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LARGE DEVIATIONS FOR RANDOM UPPER SEMICONTINUOUS FUNCTIONS

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Abstract. In this paper, we shall study large deviation principle for random upper semicontinuous functions, and obtain Cramér type theorems for those whose underlying space is a separable Banach space of type p.

1. Introduction. The large deviation principle (LDP), which characterizes asymptotic behavior of the probabilities of rare events, has been one of the most important subjects in probability since 1930's. It has advanced the theory of probability and given some useful tools in application. Cerf [2] proved a Cramér type LDP for random compact sets in a separable Banach space of type p with respect to the Hausdorff metric d_H. On the other hand, random upper semicontinuous functions, which is the extension of a random sets, is studied vigorously in recent years. Theoretically, it provides an interesting prototype in general topology and functional analysis as well as probability theory and is expected to contribute the development of those fields. It also conceives fruitful applications in the wide area such as statistics, engineering and social science. Ogura, Li and Wang [8] studied several topologies including some new ones in the space of upper semicontinuous functions, and obtained some Cramér and Sanov type LDP's for random upper semicontinuous functions with compact convex levels with respect to those topologies. However, the work in front was restricted to the case where the underlying space is a d-dimensional Euclidean space $\boldsymbol{R}^d$ and the upper semicontinuous functions have convex level sets. It is slightly immature theoretically, and has a less range of applications.

In this paper, we remove convexity from the conditions on the level sets, and prove a Cramér type LDP for random upper semicontinuous functions on the underlying separable Banach space. For this purpose, we used a property of type p space, and strengthened a condition of Ogura, Li and Wang [8] as follows: $\boldsymbol{E}[\exp\{\lambda\|X(0)\|_{\mathcal{K}}\}] < \infty$, for any $\lambda > 0$ (see Section 2 and 3 for the notation).

The rest of the paper is structured as follows. In Section 2, we give some preliminary results about random sets and random upper semicontinuous functions which are needed later. In Section 3, we first define rate functions and LDP, and then state a general version of Cramér's theorem. Finally, we show that the Cramér type LDP for random upper semicontinuous functions.

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2. Preliminaries. Throughout this paper, we assume that $(\Omega, \mathcal{A}, P)$ is a complete probability space, $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ is a separable Banach space with the dual space $\mathfrak{X}^*$. Let $\mathcal{K} = \mathcal{K}(\mathfrak{X})$ be the family of all non-empty closed subsets of $\mathfrak{X}$ and let $\mathcal{K}_k = \mathcal{K}_k(\mathfrak{X})$ (resp. $\mathcal{K}_{kc} = \mathcal{K}_{kc}(\mathfrak{X})$) be the family of all non-empty compact (resp. compact convex) subsets of $\mathfrak{X}$. For an element A of $\mathcal{K}_k$, we denote by co$A$ the closed convex hull of A. Define the Hausdorff metric d_H by

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{\mathfrak{X}}, \sup_{b\in B}\inf_{a\in A}\|b-a\|_{\mathfrak{X}}\right\},$$

for $A, B \in \mathcal{K}_k$. Then $\mathcal{K}_k$ is a complete separable metric space with respect to the Hausdorff metric d_H (see [7, Theorem 1.1.3]). It is equipped with the Minkowski addition and the scalar multiplication

$$A + B = \{a + b\,; a \in A, b \in B\}, \quad tA = \{ta\,; a \in A\},$$

for $A, B \in \mathcal{K}_k$ and $t \in \boldsymbol{R}$. We denote by cl$A$ the closure of A, and also $\|A\|_{\mathcal{K}} = d_H(\{0\}, A) = \sup_{a\in A}\|a\|_{\mathfrak{X}}$. Denote by B^* the unit ball of $\mathfrak{X}^*$. The space (B^*, w^*) is compact, separable and metrizable with respect to the weak* topology w^* in $\mathfrak{X}^*$. Denote by $C(B^*, w^*)$ the space of all real bounded continuous functions on B^* with respect to the weak* topology w^* with the uniform norm $\|\cdot\|_{C(B^*)}$ ($\|f\|_{C(B^*)} = \sup\{|f(x^*)|\,; x^* \in B^*\}$ for $f \in C(B^*, w^*)$). Then $C(B^*, w^*)$ is a separable Banach space.

For each $A \in \mathcal{K}_{kc}(\mathfrak{X})$, define the support function $s(A)(x^*) : B^* \to \boldsymbol{R}$ by

$$s(A)(x^*) = \sup\{\langle x^*, x\rangle\,; x \in A\}, \quad x^* \in B^*,$$

where $\langle\cdot,\cdot\rangle$ is the paring. Then it satisfies the following properties: for any $A_1, A_2 \in \mathcal{K}_{kc}(\mathfrak{X})$ and $t \in \boldsymbol{R}_+ = [0, \infty)$,

(1) $s(A_1) = s(A_2) \Leftrightarrow A_1 = A_2$, $s(A_1) \le s(A_2) \Leftrightarrow A_1 \subset A_2$,

(2) $s(A_1 + A_2) = s(A_1) + s(A_2)$, $s(tA_1) = ts(A_1)$,

(3) $d_H(A_1, A_2) = \|s(A_1) - s(A_2)\|_{C(B^*)}$.

Hence, the mapping $j : \mathcal{K}_{kc}(\mathfrak{X}) \to C(B^*, w^*)$ defined by $j(A) = s(A)$ for $A \in \mathcal{K}_{kc}(\mathfrak{X})$ is an isometric embedding of $(\mathcal{K}_{kc}(\mathfrak{X}), d_H)$ into a closed convex cone of the separable Banach space $(C(B^*, w^*), \|\cdot\|_{C(B^*)})$ (see [7, Theorem 1.1.12]).

DEFINITION 2.1. A set-valued mapping $F : \Omega \to \mathcal{K}(\mathfrak{X})$ is called a *set-valued random variable* or a *random set* if for each subset $A \in \mathcal{K}(\mathfrak{X})$,

$$F^{-1}(A) = \{\omega \in \Omega\,; F(\omega) \cap A \neq \emptyset\} \in \mathcal{A}.$$

A $\mathcal{K}_k(\mathfrak{X})$-valued random variable is defined through the same way.

Let $I = [0, 1]$. A function $u : \mathfrak{X} \to I$ is upper semicontinuous if and only if for any $\alpha \in (0, 1]$, the level set $[u]_\alpha = \{x \in \mathfrak{X}\,; u(x) \ge \alpha\}$ is a closed subset of $\mathfrak{X}$. For any two upper semicontinuous functions u_1, u_2 define the two operations of addition and scalar multiplication:

$$(u_1 + u_2)(x) = \sup\{\alpha \in (0, 1]\,; x \in [u_1]_\alpha + [u_2]_\alpha\}, \quad \text{for any } x \in \mathfrak{X}, \tag{2.1}$$

(2.2) $$(tu_1)(x) = \sup\{\alpha \in (0,1]\,;\, x \in t[u_1]_\alpha\}\,, \quad \text{for any } x \in \mathfrak{X},\ t \in \boldsymbol{R}\,.$$

Let $\mathcal{F} = \mathcal{F}(\mathfrak{X})$ be the family of upper semicontinuous functions $u : \mathfrak{X} \to I$ with the level set $[u]_1 = \{x \in \mathfrak{X}\,;\, u(x) = 1\}$ non-empty. $\mathcal{F}$ is not a vector space with respect to the above addition and multiplication, since we can not find any inverse of u in general. Such an upper semicontinuous function is also called a special fuzzy set in some literatures (e.g. [7]).

Let $\mathcal{F}_k = \mathcal{F}_k(\mathfrak{X})$ be the family of all functions u in $\mathcal{F}$ with their support sets $[u]_{0+} = \mathrm{cl}\{x \in \mathfrak{X}\,;\, u(x) > 0\}$ being compact subsets of $\mathfrak{X}$. An element u in $\mathcal{F}_k$ is considered as an element of the space $D(I\,;\mathcal{K}_k) = D(I\,;\mathcal{K}_k(\mathfrak{X}))$ of all functions $u : I \to \mathcal{K}_k$ which is left continuous in (0, 1] and has right limit in [0, 1). We denote by $u(\alpha)$ the α-level set $[u]_\alpha$ of u for $\alpha \in (0,1]$, and $u(0) := \mathrm{cl}u(0+) = \mathrm{cl}\{x \in \mathfrak{X};\, u(x) > 0\}$ (we use the same symbol u, since no confusion occur). Then $\mathcal{F}_k$ is identified with the subspace $D_d(I\,;\mathcal{K}_k) = D_d(I;\mathcal{K}_k(\mathfrak{X}))$ which consists of all decreasing $u \in D(I;\mathcal{K}_k)$; $u(\alpha) \supset u(\beta)$ whenever $0 \le \alpha \le \beta \le 1$. Using the addition operation in (2.1) and scalar multiplication operation in (2.2), we have the following relations:

$$[u_1 + u_2]_\alpha = [u_1]_\alpha + [u_2]_\alpha\,, \quad [tu_1]_\alpha = t[u_1]_\alpha\,,$$

for $u_1, u_2 \in \mathcal{F}_k$ and $\alpha \in I$.

A function u in $\mathcal{F}(\mathfrak{X})$ is called convex if for any $\alpha \in I$, the level set $u(\alpha)$ is a convex subset of $\mathfrak{X}$. Let $\mathcal{F}_{kc} = \mathcal{F}_{kc}(\mathfrak{X})$ be the family of all compact convex functions u in $\mathcal{F}$. It is also identified with the subspace $D_d(I;\mathcal{K}_{kc}) = D_d(I;\mathcal{K}_{kc}(\mathfrak{X}))$ of $D(I;\mathcal{K}_{kc}) = D(I;\mathcal{K}_{kc}(\mathfrak{X}))$ which consists of all functions $u : I \to \mathcal{K}_{kc}$ being decreasing and left continuous in (0, 1] and satisfying $u(0) = \mathrm{cl}u(0+)$.

DEFINITION 2.2. An $\mathcal{F}(\mathfrak{X})$-*valued random variable* or a *random upper semicontinuous function* is a function $X : \Omega \to \mathcal{F}(\mathfrak{X})$, such that $[X(\omega)]_\alpha = \{x \in \mathfrak{X};\, X(\omega)(x) \ge \alpha\}$ is a set-valued random variable for every $\alpha \in (0,1]$.

In what follows, we also denote $[X(\omega)]_\alpha$ by $X(\alpha)$. Further, we set $X(0) := X(0+)$ as in the above. An $\mathcal{F}_k(\mathfrak{X})$-valued random variable is defined through the same way. A sequence of $\mathcal{F}_k(\mathfrak{X})$-valued random variables $\{X_n\}_{n\in N}$ is called to be independent if for any $\alpha \in (0,1]$, the sequence of $\mathcal{K}_k(\mathfrak{X})$-valued random variables $\{X_n(\alpha)\}_{n\in N}$ is independent.

Now, we introduce two metrics in $\mathcal{F}_k(\mathfrak{X})$, the metric d_Q and the Hausdorff graph metric d_G. In what follows, the topology induced by a metric d is called d-topology. Moreover, denote by $\mathcal{B}_d(\mathcal{F}_k(\mathfrak{X}))$ the topological Borel field of the space $\mathcal{F}_k(\mathfrak{X})$ with respect to the d-topology.

First, we review the d_Q-metric. Let $Q = \{\alpha_k;\, k \in \boldsymbol{N}\}$ be a countable dense subset of I including 0 and 1. A d_Q-metric is defined by

$$d_Q(u_1,u_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_H(u_1(\alpha_k), u_2(\alpha_k))}{1 + d_H(u_1(\alpha_k), u_2(\alpha_k))}\,,$$

for $u_1, u_2 \in D(I;\mathcal{K}_k)$. It is a metric on $D(I;\mathcal{K}_k)$, since the relation $d_Q(u_1,u_2) = 0$ implies $u_1 = u_2$. The other conditions in the definition of metric are easily verified. Note that the space $D(I;\mathcal{K}_k)$ is embedded into the product space $\prod_{\alpha\in Q}(\mathcal{K}_k, d_H)_\alpha$, where $(\mathcal{K}_k, d_H)_\alpha$ is a copy of $(\mathcal{K}_k, d_H)$, and is identified with a subspace of $\prod_{\alpha\in Q}(\mathcal{K}_k, d_H)_\alpha$. Through this identification, d_Q induces the relative topology on the subspace of the product topology in $\prod_{\alpha\in Q}(\mathcal{K}_k, d_H)_\alpha$. Since $\prod_{\alpha\in Q}(\mathcal{K}_k, d_H)_\alpha$ is separable with respect to the product topology, so is $D(I;\mathcal{K}_k)$ with respect to that induced by d_Q. Finally, the metric d_Q also is regarded as a metric on $\mathcal{F}_k$ (resp. $\mathcal{F}_{kc}$) through the identification of $\mathcal{F}_k$ and $D_d(I;\mathcal{K}_k)$ (resp. $\mathcal{F}_{kc}$ and $D_d(I;\mathcal{K}_{kc})$) again.

We next review the Hausdorff graph metric. Let, for $u \in D_d(I;\mathcal{K}_k)$,

$$\begin{aligned} G(u) &= \mathrm{cl}\{(\alpha, x);\, \alpha \in (0,1],\, x \in u(\alpha)\} \\ &= \{\{\alpha\} \times u(\alpha);\, \alpha \in I\}\,. \end{aligned}$$

Then the graph $G(u)$ is compact in $I \times \mathcal{K}_k(\mathfrak{X})$ with the product topology. Denote

$$d_G(u_1, u_2) = d_H(G(u_1), G(u_2))\,,$$

where d_H is the Hausdorff metric on $\mathcal{K}_k(I \times \mathfrak{X})$ equipped with the metric

$$d_{I\times\mathfrak{X}}((\alpha, x), (\beta, y)) = |\alpha - \beta| \vee \|x - y\|_{\mathfrak{X}}\,,$$

which induces the product topology. We also regard the graph metric d_G as a metric on $\mathcal{F}_k$ (resp. $\mathcal{F}_{kc}$) through the identification of $\mathcal{F}_k$ and $D_d(I;\mathcal{K}_k)$ (resp. $\mathcal{F}_{kc}$ and $D_d(I;\mathcal{K}_{kc})$) again.

THEOREM 2.3 ([7, Theorem 7.2.1]). *Let* $u, u_1, u_2, \ldots \in \mathcal{F}_k(\mathfrak{X})$. *If* $\lim_{n\to\infty} d_Q(u_n, u) = 0$ *for some countable dense subset* Q *of* I *including* 0 *and* 1, *then* $\lim_{n\to\infty} d_G(u_n, u) = 0$.

Finally, let $\mathcal{C}(\mathcal{F}_k(\mathfrak{X}))$ be the cylindrical σ-field, that is, the σ-field generated by the family

$$\{u \in \mathcal{F}_k(\mathfrak{X});\, u(\alpha) \in \mathcal{U}\}\,, \quad \alpha \in I,\ \mathcal{U} \in \mathcal{B}_{d_H}(\mathcal{K}_k(\mathfrak{X}))\,.$$

Then we have $\mathcal{C}(\mathcal{F}_k(\mathfrak{X})) = \mathcal{B}_{d_Q}(\mathcal{F}_k(\mathfrak{X})) = \mathcal{B}_{d_G}(\mathcal{F}_k(\mathfrak{X}))$ (see [8, Theorem 2.10 and Lemma 2.11]).

3. Large deviations for random upper semicontinuous functions. First, we define rate functions and LDP following Dembo and Zeitouni [3]. Let $\mathcal{X}$ be a topological space so that open and closed subsets of $\mathcal{X}$ are well-defined.

DEFINITION 3.1. (1) A *rate function* J is a lower semicontinuous mapping $J : \mathcal{X} \to [0, \infty]$.

(2) A *good rate function* is a rate function for which all the level sets $\Psi_J(\alpha) = \{x;\, J(x) \le \alpha\}$ are compact subsets of $\mathcal{X}$.

DEFINITION 3.2. A family of probability measures $\{\mu_n\}$ on a measurable space $(\mathcal{X}, \mathcal{B})$ is said to satisfy the LDP with the rate function J if for all $\Gamma \in \mathcal{B}$,

$$
\begin{aligned}
-\inf_{x\in\Gamma^\circ} J(x) &\le \liminf_{n\to\infty} \frac{1}{n}\log\mu_n(\Gamma) \\
&\le \limsup_{n\to\infty} \frac{1}{n}\log\mu_n(\Gamma) \le -\inf_{x\in\bar{\Gamma}} J(x)\,,
\end{aligned}
\tag{3.1}
$$

where Γ° and $\bar{\Gamma}$ are the interior and the closure of Γ respectively.

Suppose that all the compact subsets of $\mathcal{X}$ belong to $\mathcal{B}$. A family of probability measures $\{\mu_n\}$ is said to satisfy the weak LDP with the rate function J if the upper bound of (3.1) hold for every $\alpha < \infty$ and all compact subsets of $\Psi_J(\alpha)^c$, and the lower bound of (3.1) holds for all measurable sets. A family of probability measures $\{\mu_n\}$ on $\mathcal{X}$ is exponentially tight if for every $L < \infty$, there exists a compact subset K_L of $\mathcal{X}$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\log\mu_n(K_L^c) < -L\,.$$

LEMMA 3.3 ([3, Lemma 1.2.18]). *Let $\{\mu_n\}$ be exponentially tight. If $\{\mu_n\}$ satisfies the weak LDP with a rate function J, then J is a good rate function and satisfies the LDP.*

ASSUMPTION 3.4. (a) $\mathcal{X}$ is a locally convex, Hausdorff, topological real vector space. $\mathcal{E}$ is a closed, convex subset of $\mathcal{X}$ such that $\mu(\mathcal{E}) = 1$ and $\mathcal{E}$ can be made into a Polish space with respect to the topology induced by $\mathcal{X}$.

(b) The closed convex hull of each compact subset $\mathfrak{K} \subset \mathcal{E}$ is compact.

The following is the extension of Cramér's theorem.

THEOREM 3.5 ([3, Theorem 6.1.3]). *Let Assumption* 3.4 *hold. Then $\{\mu_n\}$ satisfies in $\mathcal{X}$ (and $\mathcal{E}$) a weak LDP with rate function $\Lambda^*_{\mathcal{X}}$, where*

$$\Lambda^*_{\mathcal{X}}(x) = \sup_{\lambda\in\mathcal{X}^*}\{\langle\lambda, x\rangle - \Lambda_{\mathcal{X}}(\lambda)\}\,, \quad x \in \mathcal{X}\,,$$

$$\Lambda_{\mathcal{X}}(\lambda) = \log \boldsymbol{E}[e^{\langle\lambda, x\rangle}]\,, \quad \lambda \in \mathcal{X}^*\,.$$

We suppose that $\mathfrak{X}$ is of type $p > 1$, i.e., there exists a constant c such that

$$\boldsymbol{E}\left[\left\|\sum_{i=1}^n f_i\right\|_{\mathfrak{X}}^p\right] \le c\sum_{i=1}^n \boldsymbol{E}[\|f_i\|_{\mathfrak{X}}^p]\,,$$

for any independent random variables $f_1, f_2, \dots, f_n$ with values in $\mathfrak{X}$ and mean zero.

For each $u \in \mathcal{F}_{kc}(\mathfrak{X})$, define the support function $s(u(\alpha))(x^*) : B^* \to \boldsymbol{R}$ by

$$s(u(\alpha))(x^*) = \sup\{\langle x^*, y\rangle; y \in u(\alpha)\}\,, \quad x^* \in B^*\,.$$

Then, it satisfies the following properties:

(1) $s(u(\alpha))(x^*)$ is subadditive, i.e., for $x^*, y^* \in B^*$,

$$s(u(\alpha))(x^* + y^*) \le s(u(\alpha))(x^*) + s(u(\alpha))(y^*)\,,$$

(2) $s(u(\alpha))(x^*)$ is positive homogeneous, i.e., for $x^* \in B^*$ and $\lambda \geq 0$,

$$s(u(\alpha))(\lambda x^*) = \lambda s(u(\alpha))(x^*) .$$

Let $\mathcal{X}_Q = \prod_{\alpha \in Q}(C(B^*, w^*), \|\cdot\|_{C(B^*)})_\alpha$ be endowed with the product topology, which is induced by the metric

$$d_{\mathcal{X}_Q}(s(u_1), s(u_2)) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|s(u_1(\alpha_k)) - s(u_2(\alpha_k))\|_{C(B^*)}}{1 + \|s(u_1(\alpha_k)) - s(u_2(\alpha_k))\|_{C(B^*)}} ,$$

for $u_1 = (u_1(\alpha_k))_{k \in \boldsymbol{N}}$, $u_2 = (u_2(\alpha_k))_{k \in \boldsymbol{N}}$. Then $(\mathcal{X}_Q, d_{\mathcal{X}_Q})$ is a Hausdorff topological vector space with respect to the ordinary operations:

$$(s(u_1) + s(u_2))(\alpha_k)(x^*) = s(u_1(\alpha_k))(x^*) + s(u_2(\alpha_k))(x^*) , \quad x^* \in B^* , \quad k \in \boldsymbol{N} ,$$

$$(ts(u_1))(\alpha_k)(x^*) = ts(u_1(\alpha_k))(x^*) , \quad x^* \in B^* , \quad t \in \boldsymbol{R} , \quad k \in \boldsymbol{N} .$$

It is locally convex since the system of convex sets

$$\{s(v) \in \mathcal{X}_Q ; \|s(v(\alpha_{k_i})) - s(u(\alpha_{k_i}))\|_{C(B^*)} < \varepsilon_i , i = 1, 2, \dots, n\} ,$$

$\alpha_{k_i} \in Q$, $\varepsilon_i > 0$, $i = 1, 2, \dots, n$, $n \in \boldsymbol{N}$, is a base of the neighborhood of $s(u)$ for any $s(u) \in (\mathcal{X}_Q, d_{\mathcal{X}_Q})$.

THEOREM 3.6. *Let $X, X_1, \dots, X_n$ be $\mathcal{F}_k(\mathfrak{X})$-valued i.i.d. random variables with*

$$\boldsymbol{E}[\exp\{\lambda \|X(0)\|_{\mathcal{K}}\}] < \infty , \quad \textit{for any} \ \lambda > 0 . \tag{3.2}$$

Let

$$\Lambda(\lambda) = \log \boldsymbol{E}[e^{\langle \lambda, s(X) \rangle}] , \quad \lambda \in \mathcal{X}_Q^* ,$$

$$\Lambda^*(u) = \Lambda^*(s(u)) = \sup_{\lambda \in \mathcal{X}_Q^*} \{\langle \lambda, s(u) \rangle - \Lambda(\lambda)\} , \quad u \in \mathcal{F}_{kc}(\mathfrak{X}) .$$

For a non-convex set $u \in \mathcal{F}_k(\mathfrak{X})$ we set $\Lambda^*(u) = +\infty$. Denote by μ_n the law of $\hat{S}_n = (\sum_{i=1}^n X_i)/n$. Then $\{\mu_n\}$ satisfies the LDP on $(\mathcal{F}_k(\mathfrak{X}), \mathcal{B}_{d_Q}(\mathcal{F}_k(\mathfrak{X})))$ with the good rate function Λ^*, i.e., for any $\mathcal{U} \in \mathcal{B}_{d_Q}(\mathcal{F}_k(\mathfrak{X}))$,

$$\begin{aligned} -\inf_{u \in \mathcal{U}^\circ} \Lambda^*(u) &\leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(\mathcal{U}) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(\mathcal{U}) \leq -\inf_{u \in \bar{\mathcal{U}}} \Lambda^*(u) , \end{aligned} \tag{3.3}$$

where $\mathcal{U}^\circ$ and $\bar{\mathcal{U}}$ are the interior and the closure of $\mathcal{U}$ with respect to the d_Q-topology.

In the general case, where the sequence $X, X_1, \dots, X_n$ are not necessarily convex, we set $\hat{S}_n = (\sum_{i=1}^n X_i)/n$ and $\hat{S}_n^{\mathrm{co}} = (\sum_{i=1}^n \mathrm{co} X_i)/n$. We will use the following lemmas.

LEMMA 3.7 ([2, Lemma 2]). *Let $X_1, X_2, \dots, X_n$ be $\mathcal{K}_k(\mathfrak{X})$-valued i.i.d. random variables. Then for any $\delta > 0$,*

$$\lim_{n \to \infty} \frac{1}{n} \log P(d_H(\hat{S}_n, \hat{S}_n^{\mathrm{co}}) \geq \delta) = -\infty .$$

LEMMA 3.8. *Let $X_1, X_2, \ldots, X_n$ be $\mathcal{F}_k(\mathfrak{X})$-valued i.i.d. random variables. Then for any $\delta > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \log P(d_Q(\hat{S}_n, \hat{S}_n^{\text{co}}) \geq \delta) = -\infty .$$

PROOF. For any $\delta > 0$, there exists a natural number N such that

$$\sum_{k=N+1}^{\infty} \frac{1}{2^k} \frac{d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))}{1 + d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))} \leq \sum_{k=N+1}^{\infty} \frac{1}{2^k} < \frac{\delta}{2} .$$

Therefore,

$$\begin{aligned}\{d_Q(\hat{S}_n, \hat{S}_n^{\text{co}}) \geq \delta\} &= \left\{ \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))}{1 + d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))} \geq \delta \right\} \\ &\subset \left\{ \sum_{k=1}^{N} \frac{1}{2^k} \frac{d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))}{1 + d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))} > \frac{\delta}{2} \right\} \\ &\subset \bigcup_{k=1}^{N} \left\{ \frac{1}{2^k} \frac{d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))}{1 + d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))} > \frac{\delta}{2N} \right\} .\end{aligned}$$

Hence we have

$$\begin{aligned}P(d_Q(\hat{S}_n, \hat{S}_n^{\text{co}}) \geq \delta) &\leq P\left(\bigcup_{k=1}^{N} \left\{ \frac{1}{2^k} \frac{d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))}{1 + d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k))} > \frac{\delta}{2N} \right\} \right) \\ &\leq \sum_{k=1}^{N} P\left(d_H(\hat{S}_n(\alpha_k), \hat{S}_n^{\text{co}}(\alpha_k)) > \frac{2^k \delta}{2N} \right) .\end{aligned}$$

Since $\hat{S}_n(\alpha_k)$ and $\hat{S}_n^{\text{co}}(\alpha_k)$ are elements of $\mathcal{K}_k(\mathfrak{X})$, the proof of Lemma 3.8 is completed from Lemma 3.7. □

PROOF OF THEOREM 3.6. *Step* 1. Assume first that the sequence $X, X_1, \ldots, X_n$ are convex. Let $\mathcal{X} = \mathcal{X}_Q$, $\mathcal{E} = \overline{s(\mathcal{F}_{kc}(\mathfrak{X}))}^{d_{\mathcal{X}_Q}}$, where $\overline{s(\mathcal{F}_{kc}(\mathfrak{X}))}^{d_{\mathcal{X}_Q}}$ is the closure of $s(\mathcal{F}_{kc}(\mathfrak{X}))$ with respect to the $d_{\mathcal{X}_Q}$-topology. Then the space $\mathcal{E}$ is a closed, convex subset of $\mathcal{X}$. Since the space $\mathcal{X}$ is complete with respect to the $d_{\mathcal{X}_Q}$-topology, so $\mathcal{E}$ is complete. We will show that $\mathcal{E}$ is separable. Indeed, the space $D(I; \mathcal{K}_k)$ is separable with respect to the d_Q-topology, so is $s(\mathcal{F}_{kc}(\mathfrak{X}))$ with respect to the $d_{\mathcal{X}_Q}$-topology. Hence $\mathcal{E}$ is separable.

We now check condition (b) in Assumption 3.4. Take a compact set $\mathfrak{K} \subset \mathcal{E}$. Since $\mathcal{X}$ is a complete, locally convex, Hausdorff, topological vector space, the closed convex hull of $\mathfrak{K}$ is compact.

Denote by $\mu_n^{s,\text{co}}$ the law of $\hat{S}_n^{s,\text{co}} = (\sum_{i=1}^n s(\text{co}X_i))/n$. Now Assumption 3.4 is fulfilled and, due to Theorem 3.5, the family $\{\mu_n^{s,\text{co}}\}$ satisfy a weak LDP with the rate function

$$\Lambda_{\mathcal{X}}^*(s(u)) = \sup_{\lambda \in \mathcal{X}^*} \{\langle \lambda, s(u) \rangle - \Lambda_{\mathcal{X}}(\lambda)\} , \quad u \in \mathcal{F}_{kc}(\mathfrak{X}) .$$

From (3.2), the family of probability measures $\{\mu_n^{s,\mathrm{co}}\}$ is exponentially tight (see [4, Theorem 3.3.11]). Hence, due to Lemma 3.3, the family $\{\mu_n^{s,\mathrm{co}}\}$ satisfies the LDP with the rate function $\Lambda^*_{\mathcal{X}}(s(u))$, i.e., for any $\tilde{\mathcal{U}} \in \mathcal{B}_{d_{\mathcal{X}_Q}}(\mathcal{X})$,

$$-\inf_{s(u)\in \mathrm{Int}(\tilde{\mathcal{U}})} \Lambda^*_{\mathcal{X}}(s(u)) \le \liminf_{n\to\infty} \frac{1}{n} \log \mu_n^{s,\mathrm{co}}(\tilde{\mathcal{U}})$$
$$\le \limsup_{n\to\infty} \frac{1}{n} \log \mu_n^{s,\mathrm{co}}(\tilde{\mathcal{U}}) \le -\inf_{s(u)\in \mathrm{cl}(\tilde{\mathcal{U}})} \Lambda^*_{\mathcal{X}}(s(u)) \,,$$

where $\mathrm{Int}(\tilde{\mathcal{U}})$ and $\mathrm{cl}(\tilde{\mathcal{U}})$ are the interior and the closure of $\tilde{\mathcal{U}}$ with respect to the $d_{\mathcal{X}_Q}$-topology.

Step 2. Next, we pull back this LDP to the space $\mathcal{F}_{kc}(\mathfrak{X})$ with the help of the isometry s. We denote by μ_n^{co} the law of $\hat{S}_n^{\mathrm{co}} = (\sum_{i=1}^n \mathrm{co}X_i)/n$. Since for any $u \in \mathcal{F}_{kc}(\mathfrak{X})$,

$$\Lambda^*(u) = \Lambda^*_{\mathcal{X}}(s(u)) \,,$$

we obtain, for any $\mathcal{U} \in \mathcal{B}_{d_Q}(\mathcal{F}_{kc}(\mathfrak{X}))$,

$$-\inf_{u\in\mathcal{U}^\circ} \Lambda^*(u) \le \liminf_{n\to\infty} \frac{1}{n} \log \mu_n^{\mathrm{co}}(\mathcal{U})$$
$$\le \limsup_{n\to\infty} \frac{1}{n} \log \mu_n^{\mathrm{co}}(\mathcal{U}) \le -\inf_{u\in\bar{\mathcal{U}}} \Lambda^*(u) \,,$$

where $\mathcal{U}^\circ$ and $\bar{\mathcal{U}}$ are the interior and the closure of $\mathcal{U}$ with respect to the d_Q-topology.

Step 3. Now, we prove the lower bound of (3.3). Let $\mathcal{U} \in \mathcal{B}_{d_Q}(\mathcal{F}_k(\mathfrak{X}))$ and $u \in \mathcal{U}^\circ$ (if $\mathcal{U}^\circ \cap \mathcal{F}_{kc}(\mathfrak{X})$ is empty, there is nothing to prove). Then there exists a $\delta > 0$ such that

$$\{v \in \mathcal{F}_k(\mathfrak{X});\, d_Q(u, v) < \delta\} \subset \mathcal{U} \,.$$

We then have

$$\begin{aligned} P(\hat{S}_n \in \mathcal{U}) &\ge P(d_Q(\hat{S}_n, u) < \delta) \\ &\ge P\Big(d_Q(\hat{S}_n^{\mathrm{co}}, u) < \frac{\delta}{2},\ d_Q(\hat{S}_n, \hat{S}_n^{\mathrm{co}}) < \frac{\delta}{2}\Big) \\ &\ge P\Big(d_Q(\hat{S}_n^{\mathrm{co}}, u) < \frac{\delta}{2}\Big) - P\Big(d_Q(\hat{S}_n, \hat{S}_n^{\mathrm{co}}) \ge \frac{\delta}{2}\Big) \,. \end{aligned}$$

Thus

$$P(\hat{S}_n \in \mathcal{U}) + P\Big(d_Q(\hat{S}_n, \hat{S}_n^{\mathrm{co}}) \ge \frac{\delta}{2}\Big) \ge P\Big(d_Q(\hat{S}_n^{\mathrm{co}}, u) < \frac{\delta}{2}\Big) \,.$$

Hence

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \log \Big\{ P(\hat{S}_n \in \mathcal{U}) + P\Big(d_Q(\hat{S}_n, \hat{S}_n^{\mathrm{co}}) \ge \frac{\delta}{2}\Big)\Big\} \\ &\qquad \ge \liminf_{n\to\infty} \frac{1}{n} \log P\Big(d_Q(\hat{S}_n^{\mathrm{co}}, u) < \frac{\delta}{2}\Big) \,. \end{aligned} \tag{3.4}$$

On the other hand, from Lemma 3.8 we obtain

$$\begin{aligned}\text{LHS of (3.4)} &= \liminf_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n\in\mathcal{U})\vee\liminf_{n\to\infty}\frac{1}{n}\log P\left(d_Q(\hat{S}_n,\hat{S}_n^{\mathrm{co}})\geq\frac{\delta}{2}\right)\\ &= \liminf_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n\in\mathcal{U})\,,\end{aligned}$$

where $a\vee b$ stands for the maximum of a and b. Hence

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n\in\mathcal{U}) &\geq \liminf_{n\to\infty}\frac{1}{n}\log P\left(d_Q(\hat{S}_n^{\mathrm{co}},u)<\frac{\delta}{2}\right)\\ &= \liminf_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n^{\mathrm{co}}\in B_{\delta/2}(u))\,,\end{aligned}$$

where $B_{\delta/2}(u)$ denotes the ball of radius $\delta/2$ and at center u with respect to the d_Q-metric. Applying LDP to $\{\hat{S}_n^{\mathrm{co}}\}$, we have

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n^{\mathrm{co}}\in B_{\delta/2}(u)) &\geq -\inf_{v\in B_{\delta/2}(u)}\Lambda^*(v)\\ &\geq -\Lambda^*(u)\,.\end{aligned}$$

This implies

$$\liminf_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n\in\mathcal{U})\geq-\Lambda^*(u)\,.$$

Taking the supermum over all sets u in $\mathcal{U}^\circ$, we have

$$\liminf_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n\in\mathcal{U})\geq-\inf_{u\in\mathcal{U}^\circ}\Lambda^*(u)\,.$$

Finally, we prove the upper bound of (3.3). Let $\mathcal{U}\in\mathcal{B}_{d_Q}(\mathcal{F}_k(\mathfrak{X}))$. For any $\delta>0$, let

$$\mathcal{U}^\delta=\{A\in\mathcal{F}_k(\mathfrak{X});\,d_Q(A,\mathcal{U})\leq\delta\}\,.$$

Then

$$P(\hat{S}_n\in\mathcal{U})\leq P(\hat{S}_n^{\mathrm{co}}\in\mathcal{U}^\delta)+P(d_Q(\hat{S}_n,\hat{S}_n^{\mathrm{co}})>\delta)\,.$$

This with Lemma 3.8 ensures

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n\in\mathcal{U}) &\leq \limsup_{n\to\infty}\frac{1}{n}\log\{P(\hat{S}_n^{\mathrm{co}}\in\mathcal{U}^\delta)+P(d_Q(\hat{S}_n,\hat{S}_n^{\mathrm{co}})>\delta)\}\\ &= \limsup_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n^{\mathrm{co}}\in\mathcal{U}^\delta)\vee\limsup_{n\to\infty}\frac{1}{n}\log P(d_Q(\hat{S}_n,\hat{S}_n^{\mathrm{co}})>\delta)\\ &= \limsup_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n^{\mathrm{co}}\in\mathcal{U}^\delta)\,.\end{aligned}$$

Applying LDP to $\{\hat{S}_n^{\mathrm{co}}\}$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log P(\hat{S}_n^{\mathrm{co}}\in\mathcal{U}^\delta)\leq-\inf_{u\in\mathcal{U}^\delta}\Lambda^*(u)\,,$$

and hence

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\hat{S}_n \in \mathcal{U}) \leq - \inf_{u\in\mathcal{U}^\delta} \Lambda^*(u) . \tag{3.5}$$

In addition, $\bigcap_{\delta>0} \mathcal{U}^\delta = \bar{\mathcal{U}}$. Since Λ^* is a good rate function, we have that

$$\lim_{\delta\to 0} \inf\{\Lambda^*(u); u \in \mathcal{U}^\delta\} = \inf\{\Lambda^*(u); u \in \bar{\mathcal{U}} \cap \mathcal{F}_k(\mathfrak{X})\} .$$

Indeed, the right-hand side is clearly larger than the left-hand side. Let $\{u_n\}_{n\in \boldsymbol{N}}$ be a sequence such that u_n belongs to $\mathcal{U}^{1/n}$ for any $n \in \boldsymbol{N}$, and $\Lambda^*(u_n)$ converges to the left-hand side. Since the level sets of Λ^* is a compact subsets in $\mathcal{F}_k(\mathfrak{X})$, we can choose from $\{u_n\}_{n\in \boldsymbol{N}}$ a subsequence converging to an upper semicontinuous function u which necessarily belongs to $\bar{\mathcal{U}} \cap \mathcal{F}_k(\mathfrak{X})$. By the lower semicontinuity of Λ^*, we see that $\Lambda^*(u)$ is smaller than the left-hand side.

Hence, on letting $\delta \to 0$ in (3.5), we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\hat{S}_n \in \mathcal{U}) \leq - \inf_{u\in\bar{\mathcal{U}}} \Lambda^*(u) .$$

This completes the proof of theorem. □

COROLLARY 3.9. *Under the same assumptions and notation as in Theorem* 2.3, *the family of probability measures* $\{\mu_n\}$ *satisfies the LDP on* $(\mathcal{F}_k(\mathfrak{X}), \mathcal{B}_{d_G}(\mathcal{F}_k(\mathfrak{X})))$ *with the good rate function* Λ^*.

PROOF. This is clear from Theorem 2.3, because every d_G-open set is a d_Q-open set and every d_G-closed set is a d_Q-closed set. □

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