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NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND PAINLEVÉ TYPE WITH THE QUASI-PAINLEVÉ PROPERTY ALONG A RECTIFIABLE CURVE

SHUN SHIMOMURA

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Abstract. We present a class of nonlinear differential equations of second Painlevé type. These equations, with a single exception, admit the quasi-Painlevé property along a rectifiable curve, that is, for general solutions, every movable singularity defined by a rectifiable curve is at most an algebraic branch point. Moreover we discuss the global many-valuedness of their solutions. For the exceptional equation, by the method of successive approximation, we construct a general solution having a movable logarithmic branch point.

1. Introduction. For a general solution of the first order nonlinear differential equation

$$y' = R_1(x, y) \tag{1.1}$$

($' = d/dx$) with $R_1(x, y) \in \boldsymbol{C}(x, y)$, every movable singularity (singularity depending on initial data) is at most an algebraic branch point ([6, §§3.2, 3.3], [7, §12.5]). In particular, equation (1.1) admits the *Painlevé property*, that is, every movable singularity of a general solution is a pole, if and only if (1.1) is of Riccati type.

Consider a second order nonlinear differential equation of the form

$$y'' = R_2(x, y, y') \tag{1.2}$$

with $R_2(x, y, y') \in \boldsymbol{C}(x, y, y')$. For a general solution of (1.2), a movable singularity is not always an algebraic branch point. For example,

$$y'' = -(1 + 2y^2)(y')^2/y \qquad (\text{resp.} \quad y'' = (1 + i)(y')^2/y)$$

has the general solution $y = \sqrt{C_1 + \log(x - C_2)}$ with a logarithmic branch point at $x = C_2$ and an algebraic branch point at $x = C_2 + e^{-C_1}$ (resp. $y = C_1(x - C_2)^i$ with an essential singularity at $x = C_2$). Let $y(x)$ be a general solution of (1.2) analytic at a base point $x = x_0$. For *rectifiable* curves Γ and Γ' issuing from x_0 and terminating in a_0, suppose that $y(x)$ is analytic along Γ and Γ' except at a_0. These curves are said to be equivalent, if, for every neighbourhood U of a_0, there exists an open set Δ_U such that $a_0 \in \Delta_U \subset U$ and that the function elements of $y(x)$ at any points on $\Gamma \cap \Delta_U \setminus \{a_0\}$ and on $\Gamma' \cap \Delta_U \setminus \{a_0\}$ are analytic continuations of each other along a suitable curve in U. An equivalence class containing Γ

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defines a *singularity* of $y(x)$ at a_0, if $y(x)$ is not analytic at a_0. Let us say that equation (1.2) admits the *quasi-Painlevé property along a rectifiable curve*, if every movable singularity (defined by a rectifiable curve as above) of $y(x)$ is at most an algebraic branch point (cf. [12]). It is natural to regard the Painlevé equations (admitting the Painlevé property) as special cases belonging to some family of second order differential equations with the quasi-Painlevé property along a rectifiable curve, like Riccati equations in the category of the first order differential equations. In [12] we presented a class of differential equations of the form

$$y'' = \frac{2(2k+1)}{(2k-1)^2} y^{2k} + x \qquad (k \in \boldsymbol{N}), \tag{1.3}$$

and proved that each of them admits the quasi-Painlevé property along a rectifiable curve. If $k = 1$, this coincides with the first Painlevé equation. We stress that, for solutions of (1.2), a movable singularity treated here is defined by a rectifiable curve. As pointed out in [2] (see also [13]), in the case of a higher order equation, for solutions admitting movable *branch* points, a movable singularity defined by a curve of *infinite length* should be considered separately. For (1.3) or (E_k), which will be studied in this paper, it is not known whether a non-algebraic singularity of such type exists or not. For this reason, in this paper, we use the term 'quasi-Painlevé property along a rectifiable curve' instead of 'quasi-Painlevé property' in [12].

Let us consider differential equations of the form

$$y'' = \frac{k+1}{k^2} y^{2k+1} + xy + \alpha \qquad (k \in \boldsymbol{N}) \tag{E_k}$$

with $\alpha \in \boldsymbol{C}$. In this paper, we examine the quasi-Painlevé property along a rectifiable curve for them, and the global many-valuedness of their solutions. Equation (E_1) is nothing less than the second Painlevé equation. Equation (E_k) with $\alpha = 0$ is equivalent to a special case of

$$y'' = 2y^{2\tau+1} + xy \qquad (\tau > 0).$$

This equation was given by de Boer and Ludford ([1]) in connection with a problem in plasma physics, and Hastings and McLeod ([4]) discussed a boundary value problem on the real axis.

Our main results are stated as follows:

THEOREM 1.1. *For each* $k \in \boldsymbol{N} \setminus \{2\}$, *equation* ($E_k$) *admits the quasi-Painlevé property along a rectifiable curve*, *that is*, *every movable singularity defined by a rectifiable curve of a general solution is at most an algebraic branch point. None of the solutions of* (E_2) *have a movable algebraic branch point.*

If $k \geq 3$, a general solution can be represented by a Puiseux series around its movable singularity.

THEOREM 1.2. *Let* $y(x)$ *be a general solution of* (E_k) *with* $k \in \boldsymbol{N} \setminus \{2\}$, *and suppose that* x_0 *is a movable algebraic branch point* (*or a movable pole*) *of* $y(x)$. *Then, around* $x = x_0$,

$$
\begin{aligned}
y(x) = {} & \omega_k \xi^{-1/k} - \frac{\omega_k k x_0}{6}\xi^{2-1/k} - \frac{k^2\alpha}{3k+1}\xi^2 + c\xi^{2+1/k} \\
& + \frac{\omega_k k}{4(k-2)}\xi^{3-1/k} + \sum_{j\ge 3k} c_j \xi^{j/k}, \qquad \xi := x - x_0, \quad \omega_k = 1 \ \ or \ \ e^{\pi i/k},
\end{aligned} \tag{1.4}
$$

where c is an integration constant, c_j $(j \geq 3k)$ *are polynomials in c and* x_0, *and* $\xi^{1/k}$ *denotes an arbitrary branch of* σ *such that* $\sigma^k = \xi$.

REMARK 1.1. For (E_2) we can construct a general solution having a movable logarithmic branch point (see Theorem A.1 in Appendix).

Equation (E_k) admits the trivial solution $y \equiv 0$ if and only if $\alpha = 0$. For entire, meromorphic or algebraic nontrivial solutions we have the following:

THEOREM 1.3. (i) *Equation* (E_k) *admits no nontrivial entire solution. Moreover*, *if* $k \geq 2$, *then equation* (E_k) *admits no nontrivial meromorphic solution.*

(ii) *If* $k = 2$ *or if k is an odd integer such that* $k \geq 3$, *then* (E_k) *admits no nontrivial algebraic solution*, *that is*, *every nontrivial solution is transcendental.*

As mentioned above, if $k \geq 2$, each nontrivial solution of (E_k) is a many-valued function. In general, for a solution with movable branch points, it is not easy to know about global many-valuedness, for example, whether it is algebroid or not, because such a property depends on their global behaviour. For this question, we have the following:

THEOREM 1.4. *Suppose that* $k \geq 2$. *For every* $\nu \in \boldsymbol{N}$, *equation* (E_k) *admits a two-parameter family of solutions which are at least* ν*-valued.*

In Section 2 we first prove Theorem 1.2 by computing the coefficients of a Puiseux series expansion around an algebraic branch point. In Section 3, by using a system of equations derived from (1.4), Theorem 1.1 is established. In showing the quasi-Painlevé property along a rectifiable curve, we regard a solution of (E_k) as a function on its Riemann surface, and modify the classical method of proving Painlevé property in such a way that it is applicable to this case. Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4, respectively. In the proof of Theorem 1.4, letting a solution of (E_k) degenerate to the inverse function of a hyperelliptic integral, we apply the α-method due to Painlevé to show its many-valuedness, while he introduced the method to exclude equations admitting many-valued solutions ([10], [7]). In Appendix, we give a general solution of (E_2) expressed by a series containing logarithmic terms. To construct such a solution, we employ the method of successive approximation, by which the existence and the convergence are simultaneously shown.

Recently, for a more general class of second order equations containing (1.3) and (E_k) with $k \in \boldsymbol{N} \setminus \{2\}$, under the resonance condition, Filipuk and Halburd ([2]) proved the quasi-Painlevé property along a rectifiable curve and discussed a singularity corresponding to a curve of infinite length. The author is grateful to them for bringing their paper [2] to his attention.

2. Proof of Theorem 1.2. 2.1. Preparatory lemma. Consider the system of differential equations

$$dv_1/dt = F_1(t, v_1, v_2), \qquad dv_2/dt = F_2(t, v_1, v_2), \tag{2.1}$$

where $F_l(t, v_1, v_2)$ $(l = 1, 2)$ are analytic in a neighbourhood of $(t_0, v_{1,0}, v_{2,0}) \in \boldsymbol{C}^3$. Then we have the following lemma (cf. [3, Corollary A.4], [6, §3.2], [7, §12.3]), which will be used in the proofs of Theorems 1.1 and 1.2.

LEMMA 2.1. *Let C $(\subset \boldsymbol{C})$ be a rectifiable curve terminating in $t = t_0$. Suppose that a solution $(v_1, v_2) = (\varphi(t), \psi(t))$ of (2.1) satisfies the following:*
(i) *$\varphi(t)$ and $\psi(t)$ are analytic along C except at t_0;*
(ii) *there exists a sequence $\{t_n\}_{n\in \boldsymbol{N}} \subset C \setminus \{t_0\}$, $t_n \to t_0$ $(n \to \infty)$ such that $(\varphi(t_n), \psi(t_n)) \to (v_{1,0}, v_{2,0})$.*
Then, $\varphi(t)$ and $\psi(t)$ are analytic at $t = t_0$.

2.2. Proof of Theorem 1.2. If $k = 1$, then (1.4) coincides with a general solution of the second Painlevé equation around a movable pole (cf. [3, §2], [11, §6]). In what follows we suppose that $k \geq 2$. Let $x = x_0$ be a movable algebraic branch point (or a movable pole) of $y(x)$. If $|y(x)|$ is bounded along a segment $[x_0^*, x_0]$, then

$$y'(x) = y'(x_0^*) + \int_{x_0^*}^{x} \left(\frac{k+1}{k^2} y(t)^{2k+1} + t y(t) + \alpha\right) dt$$

is also bounded, and by Lemma 2.1, $y(x)$ is analytic at $x = x_0$, which is a contradiction. Hence $|y(x_0)| = \infty$, and we may write

$$y(x) = A_0 \xi^{\gamma}(1 + o(1)), \quad \xi = x - x_0, \quad \gamma < 0, \quad A_0 \neq 0.$$

Substitution of this into (E_k) yields $\gamma = -1/k$, $A_0^{2k} = 1$.

Consider the case where $A_0 = \omega_k = 1$ or $e^{\pi i/k}$. Then

$$y(x) = \omega_k \xi^{-1/k} + \sum_{j=m}^{\infty} c_j \xi^{j/(k\mu)}$$

for some integers m and μ satisfying $m \geq -\mu + 1$ and $\mu \geq 1$, where $\xi^{1/(k\mu)}$ denotes an arbitrary branch. Substituting this series into (E_k), and comparing the coefficients of $\xi^{-2+j/(k\mu)}$, we have

$$\left(\frac{j}{\mu} + k + 1\right)\left(\frac{j}{\mu} - (2k+1)\right) c_j = \Xi_j(x_0, c_i\,;\, i \leq j-1), \quad c_{-\mu} = \omega_k,$$

where Ξ_j are polynomials in x_0 and c_i. Suppose that $J = \{j \in \boldsymbol{Z}; c_j \neq 0,\ j/\mu \notin \boldsymbol{Z}\} \neq \emptyset$. Then $j_0 = \min J$ satisfies $\Xi_{j_0} = 0$, and hence we have $j_0 = (2k+1)\mu$ or $c_{j_0} = 0$, which contradicts the supposition. Therefore $\mu = 1$. Using the relation above, we have

$$c_{2k-1} = -\frac{\omega_k k x_0}{6}, \quad c_{2k} = -\frac{k^2 \alpha}{3k+1}$$

and $c_j = 0$ for $0 \le j \le 2k-2$. For $j = 2k+1$, we have

$$\Xi_{2k+1} = \begin{cases} \pm 4 & \text{if } k = 2\,, \\ 0 & \text{if } k \ge 3\,. \end{cases} \tag{2.2}$$

If $k \ge 3$, then $c_{2k+1} = c$, where c is an arbitrary constant. If $k = 2$, (2.2) yields $0 \cdot c_5 = \pm 4$, which implies that (E_2) does not admit a solution with a movable algebraic branch point.

In addition, for each $h \in \boldsymbol{Z}$, we get a solution expanded into a series in $(e^{2\pi i h}\xi)^{1/k} = e^{2\pi i h/k}\xi^{1/k}$ with the same coefficients as above, which corresponds to the case where $A_0 = e^{\pi i(-2h)/k}$ (if $\omega_k = 1$) or $A_0 = e^{\pi i(-2h+1)/k}$ (if $\omega_k = e^{\pi i/k}$). This fact means that, for every $l \in \boldsymbol{Z}$, the solution with $A_0 = e^{\pi i l/k}$ is an analytic continuation of the solution with $\omega_k = 1$ or $e^{\pi i/k}$. In this way we obtain the theorem.

3. Proof of Theorem 1.1. If $k = 1$, then (E_1) admits the Painlevé property. In what follows we suppose that $k \ge 3$.

3.1. System of equations. Let us find a system of equations corresponding to the integration constants x_0, c of (1.4) and equivalent to (E_k), which is a key to proving Theorem 1.1. Series expansion (1.4) is written in the form

$$y(x) = \omega_k \xi^{-1/k}\left(1 - \frac{kx}{6}\xi^2 - \frac{\omega_k^{-1}k^2\alpha}{3k+1}\xi^{2+1/k} + \omega_k^{-1}c\xi^{2+2/k} + \frac{(2k-1)k}{12(k-2)}\xi^3 + \cdots\right) \tag{3.1}$$

near $x = x_0$, since $x_0 = x - \xi$. Putting $u(x) = 1/y(x)$ around $x = x_0$, we have

$$\begin{aligned} \xi^{1/k} &= \omega_k u(x)\left(1 - \frac{kx}{6}\xi^2 - \frac{\omega_k^{-1}k^2\alpha}{3k+1}\xi^{2+1/k} + \omega_k^{-1}c\xi^{2+2/k} + \frac{(2k-1)k}{12(k-2)}\xi^3 + \cdots\right) \\ &= \omega_k u(x)\left(1 - \frac{kx}{6}u(x)^{2k} - \frac{k^2\alpha}{3k+1}u(x)^{2k+1} + \omega_k c u(x)^{2k+2} \right. \\ &\qquad \left. + \frac{(2k-1)k}{12(k-2)}\omega_k^k u(x)^{3k} + \cdots\right). \end{aligned}$$

Substituting this into

$$\begin{aligned} y'(x) = &-\frac{\omega_k}{k}\xi^{-1-1/k} + \frac{(2k-1)(3k-1)}{12(k-2)}\omega_k\xi^{2-1/k} - \frac{\omega_k kx}{6}\left(2 - \frac{1}{k}\right)\xi^{1-1/k} \\ &- \frac{\omega_k k}{6}\xi^{2-1/k} - \frac{2k^2\alpha}{3k+1}\xi + \left(2 + \frac{1}{k}\right)c\xi^{1+1/k} + \cdots\,, \end{aligned}$$

and observing that $\omega_k^k = \pm 1$, we have

$$\begin{aligned} y'(x) = \frac{k^2}{2(k-2)}u(x)^{2k-1} \mp \frac{u(x)^{-k-1}}{k}&\left(1 + \frac{k^2 x}{2}u(x)^{2k} + k^2\alpha u(x)^{2k+1} \right. \\ &\left. - (3k+2)\omega_k c u(x)^{2k+2} + \cdots\right). \end{aligned}$$

Viewing these identities, we define new unknowns u and v by

$$y = u^{-1}\,, \tag{3.2}$$

$$y' = Bu^{2k-1} \mp \frac{u^{-k-1}}{k}\Big(1 + \frac{k^2x}{2}u^{2k} + k^2\alpha u^{2k+1} + u^{2k+2}v\Big) \tag{3.3}$$

with $B = k^2/(2(k-2))$. Then, equation (E_k) is written in the form

$$\frac{du}{dx} = \pm u^{1-k}\Phi_{\pm}(x, u, v)\,, \quad \frac{dv}{dx} = \mp u^{k-2}\Psi_{\pm}(x, u, v)\,,$$

where

$$\Phi_{\pm}(x, u, v) = \frac{1}{k}\left(1 + \frac{k^2x}{2}u^{2k} + k^2\alpha u^{2k+1} + u^{2k+2}v \mp Bku^{4k-1}\right),$$

$$\Psi_{\pm}(x, u, v) = \frac{1}{k}\left(\frac{k^2x}{2} + k^2\alpha u + u^2v \mp Bku^k\right) \times \left(\frac{k^2(k-1)x}{2} + k^3\alpha u + (k+1)u^2v \mp Bk(2k-1)u^k\right).$$

For the solution $(u, v) = (u(x), v(x))$ corresponding to $y(x)$, we regard (x, v) as a function of u; which is a solution of the system

$$\frac{dx}{du} = \pm\frac{u^{k-1}}{\Phi_{\pm}(x, u, v)}\,, \quad \frac{dv}{du} = -\frac{u^{2k-3}\Psi_{\pm}(x, u, v)}{\Phi_{\pm}(x, u, v)}\,. \tag{3.4}$$

Equation (E_k) is equivalent to (3.4), whose right-hand members are analytic at $(x, u, v) = (x_0, 0, v_0)$, $v_0 \in \boldsymbol{C}$.

3.2. Auxiliary function. By (3.3) and (3.2)

$$(y' - By^{-(2k-1)})^2 = \frac{y^{2k+2}}{k^2}\left(1 + \frac{k^2x}{2}y^{-2k} + k^2\alpha y^{-2k-1} + y^{-2k-2}v\right)^2,$$

which is written in the form

$$\begin{aligned} V = {} & -B^2y^{-4k+2} + \frac{k^2x^2}{4}y^{-2k+2} + k^2\alpha xy^{-2k+1} + k^2\alpha^2y^{-2k} \\ & + \left(\frac{2}{k^2} + xy^{-2k} + 2\alpha y^{-2k-1}\right)v + \frac{y^{-2k-2}}{k^2}v^2 \end{aligned} \tag{3.5}$$

with

$$V := (y')^2 - 2By^{-2k+1}y' - \frac{y^{2k+2}}{k^2} - xy^2 - 2\alpha y\,. \tag{3.6}$$

Substituting the solution $y(x)$ of (E_k) into (3.6), we get the auxiliary function $V(x)$ associated with $y(x)$.

PROPOSITION 3.1. *If $y(x)^{-1}$ is bounded along a rectifiable curve Γ, then $V(x)$ is also bounded along Γ.*

PROOF. Differentiate $V(x)$ (cf. (3.6)) and eliminate $y''(x)$ by using (E_k) (with $y = y(x)$). Then we have

$$V'(x) - 2(2k-1)By(x)^{-2k}V(x) = 4(2k-1)B^2y(x)^{-4k+1}y'(x) + 4(k-1)Bxy(x)^{-2k+2} + 2(4k-3)B\alpha y(x)^{-2k+1}.$$

This is written in the form

$$\frac{d}{dx}\left[\left(V(x) + 2B^2y(x)^{-4k+2}\right)\exp\left(-2(2k-1)B\int_{\Gamma(x)} y(t)^{-2k}dt\right)\right]$$
$$= -2By(x)^{-2k+1}\left(2(2k-1)B^2y(x)^{-4k+1} - 2(k-1)xy(x) - (4k-3)\alpha\right)$$
$$\times \exp\left(-2(2k-1)B\int_{\Gamma(x)} y(t)^{-2k}dt\right),$$

where $\Gamma(x)$ denotes the part of Γ from its starting point to x. The boundedness of $V(x)$ immediately follows from this equality. □

3.3. Completion of the proof of Theorem 1.1. Let a_0 be a singularity of $y(x)$ defined by a rectifiable curve Γ terminating in a_0 such that $y(x)$ is analytic along $\Gamma \setminus \{a_0\}$. According to the value $A := \liminf_{x\to a_0,\, x\in\Gamma} |y(x)|$, we divide the proof into three cases:

(i) $0 < A < \infty$, (ii) $A = \infty$, (iii) $A = 0$.

Case (i). $0 < A < \infty$. Since the auxiliary function $V(x)$ is bounded as $x \to a_0$ along Γ (cf. Proposition 3.1), there exists a sequence $\{a_n\}_{n\in\boldsymbol{N}} \subset \Gamma$ such that $a_n \to a_0$ and that $y(a_n) \to y_0$ ($\neq 0, \infty$). Then, by (3.6) with $y = y(x)$, the sequence $\{y'(a_n)\}_{n\in\boldsymbol{N}}$ is also bounded, and we may choose a subsequence $\{a_{n(m)}\}_{m\in\boldsymbol{N}} \subset \Gamma$ satisfying $a_{n(m)} \to a_0$, $y(a_{n(m)}) \to y_0$ and $y'(a_{n(m)}) \to y_0'$ ($\neq \infty$). By Lemma 2.1, $y(x)$ is analytic at $x = a_0$.

Case (ii). $A = \infty$. Since $y(x) \to \infty$ as $x \to a_0$ along Γ, the function $V(x)$ is bounded along Γ near $x = a_0$. Substitution of $(y, V) = (y(x), V(x))$ into (3.5) yields a quadratic equation with respect to v. This equation admits a solution $v = v_-(x)$ which is analytic and bounded along $\Gamma \setminus \{a_0\}$. Note that one of the signs $\mp$ in (3.3) (resp. $\pm$ in (3.4)) corresponds to the branch $v_-(x)$. Let $u(x)$ be the branch corresponding to $v_-(x)$. Denote by $x = x(u)$ the inverse function of $u = u(x)$, whose existence is guaranteed by the fact that $|u'(x)| = |y'(x)/y(x)^2| \sim |y(x)^{k-1}|/k \neq 0, \infty$ along $\Gamma \setminus \{a_0\}$ (cf. (3.6)). Consider the functions $x = x(u)$ and $v = v_-(x(u))$ which are analytic in u along $u(\Gamma) \setminus \{0\} = \{u = u(x);\, x \in \Gamma \setminus \{a_0\}\}$. Then

(ii.a) $x(u) \to a_0$ as $u \to u(a_0) = 0$ along $u(\Gamma)$;

(ii.b) $v_-(x(u))$ is bounded along $u(\Gamma)$;

(ii.c) $(x, v) = (x(u), v_-(x(u)))$ satisfies (3.4).

Choosing a sequence $\{b_n\}_{n\in\boldsymbol{N}} \subset u(\Gamma)$ satisfying $b_n \to u(a_0) = 0$, $x(b_n) \to a_0$ and $v_-(x(b_n)) \to v_0$ ($\neq \infty$), and using Lemma 2.1, we deduce that $x(u)$ is analytic at $u = 0$, which implies that $x = a_0$ is at most an algebraic branch point of $y(x)$.

Case (iii). $A = 0$. In this case, we regard $y(x)$ as an analytic function on the Riemann surface $\mathfrak{R}_y$ with the projection $\pi_y : \mathfrak{R}_y \to \boldsymbol{C}$. Then $\Gamma \setminus \{a_0\}$ lies on $\mathfrak{R}_y$, and $\pi_y(a_0) :=$

$\lim_{x\to a_0, x\in\Gamma} \pi_y(x)$ is the end point of $\pi_y(\Gamma\setminus\{a_0\})$. For any curve $C \subset \mathfrak{R}_y$, denote by $\|C\|$ the length of $\pi_y(C) \subset \boldsymbol{C}$. For $a \in \mathfrak{R}_y$ and for $\rho_0 > 0$, denote by $U(a;\rho_0)$ $(\subset \mathfrak{R}_y)$ the connected component of $\pi_y^{-1}(\{\zeta \in \boldsymbol{C}\,;\ |\zeta - \pi_y(a)| < \rho_0\}) \subset \mathfrak{R}_y$ containing a. The projection $\pi_y : U(a;\rho_0) \to \{\zeta \in \boldsymbol{C}\,;\ |\zeta - \pi_y(a)| < \rho_0\}$ is a homeomorphism, provided that ρ_0 is sufficiently small.

The following fact is obtained from [11, Lemma 2.2] with $R_0 = \Delta = 1/2$, $K = 1 + |\pi_y(a_0)| + |\alpha|$.

LEMMA 3.2. *Set* $\theta_0 := (1 + |\pi_y(a_0)| + |\alpha|)^{-1}/42$. *Let* $c \in \mathfrak{R}_y$ *be a point such that* $|\pi_y(c) - \pi_y(a_0)| < 1/4$. *If the inequalities* $|y(c)| \le \theta_0/6$ *and* $|y'(c)| \ge 2$ *hold, then* $y(x)$ *is analytic in* $U(c;|y'(c)|^{-1}\theta_0)$ *and satisfies* $|y(x)| \ge \theta_0/4$ *on the boundary* $\partial U(c;|y'(c)|^{-1}\theta_0/2)$.

Put $\Gamma_0 := \{x \in \Gamma\,;\ |y(x)| \le \theta_0/6\} \subset \mathfrak{R}_y$. The supposition $A = 0$ implies $\Gamma_0 \cap \{x \in \Gamma\,;\ \|\Gamma(x,a_0)\| < \varepsilon\} \ne \emptyset$ for every $\varepsilon > 0$, where $\Gamma(x,a_0)$ denotes the part of Γ from x to a_0. We may suppose that $|y'(x)| \ge 2$ for $x \in \Gamma_0$. Indeed, if this is not the case, then $y(x)$ is analytic at $x = a_0$ (cf. Lemma 2.1). Let $a_* \in \mathfrak{R}_y$ be a point such that $\|\Gamma(a_*,a_0)\| < 1/4$. Let us start from a_* and proceed along Γ toward $x = a_0$. Let c_1 be the point in Γ_0 that we meet for the first time. By Lemma 3.2, there exists $D_1 := U(c_1;|y'(c_1)|^{-1}\theta_0/2)$ such that $|y(x)| \ge \theta_0/4$ on ∂D_1. Then $a_0 \notin D_1$, and $\partial D_1 \cap \Gamma_0 = \emptyset$. Restart from c_1 and proceed along Γ toward a_0 until we meet $c_2 \in \Gamma_0 \setminus D_1$. Then, $|y(x)| \ge \theta_0/4$ on ∂D_2, where $D_2 := U(c_2;|y'(c_2)|^{-1}\theta_0/2)$, which satisfies $a_0 \notin D_2$ and $\partial D_2 \cap \Gamma_0 = \emptyset$. Repeating this procedure, we get the sequences $\{D_n\}_{n\in\boldsymbol{N}}$ and $\{c_n\}_{n\in\boldsymbol{N}} \subset \Gamma_0$ of discs and their centres with the properties:

(iii.a) $D_n := U(c_n;r_n)$, $r_n := |y'(c_n)|^{-1}\theta_0/2$;
(iii.b) $|y(x)| \ge \theta_0/4$ on ∂D_n;
(iii.c) $a_0 \notin D_n$ and $\partial D_n \cap \Gamma_0 = \emptyset$;
(iii.d) $\|\Gamma(c_n,c_{n+1})\| > r_n$ and $\sum_{n\ge1} r_n \le \|\Gamma\|$, where $\Gamma(c_n,c_{n+1})$ is the part of Γ from c_n to c_{n+1}.

If c_n approaches some point $c_\infty \in \Gamma\setminus\{a_0\}$, then $r_n = |y'(c_n)|^{-1}\theta_0/2 \to 0$ as $n \to \infty$ (cf. (iii.a) and (iii.d)), that is, $|y'(c_\infty)| = \infty$, which contradicts the analyticity of $y(x)$ along $\Gamma\setminus\{a_0\}$. This implies that $c_n \to a_0$ as $n \to \infty$, and hence $\Gamma_0 \subset \bigcup_{n=1}^{\infty} D_n$. For each n, there exist only a finite number of D_j $(j \ne n)$ such that $D_j \cap D_n \ne \emptyset$. By (iii.d), we may choose a rectifiable curve Γ_* with the properties:

(iii.e) $\Gamma_* \subset \partial(\Gamma \cup (\bigcup_{n=1}^{\infty} D_n)) \subset \mathfrak{R}_y$;
(iii.f) Γ_* terminates in a_0;
(iii.g) $|y(x)| \ge \theta_0/6$ on $\Gamma_*\setminus\{a_0\}$;
(iii.h) $y(x)$ is analytic along $\Gamma_*\setminus\{a_0\}$.

Hence this case is reduced to either (i) or (ii). Consequently $x = a_0$ is at most an algebraic branch point of $y(x)$, which completes the proof of Theorem 1.1.

4. Proof of Theorem 1.3. Let us review some facts of value distribution theory.

For a meromorphic function $f(z)$ in $\boldsymbol{C}$, the proximity function, the counting function and the characteristic function are given by

$$m(r, f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi\,, \qquad \log^+ s := \max\{\log s, 0\}\,,$$
$$N(r, f) := \int_0^r (n(\rho, f) - n(0, f))\frac{d\rho}{\rho} + n(0, f)\log r\,,$$
$$T(r, f) := m(r, f) + N(r, f)\,,$$

respectively, where $n(r, f)$ denotes the number of poles of $f(z)$ in the disc $|z| \le r$, each counted according to its multiplicity. The characteristic function $T(r, f)$ is monotone increasing with respect to r. Furthermore $T(r, f) = O(\log r)$ if and only if $f(z)$ is a rational function (cf. [5], [9]). The following lemma is useful in the study of nonlinear differential equations (cf. [3, Lemma B.11], [9, Lemma 2.4.2]).

LEMMA 4.1. *Suppose that a meromorphic function* $w = f(z)$ *satisfies the differential equation* $w^{p+1} = P(z, w)$, $p \in \boldsymbol{N}$, *where* $P(z, w)$ *is a polynomial in* $z, w, w', \dots, w^{(q)}$. *If the total degree of* $P(z, w)$ *with respect to* w *and its derivatives does not exceed* p, *then* $m(r, f) = O(\log T(r, f) + \log r)$ *as* $r \to \infty$, $r \notin E$, *where* $E \subset (0, \infty)$ *is an exceptional set of finite linear measure.*

To prove the first assertion of Theorem 1.3, suppose that $y_*(x)$ is a nontrivial entire solution of (E_k). If $y_*(x)$ is a polynomial such that $y_*(x) = Cx^{\gamma_0} + O(x^{\gamma_0 - 1})$, $\gamma_0 \in \boldsymbol{N}$, $C \neq 0$ near $x = \infty$, then we have $(2k + 1)\gamma_0 = \gamma_0 + 1$, which is a contradiction. Hence $y_*(x)$ is transcendental and entire, so that $m(r, y_*) = T(r, y_*)$. By Lemma 4.1, for some $K_0 > 0$, we have $T(r, y_*) \le K_0 \log r$ outside an exceptional set E_0 of total length $\mu_0 < \infty$. For each r, we may choose a number $r'(r) \ge r$ satisfying $r'(r) - r \le 2\mu_0$ and $r'(r) \notin E_0$. Then

$$T(r, y_*) \le T(r'(r), y_*) \le K_0 \log(r'(r)) \le K_0 \log(r + 2\mu_0) = O(\log r)$$

for $r > 0$, which contradicts the transcendence of $y_*(x)$. This implies that (E_k) admits no nontrivial entire solution. Theorems 1.1 and 1.2 imply that each solution of (E_k) with $k \ge 2$ admits no pole. In this way we obtain the first assertion.

By Theorem 1.1 again, equation (E_2) admits no nontrivial algebraic solution. It is sufficient to show the second assertion for each odd integer $k \ge 3$. To prove by contradiction, we suppose the existence of a nontrivial algebraic solution. It is expanded into a Puiseux series of the form $(-k^2/(k + 1))^{1/(2k)}x^{1/(2k)} + \sum_{j=2k}^{\infty} b_j x^{-j/(2k)}$ around the point $x = \infty$, for which the degree of ramification is $e_\infty - 1 = 2k - 1$. By Theorem 1.2, for each branch point $x_\iota \neq \infty$, the degree of ramification is $e_\iota - 1 = k - 1$, which is even. These facts contradict the Riemann-Hurwitz formula

$$2(1 - g) = 2d - \sum_{\iota \neq \infty}(e_\iota - 1) - (e_\infty - 1)\,,$$

where d is the degree and g is the genus (see, for example [8]). Therefore (E_k) admits no nontrivial algebraic solution.

5. Proof of Theorem 1.4. 5.1. Inverse function of a hyperelliptic integral. The hyperelliptic integral

$$t - t_0 = \int_{w_0}^{w(t)} \frac{ds}{\sqrt{s^{2k+2} + C}}, \quad C \neq 0 \tag{5.1}$$

defines the function $Y = w(t)$ satisfying the differential equations

$$\dot{Y}^2 = Y^{2k+2} + C \qquad (\dot{Y} = dY/dt) \tag{5.2}$$

and

$$\ddot{Y} = (k+1)Y^{2k+1}. \tag{5.3}$$

By (5.1), every movable singularity of $w(t)$ is an algebraic branch point (see also [7, Chap. 13]).

Suppose that $k \geq 3$. We construct the Riemann surface of $\sqrt{s^{2k+2} + C}$ in the standard manner. Set $\zeta_h := (-C)^{1/(2k+2)} e^{2\pi i h/(2k+2)}$ $(h = 0, 1, \ldots, 2k+1)$, and denote by X^ε $(\varepsilon = 1, 2)$ two copies of $P^1(\boldsymbol{C}) \setminus \left(\bigcup_{j=0}^{k} \Sigma_j\right)$ cut along the segments $\Sigma_j := [\zeta_{2j}, \zeta_{2j+1}]$ $(j = 0, 1, \ldots, k)$, where $P^1(\boldsymbol{C}) = \boldsymbol{C} \cup \{\infty\}$. Let Σ_j^- and Σ_j^+ be the edges of the cut Σ_j. Gluing Σ_j^- (resp. Σ_j^+) of X^1 to Σ_j^+ (resp. Σ_j^-) of X^2, we get the Riemann surface of $\sqrt{s^{2k+2} + C}$ admitting $2k$ cycles. Let γ_1 (resp. γ_2) be the cycle lying in X^1 and surrounding only $\Sigma_0 = [\zeta_0, \zeta_1]$ (resp. $\Sigma_1 = [\zeta_2, \zeta_3]$) in the positive sense. In addition, choose another cycle $\gamma_0 = S_1 \cup S_2$, where S_1 (resp. S_2) is the segment in X^1 (resp. in X^2) from ζ_1 to ζ_2 (resp. ζ_2 to ζ_1). Now consider periods of $w(t)$ written as

$$\omega_j := \int_{\gamma_j} \frac{ds}{\sqrt{s^{2k+2} + C}} \quad (j = 0, 1, 2),$$

where the branches of the integrands are taken in such a way that they coincide at the point $s = (\zeta_1 + \zeta_2)/2 \in X^1$. It is easy to check that $\omega_1 = \omega_0 e^{-\pi i/(k+1)}$ and that $\omega_2 = \omega_0 e^{\pi i/(k+1)}$.

LEMMA 5.1. *Set* $\lambda := (\omega_1 + \omega_2)/\omega_0 = 2\cos(\pi/(k+1))$. *If* $k \geq 3$, *then there exist infinitely many pairs* $(p, q) \in \boldsymbol{N}^2$ *such that* $|q\lambda - p| < 1/q$.

PROOF. It is sufficient to show that λ is an irrational number. We write $2(k+1) = 2^d(2l+1)$, $d \in \boldsymbol{N}$, $l \in \boldsymbol{N} \cup \{0\}$. If $l = 0$, then $d \geq 3$, and hence $\lambda = 2\cos(\pi/2^{d-1})$ is an irrational number. Next suppose that $l = 1$. Since $k \geq 3$, we have $d \geq 2$, and hence $\lambda = 2\cos(\theta_1/2^d)$, $\theta_1 = 2\pi/3$ is an irrational number. Finally suppose that $l \geq 2$. Set $\varrho := e^{2\pi i/(2l+1)}$. Since $(\varrho^l + \varrho^{-l}) + \cdots + (\varrho + \varrho^{-1}) + 1 = 0$, the number $\mu = \varrho + \varrho^{-1} = 2\cos(2\pi/(2l+1))$ satisfies

$$\mu^l + \kappa_{l-1}\mu^{l-1} + \cdots + \kappa_1\mu + \kappa_0 = 0, \quad \kappa_j \in \boldsymbol{Z},$$

which implies μ is irrational. Indeed, if $\mu \in \boldsymbol{Q}$, then $\mu \in \boldsymbol{Z}$, so that $\mu = 0, \pm 1$, which contradicts $l \geq 2$. Consequently $\lambda = 2\cos(\theta_l/2^d)$ with $\theta_l = 2\pi/(2l+1)$ is also an irrational number. □

PROPOSITION 5.2. *If* $k \geq 3$ *and if* $C \neq 0$, *then* $w(t)$ *is infinitely many-valued.*

PROOF. Suppose that $w(t)$ is finitely many-valued. Consider the Riemann surface of $w(t)$ denoted by $\mathfrak{R}_w$ with the projection $\pi_w : \mathfrak{R}_w \to \boldsymbol{C}$. Choose a point $b_0 \in \boldsymbol{C}$ with the property: there exists an open set $U_0 \ni b_0$ such that, for every connected component W of $\pi_w^{-1}(U_0) \subset \mathfrak{R}_w$, the restriction of π_w to W is a homeomorphism between W and U_0. Take a point $\beta_0 \in \pi_w^{-1}(b_0)$. By Lemma 5.1, there exists a sequence $\{\sigma_n\}_{n\in \boldsymbol{N}} \subset \mathfrak{R}_w$ together with $(p_n, q_n) \in \boldsymbol{N}^2$ such that $w(\sigma_n) = w(\beta_0)$ and that $\pi_w(\sigma_n) = \pi_w(\beta_0) + q_n(\omega_1+\omega_2) - p_n\omega_0 \to b_0$ as $n \to \infty$. Since $\pi_w^{-1}(b_0)$ is a finite set, there exist a subsequence $\{\sigma_{n(m)}\}_{m\in \boldsymbol{N}}$ and a point $\beta_\infty \in \pi_w^{-1}(b_0)$ such that $w(\sigma_{n(m)}) = w(\beta_0)$ and that $\sigma_{n(m)} \to \beta_\infty$ as $m \to \infty$. Hence $w(t) \equiv w(\beta_0)$ on $\mathfrak{R}_w$, which is a contradiction. This completes the proof. □

REMARK 5.1. If $k = 2$, then (5.2) admits the general solution $w(t) = \sqrt{g(t - t_0)}$, where $g(t)$ is an elliptic function of Jacobi type satisfying $\dot{g}(t)^2 = 4g(t)^4 + 4Cg(t)$. In this case $w(t)$ is a 2-valued algebroid function.

5.2. Completion of the proof of Theorem 1.4. If $k = 2$, then Theorem A.1 in Appendix implies the existence of a general solution with a movable logarithmic branch point, from which the conclusion of Theorem 1.4 immediately follows. It is sufficient to prove the theorem under the supposition $k \geq 3$. Let $y(x)$ be a solution of (E_k) satisfying the initial condition $y(0) = y_0$, $y'(0) = y_1$. Let ε be an arbitrary small positive number. The change of variables $y = k^{1/k}\varepsilon^{-1}Y$, $x = \varepsilon^k t$ takes (E_k) into

$$\ddot{Y} = (k+1)Y^{2k+1} + \varepsilon^{3k}tY + k^{-1/k}\alpha\varepsilon^{2k+1}, \tag{5.4}$$

which admits the solution $Y_\varepsilon(t) = k^{-1/k}\varepsilon y(t)$ satisfying $Y_\varepsilon(0) = \chi_0(\varepsilon) := k^{-1/k}\varepsilon y_0$ and $\dot{Y}_\varepsilon(0) = \chi_1(\varepsilon) := k^{-1/k}\varepsilon^{k+1}y_1$. Equation (5.4) with $\varepsilon = 0$ coincides with (5.3). Let $Y_0(t)$ be the solution of (5.3) satisfying the same initial condition

$$Y_0(0) = \chi_0(\varepsilon), \quad \dot{Y}_0(0) = \chi_1(\varepsilon). \tag{5.5}$$

Then $Y_0(t)$ is also a solution of

$$\dot{Y}^2 = Y^{2k+2} + \chi_1(\varepsilon)^2 - \chi_0(\varepsilon)^{2k+2}. \tag{5.6}$$

Consider the Riemann surface of $Y_0(t)$ denoted by $\mathfrak{R}_0$ with the projection $\pi_0 : \mathfrak{R}_0 \to \boldsymbol{C}$. Let $\tau_0 \in \mathfrak{R}_0$ be a point such that $\pi_0(\tau_0) = 0$ at which initial condition (5.5) is given. Let ν be an arbitrary natural number. By Proposition 5.2 with $C = 1/2$ and the continuity with respect to initial data, we may choose $\delta = \delta(\nu) > 0$ so small that the conditions

$$|\chi_0(\varepsilon) - 2^{-1/(2k+2)}| < \delta, \quad |\chi_1(\varepsilon) - 1| < \delta \tag{5.7}$$

guarantee the existence of ν rectifiable paths $\Gamma_j \subset \mathfrak{R}_0$ $(1 \leq j \leq \nu)$ with the properties:

(i) Γ_j starts from τ_0 and terminates in τ_j, where τ_j $(1 \leq j \leq \nu)$ satisfy $\pi_0(\tau_1) = \cdots = \pi_0(\tau_\nu)$;

(ii) Γ_j is independent of $\chi_0(\varepsilon)$ and $\chi_1(\varepsilon)$;

(iii) $Y_0(t)$ continues analytically along Γ_j $(1 \leq j \leq \nu)$;

(iv) $|Y_0(\tau_j) - Y_0(\tau_{j'})| > \delta$ for every pair (j, j') such that $j \neq j'$.

Then $Y_\varepsilon(t)$ satisfying (5.4) also continues analytically along Γ_j $(1 \le j \le \nu)$ to ν different branches, provided that $\varepsilon > 0$ is sufficiently small. For such ε, as long as the initial data y_0 and y_1 satisfy (5.7), the solution $y(x)$ is a ν-valued function. This completes the proof of Theorem 1.4.

Appendix. General solution of (E$_2$). There exists a general solution of (E$_2$) with a movable logarithmic branch point described as follows:

THEOREM A.1. *For given complex numbers x_0 and c, equation* (E$_2$) *admits a solution expressible in the form*

$$y(x) = \omega_2\xi^{-1/2} - \frac{\omega_2 x_0}{3}\xi^{3/2} - \frac{4\alpha}{7}\xi^2 + \left(\frac{\omega_2}{4}\log\xi + c\right)\xi^{5/2} + \sum_{j\ge 6} \Lambda_j(\log\xi)\xi^{j/2},$$

$$\xi = x - x_0, \quad \omega_2 = 1 \text{ or } i$$

with the properties:

(i) $\Lambda_j(L) \in \boldsymbol{A}_{x_0,c}[L],\ \boldsymbol{A}_{x_0,c} := \boldsymbol{C}[x_0, c],\ 2\deg_L \Lambda_j + 7 \le j;$

(ii) *the series on the right-hand side converges for $\xi \in \mathcal{R}$ satisfying $|\xi| < r$, $|\arg\xi| < R$, where R is an arbitrary large positive number, $r = r(R)$ is a sufficiently small positive number depending on R, and $\mathcal{R}$ is the universal covering of $\boldsymbol{C} \setminus \{0\}$.*

A.1. Derivation of an integral equation. In what follows we suppose that $\omega_2 = 1$. The case $\omega_2 = i$ can be treated in a similar manner. By the same argument as in Section 2.2, we get the first three terms $\xi^{-1/2} - (x_0/3)\xi^{3/2} - (4\alpha/7)\xi^2$. Set

$$y = \xi^{-1/2} - \frac{x_0}{3}\xi^{3/2} - \frac{4\alpha}{7}\xi^2 + \xi^{5/2}v, \quad \xi = x - x_0 \tag{A.1}$$

and substitute this into (E$_2$). Then we have

$$\begin{aligned}\xi^2\frac{d^2v}{d\xi^2} + 5\xi\frac{dv}{d\xi} &= 1 + \xi g_0(\xi) + \xi^2 g_1(\xi)v + \xi^3 g_2(\xi)v^2 + \xi^6 g_3(\xi)v^3 \\ &\quad + \xi^9 g_4(\xi)v^4 + \frac{3}{4}\xi^{12}v^5\end{aligned} \tag{A.2}$$

with $g_\iota(\xi) \in \boldsymbol{A}_{x_0}[\xi^{1/2}]$, $\boldsymbol{A}_{x_0} := \boldsymbol{C}[x_0]$ $(0 \le \iota \le 4)$, $g_0(0) = x_0/2$. The change of variables

$$\xi^{1/2} = t, \quad v = \frac{1}{4}\log\xi + c + w = \frac{1}{2}\log t + c + w$$

takes (A.2) into

$$\frac{d^2w}{dt^2} + 9t^{-1}\frac{dw}{dt} = F(t, w) \tag{A.3}$$

with

$$F(t, w) = \sum_{\iota=0}^{5} P_\iota(t, \log t)w^\iota, \quad P_\iota(t, L) = \sum_{h=e(\iota)}^{m(\iota)} p_{\iota h}(L)t^h \tag{A.4}$$

satisfying

$$e(\iota) \geq 2\iota\,, \tag{A.5}$$

$$p_{\iota h}(L) \in \boldsymbol{A}_{x_0,c}[L]\,, \quad 2\deg_L p_{\iota h} \leq h\,, \quad p_{00}(L) \equiv 2x_0\,. \tag{A.6}$$

Observing that the equation $d^2w/dt^2 + 9t^{-1}dw/dt = 0$ admits the solutions $w = 1$ and $w = t^{-8}$, we consider the integral equation

$$w(t) = \frac{1}{8}\int_0^t (s - t^{-8}s^9)F(s, w(s))ds\,, \tag{A.7}$$

for $t \in \mathcal{R}$, where the path of integration is the segment joining 0 to t. The solution of (A.7) satisfies equation (A.3).

A.2. Logarithmic polynomials. Let $\mathfrak{L}$ be the set of polynomials in $(t, \log t)$ written in the form

$$P(t, \log t) = \sum_{h=0}^{m} p_h(\log t)t^h$$

with

$$p_h(L) \in \boldsymbol{A}_{x_0,c}[L]\,, \quad 2\deg_L p_h + 2 \leq h \quad (0 \leq h \leq m)\,.$$

It is easy to see that, for any $h, l \in \boldsymbol{N} \cup \{0\}$,

$$\int_0^t s^h(\log s)^l ds = t^{h+1}\varpi_{hl}(\log t)\,, \quad \varpi_{hl}(L) \in \boldsymbol{Q}[L]\,, \quad \deg_L \varpi_{hl} = l\,,$$

which implies the following:

LEMMA A.2. *If $P(t, \log t) \in \mathfrak{L}$, then*

$$P_{\mathrm{int}}(t, \log t) := \int_0^t P(s, \log s)ds \in \mathfrak{L}\,,$$

$$\deg_t P_{\mathrm{int}}(t, L) = \deg_t P(t, L) + 1\,, \quad \deg_L P_{\mathrm{int}}(t, L) = \deg_L P(t, L)\,.$$

A.3. Iterative sequence. Define the sequence $\{w_n(t)\}_{n=0}^{\infty}$ by the recursive relation

$$\begin{aligned} &w_0(t) \equiv 0\,, \\ &w_{n+1}(t) = \frac{1}{8}\int_0^t (s - t^{-8}s^9)F(s, w_n(s))ds \end{aligned} \tag{A.8}$$

for $n \geq 0$. By (A.4), (A.5) and Lemma A.2, we can inductively verify $w_n(t) \in \mathfrak{L}$ and $w_{n+1}(t) - w_n(t) \in \mathfrak{L}$ for $n \geq 0$.

For given $R > 0$, choose $r < 1$ so small that $|t \log t| < |t|^{1/2}$ holds for $|\arg(t^2)| < R$, $|t^2| < r$. By (A.4), (A.5) and (A.6),

$$|F(t, 0)| \leq M_0\,, \tag{A.9}$$

$$|F(t, w) - F(t, u)| \leq M_0|t|\,|w - u|\,, \tag{A.10}$$

for

$$|\arg(t^2)| < R\,, \quad |t^2| < r\,, \quad |w| < 1\,, \quad |u| < 1\,, \tag{A.11}$$

where $M_0 = M_0(|c|, |x_0|)$ is some positive number independent of R and r. Hence by (A.8),

$$\text{(A.12)} \qquad |w_{n+2}(t) - w_{n+1}(t)| \le \frac{M_0}{4} \int_0^t |s|^2 |w_{n+1}(s) - w_n(s)|\,|ds|\,,$$

provided that $(t, u, w) = (t, w_n, w_{n+1})$ satisfies (A.10). Then, if necessary, retaking r smaller in such a way that

$$\text{(A.13)} \qquad \exp(M_0 r^2/8) - 1 < 1/2\,,$$

we have the following:

$$\text{(A.14)} \qquad |w_n(t)| < 1\,,$$

$$\text{(A.15)} \qquad |w_{n+1}(t) - w_n(t)| \le \frac{M_0^{n+1}|t|^{2(n+1)}}{8^{n+1}(n+1)!}$$

$(n \ge 0)$ for $|\arg(t^2)| < R$, $|t^2| < r$. These are verified by induction on n. Since

$$|w_1(t) - w_0(t)| \le \frac{1}{4} \int_0^t |s|\,|F(s,0)|\,|ds| \le \frac{M_0}{8}|t|^2\,,$$

inequalities (A.14) and (A.15) are valid for $n = 0$. Moreover, supposing that (A.14) and (A.15) are valid for $n \le N$, we deduce that

$$|w_{N+1}(t)| \le |w_0(t)| + \sum_{n=0}^{N} |w_{n+1}(t) - w_n(t)|$$
$$\le \sum_{n=0}^{N} \frac{M_0^{n+1}|t|^{2(n+1)}}{8^{n+1}(n+1)!} \le \exp(M_0|t|^2/8) - 1 \le \frac{1}{2}\,,$$

and that, by (A.12),

$$|w_{N+2}(t) - w_{N+1}(t)| \le \frac{M_0}{4} \int_0^t \frac{M_0^{N+1}|s|^{2(N+1)+2}}{8^{N+1}(N+1)!}|ds| \le \frac{M_0^{N+2}|t|^{2(N+2)}}{8^{N+2}(N+2)!}\,.$$

Thus we have verified (A.14) and (A.15) for all $n \ge 0$.

A.4. Completion of the proof of Theorem A.1. By (A.15), $w(t) := \lim_{n\to\infty} w_n(t) = \sum_{n=0}^{\infty}(w_{n+1}(t) - w_n(t))$ is holomorphic for $t \in \mathcal{R}$, $|\arg(t^2)| < R$, $|t^2| < r$, and satisfies $|w(t) - w_n(t)| \le C_0|t|^{2(n+1)}$ for every n, where C_0 is a constant independent of n. Write $w_n(t) \in \mathfrak{L}$ in the form

$$w_n(t) = \sum_{h=2}^{m^*(n)} W_h^n(\log t)t^h\,, \quad W_h^n(L) \in \boldsymbol{A}_{x_0,c}[L]\,, \quad 2\deg_L W_h^n + 2 \le h\,.$$

By (A.15) again, for every pair (N, N') such that $N < N'$, we have $|w_{N'}(t) - w_N(t)| = O(|t|^{2(N+1)})$ in the domain $|\arg(t^2)| < R$, $|t^2| < r$. This implies $W_h^N(L) \equiv W_h^{N'}(L)$ for every $h \le 2N+1$, as far as $N < N'$. Therefore $w(t)$ can be expressed in the form

$$w(t) = \sum_{h=2}^{\infty} W_h(\log t)t^h\,, \quad W_h(L) \in \boldsymbol{A}_{x_0,c}[L]\,, \quad 2\deg_L W_h + 2 \le h\,,$$

whose right-hand member converges uniformly in $t^2 \in \mathcal{R}$, $|\arg(t^2)| < R$, $|t^2| < r$. Then $v(\xi) = (1/4)\log\xi + c + w(\xi^{1/2})$ satisfies (A.2). Substituting $v = v(\xi)$ into (A.1), we obtain the required expression, which completes the proof of Theorem A.1.

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DEPARTMENT OF MATHEMATICS
KEIO UNIVERSITY
3–14–1, HIYOSHI, KOHOKU-KU
YOKOHAMA 223–8522
JAPAN

E-mail address: shimomur@math.keio.ac.jp