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RICCI CURVATURE AND ALMOST SPHERICAL MULTI-SUSPENSION

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Abstract. In this paper, we give a generalization of Cheeger-Colding's suspension theorem for manifolds with almost maximal diameters. We also discuss a relationship between the eigenvalues of the Laplacian and the structure of tangent cones of non-collapsing limit spaces.

Introduction. We will study the structure of Riemannian manifolds with positive Ricci curvature satisfying an *almost maximal condition*. One of our main results of this paper is the following:

THEOREM 0.1. *Let M be an n-dimensional complete Riemannian manifold* ($n \geq 2$) *with* $\mathrm{Ric}_M \geq n-1$. *Given a sufficiently small positive number* $\varepsilon > 0$, *we assume that there exist k pairs $(p_1, q_1), \dots, (p_k, q_k)$ of points of M such that $|\overline{p_i, q_i} - \pi| < \varepsilon$ holds for each i, and that $|\overline{p_i, p_j} - \pi/2| < \varepsilon$ holds for $i \neq j$. Then we have the following*:

(1) *k is at most $n+1$.*

(2) *If $1 \leq k \leq n-1$, then there exists a compact geodesic space Z with* $\mathrm{diam}(Z) \leq \pi$ *such that $d_{GH}(M, \boldsymbol{S}^{k-1} * Z) < \Psi(\varepsilon; n)$.*

(3) *If $k = n$ or $n+1$, then $d_{GH}(M, \boldsymbol{S}^n) < \Psi(\varepsilon; n)$. In particular, M is diffeomorphic to $\boldsymbol{S}^n$.*

Here, throughout the article, we denote by $\Psi(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k; c_1, c_2, \dots, c_l)$ (more simply, Ψ) some positive function on $\boldsymbol{R}^k_{>0} \times \boldsymbol{R}^l$ satisfying

$$\lim_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \to 0} \Psi(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k; c_1, c_2, \dots, c_l) = 0$$

for each fixed $c_1, c_2, \dots, c_l$. In addition, $\boldsymbol{S}^{k-1} * Z$ denotes the k-fold spherical suspension of Z, d_{GH} is the Gromov-Hausdorff distance between compact metric spaces and $\overline{x, y}$ is the distance between x and y. In the last assertion, M is diffeomorphic to $\boldsymbol{S}^n$ by the stability theorem of Cheeger-Colding (see [6, Theorem A.1.12]).

Let us review some related results. Let M be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_M \geq n-1$. Then, it is well-known that M is compact and satisfies

$$\mathrm{diam}(M) \leq \pi\,, \quad \mathrm{rad}(M) \leq \pi\,, \quad \mathrm{vol}(M) \leq \mathrm{vol}(\boldsymbol{S}^n)\,.$$

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Moreover, M is isometric to $\boldsymbol{S}^n$ if and only if the equality holds in one of the above inequalities. A similar theorem described on the volume and the radius is proved by Colding.

THEOREM 0.2 (Colding [12, 13]). *Let M be an n-dimensional compact Riemaniann manifold* ($n \geq 2$) *with* $\mathrm{Ric}_M \geq n-1$. *Given a sufficiently small positive number* $\varepsilon > 0$, *we assume that the inequarity* $\mathrm{vol}(M) \geq \mathrm{vol}(\boldsymbol{S}^n) - \varepsilon$ (*or* $\mathrm{rad}(M) \geq \pi - \varepsilon$) *is satisfied. Then we have* $d_{GH}(M, \boldsymbol{S}^n) < \Psi(\varepsilon; n)$. *In particular, M is diffeomorphic to* $\boldsymbol{S}^n$.

We will give an alternative proof of Theorem 0.2 by using Theorem 0.1 (see Remark 1.19). An analogous statement for the diameter is known to be false (see [1] or [25] for examples). However, the following result is proved by Cheeger and Colding as one of *almost warped product theorems*.

THEOREM 0.3 (Cheeger-Colding [6, Theorem 5.12]). *Let M be an n-dimensional compact Riemaniann manifold* ($n \geq 2$) *with* $\mathrm{Ric}_M \geq n-1$. *Given a sufficiently small positive number* $\varepsilon > 0$, *we assume that* $\mathrm{diam}(M) \geq \pi - \varepsilon$. *Then there exists a compact geodesic space Z with* $\mathrm{diam}(Z) \leq \pi$ *such that* $d_{GH}(M, \boldsymbol{S}^0 * Z) < \Psi(\varepsilon; n)$.

Note that Theorem 0.3 corresponds to the case $k = 1$ of Theorem 0.1. In Section 1, we will give a simplified proof of Theorem 0.3. Then we will prove Theorem 0.1.

We will discuss in Section 2 a relationship between the first eigenvalue of the Laplacian and Theorem 0.1. We will calculate the L^2-inner product of cosine of distance functions (Proposition 2.1), and give alternative proofs of results by Aubry [2], Bertrand [3] and Petersen [28].

We will study the tangent cones on non-collapsing limit spaces in Sections 3 and 4. We prove that such tangent cones satisfy a similar property to Theorem 0.1, and study the topological structure of them. We also prove the following theorem which sharpens the conclusion in Theorem 0.1 when $k = n-1$.

THEOREM 0.4. *We assume that the assumption in Theorem* 0.1 *holds with* $k = n-1$. *Then, there exists* $0 \leq r \leq 1$ *such that* $d_{GH}(M, \boldsymbol{S}^{n-2} * \boldsymbol{S}^1(r)) < \Psi(\varepsilon; n)$ *holds. Here, we set* $\boldsymbol{S}^1(r) = \{x \in \boldsymbol{R}^2 \,;\, |x| = r\}$ *and the metric of* $\boldsymbol{S}^1(r)$ *is the standard Riemannian metric.*

We will prove Theorem 0.4 by using Theorem 0.1 and a result about singularities of non-collapsing limit spaces.

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1. Proof of Theorem 0.1.

1.1. Preliminaries. For a positive number $\varepsilon > 0$, we use the following notation;

$$a = b \pm \varepsilon \Leftrightarrow |a - b| < \varepsilon\,.$$

For a metric space Z, a point $z \in Z$ and a positive number $r > 0$, we put

$$B_r(z) = \{w \in Z\,;\, \overline{z,w} < r\}\,,\quad \bar{B}_r(z) = \{w \in Z\,;\, \overline{z,w} \le r\}\,,\quad \partial B_r(z) = \{w \in Z\,;\, \overline{z,w} = r\}\,.$$

DEFINITION 1.1. We say that Z is a *geodesic space* if for each $z_1, z_2 \in Z$, there exists an isometric embedding $c : [0, \overline{z_1, z_2}] \to Z$ such that $c(0) = z_1$, $c(\overline{z_1, z_2}) = z_2$. Also, we say that c is a *minimal geodesic from* z_1 *to* z_2.

DEFINITION 1.2. We define the metric on $[0, \pi] \times Z/\sim$ as

$$\overline{(t_1, z_1), (t_2, z_2)} = \arccos(\cos t_1 \cos t_2 + \sin t_1 \sin t_2 \cos \min\{\overline{z_1, z_2}, \pi\})\,.$$

Here, $\sim$ is the equivalence relation such that $\{0\} \times Z$ and $\{\pi\} \times Z$ go to a point, respectively. Then, this metric space is denoted by $\boldsymbol{S}^0 * Z$, and we call it the *spherical suspension* of Z. We also define

$$\boldsymbol{S}^k * Z = \overbrace{\boldsymbol{S}^0 * (\boldsymbol{S}^0 * (\cdots * (\boldsymbol{S}^0}^{k+1} * Z)) \cdots)\,.$$

REMARK 1.3. If Z is compact, then $\boldsymbol{S}^0 * Z$ is compact. Moreover if Z is a geodesic space, then $\boldsymbol{S}^0 * Z$ is also a geodesic space. We put $\mathcal{M} = \{$isometry class of compact metric spaces$\}$, then $\boldsymbol{S}^0 * : \mathcal{M} \to \mathcal{M}$ is uniformly continuous map for d_{GH}.

Next, we review the segment inequality by Cheeger and Colding. For an n-dimensional compact Riemannian manifold M $(n \ge 2)$ with $\mathrm{Ric}_M \ge n-1$ and an integrable function $g : M \to \boldsymbol{R}_{\ge 0}$, we define $\mathcal{F}_g : M \times M \to \boldsymbol{R}_{\ge 0}$ as

$$\mathcal{F}_g(x, y) = \inf_{\gamma} \int_{\gamma} g(\gamma(t))dt\,.$$

Here, the infinimum runs over all minimal geodesics γ from x to y.

THEOREM 1.4 (Cheeger-Colding [5, Theorem 2.15]). *With notation as above, we have*

$$\int_{M \times M} \mathcal{F}_g(x, y)dxdy \le C(n)\,\mathrm{vol}(M) \int_M g(x)dx\,.$$

Here, $C(n) > 0$ is a positive constant depending only on n.

REMARK 1.5. In fact, the theorem above is a special global case of the segment inequality that is proved by Cheeger and Colding. They prove a local statement on any complete Riemaniann manifold M with $\mathrm{Ric}_M \ge -(n-1)$. However, the theorem above is sufficient to prove our main result.

1.2. Almost cosine formula (a proof of Theorem 0.3). In this subsection, we will give a comparatively easy proof of Theorem 0.3. Throughout this subsection, we fix an integer $n \ge 2$, a positive number $\varepsilon > 0$, an n-dimensional compact Riemannian manifold M with $\mathrm{Ric}_M \ge n-1$ and $p, q \in M$ such that $\overline{p, q} \ge \pi - \varepsilon$ holds. We put $f(x) = \cos \overline{p, x}$.

LEMMA 1.6 (Colding [12, Lemma 1.10]). *There exists a smooth function $\tilde{f} \in C^\infty(M)$ such that*

$$\frac{1}{\mathrm{vol}(M)}\int_M |f(x)-\tilde{f}(x)|^2 dx < \Psi(\varepsilon; n)\,,$$
$$\frac{1}{\mathrm{vol}(M)}\int_M |\nabla f-\nabla \tilde{f}|^2 dx < \Psi(\varepsilon; n)\,,$$
$$\frac{1}{\mathrm{vol}(M)}\int_M |\mathrm{Hess}_{\tilde{f}}+\tilde{f}g_M|^2 dx < \Psi(\varepsilon; n)\,.$$

Here, g_M is the Riemannian metric on M.

LEMMA 1.7 (Grove-Petersen [20, Lemma 1]). *For every point x in M, we have*

$$\overline{p,x}+\overline{q,x}-\overline{p,q} < \Psi(\varepsilon; n)\,.$$

See [12], [20] for the proof of Lemmas 1.6 and 1.7.

LEMMA 1.8. *Let x be a point in M and t a number in $[-1,1]$ satisfying $f^{-1}(t) \neq \emptyset$.*

(1) *If $f(x) \le t$, then*

$$\overline{x, f^{-1}(t)}+\overline{p, f^{-1}(t)}-\overline{x,p} = 0\,.$$

(2) *If $f(x) > t$, then*

$$\overline{p,x}+\overline{x, f^{-1}(t)}-\overline{p, f^{-1}(t)} < \Psi(\varepsilon; n)\,.$$

PROOF. (1) It is easy to see that there exists a point $y \in f^{-1}(t)$ such that $\overline{p,y}+\overline{x,y}=\overline{p,x}$. On the other hand, for every point $z \in f^{-1}(t)$, we have $\overline{x,y}=\overline{p,x}-\overline{p,y}=\overline{p,x}-\overline{p,z} \le \overline{x,z}$. Thus $\overline{x,y}=\overline{x, f^{-1}(t)}$, and we have the claim.

(2) Without loss of generality, we may assume $f(q) \le t$. Then, there exists a point $y \in f^{-1}(t)$ such that $\overline{x,y}+\overline{y,q}=\overline{x,q}$. By Lemma 1.7, we have $\overline{p,x}+\overline{x,y}-\overline{p,y} < \Psi$. Thus, for every point $z \in f^{-1}(t)$, we have $\overline{x,y} \le \overline{p,y}-\overline{p,x}+\Psi = \overline{p,z}-\overline{p,x}+\Psi \le \overline{z,x}+\Psi$. Therefore, $\left|\overline{x,y}-\overline{x, f^{-1}(t)}\right| < \Psi$. This implies the claim. □

LEMMA 1.9. *We take $\tilde{f} \in C^\infty(M)$ as in Lemma 1.6. Then, for every points $x, y, z \in M$, there exist $\hat{x}, \hat{y}, \hat{z} \in M$ with the following properties.*

(1) $\overline{x,\hat{x}} < \Psi(\varepsilon; n)$, $\overline{y,\hat{y}} < \Psi(\varepsilon; n)$, $\overline{z,\hat{z}} < \Psi(\varepsilon; n)$, $|f(\hat{x})-\tilde{f}(\hat{x})| < \Psi(\varepsilon; n)$, $|f(\hat{y})-\tilde{f}(\hat{y})| < \Psi(\varepsilon; n)$, $|f(\hat{z})-\tilde{f}(\hat{z})| < \Psi(\varepsilon; n)$.

(2) $\hat{x} \notin C_{\hat{y}}$, $\hat{y} \notin C_{\hat{z}}$, $\hat{z} \notin C_{\hat{x}}$. *Here, C_m is the cut locus of $m \in M$.*

(3) *There exists an open set $U \subset [0, \overline{\hat{x},\hat{y}}]$ satisfying the following conditions*:

(a) $\mathcal{H}^1([0,\overline{\hat{x},\hat{y}}] \setminus U) = 0$. *Also, for all $u \in U$, there exists a unique minimal geodesic $\tau_u : [0, l(u)] \to M$ $(l(u) = \overline{\hat{z}, \sigma(u)})$ from $\hat{z}$ to $\sigma(u)$. Here, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure and σ is the minimal geodesic from $\hat{x}$ to $\hat{y}$.*

(b) *We have*

$$\int_U \left|f(\sigma(u))-\tilde{f}(\sigma(u))\right|^2 du < \Psi(\varepsilon; n)\,,$$
$$\int_U \left||\nabla \tilde{f}|(\sigma(u))-\sin\overline{p,\sigma(u)}\right|^2 du < \Psi(\varepsilon; n)\,,$$

$$\int_U \int_0^{l(u)} |\mathrm{Hess}_{\tilde{f}} + \tilde{f} g_M|(\tau_u(s)) ds du < \Psi(\varepsilon; n) .$$

PROOF. By Theorem 1.4 applied twice and Lemma 1.6, we have

$$\frac{1}{(\mathrm{vol}(M))^3} \int_{M^3} \mathcal{F}_{h_c}(a, b) dadbdc < \Psi(\varepsilon; n) .$$

Here, $h_c = \mathcal{F}_{|\mathrm{Hess}_{\tilde{f}} + \tilde{f} g_M|}(c, \cdot)$. Therefore, by the Tchebychev inequarity, there exists a subset $\tilde{M} \subset M^3$ such that for all $(a, b, c) \in \tilde{M}$,

- $a \notin C_b$, $b \notin C_c$, $c \notin C_a$ and, for a minimal geodesic $\sigma : [0, \overline{a, b}] \to M$ from a to b, $\mathcal{H}^1(\mathrm{Image}(\sigma) \cap C_c) = 0$;
- $|f(a) - \tilde{f}(a)| < \Psi(\varepsilon; n)$, $|f(b) - \tilde{f}(b)| < \Psi(\varepsilon; n)$, $|f(c) - \tilde{f}(c)| < \Psi(\varepsilon; n)$ and

$$\mathcal{F}_{h_c}(a, b) < \Psi(\varepsilon; n) ;$$

- for a minimal geodesic $\sigma : [0, \overline{a, b}] \to M$ from a to b, we have

$$\int_0^{\overline{a,b}} |f(\sigma(t)) - \tilde{f}(\sigma(t))|^2 dt < \Psi(\varepsilon; n) ,$$

$$\int_0^{\overline{a,b}} \big| |\nabla \tilde{f}(\sigma(t))| - \sin \overline{p, \sigma(t)} \big|^2 dt < \Psi(\varepsilon; n) ;$$

- $\mathrm{vol}(\tilde{M}) \geq (1 - \Psi(\varepsilon; n))(\mathrm{vol}(M))^3$.

By the Bishop-Gromov volume comparison theorem, we have the claim. □

LEMMA 1.10. *Let x, y, z be points in M satisfying $y, z \in f^{-1}(t)$, $t \in [-1, 1]$ and $\overline{x, y} = \overline{x, f^{-1}(t)}$. We take $\hat{x}, \hat{y}, \hat{z}$ as in Lemma* 1.9.

(1) *If $f(x) \leq t$, then*

$$\int_U |\nabla \tilde{f}(\sigma(u)) - \sin(\overline{p, x} - u)\sigma'(u)|^2 du < \Psi(\varepsilon; n) .$$

(2) *If $f(x) > t$, then*

$$\int_U |\nabla \tilde{f}(\sigma(u)) + \sin(\overline{p, x} + u)\sigma'(u)|^2 du < \Psi(\varepsilon; n) .$$

PROOF. It is easy to prove the following by Lemma 1.7.

(3) If $f(x) \leq t$, then for each $u \in U$, we have

$$|\overline{p, \sigma(u)} - (\overline{p, x} - u)| < \Psi(\varepsilon; n) .$$

(4) If $f(x) > t$, then for each $u \in U$, we have

$$|\overline{p, \sigma(u)} - (\overline{p, x} + u)| < \Psi(\varepsilon; n) .$$

We give only a proof of (1) by using (3). The proof of (2) is similar to that of (1).

$$\begin{aligned}
&\int_U |\nabla \tilde{f}(\sigma(u)) - \sin(\overline{p,x} - u)\sigma'(u)|^2 du \\
&= \int_U (|\nabla \tilde{f}|^2(\sigma(u)) - 2\sin(\overline{p,x} - u)(\tilde{f}\circ\sigma)'(u) + \sin^2(\overline{p,x} - u))du \\
&= \int_U (\sin^2(\overline{p,x} - u) - 2\sin(\overline{p,x} - u)(\tilde{f}\circ\sigma)'(u) + \sin^2(\overline{p,x} - u))du \pm \Psi \\
&= 2\int_U (\sin^2(\overline{p,x} - u) - \sin(\overline{p,x} - u)(\tilde{f}\circ\sigma)'(u))du \pm \Psi \\
&= 2\int_U \sin^2(\overline{p,x} - u)du - 2[\sin(\overline{p,x} - u)\tilde{f}\circ\sigma(u)]_0^{\overline{\hat{x},\hat{y}}} \\
&\quad + 2\int_U -\cos(\overline{p,x} - u)\tilde{f}\circ\sigma(u)du \pm \Psi \\
&= 2\int_U \sin^2(\overline{p,x} - u)du - 2(\sin(\overline{p,\hat{x}} - \overline{\hat{x},\hat{y}})\tilde{f}(\hat{y}) - \sin\overline{p,x}\ \tilde{f}(\hat{x})) \\
&\quad + 2\int_U -\cos^2(\overline{p,x} - u)du \pm \Psi \\
&= 2\int_U (\sin^2(\overline{p,x} - u) - \cos^2(\overline{p,x} - u))du \\
&\quad - 2(\sin\overline{p,\hat{y}}\cos\overline{p,y} - \sin\overline{p,x}\cos\overline{p,x}) \pm \Psi \\
&= -2\int_U \cos(2\overline{p,x} - 2u)du - \sin 2\overline{p,y} + \sin 2\overline{p,x} \pm \Psi \\
&= [\sin(2\overline{p,x} - 2u)]_0^{\overline{\hat{x},\hat{y}}} - \sin 2\overline{p,y} + \sin 2\overline{p,x} \pm \Psi = \Psi\,. \qquad \square
\end{aligned}$$

LEMMA 1.11. *With the same assumption as in Lemma* 1.9, *we have*

$$\left|\frac{\cos\overline{\hat{z},\hat{x}} - \cos\overline{p,\hat{z}}\cos\overline{p,\hat{x}}}{\sin\overline{p,\hat{x}}} - \frac{\cos\overline{\hat{y},\hat{z}} - \cos\overline{p,\hat{y}}\cos\overline{p,\hat{z}}}{\sin\overline{p,\hat{y}}}\right| \min\{\sin^2\overline{p,\hat{x}}, \sin^2\overline{p,\hat{y}}\} < \Psi(\varepsilon; n)\,.$$

PROOF. We prove the statement in the case $f(x) \le t$. The case $f(x) > t$ is similarly proved.

$$\begin{aligned}
&\left|\frac{\cos\overline{\hat{z},\hat{x}} - \cos\overline{p,\hat{z}}\cos\overline{p,\hat{x}}}{\sin\overline{p,\hat{x}}} - \frac{\cos\overline{\hat{y},\hat{z}} - \cos\overline{p,\hat{y}}\cos\overline{p,\hat{z}}}{\sin\overline{p,\hat{y}}}\right| \\
&= \left|\int_U \left(\frac{\cos l(u) - \cos\overline{p,\hat{z}}\cos(\overline{p,\hat{x}} - u)}{\sin(\overline{p,\hat{x}} - u)}\right)' du\right| \\
&= \left|\int_U \left\{\frac{(-\sin l(u)\, l'(u) - \cos\overline{p,\hat{z}}\sin(\overline{p,\hat{x}} - u))\sin(\overline{p,\hat{x}} - u)}{\sin^2(\overline{p,\hat{x}} - u)}\right.\right.
\end{aligned}$$

$$+\frac{(\cos l(u)-\cos\overline{p,\hat z}\cos(\overline{p,\hat x}-u))\cos(\overline{p,\hat x}-u)}{\sin^2(\overline{p,\hat x}-u)}\Bigg\}du\Bigg|$$

$$\le\frac{1}{\min\{\sin^2\overline{p,\hat x},\sin^2\overline{p,\hat y}\}}\Bigg\{\int_U\Bigg|-\sin l(u)<\tau_u'(l(u)),\sigma'(u)>\sin(\overline{p,\hat x}-u)$$

$$+\cos l(u)f(\sigma(u))-\cos\overline{p,\hat z}\Bigg|du\pm\Psi\Bigg\}$$

$$=\frac{1}{\min\{\sin^2\overline{p,\hat x},\sin^2\overline{p,\hat y}\}}\Bigg\{\int_U\Bigg|-\frac{d\tilde f\circ\tau_u(s)}{ds}\Big|_{s=l(u)}\sin l(u)+\cos l(u)\tilde f(\tau_u(l(u)))$$

$$-\tilde f(\tau_u(0))\Bigg|du\pm\Psi\Bigg\}$$

$$=\frac{1}{\min\{\sin^2\overline{p,\hat x},\sin^2\overline{p,\hat y}\}}\Bigg\{\int_U\Bigg|\int_0^{l(u)}\frac{d}{ds}\bigg(-\frac{d\tilde f\circ\tau_u(s)}{ds}\sin s+\cos s\ \tilde f(\tau_u(s))\bigg)ds\Bigg|du$$

$$\pm\Psi\Bigg\}$$

$$\le\frac{1}{\min\{\sin^2\overline{p,\hat x},\sin^2\overline{p,\hat y}\}}\Bigg\{\int_U\int_0^{l(u)}|\mathrm{Hess}_{\tilde f}+\tilde f g_M|(\tau_u(s))ds du\pm\Psi\Bigg\}$$

$$=\frac{1}{\min\{\sin^2\overline{p,\hat x},\sin^2\overline{p,\hat y}\}}\Psi\,.$$

□

PROPOSITION 1.12. *There exists a positive constant* $\delta=\delta(\varepsilon,n)>0$ *depending only on* ε, n *satisfying the following properties.*

(1) *We have* $\lim_{\varepsilon\to 0}\delta(\varepsilon,n)=0$ *for all* $n\in\boldsymbol{N}$.

(2) *For every point* $x\in M$, *we take* $z_x\in\partial B_{\pi/2}(p)$ *such that* $\overline{x,z_x}=\overline{x,\partial B_{\pi/2}(p)}$. *Then, for every points* $x,x'\in M\setminus(B_\delta(p)\cup B_\delta(q))$, *we have*

$$\cos\overline{x,x'}=\cos\overline{p,x}\cos\overline{p,x'}+\sin\overline{p,x}\sin\overline{p,x'}\cos\overline{z_x,z_{x'}}\pm\Psi(\varepsilon;n)\,.$$

PROOF. This follows from Lemma 1.11. □

1.3. Proof of Theorem 0.1. Let X, Z be compact metric spaces and f a map from X to Z. We say that f is *ε-Hausdorff approximation* if $|\overline{x,y}-\overline{f(x),f(y)}|<\varepsilon$ holds for all $x,y\in X$, and $B_\varepsilon(\mathrm{Image}(f))=Z$ holds. If there exists an ε-Hausdorff approximation from X to Y, then, we have $d_{GH}(X,Z)<5\varepsilon$. If $d_{GH}(X,Z)<\varepsilon$, then, there exists a 3ε-Hausdorff approximation from X to Z.

LEMMA 1.13. *Let* $\varepsilon>0$ *and let* M *be an* n*-dimensional compact Riemannian manifold* ($n\ge 2$) *with* $\mathrm{Ric}_M\ge n-1$. *We assume that there exist points* $p,q\in M$ *such that* $\overline{p,q}\ge\pi-\varepsilon$ *holds. Then, we have*

$$d_{GH}(M,\boldsymbol{S}^0*\partial B_{\pi/2}(p))<\Psi(\varepsilon;n)\,.$$

Here, the metric on $\partial B_{\pi/2}(p)$ *is the restriction of the metric on* M.

PROOF. With notation as in Proposition 1.12, we define the map

$$\phi : M \setminus (B_\delta(p) \cup B_\delta(q)) \to \boldsymbol{S}^0 * \partial B_{\pi/2}(p) = [0, \pi] \times \partial B_{\pi/2}(p)/\sim$$

by $\phi(x) = (\overline{p, x}, z_x)$. It is easy to check that ϕ is $\Psi(\varepsilon; n)$-Hausdorff approximation by Proposition 1.12. □

From now on, we will discuss the limit space.

LEMMA 1.14. *By Lemma* 1.13, *if Y is the Gromov-Hausdorff limit of a sequence $\{M_i\}_i$ of compact connected, n-dimensional Riemannian manifolds with* $\mathrm{Ric}_{M_i} \geq n-1$, *then we have*

(P) *for any $p, q \in Y$ satisfying $\overline{p, q} = \pi$, we have $Y = (\{p, q\}, d_{\boldsymbol{S}^0}) * (\partial B_{\pi/2}(p), d_Y)$.*

Moreover, $(\partial B_{\pi/2}(p), d_Y)$ is either equal to a point, equal to $\boldsymbol{S}^0$ or is a convex subspace of (Y, d_Y). In the last case, $\partial B_{\pi/2}(p)$ is itself a geodesic space for d_Y and satisfies the property (P).

PROOF. Let p, q be points in Y with $\overline{p, q} = \pi$. We assume that $\partial B_{\pi/2}(p)$ is neither equal to a point nor equal to $\boldsymbol{S}^0$. Let x, y be points in $\partial B_{\pi/2}(p)$ such that $\overline{x, y} < \pi/2$. We take a minimal geodesic $\sigma : [0, \overline{x, y}]$ from x to y.

CLAIM 1.15. *We have $\sigma(t) \in Y \setminus \{p, q\}$ for every $t \in [0, \overline{x, y}]$.*

We assume that the conclusion is false. Then we can assume that $p \in \mathrm{Image}(\sigma)$ without loss of generality. Then, we have $\overline{x, y} = \overline{x, p} + \overline{p, y} = \pi/2 + \pi/2 = \pi$. This contradicts the assumption. Therefore we have Claim 1.15.

Thus, by Proposition 1.12, for an element $z_t \in \partial B_{\pi/2}(p)$ such that $\overline{\sigma(t), z_t} = \overline{\sigma(t), \partial B_{\pi/2}(p)}$ holds, we have the equalities

$$\cos \overline{x, \sigma(t)} = \sin \overline{p, \sigma(t)} \cos \overline{x, z_t}\,,$$

$$\cos \overline{\sigma(t), y} = \sin \overline{p, \sigma(t)} \cos \overline{y, z_t}\,.$$

We shall prove that $\overline{p, \sigma(t)} = \pi/2$. We give only a proof of the case $\overline{x, \sigma(t)} \leq \pi/2$ and $\overline{\sigma(t), y} \leq \pi/2$. We can prove the other case in a similar way. Then, we have $\cos \overline{x, \sigma(t)} \leq \cos \overline{x, z_t}$ and $\cos \overline{\sigma(t), y} \leq \cos \overline{y, z_t}$. Thus, we have $\overline{x, \sigma(t)} \geq \overline{x, z_t}$ and $\overline{\sigma(t), y} \geq \overline{y, z_t}$. Especially, $\overline{x, y} = \overline{x, \sigma(t)} + \overline{\sigma(t), y} \geq \overline{x, z_t} + \overline{y, z_t}$. Therefore, $\overline{x, \sigma(t)} = \overline{x, z_t}$ and $\overline{\sigma(t), y} = \overline{y, z_t}$ hold. Hence, we have $\cos \overline{x, \sigma(t)} = \sin \overline{p, \sigma(t)} \cos \overline{x, \sigma(t)}$ and $\cos \overline{\sigma(t), y} = \sin \overline{p, \sigma(t)} \cos \overline{\sigma(t), y}$. Since $\overline{x, y} < \pi$, we have $\min\{\overline{x, \sigma(t)}, \overline{\sigma(t), y}\} < \pi/2$. Thus, we have $\overline{p, \sigma(t)} = \pi/2$. Moreover, we assume that there exist points $\hat{p}, \hat{q}$ in $\partial B_{\pi/2}(p)$ such that $\overline{\hat{p}, \hat{q}} = \pi$. By the assumption, there exists a point z in $\partial B_{\pi/2}(p) \setminus \{\hat{p}, \hat{q}\}$. By Lemma 1.7, there exists a minimal geodesic τ from $\hat{p}$ to $\hat{q}$ such that $z \in \mathrm{Image}(\tau)$. Therefore, by an argument above, we have $\mathrm{Image}(\tau) \subset \partial B_{\pi/2}(p)$ and $\partial B_{\pi/2}(p)$ is a convex subspace of (Y, d_Y). For every $x \in \partial B_{\pi/2}(p) \setminus \{\hat{p}, \hat{q}\}$, we take $z_x \in \partial B_{\pi/2}(\hat{p})$ such that $\overline{x, z_x} = \overline{x, \partial B_{\pi/2}(\hat{p})}$. Then, we have $z_x \in \partial B_{\pi/2}(p) \cap \partial B_{\pi/2}(\hat{p})$. Then, we define $\phi : \partial B_{\pi/2}(p) \to \boldsymbol{S}^0 * (\partial B_{\pi/2}(p) \cap \partial B_{\pi/2}(\hat{p}))$ by $\phi(x) = (\overline{\hat{p}, x}, z_x)$ for $x \in \partial B_{\pi/2}(p) \setminus \{\hat{p}, \hat{q}\}$, $\phi(\hat{p}) = (0, *)$ and $\phi(\hat{q}) = (\pi, *)$. By Proposition 1.12, ϕ is an isometry. Therefore, $\partial B_{\pi/2}(p)$ satisfies the property (P). □

COROLLARY 1.16. *Let Y be the Gromov-Hausdorff limit space of a sequence $\{M_i\}_i$ of compact connected, n-dimensional Riemannian manifolds with* $\mathrm{Ric}_{M_i} \geq n-1$. *We assume that there exist* 2 *pairs* $(p_1, q_1), (p_2, q_2)$ *of points of Y such that* $\overline{p_1, q_1} = \overline{p_2, q_2} = \pi$ *and* $\overline{p_1, p_2} = \pi/2$. *Then, one of the following* (1), (2), (3) *occurs.*

(1) *There exists a compact geodesic space Z with* $\mathrm{diam}(Z) \leq \pi$ *such that* $Y = \boldsymbol{S}^1 * Z$.

(2) $Y = \boldsymbol{S}^2$.

(3) $Y = \boldsymbol{S}^1$.

Similarly, we have the following.

PROPOSITION 1.17. *Let Y be the Gromov-Hausdorff limit space of a sequence $\{M_i\}_i$ of compact connected, n-dimensional Riemannian manifolds with* $\mathrm{Ric}_{M_i} \geq n-1$. *We assume that there exist k pairs $(p_1, q_1), \dots, (p_k, q_k)$ of points of Y such that $\overline{p_i, q_i} = \pi$ holds for every i, and that $\overline{p_i, p_j} = \pi/2$ for every $i \neq j$. Then, one of the following* (1), (2), (3) *occurs.*

(1) *There exists a compact geodesic space Z with* $\mathrm{diam}(Z) \leq \pi$ *such that* $Y = \boldsymbol{S}^{k-1} * Z$.

(2) $Y = \boldsymbol{S}^k$.

(3) $Y = \boldsymbol{S}^{k-1}$.

THEOREM 1.18. *Let Y be the Gromov-Hausdorff limit space of a sequence $\{M_i\}_i$ of compact connected, n-dimensional Riemannian manifolds with* $\mathrm{Ric}_{M_i} \geq n-1$. *We assume that there exist k pairs $(p_1, q_1), \dots, (p_k, q_k)$ of points of Y such that $\overline{p_i, q_i} = \pi$ holds for every i, and that $\overline{p_i, p_j} = \pi/2$ for every $i \neq j$. Then, we have the following*:

(1) *k is at most $n+1$.*

(2) *If $1 \leq k \leq n-1$, then there exists a compact geodesic space Z with* $\mathrm{diam}(Z) \leq \pi$ *such that $Y = \boldsymbol{S}^{k-1} * Z$.*

(3) *If $k = n$ or $n+1$, then we have $Y = \boldsymbol{S}^n$.*

PROOF. By Proposition 1.17 and [13, Lemma 5.10], we have (1) and (2). Note that Gromov-Hausdorff limits have Hausdorff dimension not greater than n and that $\dim_{\mathcal{H}} \boldsymbol{S}^k Z = \dim_{\mathcal{H}} Z + k + 1$ holds for every compact metric space Z (see Proposition 5.6). Thus, it suffices to prove the case $k = n$. Then, by Proposition 1.17, we have $Y = \boldsymbol{S}^n$, or $Y = \boldsymbol{S}^n_+$. Here, $\boldsymbol{S}^n_+ = \{(x_1, x_2, \dots, x_{n+1}) \in \boldsymbol{R}^{n+1};\ x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1, x_{n+1} \geq 0\}$ and the metric is the restriction of that of $\boldsymbol{S}^n$. If $Y = \boldsymbol{S}^n_+$, by [7, Theorem 6.2], we have a contradiction. Therefore we have the claim. □

Gromov's pre-compactness theorem and Theorem 1.18 imply Theorem 0.1.

REMARK 1.19. Theorem 0.2 follows from Theorem 0.1. Let M be an n-dimensional compact Riemannian manifold ($n \geq 2$) with $\mathrm{Ric}_M \geq n-1$. By the Bishop-Gromov volume comparison theorem, if $\mathrm{vol}(M) \geq \mathrm{vol}(\boldsymbol{S}^n) - \varepsilon$, then $\mathrm{rad}(M) \geq \pi - \Psi(\varepsilon; n)$. Hence, we assume that $\mathrm{rad}(M) \geq \pi - \varepsilon$. Then, for every $p \in M$, there exists $q \in M$ such that $\overline{p, q} \geq \pi - \varepsilon$

holds. First, we fix $p_1 \in M$. Then, there exists $q_1 \in M$ such that

$$\overline{p_1, q_1} \geq \pi - \varepsilon\,.$$

Thus, by Theorem 0.1, M is close to the 1-fold suspension of some compact geodesic space in the sense of Gromov-Hausdorff distance. Especially, there exists $p_2 \in M$ such that

$$\left|\overline{p_1, p_2} - \frac{\pi}{2}\right| < \Psi(\varepsilon; n)\,.$$

Similarly, there exists $q_2 \in M$ such that

$$\overline{p_2, q_2} \geq \pi - \varepsilon\,.$$

Thus, M is close to the 2-fold suspension of some compact geodesic space. Especially, $A_{\pi/2-\Psi,\pi/2+\Psi}(p_1) \cap A_{\pi/2-\Psi,\pi/2+\Psi}(p_2) \neq \emptyset$. Here, $A_{s,t}(x) = \bar{B}_t(x) \setminus B_s(x)$ for $s < t$. By iterating this argument, there exist $n+1$ pairs $(p_1, q_1), \ldots, (p_{n+1}, q_{n+1})$ of points of M such that $|\overline{p_i, q_i} - \pi| < \varepsilon$ holds for each i, and that $|\overline{p_i, p_j} - \pi/2| < \Psi(\varepsilon; n)$ holds for $i \neq j$. It implies Theorem 0.2 by Theorem 0.1.

2. First eigenvalue of the Laplacian. In this section, we explain a relationship between Theorem 0.1 and the first eigenvalue of the Laplacian. As a key tool, we shall estimate the L^2-inner product of cosine of distance functions.

PROPOSITION 2.1. *Let* $\varepsilon > 0$, M *be an* n*-dimensional compact Riemannian manifold* $(n \geq 2)$ *with* $\mathrm{Ric}_M \geq n-1$. *We assume that there exist* 2 *pairs* $(p_1, q_1), (p_2, q_2)$ *of points of* M *such that* $|\overline{p_1, q_1} - \pi| < \varepsilon$ *and* $|\overline{p_2, q_2} - \pi| < \varepsilon$. *We put* $f_i(x) = \cos \overline{p_i, x}$ $(i = 1, 2)$. *Then, we have*

$$\frac{1}{\mathrm{vol}(M)} \int_M f_1 f_2 dx = \frac{\cos \overline{p_1, p_2}}{n+1} \pm \Psi(\varepsilon; n)\,.$$

PROOF. We take $\tilde{f}_i \in C^\infty(M)$ for f_i as in Lemma 1.6. Then we have the following

$$\frac{1}{\mathrm{vol}(M)} \int_M f_i^2 dx = \frac{1}{n+1} \pm \Psi(\varepsilon; n)\,,$$

$$\frac{1}{\mathrm{vol}(M)} \int_M |\nabla f_i|^2 dx = \frac{n}{n+1} \pm \Psi(\varepsilon; n)\,,$$

$$\frac{1}{\mathrm{vol}(M)} \int_M |\Delta \tilde{f}_i(x) + n \tilde{f}_i(x)|^2 dx < \Psi(\varepsilon; n)\,.$$

See [12, Lemma 1.10] for the proof. Here, $\Delta = \mathrm{tr}(\mathrm{Hess})$.

We also take $\delta = \delta(\varepsilon, n)$ as in Proposition 1.12. We put $A_{p_1} = B_{3\delta}(p_1) \cup B_{3\delta}(q_1) \cup C_{p_1}$. For every point $x \in M \setminus A_{p_1}$ and every $s \in [0, \overline{p_1, x}]$, we define $c_x(s) \in M$ as the point on the minimal geodesic segment from p_1 to x such that $\overline{x, c_x(s)} = s$ holds. It is not difficult to see $\mathrm{vol}(A_{p_1})/\mathrm{vol}(M) \le \Psi(\varepsilon; n)$ (see [12, Lemma 1.10]). Then, we have

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(M)}\int_M g_M(\nabla f_1, \nabla f_2)dx = \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} g_M(\nabla f_1, \nabla f_2)dx \pm \Psi(\varepsilon; n)\\
&= \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} g_M(\nabla f_1, \nabla \tilde f_2)dx \pm \Psi(\varepsilon; n)\\
&= \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin \overline{p_1, x}\frac{d\tilde f_2\circ c_x(s)}{ds}\Big|_{s=0}dx \pm \Psi(\varepsilon; n)\\
&= \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \Bigg\{\sin \overline{p_1, x}\bigg(\frac{\tilde f_2\circ c_x(\delta) - \tilde f_2\circ c_x(0)}{\delta}\\
&\qquad\qquad - \frac{1}{\delta}\int_0^\delta (\delta - s)\frac{d^2\tilde f_2\circ c_x(s)}{ds^2}ds\bigg)\Bigg\}dx \pm \Psi(\varepsilon; n)\\
&= \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin \overline{p_1, x}\bigg(\frac{f_2\circ c_x(\delta) - f_2\circ c_x(0)}{\delta}\bigg)dx\\
&\quad + \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin \overline{p_1, x}\bigg(\frac{\tilde f_2\circ c_x(\delta) - f_2\circ c_x(\delta)}{\delta}\bigg)dx\\
&\quad - \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin \overline{p_1, x}\bigg(\frac{\tilde f_2\circ c_x(0) - f_2\circ c_x(0)}{\delta}\bigg)dx\\
&\quad - \frac{1}{\delta\,\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin \overline{p_1, x}\int_0^\delta (\delta - s)\bigg(\frac{d^2\tilde f_2\circ c_x(s)}{ds^2} + \tilde f_2\circ c_x(s)\bigg)dsdx\\
&\quad + \frac{1}{\delta\,\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin \overline{p_1, x}\int_0^\delta (\delta - s)\tilde f_2\circ c_x(s)dsdx \pm \Psi(\varepsilon; n)\,.
\end{aligned}$$

CLAIM 2.2. *We have*

$$\left|\frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin \overline{p_1, x}\bigg(\frac{\tilde f_2\circ c_x(\delta) - f_2\circ c_x(\delta)}{\delta}\bigg)dx\right| < \Psi(\varepsilon; n)\,.$$

PROOF. We use the next estimate:

ESTIMATE 2.3. There exists $C(n) > 0$ such that, for every $0 \le s \le \delta$ and for every integrable function $h : M \to \mathbf{R}_{\ge 0}$, we have

$$\frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} h\circ c_x(s)dx \le \frac{C(n)}{\mathrm{vol}(M)}\int_M h(x)dx\,.$$

We put $S_{p_1}(1) = \{u \in T_{p_1}M\,;\, |u| = 1\}$. For $u \in S_{p_1}(1)$, we define $t(u) > 0$ as the supremum of $t \in (0, \infty)$ such that $\exp_{p_1} su|_{[0,t]}$ is a minimal geodesic segment from p_1 to $\exp_{p_1} tu$. Also we put $\hat S_{p_1}(1) = \{u \in S_{p_1}(1)\,;\, t(u) > 3\delta\}$ and $\theta(t, u) = t^{n-1}(\det(g_{ij}|_{\exp_{p_1}(tu)}))^{1/2}$.

Here, $g_{ij} = g_M(\partial/\partial x_i,\ \partial/\partial x_j)$ for a normal coordinate $(x_1, x_2, \dots, x_n)$ around p_1. Then,

$$\begin{aligned}\int_{M\setminus A_{p_1}} h\circ c_x(s)dx &\le \int_{\hat{S}_{p_1}(1)}\int_{3\delta}^{t(u)} h\circ c_{\exp_{p_1} tu}(\delta)\theta(t,u)dtdu\\ &= \int_{\hat{S}_{p_1}(1)}\int_{3\delta}^{t(u)} h(\exp_{p_1}((t-\delta)u)\theta(t,u)dtdu\\ &= \int_{\hat{S}_{p_1}(1)}\int_{2\delta}^{t(u)-\delta} h(\exp_{p_1}(\hat{t}u))\theta(\hat{t}+\delta,u)d\hat{t}du\,.\end{aligned}$$

By the Laplacian comparison theorem, there exists $C(n) > 0$ such that

$$\theta(\hat{t}+\delta,u) \le \frac{\sin^{n-1}(\hat{t}+\delta)}{\sin^{n-1}\hat{t}}\,\theta(\hat{t},u) \le C(n)\,\theta(\hat{t},u)$$

for each $u \in \hat{S}_{p_1}(1)$ and each $\hat{t} \in [2\delta,\ t(u)-\delta]$. Thus,

$$\begin{aligned}\int_{M\setminus A_{p_1}} h\circ c_x(s)dx &\le C(n)\int_{\hat{S}_{p_1}(1)}\int_{2\delta}^{t(u)-\delta} h(\exp_{p_1}(\hat{t}u))\theta(\hat{t},u)d\hat{t}du\\ &\le C(n)\int_{S_{p_1}(1)}\int_{0}^{t(u)} h(\exp_{p_1}(\hat{t}u))\theta(\hat{t},u)d\hat{t}du\\ &= C(n)\int_M h(x)dx\,.\end{aligned}$$

Therefore, we have Estimate 2.3.
By Estimate 2.3,

$$\begin{aligned}&\left|\frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin\overline{p_1,x}\left(\frac{\tilde{f}_2\circ c_x(\delta) - f_2\circ c_x(\delta)}{\delta}\right)dx\right|\\ &\le \frac{1}{\delta\,\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} |\tilde{f}_2\circ c_x(\delta) - f_2\circ c_x(\delta)|dx\\ &\le \frac{C(n)}{\delta\,\mathrm{vol}(M)}\int_M |\tilde{f}_2 - f_2|dx\\ &\le \frac{C(n)}{\delta}\left(\frac{1}{\mathrm{vol}(M)}\int_M |\tilde{f}_2 - f_2|^2dx\right)^{1/2} < \delta(\varepsilon,n)^{-1}\Psi(\varepsilon;n)\,.\end{aligned}$$

We remark that, without loss of generality, we may assume that $\lim_{\varepsilon\to 0}\delta(\varepsilon,n)^{-1}\Psi(\varepsilon;n) = 0$ by exchanging δ if necessary. Therefore, we have Claim 2.2. □

CLAIM 2.4. *We have*

$$\left|\frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin\overline{p_1,x}\left(\frac{\tilde{f}_2\circ c_x(0) - f_2\circ c_x(0)}{\delta}\right)dx\right| < \Psi(\varepsilon;n)\,.$$

PROOF.

$$
\begin{aligned}
&\left|\frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}}\sin\overline{p_1,x}\left(\frac{\tilde{f}_2\circ c_x(0)-f_2\circ c_x(0)}{\delta}\right)dx\right|\\
&\quad\le\frac{1}{\delta\,\mathrm{vol}(M)}\int_M|\tilde{f}_2-f_2|dx\\
&\quad\le\frac{1}{\delta}\left(\frac{1}{\mathrm{vol}(M)}\int_M|\tilde{f}_2-f_2|^2dx\right)^{1/2}\\
&\quad<\delta^{-1}\Psi(\varepsilon;n)\,.
\end{aligned}
$$

Therefore, we have Claim 2.4. □

CLAIM 2.5. *We have*

$$
\left|\frac{1}{\delta\,\mathrm{vol}(M)}\int_{M\setminus A_{p_1}}\sin\overline{p_1,x}\int_0^\delta(\delta-s)\left(\frac{d^2\tilde{f}_2\circ c_x(s)}{ds^2}+\tilde{f}_2\circ c_x(s)\right)dsdx\right|<\Psi(\varepsilon;n)\,.
$$

PROOF. We use the next estimate.

ESTIMATE 2.6. There exists $C(n)>0$ such that, for every integrable function $h:M\to\boldsymbol{R}_{\ge 0}$,

$$
\frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}}\int_0^\delta h\circ c_x(s)dsdx\le\frac{C(n)\delta}{\mathrm{vol}(M)}\int_M h(x)dx\,.
$$

This Estimate 2.6 follows by integrating Estimate 2.3 with respect to s between 0 and δ.

Then,

$$
\begin{aligned}
&\left|\frac{1}{\delta\,\mathrm{vol}(M)}\int_{M\setminus A_{p_1}}\sin\overline{p_1,x}\int_0^\delta(\delta-s)\left(\frac{d^2\tilde{f}_2\circ c_x(s)}{ds^2}+\tilde{f}_2\circ c_x(s)\right)dsdx\right|\\
&\quad\le\frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}}\int_0^\delta|\mathrm{Hess}_{\tilde{f}_2}+\tilde{f}_2g_M|(c_x(s))dsdx\\
&\quad\le\frac{C(n)\delta}{\mathrm{vol}(M)}\int_M|\mathrm{Hess}_{\tilde{f}_2}+\tilde{f}_2g_M|dx\\
&\quad\le C(n)\delta\left(\frac{1}{\mathrm{vol}(M)}\int_M|\mathrm{Hess}_{\tilde{f}_2}+\tilde{f}_2g_M|^2dx\right)^{1/2}\\
&\quad<\Psi(\varepsilon;n)\,.
\end{aligned}
$$

Therefore, we have Claim 2.5. □

CLAIM 2.7. *We have*

$$
\left|\frac{1}{\delta\,\mathrm{vol}(M)}\int_{M\setminus A_{p_1}}\sin\overline{p_1,x}\int_0^\delta(\delta-s)\tilde{f}_2\circ c_x(s)dsdx\right|<\Psi(\varepsilon;n)\,.
$$

The proof is similar to that of Claim 2.5.

From these claims, we have

$$\frac{1}{\mathrm{vol}(M)}\int_M g_M(\nabla f_1, \nabla f_2)dx = \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin\overline{p_1, x}\left(\frac{f_2\circ c_x(\delta) - f_2\circ c_x(0)}{\delta}\right)dx \pm\Psi(\varepsilon; n)\,.$$

For every $x \in M \setminus A_{p_1}$, we take $z_x, z_{c_x(\delta)} \in \partial B_{\pi/2}(p_1)$ such that $\overline{x, z_x} = \overline{x, \partial B_{\pi/2}(p_1)}$ and $\overline{c_x(\delta), z_{c_x(\delta)}} = \overline{c_x(\delta), \partial B_{\pi/2}(p_1)}$. Then, by Proposition 1.12, we have

$$\cos\delta = \cos\overline{p_1, x}\cos(\overline{p_1, x} - \delta) + \sin\overline{p_1, x}\sin(\overline{p_1, x} - \delta)\cos\overline{z_x, z_{c_x(\delta)}} \pm \Psi$$

$$\cos\delta = \cos\overline{p_1, x}\cos(\overline{p_1, x} - \delta) + \sin\overline{p_1, x}\sin(\overline{p_1, x} - \delta).$$

Therefore, we have

$$\sin\overline{p_1, x}\sin(\overline{p_1, x} - \delta) = \sin\overline{p_1, x}\sin(\overline{p_1, x} - \delta)\cos\overline{z_x, z_{c_x(\delta)}} \pm \Psi.$$

Thus, we have $\overline{z_x, z_{c_x(\delta)}} < \Psi(\varepsilon; n)$. By Proposition 1.12, we have

$$\begin{aligned}
&\sin\overline{p_1, x}(f_2\circ c_x(\delta) - f_2\circ c_x(0))\\
&\quad= \sin\overline{p_1, x}\Bigg(\cos\overline{p_1, p_2}\ \cos\overline{p_1, c_x(\delta)}\\
&\qquad\qquad + \sin\overline{p_1, p_2}\ \sin\overline{p_1, c_x(\delta)}\ \frac{\cos\overline{p_2, x} - \cos\overline{p_1, p_2}\cos\overline{p_1, x}}{\sin\overline{p_1, p_2}\ \sin\overline{p_1, x}}\Bigg)\\
&\qquad - \sin\overline{p_1, x}\ \cos\overline{p_2, x} \pm \Psi(\varepsilon; n)\\
&\quad= (\sin(\overline{p_1, x} - \delta) - \sin\overline{p_1, x})\cos\overline{p_2, x}\\
&\qquad + \cos\overline{p_1, p_2}\ (\sin\overline{p_1, x}\cos(\overline{p_1, x} - \delta)) - \sin(\overline{p_1, x} - \delta)\cos\overline{p_1, x}) \pm \Psi(\varepsilon; n)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(M)}\int_M g_M(\nabla f_1, \nabla f_2)dx\\
&\quad= \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} \sin\overline{p_1, x}\left(\frac{f_2\circ c_x(\delta) - f_2\circ c_x(0)}{\delta}\right)dx \pm \Psi(\varepsilon; n)\\
&\quad= \frac{1}{\mathrm{vol}(M)}\int_{M\setminus A_{p_1}} -\cos\overline{p_2, x}\cos\overline{p_1, x}dx + \cos\overline{p_1, p_2} \pm \Psi(\varepsilon; n)\\
&\quad= -\frac{1}{\mathrm{vol}(M)}\int_M f_1 f_2 dx + \cos\overline{p_1, p_2} \pm \Psi(\varepsilon; n)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{\mathrm{vol}(M)}\int_M g_M(\nabla f_1,\nabla f_2)dx &= \frac{1}{2\mathrm{vol}(M)}\int_M |\nabla f_1|^2dx+\frac{1}{2\mathrm{vol}(M)}\int_M |\nabla f_2|^2dx\\ &\quad -\frac{1}{2\mathrm{vol}(M)}\int_M |\nabla f_1-\nabla f_2|^2dx\\ &= \frac{n}{n+1}+\frac{1}{2\mathrm{vol}(M)}\int_M(\tilde{f}_1-\tilde{f}_2)\Delta(\tilde{f}_1-\tilde{f}_2)dx\pm\Psi(\varepsilon;n)\\ &= \frac{n}{n+1}-\frac{1}{2\mathrm{vol}(M)}\int_M n(\tilde{f}_1-\tilde{f}_2)^2dx\pm\Psi(\varepsilon;n)\\ &= \frac{n}{\mathrm{vol}(M)}\int_M f_1f_2dx\pm\Psi(\varepsilon;n)\,.\end{aligned}$$

Therefore, we have Proposition 2.1

We shall give several applications of Proposition 2.1. Let M be an n-dimensional compact Riemannian manifold with $\mathrm{Ric}_M\geq n-1$, and

$$0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\lambda_{n+1}\leq\cdots$$

denote the eigenvalues of $-\Delta$ on M. By Lichnerowicz-Obata's theorem, we have the inequarity $\lambda_1\geq n$. Moreover, the equality holds if and only if M is isometric to $\boldsymbol{S}^n$. The following Corollaries 2.9 and 2.10 were first proved by Bertrand (see [3, Théorème 2.1, Théorème 3.1]). We give a new proof of them by using Proposition 2.1.

REMARK 2.8. Proposition 2.1 gives a new simplified proofs of Corollaries 2.9 and 2.10. In addition, by using Proposition 2.1, we can *explicitly* calculate H_1^2-inner product for some eigenfunctions in a limit space. For example, let Z be a compact geodesic space as in Corollary 4.8. Then there exists a Gromov-Hausdorff (renormalized) limit measure υ on Z, the cannonical Laplace operator on $L^2(Z)$ and exist linearly independent eigenfunctions $f_1,\dots,f_{n-1}$ whose eigenvalues are all n and whose norms in H_1^2 are all 1 (for example, see [7, 8, 9]). We take $p_1,\dots,p_{n-1}\in Z$ such that $f_i(p_i)=\max f_i$. Then, by Proposition 2.1 and by an argument similar to the proof of Corollary 2.10, we have

$$\int_Y f_if_jd\upsilon=\frac{\cos\overline{p_i,p_j}}{n+1}\quad\text{and}\quad\int_Y\langle df_i,df_j\rangle d\upsilon=\frac{n\cos\overline{p_i,p_j}}{n+1}\,.$$

COROLLARY 2.9 (Bertrand [3, Théorème 2.1]). *Let $\varepsilon>0$ and let M be an n-dimensional compact Riemannian manifold with $n\geq 2$ and $\mathrm{Ric}_M\geq n-1$. We assume that there exist k pairs $(p_1,q_1),\dots,(p_k,q_k)$ of points of M such that $|\overline{p_i,q_i}-\pi|<\varepsilon$ holds for each i, and that $|\overline{p_i,p_j}-\pi/2|<\varepsilon$ holds for $i\neq j$. Then, we have*

$$\lambda_k=n\pm\Psi(\varepsilon;n)\,.$$

PROOF. We put $f_i(x)=\cos\overline{p_i,x}$ for every $1\leq i\leq k$. By Proposition 2.1, $\{(n+1)^{1/2}f_i\}_i$ form an almost orthonormal family in $L_1^2(M)$. By min-max principle, we have

$$\lambda_k\leq\sup\left\{\int_M|\nabla\Sigma a_if_i|^2dx\Big/\int_M(\Sigma a_if_i)^2dx;\ (a_i)_i\in\boldsymbol{R}^k\setminus 0_k\right\}.$$

Proposition 2.1 implies

$$\int_M |\nabla \Sigma a_i f_i|^2 dx \Big/ \int_M (\Sigma a_i f_i)^2 dx \le n + \Psi(\varepsilon; n)$$

for every $(a_1, a_2) \in \boldsymbol{R}^k \setminus 0_k$. Therefore, we have Corollary 2.9. □

COROLLARY 2.10 (Bertrand [3, Théorème 3.1]). *Let* $\varepsilon > 0$ *and let* M *be an* n*-dimensional compact Riemannian manifold with* $\mathrm{Ric}_M \ge n-1$*. We assume that* $\lambda_k = n \pm \varepsilon$*. Then, there exist* k *pairs* $(p_1, q_1), \dots, (p_k, q_k)$ *of points of* M *such that* $|\overline{p_i, q_i} - \pi| < \Psi(\varepsilon; n)$ *holds for each* i*, and that* $|\overline{p_i, p_j} - \pi/2| < \Psi(\varepsilon; n)$ *holds for* $i \ne j$*.*

PROOF. Let us recall several inequalities in [28]. Let $\tilde{f}_i \in C^\infty(M)$ $(i = 1, 2, \dots, k)$ be eigenfunctions satisfying

$$-\Delta \tilde{f}_i = \lambda_i \tilde{f}_i, \quad |\lambda_i - n| < \varepsilon \ \text{ for every } i \quad \text{ and } \int_M \tilde{f}_i \tilde{f}_j dx = 0 \ \text{ for every } i \ne j\,.$$

Then, we can assume that

$$\tilde{f}_i^2 + |\nabla \tilde{f}_i|^2 \le 1\,,$$
$$\frac{1}{\mathrm{vol}(M)} \int_M \tilde{f}_i^2 dx = \frac{1}{n+1} \pm \Psi(\varepsilon; n)\,,$$
$$\frac{1}{\mathrm{vol}(M)} \int_M |\nabla \tilde{f}_i|^2 dx = \frac{n}{n+1} \pm \Psi(\varepsilon; n)\,,$$
$$\frac{1}{\mathrm{vol}(M)} \int_M |\tilde{f}_i^2 + |\nabla \tilde{f}_i|^2 - 1| dx < \Psi(\varepsilon; n)\,,$$

hold (see [28, Lemma 3.1]).

Hence, for each $p \in M$, there exists $\tilde{p} \in M$ such that $\overline{p, \tilde{p}} < \Psi(\varepsilon; n)$ and $\tilde{f}_i^2(\tilde{p}) + |\nabla \tilde{f}_i|^2(\tilde{p}) = 1 \pm \Psi(\varepsilon; n)$. We fix a function $\Psi(\varepsilon; n)$ which satisfies the inequalities above and denote it by $\psi(\varepsilon; n)$. We take $p_i, q_i \in M$ such that $\tilde{f}_i(p_i) = \max \tilde{f}_i$ and $\tilde{f}_i(q_i) = \min \tilde{f}_i$. Let $g_i(x) = \tilde{f}_i(p_i) - \tilde{f}_i(x) + \psi(\varepsilon; n)$ and $h_i(x) = \tilde{f}_i(x) - \tilde{f}_i(q_i) + \psi(\varepsilon; n)$. By Cheng-Yau's gradient estimate, we have

$$\frac{|\nabla g_i|^2}{g_i^2}, \quad \frac{|\nabla h_i|^2}{h_i^2} < \frac{C(n)}{\psi(\varepsilon; n)}\,.$$

Here, $C(n)$ is a positive constant depending only on n (see [5, 11]). Thus, if we take $\tilde{p}_i, \tilde{q}_i \in M$ as above, then we have

$$|\nabla \tilde{f}_i|^2(\tilde{p}_i)\,, \quad |\nabla \tilde{f}_i|^2(\tilde{q}_i) < \Psi(\varepsilon; n)\,.$$

Especially, we have

$$|\tilde{f}_i(p_i) - 1|, \ |\tilde{f}_i(q_i) + 1| < \Psi(\varepsilon; n)\,.$$

We put $f_i(x) = \cos \overline{p_i, x}$. By $|\nabla \arccos \tilde{f}_i| \le 1$, we have

$$\tilde{f}_i > f_i - \Psi(\varepsilon; n)\,.$$

Thus, we have

$$\Delta(\tilde{f}_i - f_i) < \Psi(\varepsilon; n)$$

in the barrier sense (see [5, Definition 4.4] for the definition of barriers). By [28, Theorem 7.2], we have

$$|\tilde{f}_i - f_i| < \Psi(\varepsilon; n)\,.$$

Especially,

$$\overline{p_i, q_i} \geq \pi - \Psi(\varepsilon; n)\,.$$

Hence, by Proposition 2.1, we have

$$\frac{1}{\mathrm{vol}(M)} \int_M \tilde{f}_i \tilde{f}_j dx = \frac{\cos \overline{p_i, p_j}}{n+1} \pm \Psi(\varepsilon; n)$$

for every $i \neq j$. Since the left-hand side is equal to 0, we have

$$\left| \overline{p_i, p_j} - \frac{\pi}{2} \right| < \Psi(\varepsilon; n)\,.$$

Therefore, we have Corollary 2.10. □

We get also a result of Petersen as a corollary of Theorem 0.1.

COROLLARY 2.11 (Petersen [28, Theorem 1.1]). *Let M be an n-dimensinal compact Riemaniann manifold with* $\mathrm{Ric}_M \geq n-1$. *We assume that* $|\lambda_{n+1} - n| < \varepsilon$. *Then, we have* $d_{GH}(M, \boldsymbol{S}^n) < \Psi(\varepsilon; n)$.

Next corollary was first proved by Aubry. It follows also by Theorem 0.1, Corollary 2.9 and Corollary 2.10. Note that he gives more *explicit* estimate for $\Psi(\varepsilon; n)$ (see [2]). Hence, next corollary is *weaker* than his theorem.

COROLLARY 2.12 (Aubry [2, Proposition 19]). *Let M be an n-dimensional compact Riemaniann manifold with* $\mathrm{Ric}_M \geq n-1$. *We assume that* $|\lambda_n - n| < \varepsilon$. *Then, we have* $|\lambda_{n+1} - n| < \Psi(\varepsilon; n)$.

Next corollary was first proved by Gallot. Note that he estimates $C(n)$ in Corollary 2.13 explicitly. Therefore, Corollary 2.13 is *weaker* than the statement proved by him. However, we could give a new proof by using the theory of limit spaces.

COROLLARY 2.13. *There exists a positive constant $C(n) > n$ such that, for every n-dimensional compact Riemannian manifold M with* $\mathrm{Ric}_M \geq n-1$,

$$\lambda_{n+2} \geq C(n) > n$$

holds.

PROOF. If the assertion is false, then there exists a sequence of compact Riemannian manifolds $\{M_k\}_k$ with $\mathrm{Ric}_{M_k} \geq n-1$ such that the $(n+2)$-th eigenvalue λ^k_{n+2} satisfies $\lim_{k\to\infty} \lambda^k_{n+2} = n$. By taking a subsequence, if necessary, we can assume that M_k converges to some compact geodesic space Y in the sense of Gromov-Hausdorff convergence. By Corollary 2.9, there exist $(n+2)$ pairs $(p_1, q_1) \cdots (p_{n+2}, q_{n+2})$ of points of Y such that $\overline{p_i, q_i} = \pi$ holds for every i, and that $\overline{p_i, p_j} = \pi/2$ holds for $i \neq j$. This contradicts Theorem 0.1. □

3. A note on the structure of tangent cones of non-collapsing limit spaces. In this section, we discuss a relationship between Theorem 0.1 and the structure of tangent cones of non-collapsing limit spaces.

DEFINITION 3.1. For a metric space Z, we define the metric on $[0, \infty) \times Z / \{0\} \times Z$ as

$$\overline{(t_1, z_1), (t_2, z_2)} = \sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos \min\{\overline{z_1, z_2}, \pi\}}\,.$$

This metric space is denoted by $C(Z)$ and is called the *metric cone* of Z. We put $z^* = [(0, z)]$.

Throughout this section, let $\{M_i\}_i$ be a sequence of n-dimensional complete Riemannian manifolds ($n \geq 2$) with $\mathrm{Ric}_{M_i} \geq -(n-1)$, $m_i \in M_i$, and Y a proper geodesic space with $y \in Y$. Here, we say that a metric space W is *proper* if every bounded closed set is compact. We assume that

- The sequence (M_i, m_i) converges to (Y, y) in the sense of pointed Gromov-Hausdorff convergence.
- There exists $v > 0$ such that $\mathrm{vol}(B_1(m_i)) \geq v > 0$ holds for each i.

We say that (Y, y) is a *non-collapsing limit space*.

DEFINITION 3.2. Let (W, w) be a pointed proper geodesic space. We say that (W, w) is a *tangent cone at* $x \in Y$ if there exists a sequence of positive numbers $\{r_i\}_i$ such that r_i converges to 0 and $(Y, r_i^{-1} d_Y, x)$ converges to (W, w) in the sense of pointed Gromov-Hausdorff convergence.

Cheeger and Colding proved the following result for tangent cones of non-collapsing limit spaces.

THEOREM 3.3 (Cheeger-Colding [7, Theorem 5.2]). *Let $(T_x Y, 0_x)$ be a tangent cone at $x \in Y$. Then, there exists a compact geodesic space Z with* $\mathrm{diam}(Z) \leq \pi$ *such that $(C(Z), z^*)$ is isometric to $(T_x Y, 0_x)$.*

We shall prove an analogous statement to Theorem 0.1 for tangent cones.

THEOREM 3.4. *Let $(T_x Y, 0_x)$ be a tangent cone at $x \in Y$ and Z a compact geodesic space with* $\mathrm{diam}(Z) \leq \pi$ *such that $(T_x Y, 0_x)$ is isometric to $(C(Z), z^*)$. We assume that there exist k pairs $(p_1, q_1), \dots, (p_k, q_k)$ of points of Z such that $\overline{p_i, q_i} = \pi$ holds for every i, and that $\overline{p_i, p_j} = \pi/2$ holds for $i \neq j$. Then, we have the following*:

(1) *k is at most n.*

(2) *If $1 \leq k \leq n-2$, then there exists a compact geodesic space X with* $\mathrm{diam}(X) \leq \pi$ *such that $Z = \boldsymbol{S}^{k-1} * X$.*

(3) *If $k = n-1$ or n, then $Z = \boldsymbol{S}^{n-1}$.*

PROOF. First, we remark the following.

(1) For every metric space X, $C(\boldsymbol{S}^{k-1} * X)$ is isometric to $\boldsymbol{R}^k \times C(X)$.

(2) If there exist $z_1, z_2 \in Z$ such that $\overline{z_1, z_2} = \pi$ holds, then $\overline{z_1, z} + \overline{z, z_2} = \pi$ for every $z \in Z$. This is a consequence of a splitting theorem for limit spaces (see [6, Theorem 6.64]).

(3) We have $Z \neq \boldsymbol{S}^k$ for every $1 \leq k \leq n-2$. It follows from $\dim_{\mathcal{H}} Z = n-1$ (see Proposition 5.6). Here $\dim_{\mathcal{H}} Z$ is the Hausdorff dimension of Z. Compare this fact with [13, Lemma 5.10].

We assume that there exist points $z_1, z_2 \in Z$ such that $\overline{z_1, z_2} = \pi$. Then, by the definition of the metric of $C(Z)$, there exists an isometric embedding $\gamma : \boldsymbol{R} \to C(Z)$ such that $\gamma(0) = z_*, \gamma(-1) = (1, z_1)$ and $\gamma(1) = (1, z_2)$. Thus, by the splitting theorem for limit space and (1) above, we have $Z = (\{z_1, z_2\}, d_{\boldsymbol{S}^0}) * (\partial B_{\pi/2}(p), p)$. Theorem 3.4 follows from this argument, (1), (2), (3) above and an argument similar to that in Section 1. □

Similarly, we have the following.

COROLLARY 3.5. *Let* $(T_x Y, 0_x)$ *be a tangent cone at* $x \in Y$ *and* Z *a compact geodesic space with* $\operatorname{diam}(Z) \leq \pi$ *such that* $(T_x Y, 0_x)$ *is isometric to* $(C(Z), z^*)$. *We assume that there exist* k *pairs* $(p_1, q_1), \ldots, (p_k, q_k)$ *of points of* Z *such that* $\overline{p_i, q_i} = \pi$ *for each* i *and that* $\det((\cos \overline{p_i, p_j})_{i,j}) \neq 0$. *Then, we have the following.*

(1) *k is at most n.*

(2) *If* $1 \leq k \leq n-2$, *then there exists a compact geodesic space* X *with* $\operatorname{diam}(X) \leq \pi$ *such that* $Z = \boldsymbol{S}^{k-1} * X$.

(3) *If* $k = n-1$ *or* n, *then* $Z = \boldsymbol{S}^{n-1}$.

PROOF. We give only a proof of the case $k = 2$. By the assumption and Theorem 3.4, there exists a compact geodesic space X such that $Z = \boldsymbol{S}^0 * X$ with $p_1 = (0, *)$ and $q_1 = (\pi, *)$. By the assumption, we have $p_2, q_2 \in \boldsymbol{S}^0 * X \setminus \{p_1, q_1\}$. Especially, we have $\operatorname{diam} X = \pi$ by the definition of the metric of $\boldsymbol{S}^0 * X$. Therefore, by Theorem 3.4, we have the assertion. □

4. The topological structure of tangent cones of non-collapsing limit spaces and a proof of Theorem 0.4. Throughout this section, we use the same notation as in Section 3. For a proper geodesic space X, we put

$$\mathcal{R}^n_\varepsilon(X) = \{x \in X \,;\, \text{There exists a positive number } r > 0 \text{ such that for every } 0 < s < r,\ d_{GH}(\bar{B}_s(x), \bar{B}_s(0_n)) \leq \varepsilon s \text{ holds.}\}$$

$$\mathcal{R}^n(X) = \bigcap\nolimits_{\varepsilon > 0} \mathcal{R}^n_\varepsilon(X)\,.$$

Here, ε is a positive number and $\bar{B}_s(0_n) \subset \boldsymbol{R}^n$. Let (Y, y) be a non-collapsing limit space of a sequence of pointed n-dimensional compact Riemaniann manifolds.

THEOREM 4.1 (Cheeger-Colding [5, Theorem 9.73]). *We have*

$$\mathcal{R}^n_\varepsilon(Y) \subset \operatorname{Int}(\mathcal{R}^n_{\Psi(\varepsilon|n)}(Y)) \text{ and } \dim_{\mathcal{H}}(Y \setminus \mathcal{R}^n(Y)) \leq n-2\,.$$

Here, for a subset $A \subset Y$, Int$A$ *is the interior of* A. *Especially, we have*

$$\mathcal{R}^n(Y) = \bigcap_{\varepsilon>0} \mathrm{Int}(\mathcal{R}^n_\varepsilon(Y)) .$$

Cheeger-Colding also proved the following important result.

THEOREM 4.2 (Cheeger-Colding [7, Theorem A.1.1]). *There exists a positive number $\varepsilon_n > 0$ satisfying the following property. For every $0 < \varepsilon < \varepsilon_n$, there exist a complete n-dimensional Riemannian manifold M, and a homeomorphism $f : \mathrm{Int}(\mathcal{R}^n_\varepsilon(Y)) \to M$ such that f, f^{-1} are $(1 - \Psi(\varepsilon; n))$-locally Hölder continuous.*

We shall prove an analogous statement to Theorem 4.2 for tangent cones.

THEOREM 4.3. *Let k be a non-negative integer, $(T_xY, 0_y)$ a tangent cone at x and X a compact geodesic space with $\mathrm{diam}(X) \le \pi$ such that $(T_xY, 0_y)$ is isometric to $(\boldsymbol{R}^k \times C(X), (0_k, x^*))$. Then, we have $\dim_{\mathcal{H}} X = n - k - 1$, $\mathcal{R}^{n-k-1}_\varepsilon(X) \subset \mathrm{Int}(\mathcal{R}^{n-k-1}_{\Psi(\varepsilon;n)}(X))$ and $\dim_{\mathcal{H}}(X \setminus \mathcal{R}^{n-k-1}(X)) \le n - k - 3$. Also, there exists a postive number $\varepsilon_n > 0$ satisfying the following property: For every $0 < \varepsilon < \varepsilon_n$, there exist a complete $(n-k-1)$-dimensional Riemannian manifold M and a homeomorphism $f : \mathrm{Int}(\mathcal{R}^{n-k-1}_\varepsilon(X)) \to M$ such that f, f^{-1} are $(1 - \Psi(\varepsilon; n))$-locally Hölder continuous.*

PROOF. First, we remark the following claims.

CLAIM 4.4. *Let X be a proper geodesic space, $x \in X$ and ε, r positive numbers. We assume that $d_{GH}(\bar{B}_r(0, x), \bar{B}_r(0_n)) \le \varepsilon r$ holds. Here,*

$$\bar{B}_r(0, x) \subset (\boldsymbol{R} \times X,\ \sqrt{(d_{\boldsymbol{R}})^2 + (d_X)^2}) .$$

Then, we have $d_{GH}(\bar{B}_r(x), \bar{B}_r(0_{n-1})) \le \Psi(\varepsilon)r$.

CLAIM 4.5. *Let Z be a compact geodesic space with $\mathrm{diam}(Z) \le \pi$ and ε, R positive numbers. We consider the next metric balls. Here be careful about the metrics.*

(1) $\bar{B}^{\boldsymbol{R}\times Z}_R(0, z) \subset (\boldsymbol{R} \times Z, \sqrt{(d_{\boldsymbol{R}})^2 + (\varepsilon^{-1} d_Z)^2})$,

(2) $\bar{B}^{C(Z)}_R(1, z) \subset (C(Z), \varepsilon^{-1} d_{C(Z)})$.

Then, we have

$$d_{GH}((\bar{B}^{\boldsymbol{R}\times Z}_R(0, z), (0, z)), (\bar{B}^{C(Z)}_R(1, z), (1, z))) \le \Psi(\varepsilon; R) .$$

We skip the proof of these claims because it is not difficult. We have $\mathcal{R}^{n-k-1}_\varepsilon(X) \subset \mathrm{Int}(\mathcal{R}^{n-k-1}_{\Psi(\varepsilon;n)}(X))$ by these claims and Theorem 4.2. Therefore, by an argument similar to the proof of Theorem 4.2, we have Theorem 4.3 (see [7, Theorem 5.14] and [7, Theorem A.1.2]). □

COROLLARY 4.6. *If the assumption in Theorem 3.4 holds with $k = n - 2$, then there exists $0 < r \le 1$ such that $Z = \boldsymbol{S}^{n-3} * \boldsymbol{S}^1(r)$ holds. Here, $\boldsymbol{S}^1(r) = \{x \in \boldsymbol{R}^2 ; |x| = r\}$, and the metric $d_{\boldsymbol{S}^1(r)}$ on $\boldsymbol{S}^1(r)$ is the standard Riemannian metric.*

PROOF. First, the next claim is straightforward.

CLAIM 4.7. *Let d be a metric on $\boldsymbol{S}^1$ such that $(\boldsymbol{S}^1, d)$ is a geodesic space homeomorphic to the standard unit sphere $(\boldsymbol{S}^1, d_{\boldsymbol{S}^1})$. Then, there exists a positive number $0 < r < \infty$ such that $(\boldsymbol{S}^1, d)$ is isometric to $(\boldsymbol{S}^1(r), d_{\boldsymbol{S}^1(r)})$.*

By Theorem 3.4, there exists a compact geodesic space X with $\operatorname{diam}(X) \le \pi$ such that Z is isometric to $\boldsymbol{S}^{n-3} * X$. By Theorem 4.3, we can prove that X is homeomorphic to some one-dimensional connected compact manifold. Namely, X is homeomorphic to $\boldsymbol{S}^1$. Therefore, by Claim 4.7, there exists $0 < r \le 1$ such that $X = \boldsymbol{S}^1(r)$ holds. □

Similarly, we have the following.

COROLLARY 4.8. *Let Z be a compact geodesic space and $\{M_i\}_i$ a sequence of compact n-dimensional Riemannian manifolds with $\operatorname{Ric}_{M_i} \ge n-1$. We assume that M_i converges to Z in the sense of Gromov-Hausdorff convergence and $\lim_{i\to\infty} \lambda^i_{n-1} = n$. Then, there exists $0 \le r \le 1$ such that Z is isometric to $\boldsymbol{S}^{n-1} * \boldsymbol{S}^1(r)$.*

PROOF. By Theorem 0.1, there exists a compact geodesic space X such that $Z = \boldsymbol{S}^{n-1} * X$. First, we assume that Z is a collapsing and X is not a point. Then, since X is a geodesic space, we have $\dim_{\mathcal{H}} Z = \dim_{\mathcal{H}} \boldsymbol{S}^{n-1} * X \ge n+1$ (see Proposition 5.6). This is a contradiction. Therefore, if Z is a collapsing, then X is a point. On the other hand, if Z is a non-collapsing, then $X = \boldsymbol{S}^1(r)$ for some $r > 0$ by an argument similar to the proof of Corollary 4.6. Therefore, we have the assertion. □

This is equivalent to Theorem 0.4 by Gromov's pre-compactness theorem.

5. Appendix: A calculation of Hausdorff dimension. In this appendix, we will prove the equality

$$\dim_{\mathcal{H}}(\boldsymbol{R}^k \times C(X)) = k + 1 + \dim_{\mathcal{H}}(X)\,,$$

for every compact metric space X. Throughout this section, we assume that a metric space X always satisfies the following property (Q).

(Q) For every positive number $\varepsilon > 0$, there exists a countable collection $\{p_i\}_i$ of points of X such that $X = \bigcup_i B_\varepsilon(p_i)$.

LEMMA 5.1. *Let l be a positive real number and A be a subset on X satisfying $\mathcal{H}^{l+1}(A) = 0$. Then, we have $\mathcal{H}^l(\partial B_r(x) \cap A) = 0$ for every $x \in X$ and for a.e. $r > 0$.*

PROOF. For $z \in X$ and $t > 0$, we put a function $\phi^t_z : \boldsymbol{R}_{>0} \to \boldsymbol{R}$ as $\phi^t_z(r) = 15t$ if $\partial B_r(x) \cap \bar{B}_t(z) \ne \emptyset$, and as $\phi^t_z(r) = 0$ if otherwise. This function is a Borel function. We put $s_1 = \inf_{w\in \bar{B}_t(z)} \overline{x, w}$ and $s_2 = \sup_{w\in\bar{B}_t(z)} \overline{x, w}$. Then, we have

$$\int_0^\infty (\phi^t_z(r))^l dr = \int_{s_1}^{s_2} (\phi^t_z(r))^l dr \le (15t)^l \int_{s_1}^{s_2} dr \le 30^{l+1} t^{l+1}\,.$$

By the assumption, for every positive numbers $\varepsilon, \delta > 0$, there exists a countable collection $\{\bar{B}_{r_i}(x_i)\}_i$ such that $A \subset \bigcup_i \bar{B}_{r_i}(x_i)$, $r_i < \delta$ and $\sum_i r_i^{l+1} < \varepsilon$. Here, we define a function $\phi^l_{\delta,\varepsilon} : \boldsymbol{R}_{>0} \to \boldsymbol{R} \cup \{\infty\}$ as $\phi^l_{\delta,\varepsilon}(r) = \sum_i (\phi^{r_i}_{x_i}(r))^l$. This function is also a Borel function. We have

$$\sum_i \int_0^\infty (\phi_{x_i}^{r_i})^l dr \leq 30^{l+1} \sum_i r_i^{l+1} < 30^{l+1}\varepsilon\,.$$

By the monotone convergence theorem, we have

$$\int_0^\infty \phi_{\delta,\varepsilon}^l(r)dr < 30^{l+1}\varepsilon\,.$$

On the other hand, by definition, we have $\mathcal{H}_\delta^l(A \cap \partial B_r(x)) \leq \phi_{\delta,\varepsilon}^l(r)$ for every $r > 0$. Since $\phi_{\delta,\varepsilon}^l \to 0$ in $L_1(\boldsymbol{R}_{>0})$ as $\varepsilon \to 0$, there exists a Borel set $V \subset \boldsymbol{R}_{>0}$ and exists a sequence $\{\varepsilon_i\}_i$ such that $\mathcal{H}^1(\boldsymbol{R}_{>0} \setminus V) = 0$ holds, $\varepsilon_i \to 0$ holds as $i \to \infty$ and that $\lim_{i\to\infty} \phi_{\delta,\varepsilon_i}^l(r) = 0$ for every $r \in V$. Therefore, we have $\mathcal{H}_\delta^l(A \cap \partial B_r(x)) = 0$ for every $r \in V$. Especially, the function, $r \to \mathcal{H}_\delta^l(A \cap \partial B_r(x))$ is Lebesgue measurable and the function $r \to \mathcal{H}^l(A \cap \partial B_r(x))$ is also Lebesgue measurable. Since

$$\int_0^\infty \mathcal{H}_\delta^l(A \cap \partial B_r(x))dr = 0\,,$$

we have

$$\int_0^\infty \mathcal{H}^l(A \cap \partial B_r(x))dr = 0\,.$$

Therefore, we have the assertion. □

LEMMA 5.2. *For every positive number $l > 0$ and every subset A in X, $\mathcal{H}^l(A) = 0$ holds if and only if $\mathcal{H}^{l+1}(\boldsymbol{R} \times A) = 0$.*

PROOF. First, we assume that $\mathcal{H}^l(A) = 0$. Since $\mathcal{H}_\delta^l(A) = 0$ for every $\delta > 0$, there exists a countable collection $\{\bar{B}_{r_i}(x_i)\}_i$ such that $r_i < \delta$, $A \subset \bigcup_i \bar{B}_{r_i}(x_i)$ and $\sum_i r_i^l < \varepsilon$. For every i and for every $0 \leq k \leq [1/r_i]+1$, we define $t_k^i \in [0,1]$ by $t_k^i = k([1/r_i]+1)^{-1}$. Here, $[r] = \sup\{s \in \boldsymbol{Z}\,;\, r \geq s\}$ for a real number r.

CLAIM 5.3. $[0,1] \times A \subset \bigcup_{i,k} \bar{B}_{100r_i}(t_k^i, x_i)$.

We will prove Claim 5.3. For every $(t,x) \in [0,1] \times A$, we chose i such that $x \in \bar{B}_{r_i}(x_i)$. We also chose k such that $|t - t_k^i| \leq [1/r_i]^{-1}$. Then, by $[1/r_i]^{-1} \leq r_i/(1-r_i)$, we have

$$\sqrt{(t_i - t)^2 + \overline{x_i, x}^2} \leq |t_i - t| + \overline{x_i, x}\left[\frac{1}{r_i}\right]^{-1} + r_i \leq \frac{r_i}{1-r_i} + r_i \leq 5r_i\,.$$

Therefore, we have Claim 5.3.

Since

$$\sum_{i,k} r_i^{l+1} \leq 2\sum_i r_i^l \leq 2\varepsilon\,,$$

we have $\mathcal{H}_\delta^{l+1}([0,1] \times A) = 0$. Thus we have the assertion.

Next, we assume that $\mathcal{H}^{l+1}(\boldsymbol{R} \times A) = 0$. By Lemma 5.1, for every $x \in X$, we have $\mathcal{H}^l(\partial B_r(0,x) \cap \boldsymbol{R} \times A) = 0$ for a.e. $r > 0$. Let $\pi : \boldsymbol{R} \times X \to X$ be the projection. Since $\bar{B}_r(x) \cap A \subset \pi(\partial B_r(0,x) \cap \boldsymbol{R} \times A)$, we have $\mathcal{H}^l(\bar{B}_r(x) \cap A) = 0$ for a.e. $r > 0$. Therefore, we have the assertion. □

COROLLARY 5.4. *For every* $A \subset X$ *and* $k \in \boldsymbol{N}$, *we have* $\dim_{\mathcal{H}}(\boldsymbol{R}^k \times A) = k + \dim_{\mathcal{H}} A$.

PROPOSITION 5.5. *For every compact metric space* Z *and for every* $l > 0$, $\mathcal{H}^{l+1}(C(Z)) = 0$ *holds if and only if* $\mathcal{H}^l(Z) = 0$ *holds. Especially, we have* $\dim_{\mathcal{H}} C(Z) = \dim_{\mathcal{H}} Z + 1$.

PROOF. Since $A_{r_1,r_2}(z_*)$ is bi-Lipshithz equivalent to $A_{s_1,s_2}(z_*)$ for every $r_1 < r_2$ and for every $s_1 < s_2$, we know that $\mathcal{H}^{l+1}(C(Z)) = 0$ holds if and only if $\mathcal{H}^{l+1}(A_{1/2,2}(z_*)) = 0$ holds. Clearly, $A_{1/2,2}(z_*)$ is bi-Lipshitz equivalent to $[1/2, 2] \times Z$. Therefore, by applying Lemma 5.2, we have the assertion. □

Corollary 5.4 and Proposition 5.5 implies

$$\dim_{\mathcal{H}}(\boldsymbol{R}^k \times C(X)) = k + 1 + \dim_{\mathcal{H}}(X)$$

for every compact metric space X. Similarly, we have the following proposition.

PROPOSITION 5.6. *For every compact metric space* X *and for every* $k \geq 0$, *we have* $\dim_{\mathcal{H}}(\boldsymbol{S}^k * X) = k + 1 + \dim_{\mathcal{H}}(X)$.

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