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CONTINUITY PROPERTIES OF RIESZ POTENTIALS OF ORLICZ FUNCTIONS

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Abstract. In this paper we are concerned with Sobolev type inequalities for Riesz potentials of functions in Orlicz classes. As an application, we study continuity properties of Riesz potentials.

1. Introduction and statement of results. For $0 < \alpha < n$ and a locally integrable function f on $\boldsymbol{R}^n$, we define the Riesz potential $U_\alpha f$ of order α by

$$U_\alpha f(x) = \int_{\boldsymbol{R}^n} |x-y|^{\alpha-n} f(y)dy\,.$$

Here it is natural to assume that $U_\alpha|f| \not\equiv \infty$, which is equivalent to

$$\int_{\boldsymbol{R}^n} (1+|y|)^{\alpha-n}|f(y)|dy < \infty\,. \tag{1.1}$$

In the present paper, we treat functions f satisfying an Orlicz condition:

$$\int_{\boldsymbol{R}^n} \Phi_{p,\varphi}(|f(y)|)dy < \infty\,. \tag{1.2}$$

Here $\Phi_{p,\varphi}(r)$ is a positive nondecreasing function on the interval $(0,\infty)$ of the form

$$\Phi_{p,\varphi}(r) = r^p\varphi(r)\,,$$

where $p > 1$ and $\varphi(r)$ is a positive monotone function on $[0,\infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$$c_1^{-1}\varphi(r) \le \varphi(r^2) \le c_1\varphi(r) \quad \text{whenever } r > 0\,. \tag{φ1}$$

We set

$$\Phi_{p,\varphi}(0) = 0\,,$$

because we will see in the proof of Lemma 2.1 below that

$$\lim_{r\to 0+} \Phi_{p,\varphi}(r) = 0 = \Phi_{p,\varphi}(0)\,.$$

For an open set $G \subset \boldsymbol{R}^n$, we denote by $L^{\Phi_{p,\varphi}}(G)$ the family of all locally integrable functions g on G such that

$$\int_G \Phi_{p,\varphi}(|g(x)|)dx < \infty\,,$$

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and define

$$\|g\|_{\Phi_{p,\varphi}} = \|g\|_{\Phi_{p,\varphi},G} = \inf\left\{\lambda > 0\,;\, \int_G \Phi_{p,\varphi}(|g(x)|/\lambda)dx \le 1\right\}.$$

This is a quasi-norm in $L^{\Phi_{p,\varphi}}(G)$.

Our first aim in the present paper is to establish integral inequalities for Riesz potentials of functions in $L^{\Phi_{p,\varphi}}$. For this purpose, if $1 < p < n/\alpha$, then we set

$$\varphi_p^*(r) = \left[\int_0^r \{t^{\alpha p-n}\varphi(t)\}^{-p'/p} t^{-1} dt\right]^{1/p'} \quad \text{for } r \ge 0\,,$$

where $1/p + 1/p' = 1$; if $p = n/\alpha > 1$, then we set

$$\varphi_p^*(r) = \left[\int_1^r \{\varphi(t)\}^{-p'/p} t^{-1} dt\right]^{1/p'} \quad \text{for } r \ge 2\,,$$

and extend it to be a (strictly) increasing continuous function on $[0,\infty)$ such that $\varphi_p^*(t) = (t/2)\varphi_p^*(2)$ for $t \in [0, 2)$. Following Alberico and Cianchi [3], we consider the Sobolev conjugate $\Psi_{p,\varphi}$ of $\Phi_{p,\varphi}$ defined by

$$\Psi_{p,\varphi}(r) = (\psi_n \circ (\varphi_p^*)^{-1})(r) \quad \text{for } r \ge 0\,,$$

where $\psi_n(r) = r^n$ and $(\varphi_p^*)^{-1}$ is the inverse of the function φ_p^*. Note that $\Psi_{p,\varphi}(r)$ is continuous on $[0,\infty)$ and $\Psi_{p,\varphi}(0) = 0$.

As an extension of Alberico and Cianchi [3, Theorem 2.3], we state our first result in the following.

THEOREM A. *Let $\alpha p \le n$ and G be a bounded open set in $\mathbf{R}^n$. Then there exists $\varepsilon_0 > 0$ such that*

$$\int_G \Psi_{p,\varphi}(\varepsilon_0 U_\alpha |f|(x))dx \le 1$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,\varphi}} \le 1$.

Cianchi [2, Theorem 2] gave a necessary and sufficient condition that the operator $f \mapsto U_\alpha f$ is bounded from one Orlicz space L^Φ to another Orlicz space L^Ψ; but our statement is straightforward and simple. Further Edmunds and Evans [4, Theorems 3.6.10, 3.6.16] discussed the boundedness of Bessel potentials in Lorenz-Karamata space setting.

Since our function $\Phi_{p,\varphi}$ may not be convex, for the reader's convenience, we give a proof of Theorem A different from Cianchi [2] in the next section.

REMARK 1.1. Theorem A implies that

$$\|U_\alpha f\|_{\Psi_{p,\varphi}} \le \varepsilon_0^{-1}\|f\|_{\Phi_{p,\varphi}} \quad \text{whenever } f \in L^{\Phi_{p,\varphi}}(G)\,,$$

where the quasi-norm $\|\cdot\|_{\Psi_{p,\varphi}}$ is defined in the same way as $\|\cdot\|_{\Phi_{p,\varphi}}$.

EXAMPLE 1.2. Consider $\Phi_{p,q}(r) = r^p(\log r)^q$ for large $r > 0$, where $p = n/\alpha > 1$ and $q \le p-1$. If $q < p-1$, then

$$\Psi_{p,q}(r) \ge C\exp(nr^{p/(p-1-q)})$$

and if $q = p - 1$, then

$$\Psi_{p,q}(r) \geq C \exp(n \exp(r^{p'}))$$

for $r \geq 1$. Hence we have the exponential integrability obtained by Edmunds, Gurka and Opic [5, Theorem 4.6], [6, Theorems 3.1 and 3.2] and the authors [12, Theorems A and B].

COROLLARY 1.3. *Let $\alpha p = n$ and G be a bounded open set in $\boldsymbol{R}^n$. Let $\Phi_{p,q}(r) = r^p(\log r)^q$ for large $r > 0$.*

(1) *If $q < p - 1$, then there exists $\varepsilon_0 > 0$ such that*

$$\int_G \{\exp(\varepsilon_0 U_\alpha |f|(x)^\beta) - 1\} dx \leq 1$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,q}} \leq 1$, where $\beta = p/(p - 1 - q)$.

(2) *If $q = p - 1$, then there exists $\varepsilon_0 > 0$ such that*

$$\int_G \{\exp(\exp(\varepsilon_0 U_\alpha |f|(x)^\beta) - e\} dx \leq 1$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,q}} \leq 1$, where $\beta = p/(p-1)$.

In the case $q > p - 1$, $U_\alpha f$ is shown to be continuous in G; see Remark 1.5.

Denote by $p^\sharp$ the Sobolev conjugate of p which is defined by

$$\frac{1}{p^\sharp} = \frac{1}{p} - \frac{\alpha}{n} > 0\,.$$

We also obtain Sobolev's type inequality for Riesz potentials in the following:

COROLLARY 1.4. *Let $\alpha p < n$. Then*

$$\int_{\boldsymbol{R}^n} \{U_\alpha |f|(x) \varphi(U_\alpha |f|(x))^{1/p}\}^{p^\sharp} dx \leq C$$

whenever f is a locally integrable function on $\boldsymbol{R}^n$ such that $\|f\|_{\Phi_{p,\varphi}} \leq 1$, where C is a positive constant independent of f.

For a measurable function u on $\boldsymbol{R}^n$, we define the integral mean over a measurable set $E \subset \boldsymbol{R}^n$ of positive measure by

$$\fint_E u(x)dx = \frac{1}{|E|} \int_E u(x)dx\,.$$

As an application of Theorem A, we discuss continuity properties for Riesz potentials of functions in $L^{\Phi_{p,\varphi}}(\boldsymbol{R}^n)$, as an extension of Adams and Hurri-Syrjänen [1, Theorem 1.6] and the authors [14, Theorems A and B].

Our main result is now stated as follows:

THEOREM B. *Let f be a locally integrable function on $\boldsymbol{R}^n$ satisfying* (1.1) *and* (1.2). *Set*

$$E_\infty = \left\{ x \in \mathbf{R}^n \,;\, \int_{\boldsymbol{R}^n} |x-y|^{\alpha-n} |f(y)| dy = \infty \right\},$$

$$E_* = \left\{ x \in \boldsymbol{R}^n \,;\, \limsup_{r \to 0} r^{\alpha p - n} \varphi(r^{-1})^{-1} \int_{B(x,r)} \Phi_{p,\varphi}(|f(y)|) dy > 0 \right\},$$

$$E^* = \left\{ x \in \boldsymbol{R}^n \,;\, \limsup_{r \to 0} r^{\alpha p - n} \int_{B(x,r)} \Phi_{p,\varphi}(|f(y)|) dy > 0 \right\}.$$

If $x_0 \in \boldsymbol{R}^n \setminus (E_\infty \cup E_* \cup E^*)$, then

$$\lim_{r \to 0+} \fint_{B(x_0,r)} \Psi_{p,\varphi}(A|U_\alpha f(x) - U_\alpha f(x_0)|) dx = 0 \tag{1.3}$$

holds for all $A > 0$.

We discuss the size of the exceptional sets after proving this theorem, in the final section.

REMARK 1.5. Suppose

$$\int_1^\infty \{t^{\alpha p - n} \varphi(t)\}^{-p'/p} t^{-1} dt < \infty \tag{1.4}$$

and set

$$\varphi_p(r) = \left(\int_r^\infty \{t^{\alpha p - n} \varphi(t)\}^{-p'/p} t^{-1} dt \right)^{1/p'}.$$

Then it is known (see [9, Theorem 1] and [10, Corollary 3.1]) that $U_\alpha f$ is continuous on $\boldsymbol{R}^n$ and

$$|U_\alpha f(x) - U_\alpha f(x_0)| = o(\varphi_p(1/|x - x_0|)) \quad \text{as } x \to x_0$$

for all $x_0 \in \boldsymbol{R}^n$, whenever f satisfies (1.1) and (1.2). On the contrary, if (1.4) does not hold, then we can find an f satisfying (1.1) and (1.2) such that $U_\alpha f$ is not continuous (see [13, Remark 3.3]).

2. Proof of Theorem A.

In spite of the fact that $\Phi_{p,\varphi}$ may not be convex, Theorem A must be a consequence of Cianchi [2] in spirit. But we here give a proof of Theorem A, because our method is straightforward and several materials are also needed for a proof of our main Theorem B. In fact, our proof is based on the boundedness of maximal functions, by use of the methods in the paper by Hedberg [7].

Throughout this paper, let C, C_1, $C_2, \dots$ denote various constants independent of the variables in question.

First we collect properties which follow from condition (φ1) (see [11] and [13]).

(φ2) φ satisfies the doubling condition, that is, there exists $c > 1$ such that

$$c^{-1}\varphi(r) \le \varphi(2r) \le c\varphi(r) \quad \text{whenever } r > 0.$$

(φ3) For each $\gamma > 0$, there exists $c = c(\gamma) \geq 1$ such that

$$c^{-1}\varphi(r) \leq \varphi(r^{\gamma}) \leq c\varphi(r) \quad \text{whenever } r > 0 .$$

(φ4) If $\gamma > 0$, then there exists $c = c(\gamma) \geq 1$ such that

$$s^{\gamma}\varphi(s) \leq ct^{\gamma}\varphi(t) \quad \text{whenever } 0 < s < t .$$

(φ5) If $\gamma > 0$, then there exists $c = c(\gamma) \geq 1$ such that

$$t^{-\gamma}\varphi(t) \leq cs^{-\gamma}\varphi(s) \quad \text{whenever } 0 < s < t .$$

LEMMA 2.1. *Let $1 < p_1 < p < p_2$. Then there exists $C > 1$ such that*

$$C^{-1}A^{p_1}\Phi_{p,\varphi}(r) \leq \Phi_{p,\varphi}(Ar) \leq CA^{p_2}\Phi_{p,\varphi}(r)$$

whenever $r > 0$ and $A > 1$.

PROOF. Let $0 < \varepsilon < 1$. In view of (φ4) and (φ5), we can find $C = C(\varepsilon)$ such that if $0 < r_1 < r_2$, then

$$C^{-1}\left(\frac{r_2}{r_1}\right)^{-\varepsilon} \leq \frac{\varphi(r_2)}{\varphi(r_1)} \leq C\left(\frac{r_2}{r_1}\right)^{\varepsilon},$$

so that

$$C^{-1}\left(\frac{r_2}{r_1}\right)^{p-\varepsilon} \leq \frac{\Phi_{p,\varphi}(r_2)}{\Phi_{p,\varphi}(r_1)} \leq C\left(\frac{r_2}{r_1}\right)^{p+\varepsilon},$$

which proves the lemma. □

COROLLARY 2.2. *Let $\alpha p \leq n$ and $1 < p_1 < p < p_2$. Let G be a bounded open set in $\boldsymbol{R}^n$. Then there exists a positive constant C such that*

$$C^{-1}\{\|f\|_{\Phi_{p,\varphi}}\}^{p_2} \leq \int_G \Phi_{p,\varphi}(|f(y)|)dy \leq C\{\|f\|_{\Phi_{p,\varphi}}\}^{p_1}$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,\varphi}} \leq 1$.

PROOF. Let $0 < \mu < \|f\|_{\Phi_{p,\varphi}} \leq 1$. Then we have by Lemma 2.1

$$\begin{aligned} 1 &< \int_G \Phi_{p,\varphi}(|f(y)/\mu|)dy \\ &\leq C\mu^{-p_2}\int_G \Phi_{p,\varphi}(|f(y)|)dy , \end{aligned}$$

so that

$$\int_G \Phi_{p,\varphi}(|f(y)|)dy \geq C^{-1}\mu^{p_2} .$$

This gives the left inequality in the present corollary. The right inequality can be proved similarly. □

LEMMA 2.3 (cf. [13, Lemma 2.5]). *Let G be a bounded open set in $\boldsymbol{R}^n$ and $\varepsilon > 0$. Let p_0 be given so that $p_0 = p$ if φ is nondecreasing, and $1 < p_0 < p$ if φ is nonincreasing. If $x \in G$, $\delta > 0$ and f is a nonnegative measurable function on G, then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\varphi_p^*(\delta^{-1})\left\{\varepsilon + c(\varepsilon)\left(\int_G \Phi_{p,\varphi}(f(y))dy\right)^{1/p_0}\right\},$$

where C and $c(\varepsilon)$ are positive constants such that C is independent of ε but $c(\varepsilon)$ may depend on ε. In case $\alpha p < n$,

$$\int_{\boldsymbol{R}^n\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\varphi_p^*(\delta^{-1})\left\{1 + \left(\int_{\boldsymbol{R}^n} \Phi_{p,\varphi}(f(y))dy\right)^{1/p_0}\right\}$$

for all $x \in \boldsymbol{R}^n$ and nonnegative measurable functions f on $\boldsymbol{R}^n$.

For a locally integrable function f on $\boldsymbol{R}^n$, define the maximal function by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{G\cap B(x,r)} |f(y)|dy,$$

where $|B(x,r)|$ denotes the n-dimensional Lebesgue measure of the ball $B(x,r)$ centered at x of radius $r > 0$.

We denote by $c(\varepsilon)$ various constants which may depend on ε.

LEMMA 2.4. *Let $\alpha p = n$ and G be a bounded open set in $\boldsymbol{R}^n$. Then, for each $\eta > 0$, there exist $\varepsilon_0 > 0$ and $c(\varepsilon_0) > 0$ such that*

$$(\varphi_p^*)^{-1}(U_\alpha f(x)) \le c(\varepsilon_0)\{\Phi_{p,\varphi}(Mf(x))\}^{1/n} + \eta$$

for all nonnegative measurable functions f on G satisfying $\int_G \Phi_{p,\varphi}(f(y))dy \le \varepsilon_0$.

PROOF. For a nonnegative measurable function f on G, set

$$F = \left(\int_G \Phi_{p,\varphi}(f(y))dy\right)^{1/p_0},$$

where p_0 is given in Lemma 2.3. Let $0 < \varepsilon < 1$. Then, noting that

$$\int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^\alpha Mf(x),$$

we have by Lemma 2.3

$$U_\alpha f(x) \le C\delta^\alpha Mf(x) + C\{\varepsilon + c(\varepsilon)F\}\varphi_p^*(\delta^{-1}). \tag{2.1}$$

First consider the case when $Mf(x) \ge 1$. Then, considering

$$\delta = \varepsilon^{1/\alpha} Mf(x)^{-1/\alpha}\{\varphi_p^*(Mf(x))\}^{1/\alpha}$$

and noting that $\varphi_p^*(r^2) \le C\varphi_p^*(r)$ for $r > 0$, we see that

$$\begin{aligned} U_\alpha f(x) &\le C\varepsilon\varphi_p^*(Mf(x)) + C\{\varepsilon + c(\varepsilon)F\}\varphi_p^*(\varepsilon^{-1/\alpha}Mf(x)^{1/\alpha}\varphi_p^*(1)^{-1/\alpha}) \\ &\le \{C\varepsilon + c(\varepsilon)F\}\varphi_p^*((\varepsilon^{-1}Mf(x))^{1/\alpha}) \\ &\le \{C\varepsilon + c(\varepsilon)F\}\varphi_p^*((\varepsilon^{-1}Mf(x))^{1/(2\alpha)}). \end{aligned}$$

Then (φ4) gives

$$\{\varepsilon^{-1} Mf(x)\}^{1/(2\alpha)} \le c(\varepsilon)\{Mf(x)\}^{1/\alpha}\{\varphi(Mf(x))\}^{1/n},$$

so that

$$U_\alpha f(x) \le \{C\varepsilon + c(\varepsilon)F\}\varphi_p^*(c(\varepsilon)Mf(x)^{1/\alpha}\varphi(Mf(x))^{1/n}).$$

If

$$C\varepsilon + c(\varepsilon)F \le 1, \tag{2.2}$$

then we have

$$U_\alpha f(x) \le \varphi_p^*(c(\varepsilon)Mf(x)^{1/\alpha}\varphi(Mf(x))^{1/n}),$$

which implies

$$(\varphi_p^*)^{-1}(U_\alpha f(x)) \le c(\varepsilon)\{Mf(x)\}^{p/n}\{\varphi(Mf(x))\}^{1/n}.$$

If $Mf(x) \le 1$, then we take $\delta = \varepsilon$ in (2.1) to see that

$$\begin{aligned} U_\alpha f(x) &\le C\varepsilon^\alpha + C\{\varepsilon + c(\varepsilon)F\}\varphi_p^*(\varepsilon^{-1}) \\ &\le C\varepsilon^\alpha + C\varepsilon\varphi_p^*(\varepsilon^{-1}) + c(\varepsilon)F. \end{aligned}$$

For given $\eta > 0$, if

$$C\varepsilon^\alpha + C\varepsilon\varphi_p^*(\varepsilon^{-1}) < \varphi_p^*(\eta)/2, \tag{2.3}$$

then we find

$$U_\alpha f(x) < \varphi_p^*(\eta)$$

whenever

$$c(\varepsilon)F < \varphi_p^*(\eta)/2. \tag{2.4}$$

Now it follows that

$$(\varphi_p^*)^{-1}(U_\alpha f(x)) \le c(\varepsilon)\{Mf(x)\}^{p/n}\{\varphi(Mf(x))\}^{1/n} + \eta$$

when (2.2), (2.3) and (2.4) are all satisfied, which completes the proof of the present lemma. □

In case $\alpha p < n$, we find from (φ4) and (φ5) that

$$C^{-1}r^{(n-\alpha p)/p}\{\varphi(r)\}^{-1/p} \le \varphi_p^*(r) \le Cr^{(n-\alpha p)/p}\{\varphi(r)\}^{-1/p}, \tag{2.5}$$

so that

$$C^{-1}r^{p/(n-\alpha p)}\{\varphi(r)\}^{1/(n-\alpha p)} \le (\varphi_p^*)^{-1}(r) \le Cr^{p/(n-\alpha p)}\{\varphi(r)\}^{1/(n-\alpha p)} \tag{2.6}$$

for $r > 0$.

LEMMA 2.5. *Let $\alpha p < n$. Then*

$$(\varphi_p^*)^{-1}(U_\alpha f(x)) \le C\{\Phi_{p,\varphi}(Mf(x))\}^{1/n}$$

for all nonnegative measurable functions f on $\boldsymbol{R}^n$ satisfying $\int_{\boldsymbol{R}^n} \Phi_{p,\varphi}(f(y))dy \le 1$.

PROOF. Let f be a nonnegative measurable function on $\boldsymbol{R}^n$ satisfying

$$\int \Phi_{p,\varphi}(f(y))dy \le 1\,.$$

As in the proof of Lemma 2.4, we have by Lemma 2.3

$$U_\alpha f(x) \le C\delta^\alpha Mf(x) + C\varphi_p^*(\delta^{-1})\,.$$

Hence it follows from (2.5) that

$$U_\alpha f(x) \le C\delta^\alpha Mf(x) + C\delta^{-(n-\alpha p)/p}\{\varphi(\delta^{-1})\}^{-1/p}$$

because $\alpha p < n$ by our assumption. Considering $\delta = \{Mf(x)\}^{-p/n}\{\varphi(Mf(x))\}^{-1/n}$, we see that

$$U_\alpha f(x) \le C\{Mf(x)\}^{1-\alpha p/n}\{\varphi(Mf(x))\}^{-\alpha/n}\,.$$

Since

$$(\varphi_p^*)^{-1}(r) \le Cr^{p/(n-\alpha p)}\{\varphi(r)\}^{1/(n-\alpha p)} = C\{r\varphi(r)^{1/p}\}^{p^\sharp/n} \tag{2.7}$$

by (2.6), we have by (φ3)

$$\begin{aligned}(\varphi_p^*)^{-1}(U_\alpha f(x)) &\le (\varphi_p^*)^{-1}(CMf(x)^{1-\alpha p/n}\,(\varphi(Mf(x)))^{-\alpha/n})\\ &\le C\{Mf(x)\}^{p/n}\{\varphi(Mf(x))\}^{-\alpha p/\{n(n-\alpha p)\}+1/(n-\alpha p)}\\ &= C\{Mf(x)\}^{p/n}\{\varphi(Mf(x))\}^{1/n}\,,\end{aligned}$$

as required. □

Note that

$$C^{-1}\frac{\Phi_{p,\varphi}(t)}{t} \le \int_0^t s^{-1}d\Phi_{p,\varphi}(s) \le C\frac{\Phi_{p,\varphi}(t)}{t} \tag{2.8}$$

for all $t > 0$ by (φ4) and (φ5).

The next lemma is an extension of Stein [15, Chapter 1], whose proof will be done along the same lines as in Stein [15, Chapter 1].

LEMMA 2.6. *For a locally integrable function f on $\boldsymbol{R}^n$,*

$$\int \Phi_{p,\varphi}(Mf(x))dx \le C\int \Phi_{p,\varphi}(|f(x)|)dx\,.$$

PROOF. Note that

$$\int \Phi_{p,\varphi}(Mf(x))dx = \int_0^\infty \lambda(t)d\Phi_{p,\varphi}(t)\,,$$

where $\lambda(t) = |\{x \in \boldsymbol{R}^n\,;\, Mf(x) > t\}|$. It follows from [15, Theorem 1, Chapter 1] that

$$\lambda(t) \le Ct^{-1}\int_{\{x\in\boldsymbol{R}^n;|f(x)|>t/2\}} |f(x)|dx$$

for $t > 0$. Hence we obtain by Fubini's Theorem and (2.8)

$$\int \Phi_{p,\varphi}(Mf(x))dx \le C \int_{\{x\in\mathbf{R}^n;|f(x)|>1/2\}} |f(x)| \left\{ \int_0^{2|f(x)|} t^{-1} d\Phi_{p,\varphi}(t) \right\} dx$$
$$\le C \int \Phi_{p,\varphi}(|f(x)|)dx.$$

Thus Lemma 2.6 is proved. □

PROOF OF THEOREM A. We give a proof of Theorem A only in case $\alpha p = n$. With the aid of Lemma 2.4, for $\eta > 0$ we find $\varepsilon_1 > 0$ such that

$$(\varphi_p^*)^{-1}(U_\alpha f(x)) \le C(\varepsilon_1)\{\Phi_{p,\varphi}(Mf(x))\}^{1/n} + \eta$$

for all nonnegative measurable functions f on G satisfying $\int_G \Phi_{p,\varphi}(f(y))dy \le \varepsilon_1$. Hence, in view of Lemma 2.6, we obtain

$$\int_G \Psi_{p,\varphi}(U_\alpha f(x))dx \le C(\varepsilon_1) \int_G \Phi_{p,\varphi}(Mf(x))dx + C\eta^n|G|$$
$$\le C(\varepsilon_1) \int_G \Phi_{p,\varphi}(f(y))dy + C\eta^n|G|$$

for all nonnegative measurable functions f on G satisfying $\int_G \Phi_{p,\varphi}(f(y))dy \le \varepsilon_1$. Now, letting $C\eta^n|G| \le 1/2$ and using Corollary 2.2, we find $0 < \varepsilon_0 < \varepsilon_1$ such that

$$\int_G \Psi_{p,\varphi}(U_\alpha f(x))dx \le 1$$

for all nonnegative measurable functions f on G satisfying $\|f\|_{\Phi_{p,\varphi}} \le \varepsilon_0$. This implies that

$$\int_G \Psi_{p,\varphi}(\varepsilon_0 U_\alpha f(x))dx \le 1$$

for all nonnegative measurable functions f on G satisfying $\|f\|_{\Phi_{p,\varphi}} \le 1$. Now the proof is completed. □

PROOF OF COROLLARY 1.4. Let $\alpha p < n$. Lemma 2.5 and (2.6) imply that

$$\{U_\alpha f(x)\varphi(U_\alpha f(x))^{1/p}\}^{p^\sharp} \le C\Phi_{p,\varphi}(Mf(x))$$

for all nonnegative measurable functions f on $\mathbf{R}^n$ satisfying $\int \Phi_{p,\varphi}(f(y))dy \le 1$. Hence we obtain by Lemma 2.6,

$$\int \{U_\alpha f(x)\varphi(U_\alpha f(x))^{1/p}\}^{p^\sharp} dx \le C \int \Phi_{p,\varphi}(Mf(x))dx \le C \int \Phi_{p,\varphi}(f(y))dy \le C$$

for all nonnegative measurable functions f on $\mathbf{R}^n$ satisfying $\int \Phi_{p,\varphi}(f(y))dy \le 1$, which proves the corollary. □

3. Proof of Theorem B. For a proof of Theorem B, we prepare a series of lemmas.

LEMMA 3.1. *Let $\alpha p \le n$. Then there exist $\beta > 1$ and $C > 0$ such that*

$$\varphi_p^*(Ar) \le CA^\beta \varphi_p^*(r)$$

for all $r > 0$ and $A > 2$.

PROOF. First note that for $r \ge 2$ and $A > 2$

$$\begin{aligned}\varphi_p^*(Ar) &\le C\left[\int_1^A \{t^{\alpha p-n}\varphi(t)\}^{-p'/p}t^{-1}dt\right]^{1/p'} + \left[\int_1^r \{(At)^{\alpha p-n}\varphi(At)\}^{-p'/p}t^{-1}dt\right]^{1/p'} \\ &= I_1 + I_2 .\end{aligned}$$

Since $\varphi(At)^{-1} \le CA^k\varphi(t)^{-1}$ with $k = \log_2 c$, c being the constant appearing in doubling property ($\varphi 2$), we have

$$I_2 \le CA^\gamma \varphi_p^*(r)$$

for $r \ge 2$ with $\gamma = (n - \alpha p + k)/p$. Similarly, since $\varphi(t)^{-1} \le CA^k\varphi(1)^{-1}$ when $1 \le t \le A$, we see that

$$I_1 \le CA^\gamma (\log A)^{1/p'} \varphi(1)^{-1/p} ,$$

so that

$$\varphi_p^*(Ar) \le CA^\gamma(\log A)^{1/p'}\varphi(1)^{-1/p} + CA^\gamma\varphi_p^*(r) \le CA^{\gamma+1/p'}\varphi_p^*(r)$$

for $r \ge 2$.

If $0 < r \le 2/A$ with $A > 2$, then

$$\varphi_p^*(Ar) \le CA^\delta \varphi_p^*(r) ,$$

where $\delta = (n - \alpha p)/p$ when $\alpha p < n$ and $\delta = 1$ when $\alpha p = n$. Finally, if $2/A < r < 2$ with $A > 2$, then

$$\varphi_p^*(Ar) \le \varphi_p^*(2A) \le CA^{\gamma+1/p'}\varphi_p^*(2) \le CA^{\gamma+1/p'+\delta}\varphi_p^*(2/A) \le CA^{\gamma+1/p'+\delta}\varphi_p^*(r) .$$

Thus the proof is completed. □

With the aid of Lemma 3.1, we establish the following result.

LEMMA 3.2. *Let G be a bounded open set in $\boldsymbol{R}^n$. Then there exist $C > 1$ and $0 < \varepsilon_0 < 1$ such that*

$$\int_G \Psi_{p,\varphi}(U_\alpha|f|(y))dy \le C\{\|f\|_{\Phi_{p,\varphi}}\}^{n/\beta}$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,\varphi}} \le \varepsilon_0$, where β is given in Lemma 3.1.

PROOF. By Theorem A we have

$$\int_G \Psi_{p,\varphi}(\varepsilon_0 F^{-1} U_\alpha|f|(y))dy \le 1$$

when $F = \|f\|_{\Phi_{p,\varphi}}$. Lemma 3.1 implies that

$$\Psi_{p,\varphi}(\varepsilon_0 F^{-1} t) \geq (\varepsilon_0 F^{-1}/C)^{n/\beta} \Psi_{p,\varphi}(t)$$

when $\varepsilon_0 F^{-1}/C > 2^\beta$. Hence

$$\int_G \Psi_{p,\varphi}(U_\alpha |f|(y))dy \leq (\varepsilon_0 F^{-1}/C)^{-n/\beta} \leq C\varepsilon_0^{-n/\beta} F^{n/\beta}$$

whenever $F < \varepsilon_0/(2^\beta C)$. □

We further need the following result.

LEMMA 3.3. *Let $\alpha p \leq n$. For a nonnegative measurable function f on $\boldsymbol{R}^n$ satisfying* (1.2), *set*

$$E_* = \left\{ x \in \boldsymbol{R}^n\,;\, \limsup_{r\to 0+} r^{\alpha p - n} \varphi(r^{-1})^{-1} \int_{B(x,r)} \Phi_{p,\varphi}(f(y))dy > 0 \right\}$$

and

$$E^* = \left\{ x \in \boldsymbol{R}^n\,;\, \limsup_{r\to 0+} r^{\alpha p - n} \int_{B(x,r)} \Phi_{p,\varphi}(f(y))dy > 0 \right\}.$$

If $x_0 \in \boldsymbol{R}^n \setminus (E_* \cup E^*)$, then

$$\lim_{r\to 0} r^{-n} \int_{B(x_0,r)} \Phi_{p,\varphi}(r^\alpha f(y))dy = 0\,.$$

PROOF. If φ is nondecreasing, then for $0 < r < 1$,

$$\Phi_{p,\varphi}(r^\alpha f(y)) = (r^\alpha f(y))^p \varphi(r^\alpha f(y)) \leq (r^\alpha f(y))^p \varphi(f(y)) = r^{\alpha p} \Phi_{p,\varphi}(f(y))\,,$$

so that

$$r^{-n} \int_{B(x_0,r)} \Phi_{p,\varphi}(r^\alpha f(y))dy \leq r^{\alpha p - n} \int_{B(x_0,r)} \Phi_{p,\varphi}(f(y))dy\,.$$

Now suppose φ is nonincreasing, and hence φ is bounded. If $r^\alpha f(y) \leq f(y)^{1/2}$, then $f(y) \leq r^{-2\alpha}$, so that

$$\begin{aligned}
\Phi_{p,\varphi}(r^\alpha f(y)) &= (r^\alpha f(y))^p \varphi(r^\alpha f(y)) \\
&\leq C(r^\alpha f(y))^p \\
&\leq C(r^\alpha f(y))^p \frac{\varphi(f(y))}{\varphi(r^{-1})} \\
&= C r^{\alpha p} \varphi(r^{-1})^{-1} \Phi_{p,\varphi}(f(y))
\end{aligned}$$

and if $r^\alpha f(y) \geq f(y)^{1/2}$, then

$$\begin{aligned}
\Phi_{p,\varphi}(r^\alpha f(y)) &= (r^\alpha f(y))^p \varphi(r^\alpha f(y)) \\
&\leq C(r^\alpha f(y))^p \varphi(f(y)) \\
&\leq C r^{\alpha p} \varphi(r^{-1})^{-1} \Phi_{p,\varphi}(f(y))
\end{aligned}$$

for $0 < r < 1$. Hence it follows that

$$r^{-n}\int_{B(x_0,r)}\Phi_{p,\varphi}(r^\alpha f(y))dy \le Cr^{\alpha p-n}\varphi(r^{-1})^{-1}\int_{B(x_0,r)}\Phi_{p,\varphi}(f(y))dy\,.$$

Now the required result follows. □

For $x_0 \in \boldsymbol{R}^n$ and $r > 0$, set $f_{x_0,r}(w) = r^\alpha f(x_0+rw)\chi_{B(0,1)}$, where χ_E denotes the characteristic function of E. Then note that

$$\int_{B(x_0,r)}|x-y|^{\alpha-n}f(y)dy = \int_{B(0,1)}|z-w|^{\alpha-n}(r^\alpha f(x_0+rw))dw \\ = U_\alpha f_{x_0,r}(z) \tag{3.1}$$

for $x = x_0 + rz$.

We are now ready to prove our main Theorem B.

PROOF OF THEOREM B. For a nonnegative measurable function f on $\boldsymbol{R}^n$ satisfying (1.1) and (1.2), it suffices to show that (1.3) holds for $x_0 \in \boldsymbol{R}^n \setminus (E_\infty \cup E_* \cup E^*)$. Write

$$\begin{aligned} U_\alpha f(x) - U_\alpha f(x_0) &= \int_{B(x_0,2|x-x_0|)}|x-y|^{\alpha-n}f(y)dy \\ &\quad + \int_{\boldsymbol{R}^n\setminus B(x_0,2|x-x_0|)}|x-y|^{\alpha-n}f(y)dy - U_\alpha f(x_0) \\ &= U_1(x) + U_2(x)\,. \end{aligned}$$

If $y \in \mathbf{R}^n \setminus B(x_0, 2|x-x_0|)$, then $|x_0 - y| \le 2|x-y|$, so that, since $U_\alpha f(x_0) < \infty$, we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x\to x_0} U_2(x) = 0\,. \tag{3.2}$$

Since $(\varphi_p^*)^{-1}$ is nondecreasing, we have

$$\begin{aligned} (\varphi_p^*)^{-1}(A|U_\alpha f(x) - U_\alpha f(x_0)|) &\le (\varphi_p^*)^{-1}(AU_1(x) + A|U_2(x)|) \\ &\le (\varphi_p^*)^{-1}(2AU_1(x)) + (\varphi_p^*)^{-1}(2A|U_2(x)|)\,, \end{aligned}$$

so that

$$\begin{aligned} \Psi_{p,\varphi}(A|U_\alpha f(x) - U_\alpha f(x_0)|) &\le C\psi_n((\varphi_p^*)^{-1}(2AU_1(x))) + C\psi_n((\varphi_p^*)^{-1}(2A|U_2(x)|)) \\ &= C\Psi_{p,\varphi}(2AU_1(x)) + C\Psi_{p,\varphi}(2A|U_2(x)|)\,. \end{aligned}$$

In view of (3.2), we have

$$\lim_{x\to x_0}\Psi_{p,\varphi}(2A|U_2(x)|) = 0\,.$$

Note that

$$U_1(x) \le \int_{B(x_0,r)}|x-y|^{\alpha-n}f(y)dy = U_\alpha f_r(x)$$

for $x \in B(x_0, r/2)$, where $f_r = f\chi_{B(x_0,r)}$. Hence, we have only to show that

$$\lim_{r\to 0+} ⨍_{B(x_0,r)} \Psi_{p,\varphi}(2AU_\alpha f_r(x))dx = 0\,.$$

Note that $U_\alpha(f_r)(x) = U_\alpha(f_r)_{x_0,r}(z)$ for $x = x_0 + rz$ and

$$\int_{B(0,1)} \Phi_{p,\varphi}((f_r)_{x_0,r}(w))dw = r^{-n}\int_{B(x_0,r)} \Phi_{p,\varphi}(r^\alpha f(y))dy$$

which tends to zero as $r \to +0$ by Lemma 3.3. Hence we have by Lemma 3.2 and Corollary 2.2

$$\begin{aligned}
⨍_{B(x_0,r)} \Psi_{p,\varphi}(2AU_1(x))dx &\le ⨍_{B(0,1)} \Psi_{p,\varphi}(U_\alpha(2A(f_r)_{x_0,r})(z))dz \\
&\le C\{\|2A(f_r)_{x_0,r}\|_{\Phi_{p,\varphi}}\}^{n/\beta} \\
&\le C(2A)^{n/\beta}\left(\int_{B(0,1)} \Phi_{p,\varphi}((f_r)_{x_0,r}(z))dz\right)^{n/(p_2\beta)} \\
&\le C(2A)^{n/\beta}\left(r^{-n}\int_{B(x_0,r)} \Phi_{p,\varphi}(r^\alpha f(y))dy\right)^{n/(p_2\beta)} .
\end{aligned}$$

Consequently it follows from Lemma 3.3 that the left-hand side tends to zero as $r \to 0+$. Thus the proof is completed. □

4. Size of exceptional sets. To evaluate the size of exceptional sets in Theorem B, we introduce the notion of capacity. For a set $E \subset \boldsymbol{R}^n$ and an open set $G \subset \boldsymbol{R}^n$, we define

$$C_{\alpha,\Phi_{p,\varphi}}(E; G) = \inf_f \int_G \Phi_{p,\varphi}(f(y))dy\,,$$

where the infimum is taken over all nonnegative measurable functions f on $\boldsymbol{R}^n$ such that f vanishes outside G and $U_\alpha f(x) \ge 1$ for every $x \in E$ (cf. Meyers [8] and the first author [11]). When $\varphi \equiv 1$, we write $C_{\alpha,p}$ for $C_{\alpha,\Phi_{p,\varphi}}$. We say that E is of $C_{\alpha,\Phi_{p,\varphi}}$-capacity zero, written as $C_{\alpha,\Phi_{p,\varphi}}(E) = 0$, if

$$C_{\alpha,\Phi_{p,\varphi}}(E \cap G; G) = 0 \quad \text{for every bounded open set } G.$$

The following can be obtained readily from the definition of $C_{\alpha,\Phi_{p,\varphi}}$; see [11, Theorem 1.1, Chapter 2].

LEMMA 4.1. *For a nonnegative measurable function f on $\boldsymbol{R}^n$ satisfying* (1.1) *and* (1.2), *set*

$$E_\infty = \left\{x \in \boldsymbol{R}^n\,;\, \int |x-y|^{\alpha-n} f(y)dy = \infty\right\}.$$

Then

$$C_{\alpha,\Phi_{p,\varphi}}(E_\infty) = 0\,.$$

As in the proof of Lemma 7.3 and Corollary 7.2 in [10], we can prove the following results.

LEMMA 4.2. *Let $\alpha p \le n$. For a nonnegative measurable function f on $\boldsymbol{R}^n$ satisfying* (1.2), *set*

$$E_* = \left\{ x \in \boldsymbol{R}^n \, ; \limsup_{r\to 0} r^{\alpha p - n} \varphi(r^{-1})^{-1} \int_{B(x,r)} \Phi_{p,\varphi}(f(y))dy > 0 \right\}.$$

Then $C_{\alpha,\Phi_{p,\varphi}}(E_*) = 0$.

LEMMA 4.3. *For a nonnegative measurable function f in $L^p(\boldsymbol{R}^n)$, set*

$$E^* = \left\{ x \in \boldsymbol{R}^n \, ; \limsup_{r\to 0} r^{\alpha p - n} \int_{B(x,r)} f(y)^p dy > 0 \right\}.$$

If $\alpha p < n$, then $C_{\alpha,p}(E^*) = 0$; and if $\alpha p = n$, then E^* is empty.

Finally, in view of Theorem B and Lemmas 4.1 through 4.3, we establish the following result.

COROLLARY 4.4. *Let $\alpha p \le n$. If f is a locally integrable function on $\boldsymbol{R}^n$ satisfying* (1.1) *and* (1.2), *then*

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi_{p,\varphi}(A|U_\alpha f(x) - U_\alpha f(x_0)|)dx = 0$$

holds for all $A > 0$ and all $x_0 \in \boldsymbol{R}^n \setminus E$, where $C_{\alpha,\Phi_{p,\varphi}}(E) = 0$ when $\alpha p = n$ or φ is nonincreasing and $C_{\alpha,p}(E) = 0$ when $\alpha p < n$ and φ is nondecreasing.

In fact, if $\alpha p = n$ or φ is nonincreasing, then $E^* \subset E_*$, so that one can take $E = E_\infty \cup E_*$; if $\alpha p < n$ and φ is nondecreasing, then $E_* \subset E^*$, so that one can take $E = E_\infty \cup E^*$.

COROLLARY 4.5. *Let $\alpha p = n$ and $\varphi(r)$ be of the form $(\log r)^{q_1}(\log\log r)^{q_2}$ for large $r > 0$, where q_1 and q_2 are real numbers. Set $\Phi(r) = \Phi_{p,\varphi}(r) = r^p\varphi(r)$. Suppose f is a locally integrable function on $\boldsymbol{R}^n$ satisfying* (1.1) *and* (1.2).

(1) *If $q_1 < p - 1$, then*

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \{\exp(A|U_\alpha f(x) - U_\alpha f(x_0)|^{\beta_1}(\log(1 + |U_\alpha f(x) - U_\alpha f(x_0)|))^{\beta_2}) - 1\}dx = 0$$

for every $A > 0$ and every $x_0 \in \boldsymbol{R}^n$ except in a set of $C_{\alpha,\Phi_{p,\varphi}}$-capacity zero, where $\beta_1 = p/(p-1-q_1)$ and $\beta_2 = q_2/(p-1-q_1)$.

(2) *If $q_1 > p - 1$, then $U_\alpha f$ is continuous on $\boldsymbol{R}^n$ and*

$$|U_\alpha f(x) - U_\alpha f(x_0)| = o((\log(1/|x-x_0|))^{1/\beta_1}(\log\log(1/|x-x_0|))^{-q_2/p}) \quad \text{as } x \to x_0$$

for every $x_0 \in \boldsymbol{R}^n$.

For the continuity of $U_\alpha f$ (case (2)), see Remark 1.5. The case $q_1 = p - 1$ is treated as follows:

COROLLARY 4.6. *Let $\alpha p = n$, $\varphi(r) = \varphi_{p-1,q}(r) = (\log r)^{p-1}(\log\log r)^q$ for large $r > 0$ and $\Phi_{p,\varphi}(r) = r^p\varphi(r)$. Suppose f is a locally integrable function on $\boldsymbol{R}^n$ satisfying* (1.1) *and* (1.2).

(1) *If* $q < p - 1$, *then*

$$\lim_{r \to 0+} \fint_{B(x_0,r)} \{\exp(\exp(A|U_\alpha f(x) - U_\alpha f(x_0)|^\beta)) - e\} dx = 0$$

for every $A > 0$ *and every* $x_0 \in \boldsymbol{R}^n$ *except in a set of* $C_{\alpha,\Phi_{p,\varphi}}$*-capacity zero, where* $\beta = p/(p-1-q)$.

(2) *If* $q = p - 1$, *then*

$$\lim_{r \to 0+} \fint_{B(x_0,r)} \{\exp(\exp(\exp(A|U_\alpha f(x) - U_\alpha f(x_0)|^\beta))) - e^e\} dx = 0$$

for every $A > 0$ *and every* $x_0 \in \boldsymbol{R}^n$ *except in a set of* $C_{\alpha,\Phi_{p,\varphi}}$*-capacity zero, where* $\beta = p/(p-1)$.

(3) *If* $q > p - 1$, *then* $U_\alpha f$ *is continuous on* $\boldsymbol{R}^n$ *and*

$$|U_\alpha f(x) - U_\alpha f(x_0)| = o((\log(\log(1/|x - x_0|)))^{(p-1-q)/p}) \quad \text{as } x \to x_0$$

for every $x_0 \in \boldsymbol{R}^n$.

REFERENCES

[1] D. R. ADAMS AND R. HURRI-SYRJÄNEN, Vanishing exponential integrability for functions whose gradients belong to $L^n(\log(e+L))^\alpha$, J. Funct. Anal. 197 (2003), 162–178.

[2] A. CIANCHI, Strong and weak type inequalities for some classical operators in Orlicz spaces, J. London Math. Soc. 60 (1999), 187–202.

[3] A. ALBERICO AND A. CIANCHI, Differentiability properties of Orlicz-Sobolev functions, Ark. Mat. 43 (2005), 1–28.

[4] D. E. EDMUNDS AND W. D. EVANS, Hardy operators, function spaces and embeddings, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004.

[5] D. E. EDMUNDS, P. GURKA AND B. OPIC, Double exponential integrability, Bessel potentials and embedding theorems, Studia Math. 115 (1995), 151–181.

[6] D. E. EDMUNDS, P. GURKA AND B. OPIC, Sharpness of embeddings in logarithmic Bessel-potential spaces, Proc. Royal Soc. Edinburgh. 126 (1996), 995–1009.

[7] L. I. HEDBERG, On certain convolution inequalities, Proc. Amer. Math. Soc. 36 (1972), 505–510.

[8] N. G. MEYERS, A theory of capacities for potentials in Lebesgue classes, Math. Scand. 8 (1970), 255–292.

[9] Y. MIZUTA, Continuity properties of Riesz potentials and boundary limits of Beppo Levi functions, Math. Scand. 63 (1988), 238–260.

[10] Y. MIZUTA, Continuity properties of potentials and Beppo-Levi-Deny functions, Hiroshima Math. J. 23 (1993), 79–153.

[11] Y. MIZUTA, Potential theory in Euclidean spaces, Gakkōtosyo, Tokyo, 1996.

[12] Y. MIZUTA AND T. SHIMOMURA, Exponential integrability for Riesz potentials of functions in Orlicz classes, Hiroshima Math. J. 28 (1998), 355–371.

[13] Y. MIZUTA AND T. SHIMOMURA, Differentiability and Hölder continuity of Riesz potentials of functions in Orlicz classes, Analysis 20 (2000), 201–223.

[14] Y. MIZUTA AND T. SHIMOMURA, Vanishing exponential integrability for Riesz potentials of functions in Orlicz classes, Illinois J. Math. 51 (2007), 1039–1060.

[15] E. M. STEIN, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.

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