Tohoku Math. J.
61 (2009), 241–252

ON THE EXISTENCE OF KÄHLER METRICS OF CONSTANT SCALAR CURVATURE

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(Received February 7, 2008, revised December 17, 2008)

Abstract. For certain compact complex Fano manifolds M with reductive Lie algebras of holomorphic vector fields, we determine the analytic subvariety of the second cohomology group of M consisting of Kähler classes whose Bando-Calabi-Futaki character vanishes. Then a Kähler class contains a Kähler metric of constant scalar curvature if and only if the Kähler class is contained in the analytic subvariety. On examination of the analytic subvariety, it is shown that M admits infinitely many nonhomothetic Kähler classes containing Kähler metrics of constant scalar curvature but does not admit any Kähler-Einstein metric.

1. Introduction. The question of whether a manifold admits a Riemannian metric of constant scalar curvature or not is a classical problem. For any real closed manifold M of dimension greater than two, Kazdan and Warner [10] proved that M admits at least a Riemannian metric of negative constant scalar curvature. On the other hand, there exists an obstruction to the existence of Kähler metrics of constant scalar curvature. Indeed, let M be an m-dimensional compact complex manifold. Denote by $\mathrm{Aut}(M)$ the complex Lie group consisting of all biholomorphic automorphisms of M and by $\mathfrak{h}(M)$ its Lie algebra consisting of all holomorphic vector fields on M. The Lie algebra $\mathfrak{h}(M)$ is called reductive if $\mathfrak{h}(M)$ is the complexification of the Lie algebra of a compact subgroup of $\mathrm{Aut}(M)$. In [14], Matsushima proved that $\mathfrak{h}(M)$ is the complexification of the real Lie algebra consisting of all infinitesimal isometries of M, and hence $\mathfrak{h}(M)$ is reductive, if M admits a Kähler-Einstein metric. Generalizing the result of Matsushima, Lichnerowicz proved in [12], [13] that $\mathfrak{h}(M)$ must satisfy a certain condition if M admits a Kähler metric of constant scalar curvature. (For details see also [11, Theorem 6.1].) When M is a compact simply connected Kähler manifold, the condition of Lichnerowicz is equivalent to that of Matsushima. For example, the one point blow-up of $\boldsymbol{CP}^2$ does not satisfy the condition (see [5, p. 100]) and hence does not admit any Kähler metric of constant scalar curvature. Thus the problem to solve is whether M with reductive $\mathfrak{h}(M)$ admits a Kähler metric of constant scalar curvature or not.

Generalizing the result of Futaki [3], Bando [1], Calabi [2] and Futaki [4] give an obstruction to the existence of a Kähler metric of constant scalar curvature whose Kähler form is contained in some particular Kähler class. Let Ω be a Kähler class, $\omega \in \Omega$ a Kähler form and s_ω the scalar curvature of ω. Let $c_1(M) \in H^2(M; \boldsymbol{Z})$ be the first Chern class of M and

2000 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C55.
Key words and phrases. Kähler manifold, constant scalar curvature, Bando-Calabi-Futaki character.
The author is deeply grateful to the referee for valuable information.

set

$$\mu_\Omega = \frac{(\Omega^{m-1} \cup c_1(M))[M]}{\Omega^m[M]},$$

where $[M]$ denotes the fundamental cycle of M. Then there exists uniquely a smooth function h_ω up to constant such that

$$s_\omega - m\mu_\Omega = \triangle_\omega h_\omega ,$$

and the integral

$$f_\Omega(X) = \int_M X h_\omega \omega^m$$

is defined for $X \in \mathfrak{h}(M)$. This integral $f_\Omega(X)$ is independent of the choice of Kähler forms $\omega \in \Omega$. Moreover, $f_\Omega : \mathfrak{h}(M) \to \boldsymbol{C}$ is a Lie algebra character and f_Ω vanishes if Ω contains a Kähler metric of constant scalar curvature. The character f_Ω is called the Bando-Calabi-Futaki character or the Futaki invariant.

When Ω is a Hodge class and a holomorphic line bundle L with $c_1(L) = \Omega$ admits a lifting of the Ω-preserving action of a subgroup G of Aut(M), in [16] Nakagawa gives a lifting of the Lie algebra character f_Ω to a group character $G \to \boldsymbol{C}/(\boldsymbol{Z} + \mu_\Omega \boldsymbol{Z})$ by using the results in [17] and [6].

Assume that there exists an inclusion $\iota : \mathrm{U}(1) \to \mathrm{Aut}(M)$ and that Ω is equal to the first Chern class of a holomorphic U(1)-line bundle L over M. For any integer $p \geq 2$ let Y denote the element $2\pi\sqrt{-1}$ of the Lie algebra of U(1) and set

$$X = \iota_* Y \in \mathfrak{h}(M), \quad X_p = \frac{1}{p} X \in \mathfrak{h}(M), \quad g_p = \exp X_p \in \mathrm{Aut}(M). \tag{1}$$

Then the order of g_p is p. We assume that the next assumption is satisfied. (See Assumption 2.2 and Lemma 2.3 in [7].)

ASSUMPTION 1.1. Assume that the fixed point set of g_p^k for $1 \leq k \leq p-1$ is independent of k and that the connected components $N_1, \ldots, N_n$ of the fixed point set, which are compact complex submanifolds of M, have cell decompositions with no codimension one cells.

Let α_p denote the primitive p-th root of unity defined by

$$\alpha_p = e^{2\pi\sqrt{-1}/p}$$

hereafter. Suppose that g_p^k acts on $K_M^{-1}|_{N_i}$ via multiplication by $\alpha_p^{kr_i}$ and acts on $L|_{N_i}$ via multiplication by $\alpha_p^{k\kappa_i}$. Suppose moreover that the normal bundle $\nu(N_i, M)$ is decomposed into the direct sum of subbundles

$$\nu(N_i, M) = \bigoplus_j \nu(N_i, \theta_j),$$

where g_p^k acts on $\nu(N_i, \theta_j)$ via multiplication by $e^{\sqrt{-1}\theta_j}$. Then a cohomology class $\Phi(\nu(N_i, M))$ is defined by

$$\Phi(\nu(N_i, M)) = \prod_j \prod_{k=1}^{R_j} \frac{1}{1 - e^{-x_k - \sqrt{-1}\theta_j}} \in H^*(N_i; \boldsymbol{C}) \quad (R_j = \mathrm{rank}_{\boldsymbol{C}}(\nu(N_i, \theta_j))),$$

where $\prod_k (1 + x_k)$ is equal to the total Chern class of $\nu(N_i, \theta_j)$. For $1 \le k \le p-1$, $\varepsilon = -1,\ 0,\ +1$ and an integer ζ, we define numbers $T_i(k, \varepsilon, \zeta)$ and $S_\varepsilon(\zeta)$ by

$$T_i(k, \varepsilon, \zeta) = \frac{1}{1 - \alpha_p^k} (\alpha_p^{k(-\varepsilon r_i + \zeta \kappa_i)} e^{-\varepsilon c_1(K_M^{-1}|_{N_i}) + \zeta c_1(L|_{N_i})} - 1)^{m+1} \mathrm{Td}(TN_i) \Phi(\nu(N_i, M))[N_i],$$

$$S_\varepsilon(\zeta) = \frac{1}{p} \sum_{i=1}^{n} \sum_{k=1}^{p-1} T_i(k, \varepsilon, \zeta),$$

where $\mathrm{Td}(TN_i)$ is the Todd class of TN_i. Then $F_L(g_p)$ is defined by

$$F_L(g_p) = (m+1) \sum_{i=0}^{m} (-1)^i \binom{m}{i} (S_{-1}(m - 2i) - S_{+1}(m - 2i))$$
$$- m\mu_\Omega \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} S_0(m + 1 - 2i).$$

The lifting of the character f_Ω given by Nakagawa is expressed by a Simons character of a certain foliation. In [7], we gives a localization formula for the Simons character under Assumption 1.1. The next theorem follows from [16, Theorem 4.7] and [7, Theorem 2.5].

THEOREM 1.2. *There exists a non-zero constant* $A(m, n)$ *determined only by* m, n *such that* $F_L(g_p) \equiv A(m, n) f_\Omega(X_p) \pmod{\boldsymbol{Z} + \mu_\Omega \boldsymbol{Z}}$.

2. Main result. For $m,\ n \ge 1$, let $H_m,\ H_n$ be the hyperplane bundles over the complex projective spaces $\boldsymbol{CP}^m$, $\boldsymbol{CP}^n$ respectively, and

$$\pi_1 : H_m \to \boldsymbol{CP}^m, \quad \pi_2 : H_n \to \boldsymbol{CP}^n$$

their projections. Let $E = \pi_1^* H_m \oplus \pi_2^* H_n$ be the rank 2 vector bundle over $\boldsymbol{CP}^m \times \boldsymbol{CP}^n$. Let M be the total space of the projective bundle of E and J_M the tautological bundle of M. Then M is an $(m + n + 1)$-dimensional simply-connected compact Kähler manifold and the same argument as in [3, Proposition 3.1] shows that M is a Fano manifold (see also [5, Proposition 4.2.1]) and the identity component of $\mathrm{Aut}(M)$ coincides with the factor group $(\mathrm{GL}(m+1, \boldsymbol{C}) \times \mathrm{GL}(n+1, \boldsymbol{C}))/\boldsymbol{C}^*$, where $\boldsymbol{C}^*$ is the center of $\mathrm{GL}(m+n+2, \boldsymbol{C})$. Hence the Lie algebra $\mathfrak{h}(M)$ is isomorphic to

$$\{(A, B) \in \mathfrak{gl}(m+1, \boldsymbol{C}) \oplus \mathfrak{gl}(n+1, \boldsymbol{C})\,;\ \mathrm{Tr}\, A + \mathrm{Tr}\, B = 0\},$$

which satisfies the condition of Matsushima.

Applying the Gysin exact sequence to the fibration

$$F = \boldsymbol{CP}^1 \to M \overset{p}{\to} B = \boldsymbol{CP}^m \times \boldsymbol{CP}^n,$$

we have the split exact sequence

$$H^{-1}(B; \boldsymbol{Z}) = 0 \to H^2(B; \boldsymbol{Z}) \simeq H^2(\boldsymbol{CP}^m; \boldsymbol{Z}) \oplus H^2(\boldsymbol{CP}^n; \boldsymbol{Z})$$
$$\xrightarrow{p^*} H^2(M; \boldsymbol{Z}) \xrightarrow{f} H^0(B; \boldsymbol{Z}) \simeq \boldsymbol{Z} \to H^3(B; \boldsymbol{Z}) = 0\,,$$

where f is the integration along the fiber. Then H_m, H_n are naturally regarded as vector bundles over $\boldsymbol{CP}^m \times \boldsymbol{CP}^n$, and since $f(c_1(J_M^*)) = 1$, it follows that

$$H^2(M; \boldsymbol{Z}) = \{\lambda\tilde{u} + \mu\tilde{v} + \nu\tilde{w}\,;\ \lambda,\ \mu,\ \nu \in \boldsymbol{Z}\} \simeq \boldsymbol{Z}^3\,,$$

where $\tilde{u} = c_1(p^*H_m)$, $\tilde{v} = c_1(p^*H_n)$ and $\tilde{w} = c_1(J_M^*)$.

REMARK 2.1. Let $\hat{u}$, $\hat{v}$ be the first Chern forms of H_m, H_n, respectively. Then $x\hat{u}+y\hat{v}$ is a Kähler form on $\boldsymbol{CP}^m \times \boldsymbol{CP}^n$ for $x,\ y > 0$, and hence $x\tilde{u} + y\tilde{v} + z\tilde{w}$ is a Kähler class of M for $x,\ y > 0$ and sufficiently small $z > 0$. Therefore the set of Kähler classes of M is contained in the subset $\{x\tilde{u} + y\tilde{v} + z\tilde{w}\,;\ x,\ y,\ z > 0\}$ of $H^2(M; \boldsymbol{R}) \simeq \boldsymbol{R}^3$.

Now, let $F(x, y, z)$ be an integral homogeneous polynomial of degree $m+n+4$ defined by

$$F(x, y, z) = -(m(m+2)yz + n(n+2)xz + 2xy)g(x, y, z) + xyz\,h(x, y, z)\,,$$

where

$$g(x, y, z) = \sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^{m+n+s+q+1}$$
$$((x-z)^{m-q}y^{n+q+2} - x^{m-q}(y-z)^{n+q+2})\,,$$

$$h(x, y, z) = \sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^{m+n+s+q+1}$$
$$\begin{pmatrix} \{(m+n+2-s) + (n+2)(s-m+q)\}(x-z)^{m-q}y^{n+q+1} \\ +\, m(m-q)(x-z)^{m-q-1}y^{n+q+2} \\ +\{(m+n+2-s) - n(s-m+q)\}x^{m-q}(y-z)^{n+q+1} \\ -\,(m+2)(m-q)x^{m-q-1}(y-z)^{n+q+2} \end{pmatrix}.$$

For example, if $(m, n) = (1, 2)$, we have

$$F(x, y, z) = 120x^2y^3z^2 - 420x^2y^2z^3 + 390x^2yz^4 - 120x^2z^5 + 60xy^4z^2 - 90xy^3z^3$$
$$+ 150xy^2z^4 - 99xyz^5 + 24xz^6 - 90y^4z^3 + 90y^3z^4 - 45y^2z^5 + 9yz^6\,.$$

Our main result is the next theorem.

THEOREM 2.2. *The character f_Ω for $\Omega = x\tilde{u} + y\tilde{v} + z\tilde{w}$ vanishes if and only if $F(x, y, z) = 0$. Hence the open subset of $H^2(M; \boldsymbol{R}) \simeq \boldsymbol{R}^3$ defined by $F(x, y, z) \neq 0$ does not contain any Kähler metric of constant scalar curvature.* (*See Remark* 3.2.)

REMARK 2.3. The group $\mathrm{Aut}(M)$ contains an $(m+n+1)$-dimensional algebraic torus. Hence M is toric and the character can be calculated also by the formula of Nakagawa [15].

3. Proof of the Theorem. Let $q \in M$, $q_m \in p^* H_m$, $q_n \in p^* H_n$ and $q_J \in J_M^*$ be points. Then the point q and the set (q_m, q_n, q_J) are expressed as follows:

$$\begin{aligned} q &= [(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)] \\ &= [(az_0, \dots, az_m), (bw_0, \dots, bw_n), (ca\eta_0, cb\eta_1)], \\ (q_m, q_n, q_J) &= [[(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)], h_m, h_n, \xi] \\ &= [[(az_0, \dots, az_m), (bw_0, \dots, bw_n), (ca\eta_0, cb\eta_1)], ah_m, bh_n, c\xi] \end{aligned}$$

for $a, b, c \in \boldsymbol{C}^*$.

REMARK 3.1. Since f_Ω vanishes on $[\mathfrak{h}(M), \mathfrak{h}(M)]$ and $\mathfrak{h}(M)/[\mathfrak{h}(M), \mathfrak{h}(M)]$ is represented by the vector field along the fiber $\boldsymbol{CP}^1$, the character f_Ω vanishes if and only if $f_\Omega(X) = 0$ for the vector field X along the fiber.

Now we assume that p is an odd prime number hereafter. Then an action of $\boldsymbol{Z}_p = \langle g_p \rangle \subset (\mathrm{GL}(m+1, \boldsymbol{C}) \times \mathrm{GL}(n+1, \boldsymbol{C}))/\boldsymbol{C}^*$ on M is defined by

$$\begin{aligned} & g_p \cdot [(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)] \\ & \qquad = [(z_0, \dots, z_m), (\alpha_p w_0, \dots, \alpha_p w_n), (\eta_0, \eta_1)]. \end{aligned} \tag{2}$$

This action naturally extends to an inclusion $\iota : \mathrm{U}(1) \to \mathrm{Aut}(M)$, which defines vector fields $X,\ X_p \in \mathfrak{h}(M)$ along the fiber as in (1) and we have $g_p = \exp(X_p)$. The fixed point set of g_p^k has the following two connected components

$$N_1 = [(z_0, \dots, z_m), (w_0, \dots, w_n), (1, 0)], \quad N_2 = [(z_0, \dots, z_m), (w_0, \dots, w_n), (0, 1)]$$

for $1 \le k \le p-1$, which are isomorphic to $\boldsymbol{CP}^m \times \boldsymbol{CP}^n$ and have cell decompositions with no codimension one cells. Let $\nu(N_i, M)$ be the normal bundle of N_i $(i = 1, 2)$ in M. Then, since

$$\begin{aligned} &[(z_0, \dots, z_m), (w_0, \dots, w_n), (1, \tau)] = [(az_0, \dots, az_m), (bw_0, \dots, bw_n), (1, a^{-1}b\tau)], \\ &g_p \cdot [(z_0, \dots, z_m), (w_0, \dots, w_n), (1, \tau)] = [(z_0, \dots, z_m), (w_0, \dots, w_n), (1, \alpha_p^{-1}\tau)], \end{aligned}$$

we have

$$\nu(N_1, M) \simeq H_m^{-1} \otimes H_n, \quad g_p|\nu(N_1, M) = g_p|(K_M^{-1}|_{N_1}) = \alpha_p^{-1}.$$

The same argument shows that

$$\nu(N_2, M) \simeq H_m \otimes H_n^{-1}, \quad g_p|\nu(N_2, M) = g_p|(K_M^{-1}|_{N_2}) = \alpha_p.$$

Hence it follows from the equality $c_1(K_M^{-1}|_{N_i}) = c_1(M)|_{N_i} = c_1(TN_i) + c_1(\nu(N_i, M))$ that

$$\begin{gathered} c_1(\nu(N_1, M)) = -u + v, \quad c_1(\nu(N_2, M)) = u - v, \\ c_1(K_M^{-1}|_{N_1}) = mu + (n+2)v, \quad c_1(K_M^{-1}|_{N_2}) = (m+2)u + nv, \end{gathered}$$

where $u = c_1(H_m),\ v = c_1(H_n)$. It is obvious that $\tilde{u}|_{N_i} = u$, $\tilde{v}|_{N_i} = v$ for $i = 1, 2$. Also, since

$$\begin{aligned} &[[(z_0, \dots, z_m), (w_0, \dots, w_n), (1, 0)], \xi] \\ &\qquad = [[(az_0, \dots, az_m), (bw_0, \dots, bw_n), (1, 0)], a^{-1}\xi], \end{aligned}$$

it follows that $\tilde{w}|_{N_1} = -u$. The same argument shows that $\tilde{w}|_{N_2} = -v$. Using the equalities above, we see that

$$c_1(M) = (m+2)\tilde{u} + (n+2)\tilde{v} + 2\tilde{w}\,,$$

and hence for $\Omega = x\tilde{u} + y\tilde{v} + z\tilde{w}$ it follows that

$$\mu_\Omega = \frac{m(m+2)yz + n(n+2)xz + 2xy}{(m+n+1)xyz}\,. \tag{3}$$

Let $\lambda,\ \mu,\ \nu$ be integers. Then $\Omega = \lambda\tilde{u} + \mu\tilde{v} + \nu\tilde{w}$ coincides with the first Chern class of the complex line bundle L defined by

$$L = p^*H_m^\lambda \otimes p^*H_n^\mu \otimes (J_M^*)^\nu\,.$$

The action (2) lifts to actions on $p^*H_m,\ p^*H_n,\ J_M^*$ as follows:

$$\begin{aligned} g_p \cdot [[(z_0,\ldots,z_m),(w_0,\ldots,w_n),(\eta_0,\eta_1)],h_m,h_n,\xi] \\ = [[(z_0,\ldots,z_m),(\alpha_p w_0,\ldots,\alpha_p w_n),(\eta_0,\eta_1)],h_m,h_n,\xi]\,. \end{aligned}$$

This action defines a lift of the action (2) to L and we can show that

$$\begin{gathered} g_p|(p^*H_m|_{N_i}) = 1\,,\quad g_p|(p^*H_n|_{N_i}) = \alpha_p^{-1}\quad (i=1,\ 2) \\ g_p|(J_M^*|_{N_1}) = 1\,,\quad g_p|(J_M^*|_{N_2}) = \alpha_p\,, \end{gathered}$$

and hence that

$$g_p|(L|_{N_1}) = \alpha_p^{-\mu}\,,\quad g_p|(L|_{N_2}) = \alpha_p^{-\mu+\nu}\,. \tag{4}$$

Using the results above, we have

$$\begin{aligned} T_i(k,\varepsilon,\zeta) = u^m v^n\text{-coeff. of} \\ \frac{1}{1-\alpha_p^k}\left(\alpha_p^{k(-\varepsilon r+\zeta\kappa)}e^{-\varepsilon(au+bv)+\zeta(\rho u+\tau v)} - 1\right)^{m+n+2} \\ \left(\frac{u}{1-e^{-u}}\right)^{m+1}\left(\frac{v}{1-e^{-v}}\right)^{n+1}\frac{1}{1-\alpha_p^{-k\delta}e^{-\delta(u-v)}}\,, \end{aligned}$$

where $r,\ \kappa,\ a,\ b,\ \rho,\ \tau,\ \delta$ are numbers determined by i as follows:

	r	κ	a	b	ρ	τ	δ
$i=1$	-1	$-\mu$	m	$n+2$	$\lambda-\nu$	μ	-1
$i=2$	1	$-\mu+\nu$	$m+2$	n	λ	$\mu-\nu$	1

Then, using the substitution $x = e^u - 1,\ y = e^v - 1$, we have

$$\begin{aligned}
T_i(k,\varepsilon,\zeta) &= u^{-1}v^{-1}\text{-coeff. of}\\
&\quad \frac{1}{1-\alpha_p^k}(\alpha_p^{k(-\varepsilon r+\zeta\kappa)}e^{u(\zeta\rho-\varepsilon a)}e^{v(\zeta\tau-\varepsilon b)}-1)^{m+n+2}\\
&\quad \left(\frac{e^u}{e^u-1}\right)^{m+1}\left(\frac{e^v}{e^v-1}\right)^{n+1}\frac{1}{1-\alpha_p^{-k\delta}e^{-\delta u}e^{\delta v}}\\
&= \left(\frac{1}{2\pi i}\right)^2\oint_{C(u)}\oint_{C(v)}\frac{1}{1-\alpha_p^k}(\alpha_p^{k(-\varepsilon r+\zeta\kappa)}e^{u(\zeta\rho-\varepsilon a)}e^{v(\zeta\tau-\varepsilon b)}-1)^{m+n+2}\\
&\qquad \frac{(e^u)^m}{(e^u-1)^{m+1}}\frac{(e^v)^n}{(e^v-1)^{n+1}}\frac{1}{1-\alpha_p^{-k\delta}e^{-\delta u}e^{\delta v}}e^u e^v\,dv\,du
\end{aligned}$$

(where $C(u),\ C(v)$ are sufficiently small counterclockwise loops around the origin)

$$\begin{aligned}
&= \left(\frac{1}{2\pi i}\right)^2\oint_{C(x)}\oint_{C(y)}\frac{1}{1-\alpha_p^k}(\alpha_p^{k(-\varepsilon r+\zeta\kappa)}(1+x)^{\zeta\rho-\varepsilon a}(1+y)^{\zeta\tau-\varepsilon b}-1)^{m+n+2}\\
&\qquad \frac{(1+x)^m}{x^{m+1}}\frac{(1+y)^n}{y^{n+1}}\frac{1}{1-\alpha_p^{-k\delta}(1+x)^{-\delta}(1+y)^{\delta}}\,dy\,dx
\end{aligned}$$

(where $C(x),\ C(y)$ are sufficiently small counterclockwise loops around the origin).

Here we set $\beta = \zeta\rho - \varepsilon a, \gamma = \zeta\tau - \varepsilon b$ and

$$\begin{aligned}
\Phi &= (1+x)^{-\delta}(1+y)^{\delta} - 1 = -\delta x + \delta y + Q(x,y)\,,\\
\Psi &= (1+x)^{\beta}(1+y)^{\gamma} - 1 = \beta x + \gamma y + R(x,y)\,,
\end{aligned}$$

where the total degrees of $Q(x,y),\ R(x,y)$ are greater than 1. Then we have

$$\begin{aligned}
&T_i(k,\varepsilon,\zeta)\\
&\quad = x^m y^n\text{-coeff. of}\\
&\qquad \frac{1}{1-\alpha_p^k}(\alpha_p^{k(\zeta\kappa-\varepsilon r)}-1+\alpha_p^{k(\zeta\kappa-\varepsilon r)}\Psi)^{m+n+2}(1+x)^m(1+y)^n(1-\alpha_p^{-k\delta}-\alpha_p^{-k\delta}\Phi)^{-1}\\
&\quad = x^m y^n\text{-coeff. of}\\
&\qquad \frac{1}{1-\alpha_p^k}\sum_{s=0}^{m+n}\binom{m+n+2}{s}(\alpha_p^{k(\zeta\kappa-\varepsilon r)}-1)^{m+n+2-s}\alpha_p^{ks(\zeta\kappa-\varepsilon r)}\Psi^s(1+x)^m(1+y)^n\\
&\qquad\qquad \sum_{j=0}^{m+n}\frac{\alpha_p^{-kj\delta}\Phi^j}{(1-\alpha_p^{-k\delta})^{j+1}}\\
&\quad = x^m y^n\text{-coeff. of}\\
&\qquad \sum_{s=0}^{m+n}\sum_{j=0}^{m+n-s}\binom{m+n+2}{s}(-1)\Lambda_j(\alpha_p^k)(1+x)^m(1+y)^n\Phi^j\Psi^s\,,
\end{aligned}$$

where $\Lambda_j(t)$ is an element of $\mathbf{Z}[t, t^{-1}]$ defined by

$$\Lambda_j(t) = \frac{t^{s(\zeta\kappa-\varepsilon r)+\delta}(t^{\zeta\kappa-\varepsilon r}-1)^{m+n+2-s}}{(t-1)(t^{\delta}-1)^{j+1}} .$$

Here, since

$$\sum_{k=1}^{p-1} \alpha_p^{kl} \equiv -1 \pmod{p}$$

for any integer l, we have

$$(-1)\sum_{k=1}^{p-1} \Lambda_j(\alpha_p^k) \equiv \Lambda_j(1) \pmod{p}$$
$$= \begin{cases} 0 & \text{if } j < m+n-s \\ \delta^{m+n-s+1}(\zeta\kappa-\varepsilon r)^{m+n+2-s} & \text{if } j = m+n-s \end{cases} .$$

Therefore we have

$$\begin{aligned}
&\sum_{k=1}^{p-1} T_i(k, \varepsilon, \zeta) \\
&\quad \equiv x^m y^n\text{-coeff. of} \\
&\qquad \sum_{s=0}^{m+n} \binom{m+n+2}{s} \delta^{m+n-s+1}(\zeta\kappa-\varepsilon r)^{m+n+2-s}(-\delta(x-y))^{m+n-s}(\beta x+\gamma y)^s \\
&\qquad\qquad \pmod{p} \\
&\quad = x^m y^n\text{-coeff. of} \\
&\qquad \sum_{s=0}^{m+n} \binom{m+n+2}{s} \delta^{m+n-s+1}(\zeta\kappa-\varepsilon r)^{m+n+2-s}(-\delta)^{m+n-s} \\
&\qquad \sum_{h=0}^{s} \binom{s}{h} \beta^h x^h \gamma^{s-h} y^{s-h} \sum_{q=0}^{m} \binom{m+n-s}{q} x^q (-y)^{m+n-s-q} \\
&\quad = \sum_{s=0}^{m+n} \sum_{q=0}^{m} \binom{m+n+2}{s} \binom{s}{m-q} \binom{m+n-s}{q} (-1)^q \\
&\qquad\qquad \delta(\kappa\zeta-r\varepsilon)^{m+n+2-s}(\rho\zeta-a\varepsilon)^{m-q}(\tau\zeta-b\varepsilon)^{s-m+q} ,
\end{aligned}$$

and hence it follows that

$$S_\varepsilon(\zeta) \equiv \frac{1}{p}\sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^q$$

$$\begin{pmatrix} (-1)^{m+n+s+1}(\mu\zeta-\varepsilon)^{m+n+2-s}((\lambda-\nu)\zeta-m\varepsilon)^{m-q}(\mu\zeta-(n+2)\varepsilon)^{s-m+q} \\ +((-\mu+\nu)\zeta-\varepsilon)^{m+n+2-s}(\lambda\zeta-(m+2)\varepsilon)^{m-q}((\mu-\nu)\zeta-n\varepsilon)^{s-m+q} \end{pmatrix} \pmod{\boldsymbol{Z}}$$

$$= \frac{1}{p}g(\lambda,\mu,\nu)\zeta^{m+n+2} - \varepsilon\frac{1}{p}h(\lambda,\mu,\nu)\zeta^{m+n+1} + \varphi(\zeta)\,,$$

where the degree of $\varphi(\zeta)$ is less than $m+n+1$.

Here for $f(x) = (\sinh x)^k$ we have

$$f(x) = \frac{1}{2^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}e^{(k-2i)x}\,, \qquad f(x) = x^k + \frac{k}{6}x^{k+2} + \text{higher order terms}$$

and hence it follows that

$$2^k f^{(l)}(0) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-2i)^l = \begin{cases} 0 & \text{if } 0 \le l < k \text{ or } l = k+1 \\ 2^k k! & \text{if } l = k \end{cases}\,.$$

Therefore it follows from (3) that

$$\begin{aligned}
&\lambda\mu\nu F_L(g_p) \\
&\quad = (m+n+2)\lambda\mu\nu \\
&\qquad \sum_{i=0}^{m+n+1}(-1)^i\binom{m+n+1}{i}(S_{-1}(m+n+1-2i) - S_{+1}(m+n+1-2i)) \\
&\qquad - (m(m+2)\mu\nu + n(n+2)\lambda\nu + 2\lambda\mu)\sum_{i=0}^{m+n+2}(-1)^i\binom{m+n+2}{i}S_0(m+n+2-2i) \\
&\quad \equiv \frac{2^{m+n+2}(m+n+2)!}{p}F(\lambda,\mu,\nu) \pmod{\boldsymbol{Z}}\,.
\end{aligned}$$

Hence, for any odd prime number p, it follows from Theorem 1.2 that

$$\begin{aligned}
\frac{1}{p}A(m,n)\lambda\mu\nu f_{\Omega(\lambda,\mu,\nu)}(X) &= A(m,n)\lambda\mu\nu f_{\Omega(\lambda,\mu,\nu)}(X_p) \\
&\equiv \frac{1}{p}2^{m+n+2}(m+n+2)!F(\lambda,\mu,\nu) \pmod{\boldsymbol{Z}}\,,
\end{aligned}$$

where $\Omega(\lambda,\mu,\nu) = \lambda\tilde{u} + \mu\tilde{v} + \nu\tilde{w}$, which implies that

$$A(m,n)\lambda\mu\nu f_{\Omega(\lambda,\mu,\nu)}(X) = 2^{m+n+2}(m+n+2)!F(\lambda,\mu,\nu)\,. \tag{5}$$

Now, since $\triangle_{k\omega} = k^{-1}\triangle_\omega$, it follows that $xyzf_{\Omega(x,y,z)}(X)$ is a homogeneous function in $x,\ y,\ z$ of degree $m+n+4$ as well as $F(x,y,z)$. Moreover, since the set

$$\{(x,y,z) \in \boldsymbol{R}^3\,;\ (rx,ry,rz) \in \boldsymbol{Z}^3 \text{ for some } r > 0\}$$

is dense in $\boldsymbol{R}^3$, the equality (5) implies that for any $(x, y, z) \in \boldsymbol{R}^3$

$$A(m, n)xyzf_{\Omega(x,y,z)}(X) = 2^{m+n+2}(m+n+2)!F(x, y, z) .$$

The result in Theorem 2.2 follows immediately from the equality above.

REMARK 3.2. Let $G = (U(m+1) \times U(n+1))/U(1)$ be the maximal compact subgroup of $\mathrm{Aut}(M)$ and $q = [(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)]$ a point in M. Then we can see that the real dimension of the isotropy subgroup of G at q is equal to m^2+n^2 if $\eta_0\eta_1 \neq 0$ and is equal to m^2+n^2+1 if $\eta_0\eta_1 = 0$, which implies that the real codimension of the principal orbit of G in M is one. Hence it follows from Corollary 1.1 in [8] that each Kähler class of M contains an extremal metric, and therefore it follows from [2, Theorem 4] (see also [5, Theorem 3.3.1]) that a Kähler class contains a Kähler metric of constant scalar curvature if the character for the Kähler class vanishes. Hence a Kähler class $\Omega = x\tilde{u} + y\tilde{v} + z\tilde{w}$ contains a Kähler metric of constant scalar curvature if and only if $F(x, y, z) = 0$. Moreover we can see that the $\mathrm{Aut}(M)$-orbit of q with $\eta_0\eta_1 \neq 0$ coincides with the open subset $M \setminus (N_1 \cup N_2)$ of M. Hence M is an almost-homogeneous manifold (see [9]) and therefore it follows from [8, Theorem 4] that M admits a Kähler metric of constant scalar curvature.

4. Examples. In this section, we consider the cases $1 \leq m < n \leq 10$. Since $F(x, y, z)$ is a homogeneous polynomial, $F(x, y, z)$ for $x,\ y,\ z > 0$ is determined by its restriction to the face f of a regular octahedron defined by

$$f = \{(x, y, z)\,;\ x + y + z = 1\,,\ x,\ y,\ z > 0\} .$$

Let C be a point in f defined by

$$C = \frac{1}{m+n+6}(m+2, n+2, 2)$$

and set $A = (1, 0, 0),\ B = (0, 1, 0)$. Then, since C is homothetic to $c_1(M) > 0$, C is a Kähler class and hence the interior of the triangle ABC is contained in the set of Kähler classes of M (see Remark 2.1). Let $l_1,\ l_2$ be lines in f defined by

$$l_1(t) = (x_1(t), y_1(t), z_1(t)) = (1-t)A + tC\ ,$$
$$l_2(t) = (x_2(t), y_2(t), z_2(t)) = (1-t)\left(\frac{1}{2}, \frac{1}{2}, 0\right) + t(0, 0, 1)$$

for $0 < t < 1$. Then we have

$$\begin{aligned}
&\lim_{t\to+0} F(l_1(t))/y_1(t)^{n+3} \\
&\quad = \lim_{t\to+0} \sum_{s=0}^{m+n}\sum_{q=0}^{m} \binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^{m+n+s+q+1}t^q \\
&\qquad\qquad 2(n+2)^{-n-2}(m+n+6)^{-q} \\
&\qquad\qquad \{((n+1)s+(n+2)q-m-mn-2n-n^2)(n+2)^{n+1+q} \\
&\qquad\qquad\quad -((n+1)s+nq-m-mn-2n-n^2-2)n^{n+1+q}\} \\
&\quad = \sum_{s=0}^{m+n}\binom{m+n+2}{s}\binom{s}{m}(-1)^{m+n+s+1}2(n+2)^{-n-2} \\
&\qquad\quad \{((n+1)s-m-mn-2n-n^2)(n+2)^{n+1} \\
&\qquad\quad -((n+1)s-m-mn-2n-n^2-2)n^{n+1}\}, \\
&\lim_{t\to+0} F(l_2(t))/z_2(t)^2 \\
&= \sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^{m+n+s+q+1}2^{-m-n-2} \\
&\qquad \{2(n-m)q^2+(2(n+1)s+(m^2-4mn-n^2-7m-3n-2))q \\
&\qquad +(-mn+n^2-m+2n+1)s+3m^2+m^2n-n^3-mn-4n^2-2m-4n\}.
\end{aligned}$$

Direct computation using the equalities above shows that

$$\lim_{t\to+0} F(l_1(t))/y_1(t)^{n+3} < 0\,, \quad \lim_{t\to+0} F(l_2(t))/z_2(t)^2 > 0\,,$$

which imply that there exist points P_1, P_2 in the interior of the triangle ABC such that $F(P_1) < 0$ and $F(P_2) > 0$. Therefore there exist infinitely many Kähler classes Ω such that f_Ω vanishes and hence that Ω contains a Kähler metric of constant scalar curvature (see Remark 3.2).

On the other hand, direct computation also shows that

$$F(m+2, n+2, 2) \neq 0\,,$$

which implies that $c_1(M)$ does not contain any Kähler metric of constant scalar curvature. This result shows that M does not admit any Kähler-Einstein metric. (See [3].)

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