begin{aligned} & \hat{\rho} = \rho, \quad \hat{a} = a\,, \quad \hat{b} = b\,, \quad \hat{c} = c + 2\lambda\,, \quad d\hat{c} = dc\,, \\ & \check{\rho} = \rho\,, \quad \check{a} = a, \quad \check{b} = b + \lambda'\,, \quad \check{c} = c\,, \quad d\check{b} = db\,.\end{aligned}$$

Hence, the lemma is proved. □

Thinking it as the Cauchy problem is not a good idea to solve (1). Let us explain the reason why. A solution (u_1, u_2) of (1) annihilates two forms

$$F_1(u_1, u_2) = (\partial_x\partial_y u_1)^2 + (\partial_x\partial_y u_2)^2 - (\partial_x{}^2 u_1)(\partial_y{}^2 u_1) - (\partial_x{}^2 u_2)(\partial_y{}^2 u_2),$$
$$F_2(u_1, u_2) = (\partial_x{}^2 u_1 - \partial_y{}^2 u_1)\partial_x\partial_y u_2 - (\partial_x{}^2 u_2 - \partial_y{}^2 u_2)\partial_x\partial_y u_1 .$$

We define linear partial differential operators l_{jk} $(j, k = 1, 2)$ to be

$$l_{j1}\phi = \lim_{t\to 0} \frac{F_j(u_1 + t\phi, u_2) - F_j(u_1, u_2)}{t}, \quad l_{j2}\phi = \lim_{t\to 0} \frac{F_j(u_1, u_2 + t\phi) - F_j(u_1, u_2)}{t}.$$

They are

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} 2p_{12}\partial_x\partial_y - p_{13}\partial_x{}^2 - p_{11}\partial_y{}^2 & 2p_{22}\partial_x\partial_y - p_{23}\partial_x{}^2 - p_{21}\partial_y{}^2 \\ p_{22}(\partial_x{}^2 - \partial_y{}^2) - (p_{21} - p_{23})\partial_x\partial_y & (p_{11} - p_{13})\partial_x\partial_y - p_{12}(\partial_x{}^2 - \partial_y{}^2) \end{pmatrix}.$$

Regarding p_{jk} as constants and $\xi = \partial_x$, $\eta = \partial_y$ as indeterminates,

$$\text{(22)} \qquad l_{11}l_{22} - l_{12}l_{21} = \rho^2(\xi\sin\psi_1 - \eta\cos\psi_1)(\xi\sin\psi_2 - \eta\cos\psi_2) \times (\xi\sin\psi_3 - \eta\cos\psi_3)(\xi\cos\psi_3 + \eta\sin\psi_3)$$

is said to be the characteristic polynomial of (1) (see Petrowsky [13, Kap. III, §2]), where $\psi_1 = (c + a)/2$, $\psi_2 = (c - a)/2$, $\psi_3 = c/2$. Four factors $s_j\xi + t_j\eta$ on the right hand side of (22) depend on unknowns, they are distinct if the rank of P is equal to 2, while three of them are the same if the rank of P is equal to 1. So, the Cauchy problem is not easy to solve. From now on, our unknown functions are rather ρ, a, b, c than u_1, u_2.

We can construct isometric immersions f and (u_1, u_2) if we know ρ, a, b, c as functions of x, y. To do this, we take the rank of P into account (see (a) above). The rank of our P is equal to the dimension of the first normal space in the notations of do Carmo and Dajzcer [2]. They assumed this number to be constant.

If $P = O$ in a non-empty simply-connected domain, $\hat{e}_j$ are constant (see (7)) and the image is a domain of a fixed two-dimensional Euclidean subspace of $\boldsymbol{R}^4$. This case apart, we restrict ourselves in this article to the following two cases.

1. Rank $P = 1$ (a/π is an integer) in a simply-connected domain;
2. Rank $P = 2$ (a/π is never an integer) in a simply-connected domain.

We reduce (1) to two single equations (24), (25) of the first order in the case 1 (see §3), and to a single equation (36) of the second order in the case 2 (see §4).

3. The rank one case. Suppose that the rank P be identically equal to 1 for a solution (u_1, u_2) of (1) in a non-empty simply-connected domain of the xy-plane. We can represent p_{jk}'s as

$$\text{(23)} \qquad p_{j1} = \xi_j \sin^2\theta, \quad p_{j2} = -\xi_j\cos\theta\sin\theta, \quad p_{j3} = \xi_j\cos^2\theta \quad (j = 1, 2)$$

if we set $p_{j1}/p_{j2} = p_{j2}/p_{j3} = -\tan\theta$. And then, we have

$$\theta \equiv (c + a + \pi)/2 \pmod{\pi\mathbf{Z}}, \quad \xi_1 = \rho\cos b \quad \text{and} \quad \xi_2 = \rho\sin b.$$

PROPOSITION 3.1. *Any triplet* (θ, ξ_1, ξ_2) *of real-valued solutions of partial differential equations*

$$\partial_x \sin\theta = \partial_y \cos\theta\,, \tag{24}$$

$$\partial_x(\xi_j \cos\theta) + \partial_y(\xi_j \sin\theta) = 0 \quad (j = 1, 2) \tag{25}$$

gives rise to a solution of (1).

PROOF. The equalities (16) hold if p_{jk} are represented as (23) by means of a triplet (θ, ξ_1, ξ_2). Also, the integrability condition (17) holds if the triplet satisfies (24) and (25). So, p_{jk} are the second order derivatives of a solution of (1). □

COROLLARY 3.2. *In the notations of Proposition* 3.1, θ *is of class* C^2 *in the whole plane if and only if* θ *is a constant and if* u_1, u_2 *are functions of single variable* $y\cos\theta - x\sin\theta$ *up to polynomial functions of degree one.*

PROOF. First, let us show that a solution θ of (24) of class C^2 in the closed disk D of radius R with center at (x^0, y^0) satisfies the inequality

$$\{\partial_x\theta(x^0, y^0)\}^2 + \{\partial_y\theta(x^0, y^0)\}^2 < 1/R^2\,. \tag{26}$$

In fact, suppose $(x^0, y^0) = (0, 0)$ and $\theta(0, 0) = 0$ (see (21)). Then, $\theta(x, 0) = 0$ for $-R \le x \le R$ because $\theta_x(x, 0) = -\theta_y(x, 0)\tan\theta(x, 0)$. The function

$$\psi(x, y) = \theta_y \cos\theta - \theta_x \sin\theta$$

is of class C^1, $\theta_x = -\psi\sin\theta$ and $\theta_y = \psi\cos\theta$. The equality $\theta_{yx} = \theta_{xy}$ yields

$$\psi_x \cos\theta + \psi_y \sin\theta = -\psi^2\,.$$

Restricting this to $y = 0$, we have

$$\psi(x, 0) = s/(1 + sx)$$

for $-R \le x \le R$, where $s = \psi(0, 0)$. So, $|s|R < 1$, proving (26).

θ is a constant if θ is of class C^2 in the whole plane, as is shown by making R go to $+\infty$ in (26). Let $\theta = -\pi/2$, for example. Then, $p_{j2} = p_{j3} = 0$ and $\partial_y\xi_j = 0$ for $j = 1, 2$ (see (23) and (25)). So, u_1, u_2 are functions only of x up to polynomial functions of degree one.

Conversely, a pair of arbitrary functions $(u_1(x), u_2(x))$ is a solution of (1), and the equations (24) and (25) hold if we set $\theta = -\pi/2, \xi_j = u_j''(x)$ $(j = 1, 2)$ (see Example 1 in §5). □

A general algorithm to solve (24) is as follows. Draw an arbitrary smooth curve in the plane which is assumed to be

$$C = \{(x, y) \in \boldsymbol{R}^2;\ h(x, y) = 0\}\,,$$

where h is a real-valued smooth function such that $(h_x, h_y) \ne (0, 0)$ everywhere on C (we denote $h_x = \partial_x h, h_y = \partial_y h$). Given an arbitrary point (x, y) of $\boldsymbol{R}^2$ (not necessarily on C), suppose that through (x, y) passes a straight line which is normal to C at (x', y'), say. Then, we have

$$h(x', y') = 0\,, \quad h_y(x', y')(x - x') = h_x(x', y')(y - y') \tag{27}$$

and there exists a real-valued function $\theta = \theta(x, y)$ $(\operatorname{mod} \pi \boldsymbol{Z})$ such that

$$h_y(x', y') \cos\theta = h_x(x', y') \sin\theta\,. \tag{28}$$

Let us show that x' and y' are piecewise smooth functions of (x, y), multiple-valued in general. We define the signed arclength s on C measured from a fixed point according to an orientation of C and denote the range of s by I when we run over C. Then, a point of C is a function of s which we denote by $(\xi, \eta) = (\xi(s), \eta(s))$ $(s \in I)$.

If the normal line to C at $(\xi(s_0), \eta(s_0))$ passes through (x, y), the length of the vector $(x - \xi(s), y - \eta(s))$ is extremal at $s = s_0$ and this vector is perpendicular to the unit tangent vector $(d\xi/ds(s_0), d\eta/ds(s_0))$. By deleting the suffix 0, we have

$$x = \xi(s) - l\frac{d\eta}{ds}(s)\,, \quad y = \eta(s) + l\frac{d\xi}{ds}(s) \tag{α}$$

with a real number l if the normal line to C at $(\xi(s), \eta(s))$ passes through (x, y). l is the signed distance from (x, y) to C along the normal line. Two equalities (α) define in turn a mapping $(s, l) \mapsto (x, y)$ of $I \times \boldsymbol{R}$ into $\boldsymbol{R}^2$. The exterior differentiation yields

$$dx \wedge dy = \{1 - l\kappa(s)\} ds \wedge dl \qquad \left(\kappa(s) = \frac{d\xi}{ds}(s)\frac{d^2\eta}{ds^2}(s) - \frac{d\eta}{ds}(s)\frac{d^2\xi}{ds^2}(s)\right). \tag{β}$$

κ is equal to the curvature of C at (ξ, η). The focal set

$$\Gamma = \left\{(x, y) \in \boldsymbol{R}^2\,;\, x = \xi(s) - \frac{1}{\kappa(s)}\frac{d\eta}{ds}(s),\ y = \eta(s) + \frac{1}{\kappa(s)}\frac{d\xi}{ds}(s),\ s \in I,\ \kappa(s) \neq 0\right\} \tag{γ}$$

is independent of the orientation of C, because $l\kappa$ is unchanged if we reverse the orientaion of C and change the sign of l at the same time.

Let U be a connected component of the complement of Γ. The inverse mapping $(x, y) \mapsto (s, l)$ of (α) of U into $I \times \boldsymbol{R}$ is constructed in the following way. We apply the implicit function theorem to the equation

$$\{x - \xi(s)\}\frac{d\xi}{ds}(s) + \{y - \eta(s)\}\frac{d\eta}{ds}(s) = 0 \tag{δ_1}$$

to find $s = s(x, y)$ and substitute the solution into the equation

$$l = \{\xi(s) - x\}\frac{d\eta}{ds}(s) + \{y - \eta(s)\}\frac{d\xi}{ds}(s) \tag{δ_2}$$

to have $l = l(x, y)$. The solution $s(x, y)$ will not necessarily be unique, if there exists any. So, the mapping $(x, y) \mapsto (s, l)$ will be multiple-valued in general. This followed by

$$s \mapsto (x', y') = (\xi(s), \eta(s)) \tag{ε}$$

gives rise to a mapping $(x, y) \mapsto (x', y')$ of U into C. Conformally to (27) and (28), we can define $\theta = \theta(x, y) \pmod{\pi \mathbf{Z}}$ by setting

$$(\zeta) \qquad \sin\theta(x, y) = -\sigma \frac{d\xi}{ds}(s), \quad \cos\theta(x, y) = \sigma \frac{d\eta}{ds}(s) \quad (\sigma = \pm 1).$$

Suppose that the function h be of class C^k with $k \geq 4$. Then, $s \mapsto (\xi(s), \eta(s))$ is of class C^k and the mapping $(s, l) \mapsto (x, y)$ (see (α)) is of class C^{k-1}. The inverse $(x, y) \mapsto (s, l)$ (see (δ)) is also of class C^{k-1} in U. So, the composed mapping $(x, y) \mapsto (x', y')$ is of class C^{k-1} in U. Finally, $\theta(x, y)$ is of class C^{k-2} in U. If in particular h is real analytic, θ is real analytic in U.

If, through every point (x, y) of U, there pass p straight lines each of which is normal to C at $(x'_{(\nu)}, y'_{(\nu)})$, say $(\nu = 1, \dots, p)$, then for every ν, we have the ν-th function elements $\theta_{(\nu)}(x, y) \pmod{\pi \mathbf{Z}}$ of θ in U (see Examples 3, 4 in §5).

LEMMA 3.3. (i) *$\theta(x, y)$ obtained by the formulas* (27) *and* (28) *is a multiple-valued solution of* (24) *in the whole xy-plane.*

(ii) *Conversely, any solution θ of* (24) *is obtained by* (27) *and* (28) *involving a function h.*

(iii) *If h is an irreducible polynomial of degree $n \geq 2$, the number of function elements of θ does not exceed n^4.*

PROOF. (i) Given any point (x, y) of the plane, the point (x', y') of C attaining the minimal distance of (x, y) to C is a solution of (27). So, there exists at least one function element of θ defined by (27) in the whole xy-plane. For an arbitrary solution $(x'(x, y), y'(x, y))$ of (27), there exist real numbers $l = l(x, y)$, $l' = l'(x, y)$ such that

$$(x - x', y - y') = l(h_x(x', y'), h_y(x', y')), \quad (h_x(x', y'), h_y(x', y')) = l'(\cos\theta, \sin\theta).$$

So, $(x - x', y - y') = (r\cos\theta, r\sin\theta)$ $(r = ll')$. r is a signed distance of (x', y') to (x, y), that is to say, $r > 0$ locally in one side of C, $r < 0$ in the other side and $r = 0$ on C. If an infinitesimal variation (dx, dy) of (x, y) bears an infinitesimal variation (dx', dy') of (x', y'), we have $(dx - dx', dy - dy') = (dr)(\cos\theta, \sin\theta) + (d\theta)(-r\sin\theta, r\cos\theta)$, so

$$dr = (dx - dx')\cos\theta + (dy - dy')\sin\theta = (dx)\cos\theta + (dy)\sin\theta$$

because $(dx')\cos\theta + (dy')\sin\theta = 0$. So, $(\cos\theta, \sin\theta) = (\partial_x r, \partial_y r)$, proving (24).

(ii) Prescribing an arbitrary point (x^0, y^0), look at a fixed function element of θ which is smooth in a simply-connected neighborhood V of (x^0, y^0). We set

$$h(x, y) = \int_{(x^0, y^0)}^{(x, y)} \{(\cos\theta)dx + (\sin\theta)dy\}.$$

The curvi-linear integral is well-defined in V if θ is a solution of (24). We can verify that (27) and (28) hold, where h is the same as r in the proof of (i).

(iii) Assume, after an isometry if necessary, that the coefficients neither of x'^n nor of y'^n in $h(x', y')$ be zero. Set $k(x', y') = (x' - x^0)h_y(x', y') - (y' - y^0)h_x(x', y')$ for a fixed point (x^0, y^0) and rewrite h and k as

$$\text{(a)} \qquad h(x', y') = \sum_{p=0}^{n} h_p(x') y'^{n-p}, \quad k(x', y') = \sum_{p=0}^{n} k_p(x') y'^{n-p}.$$

$h_p(x')$ and $k_p(x')$ are polynomials of degree at most p. Next,

$$\text{(b)} \qquad R_1(x', y') = \sum_{t=1}^{n} \sum_{p=0}^{n-t} v_t y'^{n-t-p} \{h_p(x') k(x', y') - k_p(x') h(x', y')\}$$

is of degree at most $n-1$ as a polynomial of y' if v_t are independent of y' and

$$R_1(x', y') = \sum_{s,t=1}^{n} v_t c_{st}(x') y'^{s-1},$$

$$\text{(c)} \qquad c_{st}(x') = \sum_{p=\max(0,\, n+1-s-t)}^{n-\max(s,\, t)} \{h_p(x') k_{2n+1-s-t-p}(x') - k_p(x') h_{2n+1-s-t-p}(x')\}.$$

$R_1(x', y')$ is independent of y' if v_t is the $(1, t)$-cofactor of $(c_{st}(x'))_{s,t=1}^{n}$, and then

$$\text{(d)} \qquad R_1(x') = \det(c_{st}(x'))_{s,t=1}^{n}.$$

$R_1(x')$ is a polynomial of degree at most n^2 because $c_{st}(x')$ are of degree at most $2n+1-s-t$. This is equal to the resultant of h, k within a sign (see van der Waerden [16, §27]). We obtain also a polynomial $R_2(y')$ of degree at most n^2 by replacing x' by y'. From (b), there exist polynomials $A_1(x', y')$, $B_1(x', y')$ of degree at most $n-1$ with respect to y' and $A_2(x', y')$, $B_2(x', y')$ of degree at most $n-1$ with respect to x' such that

$$\text{(29)} \qquad \begin{aligned} R_1(x') &= A_1(x', y') h(x', y') + B_1(x', y') k(x', y'), \\ R_2(y') &= A_2(x', y') h(x', y') + B_2(x', y') k(x', y'). \end{aligned}$$

Suppose at first that neither $R_1(x')$ nor $R_2(y')$ be identically equal to 0. Then, we have $R_1(x'(x^0, y^0)) = R_2(y'(x^0, y^0)) = 0$ if $(x'(x^0, y^0), y'(x^0, y^0))$ is a *complex* solution of (27), whose number does not exceed $(n^2)^2 = n^4$. So, the number of function elements of θ at (x^0, y^0), which is the number of *real* solutions of (27), does not exceed n^4.

Suppose on the contrary that one of $R_1(x')$ or $R_2(y')$ be identically equal to 0. Then, h divides k and $k = \alpha h$ with a constant α because h is irreducible, B_1 is of degree at most $n-1$ with respect to y', B_2 is of degree at most $n-1$ with respect to x' and k is of degree at most n. Therefore, $h = h_0(r^2)\exp[\alpha \arg\{x' - x^0 + i(y' - y^0)\}]$, where $r^2 = (x' - x^0)^2 + (y' - y^0)^2$. So, $\alpha = 0$ because α is real. Also, $h(x', y')$ is a polynomial of single variable r^2, that is to say, C is a circle with center at (x^0, y^0). □

REMARK. If for example $h(x', y') = ax'^2 + by'^2 + 2sx' + 2ty' - c$ and $ab \neq 0$, we have

$$\text{(e)}\quad \begin{aligned} R_1(x')/(4b) = &-ae^2x'^4 - 2e(as_1 + es)x'^3 - (as_1{}^2 + bt_1{}^2 - ce^2 + 2et_1t + 4es_1s)x'^2 \\ &+ 2\{e(cs_1 + t^2x^0 - sty^0) - ss_1{}^2 - s_1tt_1 + bt_1(tx^0 - sy^0)\}x' \\ &- cs_1{}^2 + 2s_1t(tx^0 - sy^0) - b(s_1y^0 - t_1x^0 + ex^0y^0)^2 , \\ R_2(y', a, b, c, s, t, x^0, y^0) = &-R_1(y', b, a, c, t, s, y^0, x^0) , \end{aligned}$$

where $e = a - b$, $s_1 = s + bx^0$, $t_1 = t + ay^0$. One of $R_1(x')$ or $R_2(y')$ vanishes identically if and only if $a - b = s_1 = t_1 = 0$. Also, $h(x', y') = a(x' - x^0)^2 + a(y' - y^0)^2 - c - a(x^0)^2 - a(y^0)^2$ if the last condition is satisfied. Any straight line passing through (x^0, y^0) is normal to C in this case (see Example 3 in §5).

(25) is linear and solved in the domain of definition of θ. Once we know θ, ξ_1, ξ_2, we obtain μ, ν from (9) and f by integrating (12). We reduced therefore (1) to a global question of finding implicit functions from (27). If θ is multiple-valued, ξ_1, ξ_2, the mapping f and u_1, u_2 are also multiple-valued (see Examples 3, 4 in §5).

4. The rank two case. In this section, the rank of P is supposed to be 2 for a solution (u_1, u_2) of (1) identically in a simply-connected domain D containing the origin in the interior of the xy-plane. Then, $\rho \sin a \neq 0$ in D (see (20)).

Let us introduce two functions s_1, s_2 of x, y such that g_{11} and g_{12} depend only on s_1 and that g_{21} and g_{22} depend only on s_2. To do this, we set in this section

$$\text{(30)}\qquad \psi_1 = (c + a)/2 , \quad \psi_2 = (c - a)/2 , \quad \kappa_1 = \psi_1 + b , \quad \kappa_2 = \psi_2 - b .$$

(We studied in §3 the case $\psi_1 \equiv \psi_2 \equiv \theta + (\pi/2) \pmod{\pi \mathbf{Z}}$). (10), (18) and (19) yield

$$dg_{11} = (\cos\kappa_1)\Omega_1 , \quad dg_{12} = (\sin\kappa_1)\Omega_1 , \quad dg_{21} = (\cos\kappa_2)\Omega_2 , \quad dg_{22} = (\sin\kappa_2)\Omega_2 ,$$

where $\Omega_j = \rho\cos\psi_j\, dx + \rho\sin\psi_j\, dy$ $(j = 1, 2)$. $d(dg_{jk}) = 0$ implies $d\Omega_j = d\kappa_j \wedge \Omega_j = 0$, or

$$\text{(31)}\qquad \partial_x(\rho\sin\psi_j) = \partial_y(\rho\cos\psi_j) , \quad (\sin\psi_j)\partial_x\kappa_j = (\cos\psi_j)\partial_y\kappa_j \quad (j = 1, 2) .$$

Let us define real-valued functions s_1, s_2 in D to be

$$\text{(32)}\quad s_1 = \int_{(0,0)}^{(x,y)} (\rho\cos\psi_1\, dx + \rho\sin\psi_1\, dy) , \quad s_2 = \int_{(0,0)}^{(x,y)} (-\rho\cos\psi_2\, dx - \rho\sin\psi_2\, dy) .$$

Then, $\Omega_1 = ds_1$ and $\Omega_2 = -ds_2$. The curvi-linear integrals are well-defined by (31).

(32) yields $\partial(s_1, s_2)/\partial(x, y) = \rho^2 \sin a \neq 0$ in D. So, we have a mapping T of D into a simply-connected domain, denoted by Δ, of the s_1s_2-plane containing the origin in the interior. The inverse mapping $T^{-1} : \Delta \to D$ is defined to be

$$\text{(33)}\quad x = -\int_{(0,0)}^{(s_1,s_2)} \frac{\sin\psi_2\, ds_1 + \sin\psi_1\, ds_2}{\rho\sin a} , \quad y = \int_{(0,0)}^{(s_1,s_2)} \frac{\cos\psi_2\, ds_1 + \cos\psi_1\, ds_2}{\rho\sin a} .$$

We make use of (s_1, s_2) as a local coordinate system in D as well as (x, y).

Given a real-, complex- or vector-valued function $g(x, y)$ defined in D, we define a function $\hat{g}(s_1, s_2)$ in Δ to be $\hat{g} = g \circ T^{-1}$. Reciprocally, given a function $\hat{g}(s_1, s_2)$ defined in Δ, we define a function $g(x, y)$ in D to be $g = \hat{g} \circ T$. Briefly,

$$\hat{g}(s_1, s_2) = g(x, y) \quad \text{if} \quad (s_1, s_2) = T(x, y)\,. \tag{34}$$

LEMMA 4.1. *ds_1 and ds_2 are invariant by isometries of the xy-plane and by isometries of the u_1u_2-plane.*

The proof is immediate from (21). We set

$$\begin{aligned} \hat{A} &= \log\left|\tan\frac{\hat{a}}{2}\right|, \\ \hat{B} &= \frac{\partial}{\partial s_1}\left(\frac{1}{\sin\hat{a}}\frac{\partial\hat{\kappa}_1}{\partial s_1} - \frac{\cos\hat{a}}{\sin\hat{a}}\frac{\partial\hat{\kappa}_2}{\partial s_2}\right) + \frac{\partial}{\partial s_2}\left(\frac{1}{\sin\hat{a}}\frac{\partial\hat{\kappa}_2}{\partial s_2} - \frac{\cos\hat{a}}{\sin\hat{a}}\frac{\partial\hat{\kappa}_1}{\partial s_1}\right). \end{aligned} \tag{35}$$

$a \mapsto A = \log|\tan(a/2)|$ (that is to say, $dA/da = 1/\sin a$) is a real-analytic mapping of

$$J_m = (m\pi, (m+1)\pi)$$

onto $\boldsymbol{R}$ for every integer m and monotone increasing or monotone decreasing. The inverse mapping $A \mapsto a$ is multiple-valued. So, let $a(A)$ be a function element with values in a fixed J_m. The function $\hat{B}$ depends on $\hat{a}$. We regard $\hat{B}$ as depending rather on $\hat{A}$ than $\hat{a}$. We emphasize this by writing sometimes $\hat{B}_{\hat{A}}$ in what follows.

PROPOSITION 4.2. *Any real-valued solution $\hat{A}$ of a partial differential equation*

$$\frac{\partial^2\hat{A}}{\partial {s_1}^2} - \frac{\partial^2\hat{A}}{\partial {s_2}^2} + \hat{B} = 0 \tag{36}$$

gives rise to a solution (u_1, u_2) of (1).

PROOF. $\hat{\kappa}_1$ depends only on s_1 and $\hat{\kappa}_2$ depends only on s_2 because $d\hat{\kappa}_1 \wedge ds_1 = 0$ and $d\hat{\kappa}_2 \wedge ds_2 = 0$. So, we set

$$\hat{b}(s_1, s_2) = \{\hat{\kappa}_1(s_1) - \hat{\kappa}_2(s_2) - \hat{a}(s_1, s_2)\}/2\,, \quad \hat{c}(s_1, s_2) = \hat{\kappa}_1(s_1) + \hat{\kappa}_2(s_2)\,. \tag{37}$$

The first equations of (31) are interpreted as

$$d\log(\sqrt{|\sin\hat{a}|}\,\hat{\rho}) = \left(\frac{-1}{\sin\hat{a}}\frac{\partial\hat{\psi}_2}{\partial s_2} + \frac{\cot\hat{a}}{2}\frac{\partial\hat{c}}{\partial s_1}\right)ds_1 + \left(\frac{1}{\sin\hat{a}}\frac{\partial\hat{\psi}_1}{\partial s_1} - \frac{\cot\hat{a}}{2}\frac{\partial\hat{c}}{\partial s_2}\right)ds_2\,. \tag{38}$$

(36) is actually the integrability condition of (38) and it is one of the best interpretations of (1) if the rank of P is assumed to be 2 (see Lemma 4.1).

We prescribe arbitrary real-valued functions $\hat{\kappa}_1(s_1)$, $\hat{\kappa}_2(s_2)$, $\hat{A}_1(s_1)$, $\hat{A}_2(s_2)$ of class C^2 and define $\hat{c}$ by (37). The Goursat problem for (36) is to find a real-valued function $\hat{A}(s_1, s_2)$ satisfying (36) and

$$\hat{A}(s_1, s_1) = \hat{A}_1(s_1) + \hat{A}_2(0)\,, \quad \hat{A}(-s_2, s_2) = \hat{A}_1(0) + \hat{A}_2(s_2)\,. \tag{39}$$

To do this, define a sequence of functions $\{\hat{A}_{(n)}(s_1, s_2)\}_{n=0}^{\infty}$ as follows (see [11, no. 50]).

$$(40)\quad \begin{aligned} \hat{A}_{(0)}(s_1, s_2) &= \hat{A}_1\left(\frac{s_1+s_2}{2}\right) + \hat{A}_2\left(\frac{s_2-s_1}{2}\right), \\ \hat{A}_{(n+1)}(s_1, s_2) &= \hat{A}_{(0)}(s_1, s_2) - \frac{1}{4}\int_0^{s_1+s_2} d\sigma_2 \int_0^{s_1-s_2} \hat{B}_{(n)}\left(\frac{\sigma_1+\sigma_2}{2}, \frac{\sigma_2-\sigma_1}{2}\right) d\sigma_1 \end{aligned}$$

for $n \geq 0$, where $\hat{B}_{(n)}$ stands for $\hat{B}_{\hat{A}}$ with $\hat{A} = \hat{A}_{(n)}$. Given any bounded convex closed subset F of the $s_1 s_2$-plane containing the origin, we have

$$\begin{aligned} &|\hat{A}_{(n+1)} - \hat{A}_{(n)}| \leq l_1 (C l_2 l_3)^n |s_1{}^2 - s_2{}^2| (|s_1 - s_2| + |s_1 + s_2|)^n / n!, \\ &|\partial_{s_j} \hat{A}_{(n+1)} - \partial_{s_j} \hat{A}_{(n)}| \leq l_1 (C l_2 l_3)^n (|s_1 - s_2| + |s_1 + s_2|)^{n+1} / n! \quad (j = 1, 2) \end{aligned}$$

for $n \geq 0$ in F, where

$$\begin{aligned} &l_1 = \max(|\hat{A}_{(0)}|, |\partial_{s_1}\hat{A}_{(0)}|, |\partial_{s_2}\hat{A}_{(0)}|, |\hat{B}_{\hat{A}_{(0)}}|), \quad l_2 = \max(|s_1 - s_2|, |s_1 + s_2|, 1), \\ &l_3 = \max\{|\hat{B}_{\hat{A}} - \hat{B}_{\hat{A}'}| / (|\hat{A} - \hat{A}'| + |\partial_{s_1}\hat{A} - \partial_{s_1}\hat{A}'| + |\partial_{s_2}\hat{A} - \partial_{s_2}\hat{A}'|)\}, \end{aligned}$$

and C is a positive number independent of n, s_1, s_2 and of F. So, $\{\hat{A}_{(n)}\}_{n=0}^{\infty}$ converges uniformly in F, the limit function $\hat{A}$ satisfies

$$\hat{A}(s_1, s_2) = \hat{A}_1\left(\frac{s_1+s_2}{2}\right) + \hat{A}_2\left(\frac{s_2-s_1}{2}\right) - \frac{1}{4}\int_0^{s_1+s_2} d\sigma_2 \int_0^{s_1-s_2} \hat{B}\left(\frac{\sigma_1+\sigma_2}{2}, \frac{\sigma_2-\sigma_1}{2}\right) d\sigma_1$$

and it is a unique solution in F to the Goursat problem given in (36) and (39). We substitute the solution into the inverse function $\hat{a}$ with values in a fixed J_m. We define $\hat{b}$ by (37) and $\hat{\rho}$ by integrating (38). Now, we set

$$(41)\quad \begin{aligned} \hat{g}_{11} &= \int_0^{s_1} \cos\hat{\kappa}_1(\eta) d\eta, \quad \hat{g}_{12} = \int_0^{s_1} \sin\hat{\kappa}_1(\eta) d\eta, \\ \hat{g}_{21} &= -\int_0^{s_2} \cos\hat{\kappa}_2(\eta) d\eta, \quad \hat{g}_{22} = -\int_0^{s_2} \sin\hat{\kappa}_2(\eta) d\eta. \end{aligned}$$

Then, $g_{jk} = \hat{g}_{jk} \circ T$ satisfy $\partial_y(g_{11} + g_{21}) = \partial_x(g_{12} + g_{22})$, $\partial_y(g_{12} - g_{22}) = \partial_x(g_{21} - g_{11})$. Finally, the pair of

$$(42)\quad \begin{aligned} u_1 &= \int_{(0,0)}^{(x,y)} \left(\frac{g_{11}+g_{21}}{2} dx + \frac{g_{12}+g_{22}}{2} dy\right), \\ u_2 &= \int_{(0,0)}^{(x,y)} \left(\frac{g_{12}-g_{22}}{2} dx + \frac{g_{21}-g_{11}}{2} dy\right) \end{aligned}$$

is a solution of (1) in D. □

The equation (9) is rewritten as

$$(43)\quad \frac{d\hat{\mu}}{ds_1} = -\frac{1}{2}\hat{\mu}\, \exp(\hat{\kappa}_1 \mathbf{i})\mathbf{k}, \quad \frac{d\hat{\nu}}{ds_2} = -\frac{1}{2}\hat{\nu}\, \exp(-\hat{\kappa}_2 \mathbf{i})\mathbf{k}.$$

So, $\hat{\mu}$ depends only on s_1 and $\hat{\nu}$ depends only on s_2. We obtain f from (12) and (33).

The transformation T is multiple-valued in general (see (32), (33) and (34)). We reduced therefore the system (1) to a global question of finding inverse functions. If T is multiple-valued, then the mapping f and (u_1, u_2) are also multiple-valued (see Examples 5, 6 in §5).

5. Examples.

EXAMPLE 1. A pair of arbitrary functions $(u_1(x), u_2(x))$ of class C^2 of the variable x is a solution of (1) (see Corollary 3.2 in §3). μ, ν are obtained if we integrate (not always by quadrature)

$$\mu'(x) = \mu(x)\{u_2''(x)\mathbf{j} - u_1''(x)\mathbf{k}\}/2\,, \quad \nu'(x) = \nu(x)\{u_1''(x)\mathbf{k} - u_2''(x)\mathbf{j}\}/2\,.$$

$e_1 = \mu(x)\overline{\nu(x)}$ is independent of y and $e_2 = \mu(x)\mathbf{i}\overline{\nu(x)}$ is constant. The image by f is the product of a curve in a three-dimensional Euclidean subspace E and a straight line perpendicular to E. x is the arc length of the curve and y is the arc length of the straight line. This is an application of formulas in §3 with $2a \equiv a - c \equiv 0 \pmod{2\pi\mathbf{Z}}$.

EXAMPLE 2. A pair of arbitrary functions $(u_1(x), u_2(y))$ of class C^2 of the variable x and of the variable y, respectively, is a solution of (1) and

$$df = (dx)\mu_0 \exp\{-u_1'(x)\mathbf{k}\}\bar{\nu}_0 + (dy)\mu_0\mathbf{i}\exp\{-u_2'(y)\mathbf{k}\}\bar{\nu}_0$$

(μ_0, ν_0 are constant unit quaternions). The image is the product of a curve in a two-dimensional Euclidean subspace E_1 and a curve in a two-dimensional Euclidean subspace E_2 perpendicular to E_1. x, y are the arc lengths of the curves. This is an application of formulas in §4 with $a + 2b + c \equiv 2c \equiv 0 \pmod{2\pi\mathbf{Z}}$.

EXAMPLE 3. Let C be the ellipse $(x/a)^2 + (y/b)^2 = 1$, where a, b are constants satisfying $0 < b < a$. An angle $\phi \in [0, 2\pi]$ represents a point of C as $(\xi, \eta) = (a\cos\phi, b\sin\phi)$. The arc length s in positive sense is such that $ds = \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\, d\phi$. The curvature is equal to $\kappa = ab(d\phi/ds)^3$. The normal line at (ξ, η) passes through (x, y) if and only if

$$x = \xi - (lb^2\xi/\sqrt{b^4\xi^2 + a^4\eta^2})\,, \quad y = \eta - (la^2\eta/\sqrt{b^4\xi^2 + a^4\eta^2})\,.$$

Equations (δ_1), (δ_2) in §3 are

$$(\delta_1) \qquad a^2(x - \xi)\eta = b^2(y - \eta)\xi\,,$$

$$(\delta_2) \qquad l = \{a^2(\eta - y)\eta + b^2(\xi - x)\xi\}/\sqrt{b^4\xi^2 + a^4\eta^2}\,,$$

respectively. We obtain the angle $\phi \pmod{2\pi\mathbf{Z}}$ from (δ_1) and the signed length l of the normal segment from (δ_2).

The focal set Γ is the simple closed curve $|ax|^{2/3} + |by|^{2/3} = c^{2/3}$ which is parametrized as $x = c\xi^3/a^4$, $y = -c\eta^3/b^4$ with $c = a^2 - b^2$. We can draw four normal lines from a point inside of Γ, two from a point outside of Γ, three from a point of Γ except for cusps and two from each one of cusps. At any point (x, y) not lying on Γ, we can define the multiple-valued function $\theta(x, y) \pmod{\pi\mathbf{Z}}$ to be

$$\tan\theta(x, y) = (a/b)\tan\phi$$

(see (δ_1) above). θ has four function elements in the interior of Γ and two in the exterior.

EXAMPLE 4. Let C be the curve $y = \cos x$ $(-\infty < x < +\infty)$. The equation of the normal line to C at $(x', \cos x')$ is

$$x' - x + y\sin x' = \sin x' \cos x' . \tag{44}$$

Set for the moment $F(x') = x' - x + y\sin x' - \sin x' \cos x'$. Given a point (x, y) of the plane, let $n = n(x, y)$ be the number of real solutions x' of the equation $F(x') = 0$. We have $n(x + 2\pi, y) = n(x, y)$ because of the periodicity.

n is finite at every point because F is an analytic function of three variables and does not vanish if $|x'| > |y| + |x| + 1$. Next, for any positive integer N, we have

$$F(k\pi - \pi/2)F(k\pi + \pi/2) < 0 \quad (k = -N, 1 - N, \dots, N - 1, N)$$

if $|x| \le \pi$ and if $|y| \ge (N + 2)\pi$. So, F has a zero in every interval $(k\pi - \pi/2, k\pi + \pi/2)$. Since n is a periodic function of x of period 2π, we have

$$n(x, y) \ge 2N \quad \text{if} \quad |y| \ge (N + 2)\pi \quad (N = 1, 2, 3, \dots) .$$

Therefore, through an arbitrary point (x, y), there passes only a finite number of normal lines, but the number of normal lines is not limited as $|y|$ goes to infinity.

If the normal line at $(x', \cos x')$ passes through a given point (x, y), the formulas (27), (28) are interpreted as (44) and

$$\cos\theta = \sin\theta \sin x' . \tag{45}$$

θ is always assumed to satisfy $\pi/4 \le \theta \le 3\pi/4$. If r is the signed distance of (x', y') to (x, y) such that $(y - \cos x)r \ge 0$, we have $x - x' = r\cos\theta$, $y - \cos x' = r\sin\theta$ and

$$dx\mathbf{1} + dy\mathbf{i} = \exp(\theta\mathbf{i})[dr\mathbf{1} + \{r\,d\theta - (dx'/\sin\theta)\}\mathbf{i}] .$$

Change the coordinate system from (x, y) to (r, θ) supposing that $2\sin^2 x' + y\cos x' \ne 0$. Then, x' depends only on θ and $\xi_j = \varphi_j(\theta)\sin\theta\cos x'/(2\sin^2 x' + y\cos x')$ satisfy (25), where $\varphi_j(\theta)$ $(j = 1, 2)$ are arbitrary real-valued functions. (u_1, u_2) is obtained from

$$d(\partial_x u_j) = -\varphi_j(\theta)\sin\theta\,d\theta , \quad d(\partial_y u_j) = \varphi_j(\theta)\cos\theta\,d\theta \quad (j = 1, 2) .$$

μ, ν depend only on θ, and the equation (9) is rewritten as

$$\mu' = (\mu/2)\exp(\theta\mathbf{i})\{\varphi_1(\theta)\mathbf{j} + \varphi_2(\theta)\mathbf{k}\} , \quad \nu' = (\nu/2)\exp(-\theta\mathbf{i})\{\varphi_1(\theta)\mathbf{j} + \varphi_2(\theta)\mathbf{k}\} .$$

If $\varphi_1 = \alpha$ (real constant), $\varphi_2 = 0$, $\mu|_{\theta=0} = \mathbf{1}$ and $\nu|_{\theta=0} = \mathbf{i}$, for example, we have

$$\mu = \left(\frac{\varepsilon}{\beta}\mathbf{1} - \frac{\alpha}{2\beta}\mathbf{k}\right)\exp(-\gamma\theta\mathbf{i}) + \left(\frac{\gamma}{\beta}\mathbf{1} + \frac{\alpha}{2\beta}\mathbf{k}\right)\exp(\varepsilon\theta\mathbf{i}) ,$$

$$\nu = \left(\frac{\varepsilon}{\beta}\mathbf{i} - \frac{\alpha}{2\beta}\mathbf{j}\right)\exp(\gamma\theta\mathbf{i}) + \left(\frac{\gamma}{\beta}\mathbf{i} + \frac{\alpha}{2\beta}\mathbf{j}\right)\exp(-\varepsilon\theta\mathbf{i}) ,$$

$$\text{where} \quad \beta = \sqrt{1 + \alpha^2} , \ \gamma = (\beta + 1)/2 , \ \varepsilon = (\beta - 1)/2 .$$

We set, for simplicity, $\mathbf{i}' = (\mathbf{i} + \alpha\mathbf{j})/\beta$, $\mathbf{j}' = (\mathbf{j} - \alpha\mathbf{i})/\beta$. Then, the mapping is

$$f(x, y) = \frac{\alpha}{\beta} r\mathbf{j}' - \frac{1}{\beta} r \exp(\beta\theta\mathbf{i}')\mathbf{i}' - \int_{x_0'}^{x'} \exp(\beta\theta\mathbf{i}')\frac{dx'}{\sin\theta}. \tag{46}$$

The lower limit x_0' of integration is a constant depending on the function element under consideration.

EXAMPLE 5. We apply the formulas in §4 to

$$\rho = 2e^{s_2/2}, \quad a = \pi/2, \quad b = s_1/2, \quad c = s_1 + (\pi/2).$$

Then, $x = e^{-s_2/2}\cos(s_1/2)$, $y = e^{-s_2/2}\sin(s_1/2)$ if we modify (33). From (41), we have

$$g_{11} = \frac{x^2 - y^2}{x^2 + y^2}, \quad g_{12} = \frac{2xy}{x^2 + y^2}, \quad g_{21} = \log(x^2 + y^2), \quad g_{22} = 0.$$

Substituting these into (42), a solution of (1) is

$$u_1 = x \log\sqrt{x^2 + y^2} - (x/2), \quad u_2 = y \log\sqrt{x^2 + y^2} - (y/2). \tag{47}$$

The equation for $\hat{\nu}$ in (43) is $\hat{\nu}' = -\hat{\nu}\mathbf{k}/2$, so $\hat{\nu} = \beta\exp(-s_2\mathbf{k}/2)$. The equation for $\hat{\mu}$ is $\hat{\mu}' = (\hat{\mu}/2)\exp(s_1\mathbf{i})\mathbf{j}$ which is equivalent to $\hat{\mu}'' + \hat{\mu}'\mathbf{i} + (\hat{\mu}/4) = 0$ and $\hat{\mu}'(0) = \hat{\mu}(0)\mathbf{j}/2$. So, $\hat{\mu}$ is the sum of two exponential functions multiplied by constants from the left.

$$\mu = \frac{(\sqrt{2}+1)\alpha}{\sqrt{4+2\sqrt{2}}}\mathbf{k}\left(\frac{x\mathbf{1} + y\mathbf{i}}{\sqrt{x^2+y^2}}\right)^{\sqrt{2}-1} + \frac{\alpha}{\sqrt{4+2\sqrt{2}}}\left(\frac{x\mathbf{1} - y\mathbf{i}}{\sqrt{x^2+y^2}}\right)^{\sqrt{2}+1}, \ \nu = \beta(x^2+y^2)^{\mathbf{k}/2},$$

where α, β are unit quaternions. For a particular choice of α and β, we obtain the following mapping.

$$f(x, y) = f(0, 0) + \frac{1}{\sqrt{2}}\left(\frac{x\mathbf{1} - y\mathbf{i}}{\sqrt{x^2+y^2}}\right)^{\sqrt{2}}(x^2 + y^2)^{(\mathbf{1}-\mathbf{k})/2}. \tag{48}$$

If we set $x = r\cos\theta$, $y = r\sin\theta$ ($r = \sqrt{x^2 + y^2}$), this is rewritten as

$$f(x, y) = f(0, 0) + (1/\sqrt{2})\exp(-\sqrt{2}\theta\mathbf{i})r^{\mathbf{1}-\mathbf{k}}.$$

It is quite elementary to verify that f is an isometric immersion. In fact, we have

$$\begin{aligned} df &= \exp(-\sqrt{2}\theta\mathbf{i})\{-(rd\theta)\mathbf{i} + (dr/\sqrt{2})(\mathbf{1} - \mathbf{k})\}r^{-\mathbf{k}}, \\ |df|^2 &= |-(rd\theta)\mathbf{i} + (dr/\sqrt{2})(\mathbf{1} - \mathbf{k})|^2 = r^2(d\theta)^2 + (dr)^2 = (dx)^2 + (dy)^2. \end{aligned}$$

We see that f is multiple-valued with an infinite number of function elements.

EXAMPLE 6. The rest of the present paper will be devoted to the study of this example. We make use of quaternions as above and also of ordinary complex numbers with imaginary unit $i = \sqrt{-1}$. We apply the formulas (35) through (38) to

$$\hat{\rho} = e^{2s_1s_2}, \quad \hat{a} = \pi/2, \quad \hat{b} = {s_1}^2 + {s_2}^2 - (\pi/4), \quad \hat{c} = 2{s_1}^2 - 2{s_2}^2. \tag{49}$$

Then, $\psi_1 = {s_1}^2 - {s_2}^2 + (\pi/4)$, $\psi_2 = {s_1}^2 - {s_2}^2 - (\pi/4)$, $\kappa_1 = 2{s_1}^2$, $\kappa_2 = -2{s_2}^2$. The mapping T^{-1} (see (33)) is defined to be

$$z = x + iy = \int_0^t e^{\eta^2} d\eta = t\, {}_1F_1(1/2, 3/2; t^2)\,, \quad \text{where } t = e^{\pi i/4}(s_1 + is_2)\,. \tag{50}$$

We discuss the mapping $T : (x, y) \mapsto (s_1, s_2)$ in Lemma 5.2 below.

Equations (43) are interpreted as

$$\hat{\mu}' = -(\hat{\mu}/2)\exp(2{s_1}^2\mathbf{i})\mathbf{k}\,, \quad \hat{\nu}' = -(\hat{\nu}/2)\exp(2{s_2}^2\mathbf{i})\mathbf{k}\,. \tag{51}$$

Two equations are of the same type and the former is equivalent to

$$\hat{\mu}'' + 4s_1\hat{\mu}'\mathbf{i} + (\hat{\mu}/4) = 0 \quad \text{and} \quad \hat{\mu}'(0) = -\hat{\mu}(0)\mathbf{k}/2\,.$$

We have the following unique power series solution satisfying $\hat{\mu}(0) = \mathbf{1}$.

$$\begin{aligned} &\hat{\mu}(s_1) = \Phi(s_1) - \mathbf{k}\Psi(s_1)\,, \quad \hat{\nu}(s_2) = \Phi(s_2) - \mathbf{k}\Psi(s_2)\,, \\ &\Phi(l) = {}_1F_1\left(-\frac{1}{32}\mathbf{i}, \frac{1}{2}\mathbf{1}; -2l^2\mathbf{i}\right), \quad \Psi(l) = {}_1F_1\left(\frac{1}{2}\mathbf{1} - \frac{1}{32}\mathbf{i}, \frac{3}{2}\mathbf{1}; -2l^2\mathbf{i}\right)\frac{l}{2}\,, \end{aligned} \tag{52}$$

where we set for quaternions α, γ, λ which commute one another

$${}_1F_1(\alpha, \gamma; \lambda) = \sum_{p=0}^{\infty} \frac{\Gamma(\alpha + p\mathbf{1})\Gamma(\gamma)}{p!\Gamma(\alpha)\Gamma(\gamma + p\mathbf{1})}\lambda^p\,.$$

The gamma function is defined to be

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha - \mathbf{1}}du \quad (\Re\alpha > 0)\,, \quad \Gamma(\alpha + \mathbf{1}) = \alpha\Gamma(\alpha)\,.$$

In view of (50), (52) and (12), we define $\hat{f}$ to be

$$\hat{f}(s_1, s_2) = \int_{(0,0)}^{(s_1,s_2)} \hat{\mu}(s_1')\{\exp(\mathbf{t}'^2)d\mathbf{t}'\}\overline{\hat{\nu}(s_2')}\,, \quad \mathbf{t}' = \exp(\pi\mathbf{i}/4)(s_1'\mathbf{1} + s_2'\mathbf{i})\,, \tag{53}$$

and an isometric immersion to be $f = \hat{f} \circ T$. Let us state the conclusion as Proposition 5.1 in advance and a detailed analysis as Lemmas 5.2 and 5.3 afterwards.

PROPOSITION 5.1. *Let t be the inverse function of $z = t\, {}_1F_1(1/2, 3/2; t^2)$ and $\mathcal{R}$ be the Riemann surface of the field of meromorphic functions of the variable t. Then, f realizes $\mathcal{R}$ isometrically in $\boldsymbol{R}^4$ and t is a uniformizing variable.*

PROOF. z is an entire function of t, so x, y are single-valued real-analytic functions in the whole s_1s_2-plane (see (50)). $\hat{f}$ is a single-valued real-analytic mapping of the s_1s_2-plane into $\boldsymbol{R}^4$ (see (53)). So, the variable t uniformizes both z and f. On the other hand, a function of t is a function on $\mathcal{R}$ if and only if it is single-valued and meromorphic in the whole t-plane possibly except at $t = \infty$. Since f is an isometric immersion of the z-plane into $\boldsymbol{R}^4$, f realizes $\mathcal{R}$ isometrically in $\boldsymbol{R}^4$. □

LEMMA 5.2. *Let $z = z(t) = t\, {}_1F_1(1/2, 3/2; t^2)$ (see (50)). Then,*

(i) *For every complex number z^0, there exists an infinite number of complex numbers t such that $z(t) = z^0$.*

(ii) *$z(t)$ tends to $\sqrt{\pi}i/2$ as t goes to ∞ in the sector $\pi/4 \le \arg t \le 3\pi/4$, and, $z(t)$ tends to $-\sqrt{\pi}i/2$ as t goes to ∞ in the sector $-3\pi/4 \le \arg t \le -\pi/4$.*

PROOF. (i) This is a consequence of a theorem of Picard on the value distribution (see Shimomura [14, p. 23]). However, we prove it directly because we need (57) below. Set $\arg t = \pi/2$ for a moment. Then,

$$z = I_1 + I_2 ,$$

where I_1 is the integral of e^{η^2} from $\eta = 0$ to ∞i and I_2 is that from ∞i to $|t|i$. First, we have $I_1 = \sqrt{\pi}i/2$. Rewrite I_2 as an integral with respect to $v = t^2 - \eta^2$ to have

$$I_2 = \frac{e^{t^2}}{2t}\int_0^{+\infty} \frac{e^{-v}dv}{\sqrt{1-v'}} = \frac{e^{t^2}}{2t}(1+E), \quad \text{where } E = \frac{1}{t^2}\int_0^{+\infty} \frac{v\,e^{-v}dv}{\sqrt{1-v'}+1-v'}, \quad v' = \frac{v}{t^2}.$$

z, I_1, I_2, E being holomorphic in the upper half-plane, the theorem of identity implies

$$z(t) = (\sqrt{\pi}i/2) + e^{t^2}(1+E)/(2t) \quad \text{if } 0 < \arg t < \pi . \tag{54}$$

For any δ satisfying $0 < \delta < \pi/2$, there exists a positive constant $A = A(\delta)$ such that

$$|E| \le A/|t|^2 \quad \text{if } \delta \le \arg t \le \pi - \delta \text{ and } |t| \ge 1 , \tag{55}$$

because $|\sqrt{1-v'} + 1 - v'|$ ($v' = v/t^2$) is greater than a positive constant.

For an arbitrary z^0 ($\ne \sqrt{\pi}i/2$), we set

$$\zeta = 2z^0 - \sqrt{\pi}i = |\zeta|e^{i\alpha}, \quad -\pi < \alpha \le \pi .$$

The equation $z = z^0$ implies

$$e^{t^2}(1+E) = \zeta t = \zeta t\, e^{2n\pi i}$$

if $\delta \le \psi = \arg t \le \pi - \delta$ and $|t| \ge 1$, or

$$|t|^2 \cos 2\psi = \log(|\zeta||t|) + E_1(t), \quad |t|^2 \sin 2\psi = \psi + \alpha + 2n\pi + E_2(t), \tag{56}$$

where $E_1(t)$ and $E_2(t)$ are real-valued, $|E_1(t)| + |E_2(t)| \le C_1/|t|^2$, n is an integer and $C_1 = C_1(z^0, \delta)$ is a positive constant independent of t, n. Take a large T (≥ 1) and a δ, $0 < \delta < \pi/4$. Then, for every τ ($\ge T$), there exists a unique $\psi(\tau)$ such that $\delta \le \psi(\tau) \le \pi/2$ and

$$\tau^2 \cos 2\psi(\tau) = \log(|\zeta|\tau) + E_1(\tau e^{\psi(\tau)i}).$$

It satisfies

$$\left|\psi(\tau) - \frac{\pi}{4} + \frac{\log\tau}{2\tau^2}\right| \le \frac{C_2}{\tau^2}, \tag{a}$$

where $C_2 = C_2(z^0)$ is a positive constant independent of τ. We can then choose a positive integer $n_0 = n_0(z^0)$ such that there exists a unique $\tau_n \ge T$ such that

$$\tau_n{}^2 \sin 2\psi(\tau_n) = \psi(\tau_n) + \alpha + 2n\pi + E_2(\tau_n e^{\psi(\tau_n)i})$$

for every integer $n \geq n_0$. It satisfies

$$\text{(b)} \qquad |\tau_n{}^2 - \psi(\tau_n) - \alpha - 2n\pi| \leq \frac{C_3(\log n)^2}{n} \quad \text{for } n \geq n_0 \,,$$

where $C_3 = C_3(z^0)$ is a positive number independent of n.

We have $z(t_n(z^0)) = z^0$ if we set $t_n(z^0) = \tau_n e^{\psi(\tau_n)i}$ $(n \geq n_0)$ provided that $z^0 \neq \sqrt{\pi}i/2$. (a) and (b) imply

$$\text{(57)} \qquad \left| |t_n(z^0)|^2 - \alpha - \left(2n + \frac{1}{4}\right)\pi \right| \leq \frac{B(\log n)^2}{n} \,, \qquad \left| \arg t_n(z^0) - \frac{\pi}{4} + \frac{\log n}{8\pi n} \right| \leq \frac{B}{n} \,,$$

for $n \geq n_0$, where $B = B(z^0)$ is a positive constant independent of n. There exists also an infinite number of t such that $z(t) = \sqrt{\pi}i/2$ because $z(\overline{t_n(-\sqrt{\pi}i/2)}) = \sqrt{\pi}i/2$ in view of $z(\bar{t}) = \overline{z(t)}$ and $z(-t) = -z(t)$. Consequently, there exists an infinite number of t such that $z(t) = z^0$ for every value of z^0.

We have an infinite number of zeros $\{t_n(0), -t_n(0), \bar{t}_n(0), -\bar{t}_n(0)\}_{n=n_0}^{\infty}$ of z. There exists at most a finite number of zeros other than this sequence. (57) is an extended version of the formulas in the book of Buchholz [1, p. 180, (3a), (3b)].

(ii) Obvious from (54), (55) and by $z(-t) = -z(t)$. □

LEMMA 5.3. (i) *For given* $t_1 \neq \infty$, $t_2 \neq \infty$ *and* $z^0 \neq \infty$ *satisfying*

$$z(t_1) = z(t_2) = z^0 \,,$$

let f_1 *and* f_2 *be two function elements of* f *such that*

$$f_1(z^0) = \hat{f}(t_1) \quad \text{and} \quad f_2(z^0) = \hat{f}(t_2) \,.$$

Then, $t_1 = t_2$ *if* $f_1 = f_2$.

(ii) f *has an infinite number of function elements each of which is real-analytic in the whole* xy*-plane.*

(iii) f *has a unique function element* f_0 *whose first order derivatives are discontinuous at two finite points* $(x, y) = (0, \sqrt{\pi}/2)$ *and* $(0, -\sqrt{\pi}/2)$.

(iv) *The image of the* xy*-plane by* f *is arcwise-connected.*

PROOF. (i) Denote by $\tilde{t} = \tilde{t}(t)$ the solution of

$$\text{(c)} \qquad \exp(\tilde{t}^2)d\tilde{t} = \exp(t^2)dt \,, \quad \tilde{t}(t_1) = t_2$$

in a disk $|t - t_1| < \delta$ $(\delta > 0)$. Then, $z(\tilde{t}) = z(t)$, $f_2(z(t)) = f_2(z(\tilde{t})) = \hat{f}(\tilde{t})$, $f_1(z(t)) = \hat{f}(t)$ if $|t - t_1| < \delta$. For the proof, it suffices to show that $t_2 = t_1$ if

$$\text{(d)} \qquad \hat{f}(\tilde{t}) - \hat{f}(t_2) = \hat{f}(t) - \hat{f}(t_1)$$

in a disk $|t - t_1| < \delta'$ $(0 < \delta' \leq \delta)$. By regarding $t = e^{\pi i/4}(s_1 + s_2 i)$ as $\mathbf{t} = \exp(\pi\mathbf{i}/4)(s_1\mathbf{1} + s_2\mathbf{i})$ and $\tilde{t} = e^{\pi i/4}(\tilde{s}_1 + \tilde{s}_2 i)$ as $\tilde{\mathbf{t}} = \exp(\pi\mathbf{i}/4)(\tilde{s}_1\mathbf{1} + \tilde{s}_2\mathbf{i})$, we know (c), (d) and (53) imply

$$\exp(\tilde{\mathbf{t}}^2)d\tilde{\mathbf{t}} = \exp(\mathbf{t}^2)d\mathbf{t} \,, \quad \hat{\mu}(\tilde{s}_1)\{\exp(\tilde{\mathbf{t}}^2)d\tilde{\mathbf{t}}\}\overline{\hat{\nu}(\tilde{s}_2)} = \hat{\mu}(s_1)\{\exp(\mathbf{t}^2)d\mathbf{t}\}\overline{\hat{\nu}(s_2)}.$$

Since $\exp(\mathbf{t}^2)d\mathbf{t}$ generates $\mathbf{1}$ and $\mathbf{i}$ at every point in the disk $|t-t_1|<\delta'$, we have

$$\hat{\mu}(\tilde{s}_1)\overline{\hat{\nu}(\tilde{s}_2)}=\hat{\mu}(s_1)\overline{\hat{\nu}(s_2)}\,,\quad \hat{\mu}(\tilde{s}_1)\mathbf{i}\,\overline{\hat{\nu}(\tilde{s}_2)}=\hat{\mu}(s_1)\mathbf{i}\,\overline{\hat{\nu}(s_2)}\,.$$

So, there exists a real-valued smooth function $\omega=\omega(t)$ such that

$$\text{(e)}\qquad \hat{\mu}(\tilde{s}_1)=\hat{\mu}(s_1)\exp(\omega\mathbf{i})\,,\quad \hat{\nu}(\tilde{s}_2)=\hat{\nu}(s_2)\exp(\omega\mathbf{i})\,.$$

We differentiate both sides of two equalities and cancel $\hat{\mu}$ and $\hat{\nu}$ by making use of (51) and (e) itself. Then,

$$\{(d\tilde{s}_j)\exp(2\tilde{s}_j^2\mathbf{i}+\omega\mathbf{i})-(ds_j)\exp(2s_j^2\mathbf{i}-\omega\mathbf{i})\}\mathbf{k}=-2d\exp(\omega\mathbf{i})\quad (j=1,2)\,.$$

The left-hand sides are orthogonal to the right-hand sides, so ω is a constant and

$$(d\tilde{s}_j)\exp\{(2\tilde{s}_j^2-2s_j^2+2\omega)\mathbf{i}\}=(ds_j)\mathbf{1}\quad (j=1,2)\,.$$

Since ds_j and $d\tilde{s}_j$ are real, there exist integers n_1,n_2 such that

$$2\tilde{s}_j^2-2s_j^2+2\omega=n_j\pi,\quad d\tilde{s}_j=(-1)^{n_j}ds_j\quad (j=1,2)\,.$$

Since $\tilde{s}_jd\tilde{s}_j=s_jds_j$ by the first equations, $\tilde{s}_j=(-1)^{n_j}s_j$ by the second equations, and then $n_1=n_2\ (=2\omega/\pi)$ again by the first ones. We denote it by n to have

$$\text{(f)}\qquad \tilde{s}_j=(-1)^n s_j\quad (j=1,2)\,.$$

Finally, the equation $\exp(\tilde{\mathbf{t}}^2)d\tilde{\mathbf{t}}=\exp(\mathbf{t}^2)d\mathbf{t}$ is reduced to $(-1)^n=1$, so n is even and $(\tilde{s}_1,\tilde{s}_2)=(s_1,s_2)$, or $\tilde{t}=t$ identically and $t_2=t_1$.

(ii) We have just shown that distinct zeros of $z(t)$ give rise to distinct function elements of f. But, we discuss once more in another way to obtain (59) below.

For zeros $t_n(0)=e^{\pi i/4}(s_{n,1}+s_{n,2}i)$ of $z(t)$ (see (57)), we have $s_{n,1}>0>s_{n,2}$ and there exists a positive number C_4 independent of n such that

$$\text{(g)}\qquad |s_{n,1}-\sqrt{2\pi n}|\le\frac{C_4}{\sqrt{n}}\,,\quad \left|s_{n,2}+\frac{\log n}{\sqrt{32\pi n}}\right|\le\frac{C_4}{\sqrt{n}}\quad\text{for}\quad n\ge n_0\,.$$

From this and (52), we have at first

$$\text{(h)}\qquad |\hat{\nu}(s_{n,2})-\mathbf{1}|\le\frac{C_5\log n}{\sqrt{n}}$$

with a positive number C_5 independent of n. Next, we apply the inequalities

$$\text{(58)}\qquad \begin{aligned}&\left|{}_1F_1\Big(-\frac{1}{32}\mathbf{i},\frac{1}{2}\mathbf{1};-\lambda\mathbf{i}\Big)-\frac{\sqrt{\pi}}{\Gamma((\mathbf{1}/2)+(\mathbf{i}/32))}(\lambda\mathbf{i})^{\mathbf{i}/32}\right|\le\frac{D}{\sqrt{\lambda}}\qquad(\lambda\ge1)\,,\\ &\left|{}_1F_1\Big(\frac{1}{2}\mathbf{1}-\frac{1}{32}\mathbf{i},\frac{3}{2}\mathbf{1};-\lambda\mathbf{i}\Big)-\frac{\sqrt{\pi}}{2\Gamma(\mathbf{1}+(\mathbf{i}/32))}(\lambda\mathbf{i})^{(\mathbf{i}/32)-(\mathbf{1}/2)}\right|\le\frac{D}{\lambda}\quad(\lambda\ge1)\end{aligned}$$

to $\lambda=2s_{n,1}^2$ (see [1, p. 91, (3)]), where D is a positive number independent of λ. Then, there exists a positive number C_6 independent of n such that

$$\text{(i)}\quad |\hat{\mu}(s_{n,1})-\mathbf{u}(4\pi n)^{\mathbf{i}/32}|\le\frac{C_6}{\sqrt{n}}\ \text{ with }\ \mathbf{u}=\frac{\sqrt{\pi}\,\mathbf{i}^{\mathbf{i}/32}}{\Gamma((\mathbf{1}/2)+(\mathbf{i}/32))}-\mathbf{k}\frac{\sqrt{\pi}\,\mathbf{i}^{(\mathbf{i}/32)-(\mathbf{1}/2)}}{\sqrt{32}\Gamma(\mathbf{1}+(\mathbf{i}/32))}\,.$$

$\mathbf{u}$ and $(4\pi n)^{\mathbf{i}/32}$ are of unit length. Denote

$$\hat{e}_{n,1} = \hat{\mu}(s_{n,1})\overline{\hat{\nu}(s_{n,2})}\,, \quad \hat{e}_{n,2} = \hat{\mu}(s_{n,1})\mathbf{i}\overline{\hat{\nu}(s_{n,2})}\,.$$

Then, by (h), (i), there exists a positive number E independent of n such that

$$|\hat{e}_{n,1} - \mathbf{u}(4\pi n)^{\mathbf{i}/32}| + |\hat{e}_{n,2} - \mathbf{ui}(4\pi n)^{\mathbf{i}/32}| \le \frac{E \log n}{\sqrt{n}} \quad \text{for } n \ge n_0\,. \tag{59}$$

Since e^{π} is a transcendental number (see Gelfond [10] and Siegel [15, p. 84]), $(\log 2)/\pi$ is irrational and the set of fractional parts of $\{(\log(2^p\pi))/(64\pi)\}_{p=2}^{\infty}$ is everywhere dense in the interval $(0, 1)$. So, given any integer $L \ge 2$, there exist positive integers $n_1, \dots, n_{2L}$ such that

$$\left|\hat{e}_{n_p,1} - \mathbf{u} \exp\frac{p\pi\mathbf{i}}{L}\right| < \frac{\pi}{4L}\,, \quad \left|\hat{e}_{n_p,2} - \mathbf{ui} \exp\frac{p\pi\mathbf{i}}{L}\right| < \frac{\pi}{4L}$$

for $1 \le p \le 2L$. Let $f_p(x, y)$ be the function element such that $f_p(0,0) = \hat{f}(s_{n_p,1}, s_{n_p,2})$. Then, $f_1, \dots, f_{2L}$ are distinct because $\partial_x f_p(0,0) = \hat{e}_{n_p,1}$ and $\partial_y f_p(0,0) = \hat{e}_{n_p,2}$ ($p = 1, \dots, 2L$) are distinct. L being arbitrary, f has an infinite number of function elements.

If $z(t^0) = z^0$ for a finite t^0 and a finite z^0, basic equalities

$$\frac{dz}{dt} = e^{t^2}\,, \quad \frac{dt}{dz} = e^{-t^2} \tag{60}$$

show that the function element of $t(z)$ satisfying $t(z^0) = t^0$ is holomorphic in a neighborhood of z^0. So, every function element of $(s_1(x, y), s_2(x, y))$ is real analytic provided that a finite (s_1, s_2) corresponds to a finite (x, y). And hence, every function element $f(x, y) = \hat{f}(s_1(x, y), s_2(x, y))$ is real analytic in the whole xy-plane with the only exception shown just below.

(iii) The sector $\pi/4 < \arg t < 3\pi/4$ corresponds to the sector $s_1 > 0, s_2 > 0$. In the domain $(\pi/4) + \delta < \arg t < (3\pi/4) - \delta$, $|t| \ge 1$ $(0 < \delta < \pi/4)$, there exists a positive constant $C_7 = C_7(\delta)$ independent of t such that $C_7|t| < s_1 < |t|$ and $C_7|t| < s_2 < |t|$. Applying again (54), (55) and (58) with $\lambda = 2{s_1}^2$ or $\lambda = 2{s_2}^2$, we have

(j) $\hat{\mu}(s_1) = \mathbf{u}(\sqrt{2}s_1)^{\mathbf{i}/16} + E_5\,, \quad \hat{\nu}(s_2) = \mathbf{u}(\sqrt{2}s_2)^{\mathbf{i}/16} + E_6\,, \quad |E_5| + |E_6| \le C_8/|t|\,,$

where $C_8 = C_8(\delta)$ is a positive constant independent of t. Let f_0 be the function element of f obtained from (53) by integrating on the line segment from $(0, 0)$ to (s_1, s_2) in this sector. (j) implies

$$(\partial_x f_0)\hat{} = \mathbf{u}(s_1/s_2)^{\mathbf{i}/16}\,\bar{\mathbf{u}} + \cdots\,, \quad (\partial_y f_0)\hat{} = \mathbf{u}(s_1/s_2)^{\mathbf{i}/16}\,\mathbf{i}\bar{\mathbf{u}} + \cdots\,. \tag{61}$$

As $|t|$ goes to $+\infty$ in this sector, $x + iy$ tends to $\sqrt{\pi}i/2$ and s_1/s_2 assumes any value of the interval $[\tan\delta, \cot\delta]$. So, $\partial_x f_0$ and $\partial_y f_0$ are discontinuous at $(x, y) = (0, \sqrt{\pi}/2)$. f_0 has the same property at $(x, y) = (0, -\sqrt{\pi}/2)$ by symmetry.

(iv) Denote by ∞_+ and ∞_- the limits of t in the sectors $\pi/4 < \arg t < 3\pi/4$ and $-3\pi/4 < \arg t < -\pi/4$, respectively, as $|t|$ goes to $+\infty$. Let S be the union of the s_1s_2-plane, ∞_+ and of ∞_-. Then, $\hat{f}$ is a one-to-one mapping of S onto the image of the xy-plane by f. Given any point $\hat{f}(\sigma)$ $(\sigma \in S)$ of the image, $\hat{f}(\sigma)$ is equal to the integral (53) on the line segment or the half-line in S from $(0,0)$ to σ. So, any point $\hat{f}(\sigma)$ can be connected with $\hat{f}(0,0)=0$ by an arc. Also, two arbitrary points $\hat{f}(\sigma)$, $\hat{f}(\sigma')$ can be connected by an arc in the image of the xy-plane by f. □

REMARK 1. We restrict ourselves to the half-line $\arg t = \pi/4$ parametrized as $t = e^{\pi i/4}s_1$, $0 \le s_1 < +\infty$. As s_1 goes to $+\infty$, the first order derivatives of f behave like

$$(\partial_x f)\hat{}(s_1,0) = \mathbf{u}(\sqrt{2}s_1)^{\mathbf{i}/16} + \cdots, \quad (\partial_y f)\hat{}(s_1,0) = \mathbf{ui}(\sqrt{2}s_1)^{\mathbf{i}/16} + \cdots. \tag{62}$$

This is geometrically interpreted as follows. As s_1 goes to $+\infty$ on this half-line, the tangent space tends to the fixed linear subspace spanned by **u** and **ui**, and the image of the standard orthonormal frame of the xy-plane rotates by an angle $(\log s_1)/16$ in this subspace within an error. Analogous interpretations may be possible for (59) and (61). But we cannot evaluate $\hat{f}$ themselves as $|t|$ goes to $+\infty$.

REMARK 2. Independently of the notation in the proofs of Lemmas 5.2 and 5.3, we enumerate all zeros of $z(t)$ to be $t_p = e^{\pi i/4}(s_{p,1} + s_{p,2}i)$ $(p = 0, 1, 2, \dots)$ with $t_0 = 0$. For each $p \ge 1$, let $\hat{\gamma}_p$ be a fixed oriented rectifiable curve from t_0 to t_p passing through no other zero in the t-plane and Γ_p be the corresponding curve in $\mathcal{R}$. Then, $\hat{\gamma}_p \circ T$ is a loop in the z-plane, while Γ_p is not closed and $\{\Gamma_p\}_{p=1}^{\infty}$ are distinct due to Lemma 5.3, (i). With $\hat{\gamma}_p$ or Γ_p, we can associate an element A_p of $SO(4)$ defined to be

$$A_p\mathbf{q} = \hat{\mu}(s_{p,1})\mathbf{q}\overline{\hat{\nu}(s_{p,2})} \tag{63}$$

(see (52)). Then, the subgroup of $SO(4)$ generated by $\{A_p\}_{p=1}^{\infty}$ can be regarded as the monodromy group of $\mathcal{R}$.

On the other hand, in view of (42), (50), we define $(\hat{u}_1, \hat{u}_2)$ to be

$$\hat{u}_1 + i\hat{u}_2 = \frac{1}{2}\int_{(0,0)}^{(s_1,s_2)} \{{}_1F_1(1/2, 3/2; 2s_1'^{\,2}i)\exp(\bar{t}'^2)s_1' d\bar{t}' \\ - {}_1F_1(1/2, 3/2; 2s_2'^{\,2}i)\exp(t'^2)s_2' dt'\} \tag{64}$$

and (u_1, u_2) to be $(\hat{u}_1, \hat{u}_2) \circ T$. Then, $(\hat{u}_1, \hat{u}_2)$ is a single-valued real analytic function.

COROLLARY 5.4. (i) *The solution* (u_1, u_2) *of* (1) *has an infinite number of function elements each of which is real analytic in the whole* xy*-plane.*

(ii) (u_1, u_2) *has a unique function element which is non-differentiable at two finite points* $(x, y) = (0, \sqrt{\pi}/2)$ *and* $(0, -\sqrt{\pi}/2)$.

PROOF. (i) Set $\rho_n = \hat{\rho}(s_{n,1}, s_{n,2}) = e^{2s_{n,1}s_{n,2}}$ (see (19)) in the notation of (g). Then, there exists a positive number F independent of n such that

$$|\rho_n - (1/\sqrt{n})| \le F(\log n)^2/n \quad \text{for} \quad n \ge n_0 . \tag{65}$$

So, ρ_n are distinct for an infinite number of n. And hence, (u_1, u_2) has an infinite number of function elements.

(ii) In the domain $(\pi/4)+\delta<\arg t<(3\pi/4)-\delta$, $|t|\geq 1$ $(0<\delta<\pi/4)$, we apply again (54), (55) to t replaced by $\sqrt{2}s_j e^{\pi i/4}$ $(j=1,2)$. Then,

$$d\left(\hat{u}_1-\frac{\sqrt{\pi}}{4}y\right)+i\,d\left(\hat{u}_2-\frac{\sqrt{\pi}}{4}y\right)$$
$$=e^{i|t|^2-2s_1s_2}\left(\frac{1+E_7}{8s_1}d\bar{t}-\frac{1+E_8}{8s_2}dt\right),\quad |E_7|+|E_8|\leq\frac{C_{10}}{|t|^2},$$

where $C_{10}=C_{10}(\delta)$ is a positive constant independent of t. The corresponding function element of (u_1,u_2) is non-differentiable at $(x,y)=(0,\sqrt{\pi}/2)$. By symmetry, the same function element is non-differentiable at $(x,y)=(0,-\sqrt{\pi}/2)$, too. □

REMARK. We enumerate some of problems to be investigated hereafter.

(a) Value distribution of an isometric immersion. In Example 6, we do not know the behavior of $\hat{f}(s_1,s_2)$ at infinity.

(b) Smoothness of an isometric immersion at a point of ramification. In Example 4, we do not know how to choose the lower limits of integration in (46) in such a way that f be continuous in the whole xy-plane.

(c) Solution of the system (1) when the rank of the matrix P varies. The local coordinate system (s_1,s_2) introduced in §4 is no more useful. See the remark after (22).

(d) Isometric immersions whose domains of definition are proper subsets of $\boldsymbol{R}^2$. The mapping associated with $(u_1,u_2)=(\sqrt{x^2-y^2},\sqrt{2xy})$ will be an example.

(e) Isometric immersions with non-vanishing normal curvature. In the present article, we do not study at all isometric immersions of "composition" type.

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