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CLASSIFICATION OF MÖBIUS ISOPARAMETRIC HYPERSURFACES IN THE UNIT SIX-SPHERE

Dedicated to Professors Udo Simon and Seiki Nishikawa on the occasion of
their seventieth and sixtieth birthday

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Abstract. An immersed umbilic-free hypersurface in the unit sphere is equipped with three Möbius invariants, namely, the Möbius metric, the Möbius second fundamental form and the Möbius form. The fundamental theorem of Möbius submanifolds geometry states that a hypersurface of dimension not less than three is uniquely determined by the Möbius metric and the Möbius second fundamental form. A Möbius isoparametric hypersurface is defined by two conditions that it has vanishing Möbius form and has constant Möbius principal curvatures. It is well-known that all Euclidean isoparametric hypersurfaces are Möbius isoparametrics, whereas the latter are Dupin hypersurfaces. In this paper, combining with previous results, a complete classification for all Möbius isoparametric hypersurfaces in the unit six-sphere is established.

1. Introduction. For a hypersurface $x : M^n \to \boldsymbol{S}^{n+1}$ in the $(n+1)$-dimensional unit sphere $\boldsymbol{S}^{n+1}$ without umbilic points, we choose a local orthonormal basis $\{e_1, \dots, e_n\}$ with respect to the induced metric $I = dx \cdot dx$ and the dual basis $\{\theta_1, \dots, \theta_n\}$. Let $h = \sum_{i,j} h_{ij}\theta_i \otimes \theta_j$ be the second fundamental form of x, with the squared length $\|h\|^2 = \sum_{i,j}(h_{ij})^2$ and the mean curvature $H = (1/n)\sum_i h_{ii}$, respectively. Define $\rho^2 = n/(n-1) \cdot (\|h\|^2 - nH^2)$. Then the positive definite form $g = \rho^2 dx \cdot dx$ is Möbius invariant, which is called the Möbius metric of $x : M^n \to \boldsymbol{S}^{n+1}$. The Möbius second fundamental form $\mathbf{B}$, another basic Möbius invariant of x, together with g completely determine a hypersurface of $\boldsymbol{S}^{n+1}$ up to Möbius equivalence, see Theorem 2.1 below.

An important class of hypersurfaces for Möbius differential geometry is the class of so-called Möbius isoparametric hypersurfaces in $\boldsymbol{S}^{n+1}$. According to [12], an umbilic-free hypersurface of $\boldsymbol{S}^{n+1}$ is called Möbius isoparametric if it satisfies the condition that the Möbius invariant 1-form

$$\boldsymbol{\Phi} = -\rho^{-1}\sum_i \left\{ e_i(H) + \sum_j (h_{ij} - H\delta_{ij}) e_j(\log\rho) \right\} \theta_i \tag{1.1}$$

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vanishes and all its Möbius principal curvatures are constant. Recall that the Möbius principal curvatures are the eigenvalues of the so-called Möbius shape operator $\boldsymbol{\Psi} := \rho^{-1}(\mathbf{S} - H\,\mathrm{id})$, where $\mathbf{S}$ denotes the standard shape operator of $x : M^n \to \boldsymbol{S}^{n+1}$. This definition of Möbius isoparametric hypersurfaces is meaningful. Indeed, if we compare it with that of (Euclidean) isoparametric hypersurfaces in $\boldsymbol{S}^{n+1}$, then we see that under Möbius transformation the images of all hypersurfaces of the sphere with constant mean curvature and constant scalar curvature satisfy $\boldsymbol{\Phi} \equiv 0$, and the Möbius invariant operator $\boldsymbol{\Psi}$ plays the same role in Möbius geometry as $\mathbf{S}$ does in the Euclidean situation (see Theorem 2.1 below). Standard examples of Möbius isoparametric hypersurfaces are the images of (Euclidean) isoparametric hypersurfaces in $\boldsymbol{S}^{n+1}$ under Möbius transformations. However, there are other examples which cannot be obtained in this way. For example, it occurs among our classification for hypersurfaces of $\boldsymbol{S}^{n+1}$ with *parallel* Möbius second fundamental form, meaning that the Möbius second fundamental form is parallel with respect to the Levi-Civita connection of the Möbius metric g. For more details, we refer to [8]. On the other hand, it was proved in [12] that any Möbius isoparametric hypersurface is in particular a Dupin hypersurface, which is a consequence of [21] that for a compact Möbius isoparametric hypersurface embedded in $\boldsymbol{S}^{n+1}$, the number γ of distinct principal curvatures can only take the values $\gamma = 2,\ 3,\ 4,\ 6$.

In [12], the authors classified locally all Möbius isoparametric hypersurfaces of $\boldsymbol{S}^{n+1}$ with $\gamma = 2$. In [9] and [11], by relaxing the restriction of $\gamma = 2$, we established the classification for all Möbius isoparametric hypersurfaces in $\boldsymbol{S}^4$ and $\boldsymbol{S}^5$, respectively. More precisely, we showed that a Möbius isoparametric hypersurface in $\boldsymbol{S}^4$ is either of parallel Möbius second fundamental form or it is Möbius equivalent to a Euclidean isoparametric hypersurface in $\boldsymbol{S}^4$ with three distinct principal curvatures, that is, a tube of constant radius over a standard Veronese embedding of $\boldsymbol{R}P^2$ into $\boldsymbol{S}^4$. Also, a hypersurface in $\boldsymbol{S}^5$ is Möbius isoparametric if and only if either, it has parallel Möbius second fundamental form, or it is Möbius equivalent to the preimage of the stereographic projection of the cone $\tilde{x} : N^3 \times \boldsymbol{R}^+ \to \boldsymbol{R}^5$ defined by $\tilde{x}(x,t) = tx$, where $t \in \boldsymbol{R}^+$ and $x : N^3 \to \boldsymbol{S}^4 \hookrightarrow \boldsymbol{R}^5$ is the Cartan isoparametric immersion in $\boldsymbol{S}^4$ with three distinct principal curvatures, or it is Möbius equivalent to a Euclidean isoparametric hypersurface in $\boldsymbol{S}^5$ with four distinct principal curvatures. In a very recent effort [10], we established a complete classification for Möbius isoparametric hypersurfaces in $\boldsymbol{S}^{n+1}$, $n \geq 5$, with three distinct Möbius principal curvatures such that one of them is simple. Hence, as an immediately consequence, we have classified the Möbius isoparametric hypersurfaces in $\boldsymbol{S}^6$ with three distinct Möbius principal curvatures.

In this paper, we will classify Möbius isoparametric hypersurfaces in $\boldsymbol{S}^6$ with four and five distinct Möbius principal curvatures. Combining this with the previous results, we have completed the classification for all Möbius isoparametric hypersurfaces in $\boldsymbol{S}^6$. To state our results, let us recall that for the n-dimensional hyperbolic space of constant sectional curvature $-c < 0$,

$$\begin{aligned}\boldsymbol{H}^n(-c) = \{(y_0, y_1, \ldots, y_n) \in \boldsymbol{L}^{n+1};\ \langle y, y\rangle &= -y_0^2 + y_1^2 + \cdots + y_n^2 \\ &= -1/c,\ y_0 \geq 1/\sqrt{c})\}\end{aligned}$$

and the hemisphere $\boldsymbol{S}^n_+ = \{(x_1, \dots, x_{n+1}) \in \boldsymbol{S}^n; \sum_i x_i^2 = 1, x_1 \geq 0)\}$, one can define the conformal diffeomorphisms $\sigma : \boldsymbol{R}^n \to \boldsymbol{S}^n \setminus \{(-1, 0, \dots, 0)\}$ and $\tau : \boldsymbol{H}^n(-1) \to \boldsymbol{S}^n_+$ by

$$\sigma(u) = \left(\frac{1- \mid u \mid^2}{1+ \mid u \mid^2}, \frac{2u}{1+ \mid u \mid^2} \right), \quad u \in \boldsymbol{R}^n$$

and

$$\tau(y) = (1/y_0, y'/y_0), \quad y_0 \geq 1, \quad y = (y_0, y_1, \dots, y_n) =: (y_0, y') \in \boldsymbol{H}^n(-1),$$

respectively. Then we can state our main result as follows.

CLASSIFICATION THEOREM. *Let $x : M^5 \to \boldsymbol{S}^6$ be a Möbius isoparametric hypersurface. Then x is Möbius equivalent to an open subset of one of the following hypersurfaces in $\boldsymbol{S}^6$:*

(1) *The standard torus $\boldsymbol{S}^k(a) \times \boldsymbol{S}^{5-k}(b)$ with $k = 1, 2, 3, 4$ and $a^2 + b^2 = 1$.*

(2) *The image of σ of the standard cylinder $\boldsymbol{S}^k(1) \times \boldsymbol{R}^{5-k}$ with $k = 1, 2, 3, 4$.*

(3) *The image of τ of the standard hyperbolic cylinder $\boldsymbol{S}^k(r) \times \boldsymbol{H}^{5-k}(-1/(1+r^2)) \subset \boldsymbol{H}^6(-1)$ with $k = 1, 2, 3, 4$ and $r > 0$.*

(4) *The preimage of the stereographic projection of the warped product embedding*

$$\tilde{x} : \boldsymbol{S}^p(a) \times \boldsymbol{S}^q(\sqrt{1-a^2}) \times \boldsymbol{R}^+ \times \boldsymbol{R}^{4-p-q} \to \boldsymbol{R}^6$$

with $p \geq 1, q \geq 1, p+q \leq 4, 0 < a < 1$, defined by

$$\tilde{x}(u', u'', t, u''') = (tu', tu'', tu'''),$$

$$u' \in \boldsymbol{S}^p(a), \quad u'' \in \boldsymbol{S}^q(\sqrt{1-a^2}), \quad t \in \boldsymbol{R}^+, \quad u''' \in \boldsymbol{R}^{4-p-q}.$$

(5) *Minimal hypersurfaces defined by $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^5 = N^3 \times \boldsymbol{H}^2(-2/15) \to \boldsymbol{S}^6$ with*

$$\tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0, \quad y_0 \in \boldsymbol{R}^+, \quad y_1 \in \boldsymbol{R}^5, \quad y_2 \in \boldsymbol{R}^2.$$

Here $y_1 : N^3 \to \boldsymbol{S}^4(\sqrt{30}/2) \hookrightarrow \boldsymbol{R}^5$ is Cartan's minimal isoparametric hypersurface with vanishing scalar curvature and principal curvatures $\sqrt{10}/5, 0, -\sqrt{10}/5$, and $(y_0, y_2) : \boldsymbol{H}^2(-2/15) \hookrightarrow \boldsymbol{L}^3$ is the standard embedding of the hyperbolic space of sectional curvature $-2/15$ into the 3-dimensional Lorentz space with $-y_0^2 + y_2^2 = -15/2$.

(6) *Hypersurfaces defined by $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^5 = N^3 \times \boldsymbol{H}^2(-r^{-2}) \to \boldsymbol{S}^6$ with*

$$\tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0, \quad y_0 \in \boldsymbol{R}^+, \quad y_1 \in \boldsymbol{R}^5, \quad y_2 \in \boldsymbol{R}^2, \quad r > 0.$$

Here $y_1 : N^3 \to \boldsymbol{S}^4(r) \hookrightarrow \boldsymbol{R}^5$ is an isoparametric immersion in $\boldsymbol{S}^4(r)$ with three distinct principal curvatures whose mean curvature H_1 and constant scalar curvature R_1 are given by

$$H_1 = -\frac{5}{3}\lambda \neq 0, \quad R_1 = \frac{6}{r^2} - \frac{4}{5} + 20\lambda^2,$$

and $(y_0, y_2) : \boldsymbol{H}^2(-r^{-2}) \hookrightarrow \boldsymbol{L}^3$ is the standard embedding of the hyperbolic space of sectional curvature $-r^{-2}$ into the 3-dimensional Lorentz space with $-y_0^2 + y_2^2 = -r^2$.

(7) *Hypersurfaces defined by* $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^5 = N^4 \times \boldsymbol{H}^1(-r^{-2}) \to \boldsymbol{S}^6$ *with*

$$\tilde{x}_1 = y_1/y_0\,, \quad \tilde{x}_2 = y_2/y_0\,, \quad y_0 \in \boldsymbol{R}^+\,, \quad y_1 \in \boldsymbol{R}^6\,, \quad y_2 \in \boldsymbol{R}\,, \quad r > 0\,.$$

Here $y_1 : N^4 \to \boldsymbol{S}^5(r) \hookrightarrow \boldsymbol{R}^6$ *is an Euclidean isoparametric hypersurfaces with four distinct principal curvatures whose mean curvature* H_1 *and scalar curvature* R_1 *are given by*

$$H_1 = -\frac{5}{4}\lambda\,, \quad R_1 = \frac{12}{r^2} - \frac{4}{5} + 20\lambda^2\,,$$

and $(y_0, y_2) : \boldsymbol{H}^1(-r^{-2}) \hookrightarrow \boldsymbol{L}^2$ *is the standard embedding with* $-y_0^2 + y_2^2 = -r^2$.

Our study of Möbius isoparametric hypersurfaces is closely connected to that of Dupin hypersurfaces. More concretely, the above mentioned results are counterparts of Dupin hypersurfaces, cf. [4–6, 18–21]. For further background, we note that the classification of hypersurfaces with four principal curvatures in $\boldsymbol{S}^{n+1}$ under the Möbius transformation group can be compared with that of Dupin hypersurfaces with four principal curvature under the Lie sphere transformation group; that was established by Cecil, Chi and Jensen in [5, 6]. It is interesting to point out that the Lie sphere transformation group contains the Möbius transformation group in $\boldsymbol{S}^{n+1}$ as a subgroup; the dimension difference is $n+3$. Therefore, the Möbius differential geometry for hypersurfaces in spheres seems to be essentially different from the Lie sphere geometry, and therefore more attention should be deserved to their geometry.

The organization of this paper is as follows. In Section 2, we first review some elementary facts of Möbius geometry for hypersurfaces in $\boldsymbol{S}^{n+1}$, then present known results achieved in [10], [12] and [23], respectively. In Section 3, investigating Möbius isoparametric hypersurfaces of $\boldsymbol{S}^6$ with four distinct Möbius principal curvatures, we first show that for some constant λ the linear combination $\mathbf{A} + \lambda\mathbf{B}$ of the Blaschke tensor $\mathbf{A}$ and the Möbius second fundamental form $\mathbf{B}$ has two distinct constant eigenvalues. Then we prove Theorem 3.1 that gives a preliminary classification for such hypersurfaces. In Section 4, we prove Theorem 4.1 that gives a preliminary classification for Möbius isoparametric hypersurfaces of $\boldsymbol{S}^6$ with five distinct Möbius principal curvatures. In Section 5, we prove Theorem 5.1 and Theorem 5.2 by calculating the Möbius invariants of the hypersurfaces which appear in Theorem 3.1 and Theorem 4.1. Finally, in Section 6, we complete the proof of the Classification Theorem.

2. Möbius invariants for hypersurfaces in $\boldsymbol{S}^{n+1}$. In this section we define Möbius invariants and recall the structure equations for hypersurfaces in $\boldsymbol{S}^{n+1}$. For details we refer to [22]. Let $\boldsymbol{L}^{n+3}$ be the Lorentz space, namely, $\boldsymbol{R}^{n+3}$ with inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, w\rangle = -x_0w_0 + x_1w_1 + \cdots + x_{n+2}w_{n+2}\,,$$

for $x = (x_0, x_1, \dots, x_{n+2})$, $w = (w_0, w_1, \dots, w_{n+2}) \in \boldsymbol{R}^{n+3}$.

For an immersed hypersurface $x : M^n \to \boldsymbol{S}^{n+1} \hookrightarrow \boldsymbol{R}^{n+2}$ of $\boldsymbol{S}^{n+1}$ without umbilics, we define its Möbius position vector $Y : M^n \to \boldsymbol{L}^{n+3}$ by

$$Y = \rho(1, x)\,, \quad \rho^2 = n/(n-1)\cdot(\|h\|^2 - nH^2) > 0\,. \tag{2.1}$$

Then two hypersurfaces $x, \tilde{x} : M^n \to \boldsymbol{S}^{n+1}$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(n+2, 1)$ in $\boldsymbol{L}^{n+3}$ such that $Y = \tilde{Y}T$. It follows immediately that

$g = \langle dY, dY\rangle = \rho^2 dx \cdot dx$ is a Möbius invariant, which is defined as the *Möbius metric* of $x : M^n \to \boldsymbol{S}^{n+1}$.

Let Δ be the Laplace-Beltrami operator of g and define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle Y . \tag{2.2}$$

Then it holds that

$$\langle \Delta Y, Y\rangle = -n , \quad \langle \Delta Y, dY\rangle = 0 , \quad \langle \Delta Y, \Delta Y\rangle = 1 + n^2 R , \tag{2.3}$$

$$\langle Y, Y\rangle = 0 , \quad \langle N, Y\rangle = 1 , \quad \langle N, N\rangle = 0 , \tag{2.4}$$

where R is the normalized scalar curvature of g, which is called the normalized Möbius scalar curvature of $x : M^n \to \boldsymbol{S}^{n+1}$.

Let $\{E_1, \dots, E_n\}$ be a local orthonormal basis for (M^n, g), and $\{\omega_1, \dots, \omega_n\}$ the dual basis. Write $Y_i = E_i(Y)$. Then it follows from (2.1), (2.3) and (2.4) that

$$\langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0 , \quad \langle Y_i, Y_j\rangle = \delta_{ij} , \quad 1 \le i, j \le n . \tag{2.5}$$

Let V be the orthogonal complement to the subspace $\mathrm{Span}\{Y, N, Y_1, \dots, Y_n\}$ in $\boldsymbol{L}^{n+3}$. Then along M^n we have the orthogonal decomposition

$$\boldsymbol{L}^{n+3} = \mathrm{Span}\{Y, N\} \oplus \mathrm{Span}\{Y_1, \dots, Y_n\} \oplus V . \tag{2.6}$$

A local unit vector basis $E = E_{n+1}$ for V can be written as $E = E_{n+1} := (H, Hx + e_{n+1})$. Then, along M^n, $\{Y, N, Y_1, \dots, Y_n, E\}$ forms a moving frame in $\boldsymbol{L}^{n+3}$. Unless otherwise stated, we will use the following range of indices throughout this paper: $1 \le i, j, k, l, t \le n$.

We can write the structure equations as follows:

$$dY = \sum_i Y_i \omega_i , \tag{2.7}$$

$$dN = \sum_{i,j} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i E , \tag{2.8}$$

$$dY_i = -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j E , \tag{2.9}$$

$$dE = -\sum_i C_i \omega_i Y - \sum_{i,j} B_{ij}\omega_j Y_i , \tag{2.10}$$

where ω_{ij} is the connection form of the Möbius metric g, which is defined by the structure equations $d\omega_i = \sum_j \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$. The tensors $\mathbf{A} = \sum_{i,j} A_{ij}\omega_i \otimes \omega_j$, $\boldsymbol{\Phi} = \sum_i C_i \omega_i$ and $\mathbf{B} = \sum_{i,j} B_{ij}\omega_i \otimes \omega_j$ are called the Blaschke tensor, the Möbius form and the Möbius second fundamental form of $x : M^n \to \boldsymbol{S}^{n+1}$, respectively. The relations between $\boldsymbol{\Phi}$, $\mathbf{B}$, $\mathbf{A}$ and the Euclidean invariants of x are given by (cf. [22])

$$C_i = -\rho^{-2}\left[e_i(H) + \sum_j (h_{ij} - H\delta_{ij})e_j(\log \rho)\right] , \tag{2.11}$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij})\,, \tag{2.12}$$

$$\begin{aligned} A_{ij} =& -\rho^{-2}[\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}] \\ & -\frac{1}{2}\rho^{-2}(|\nabla\log\rho|^2 - 1 + H^2)\delta_{ij}\,, \end{aligned} \tag{2.13}$$

where Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the orthonormal basis $\{e_i\}$ of $dx \cdot dx$, respectively.

Let $C_{i,j}$, $A_{ij,k}$ and $B_{ij,k}$ denote the components of the covariant derivative of C_i, A_{ij}, B_{ij}, respectively. Then the integrability conditions for the structure equations (2.7) through (2.10) are given by

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k\,, \tag{2.14}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - A_{ik}B_{kj})\,, \tag{2.15}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j\,, \tag{2.16}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}\,, \tag{2.17}$$

$$R_{ij} := \sum_k R_{ikjk} = -\sum_k B_{ik}B_{jk} + (\mathrm{tr}\,A)\delta_{ij} + (n-2)A_{ij}\,, \tag{2.18}$$

$$\sum_i B_{ii} = 0,\ \ \sum_{i,j}(B_{ij})^2 = \frac{n-1}{n}\,, \quad \mathrm{tr}\,A = \sum_i A_{ii} = \frac{1}{2n}(1 + n^2R)\,. \tag{2.19}$$

Here R_{ijkl} denote the components of the curvature tensor of g, which are defined by the structure equations

$$d\omega_{ij} - \sum_k \omega_{ik}\wedge\omega_{kj} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l\,, \tag{2.20}$$

$R = 1/n(n-1)\sum_{i,j} R_{ijij}$ being the normalized Möbius scalar curvature of $x : M^n \to \boldsymbol{S}^{n+1}$.

The second covariant derivative of B_{ij} is defined by

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k} + \sum_l B_{lj,k}\omega_{li} + \sum_l B_{il,k}\omega_{lj} + \sum_l B_{ij,l}\omega_{lk}\,. \tag{2.21}$$

Then the following Ricci identity holds:

$$B_{ij,kl} - B_{ij,lk} = \sum_t B_{tj}R_{tikl} + \sum_t B_{it}R_{tjkl}\,. \tag{2.22}$$

From (2.12), we see that the Möbius shape operator of $x : M^n \to \boldsymbol{S}^{n+1}$ takes the form

$$\boldsymbol{\Psi} = \rho^{-1}(\mathbf{S} - H\mathrm{id}) = \sum_{i,j} B_{ij}\omega_i E_j\,, \tag{2.23}$$

which implies that for an umbilic free hypersurface in $\boldsymbol{S}^{n+1}$, the number of distinct Möbius principal curvatures is identical to that of its distinct Euclidean principal curvatures.

One can easily show that all coefficients in (2.7) through (2.10) are determined by $\{g, \boldsymbol{\Psi}\}$. Thus we obtain

THEOREM 2.1 ([22], see also [1]). *Two hypersurfaces $x : M^n \to \boldsymbol{S}^{n+1}$ and $\tilde{x} : \tilde{M}^n \to \boldsymbol{S}^{n+1}$, $n \geq 3$, are Möbius equivalent if and only if there exists a diffeomorphism $F : M^n \to \tilde{M}^n$ which preserves the Möbius metric and the Möbius shape operator.*

The following results will be needed later.

THEOREM 2.2 ([12]). *Let $x : M^n \to \boldsymbol{S}^{n+1}$ be a Möbius isoparametric hypersurface with two distinct principal curvatures. Then x is Möbius equivalent to an open subset of one of the following hypersurfaces in $\boldsymbol{S}^{n+1}$:*

(1) *The standard torus $\boldsymbol{S}^k(a) \times \boldsymbol{S}^{n-k}(b)$ with $1 \leq k \leq n-1$ and $a^2 + b^2 = 1$.*

(2) *The image of σ of the standard cylinder $\boldsymbol{S}^k(1) \times \boldsymbol{R}^{n-k}$ with $1 \leq k \leq n-1$.*

(3) *The image of τ of the standard hyperbolic cylinder $\boldsymbol{S}^k(r) \times \boldsymbol{H}^{n-k}(-1/(1+r^2)) \subset \boldsymbol{H}^{n+1}(-1)$ with $1 \leq k \leq n-1$ and $r > 0$.*

THEOREM 2.3 ([8, 10]). *Let $x : M^5 \to \boldsymbol{S}^6$ be a Möbius isoparametric hypersurface with three distinct principal curvatures. Then x is Möbius equivalent to an open subset of one of the following hypersurfaces:*

(1) *The preimage of the stereo-graphic projection of the warped product embedding*

$$\tilde{x} : \boldsymbol{S}^p(a) \times \boldsymbol{S}^q(\sqrt{1-a^2}) \times \boldsymbol{R}^+ \times \boldsymbol{R}^{4-p-q} \to \boldsymbol{R}^6$$

with $p \geq 1,\ q \geq 1,\ p+q \leq 4,\ 0 < a < 1$, defined by

$$\tilde{x}(u', u'', t, u''') = (tu', tu'', tu''')\,,$$

$$u' \in \boldsymbol{S}^p(a)\,, \quad u'' \in \boldsymbol{S}^q(\sqrt{1-a^2})\,, \quad t \in \boldsymbol{R}^+\,, \quad u''' \in \boldsymbol{R}^{4-p-q}\,.$$

(2) *Minimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^5 = N^3 \times \boldsymbol{H}^2(-2/15) \to \boldsymbol{S}^6$$

with

$$\tilde{x}_1 = y_1/y_0\,, \quad \tilde{x}_2 = y_2/y_0\,, \quad y_0 \in \boldsymbol{R}^+\,, \quad y_1 \in \boldsymbol{R}^5\,, \quad y_2 \in \boldsymbol{R}^2\,.$$

Here $y_1 : N^3 \to \boldsymbol{S}^4(\sqrt{30}/2) \hookrightarrow \boldsymbol{R}^5$ is Cartan's minimal isoparametric hypersurface with vanishing scalar curvature and principal curvatures $\sqrt{10}/5,\ 0,\ -\sqrt{10}/5$, and $(y_0, y_2) : \boldsymbol{H}^2(-2/15) \hookrightarrow \boldsymbol{L}^3$ is the standard embedding of the hyperbolic space of sectional curvature $-2/15$ into the 3-dimensional Lorentz space with $-y_0^2 + y_2^2 = -15/2$.

THEOREM 2.4 ([23]). *Let $x : M^n \to \boldsymbol{S}^{n+1}$ be an immersed umbilic-free hypersurface with vanishing Möbius form and such that for some constant λ, the linear combination $\mathbf{A}+\lambda\mathbf{B}$ of the Blaschke tensor $\mathbf{A}$ and the Möbius second fundamental form $\mathbf{B}$ has two distinct constant eigenvalues. Moreover, if x has at least three distinct Möbius principal curvatures, then it is locally Möbius equivalent to one of the following families of hypersurfaces in $\boldsymbol{S}^{n+1}$:*

(1) *Hypersurfaces defined by $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^p \times \boldsymbol{H}^{n-p}(-r^{-2}) \to \boldsymbol{S}^{n+1}$ with*

$$\tilde{x}_1 = y_1/y_0\,, \quad \tilde{x}_2 = y_2/y_0\,, \quad y_0 \in \boldsymbol{R}^+\,, \quad y_1 \in \boldsymbol{R}^{p+2}\,,$$
$$y_2 \in \boldsymbol{R}^{n-p}\,, \quad 2 \le p \le n-1\,, \quad r > 0\,.$$

Here $y_1 : N^p \to \boldsymbol{S}^{p+1}(r) \hookrightarrow \boldsymbol{R}^{p+2}$ is an immersed umbilic-free hypersurface with constant mean curvature H_1 and constant scalar curvature R_1 in the $(p+1)$-dimensional sphere of radius r, where

$$H_1 = -\frac{n}{p}\lambda\,, \quad R_1 = \frac{np(p-1)-(n-1)r^2}{nr^2} + n(n-1)\lambda^2\,,$$

whereas $(y_0, y_2) : \boldsymbol{H}^{n-p}(-r^{-2}) \hookrightarrow \boldsymbol{L}^{n-p+1}$ is the standard embedding of the hyperbolic space of sectional curvature $-r^{-2}$ into the $(n-p+1)$-dimensional Lorentz space with $-y_0^2 + y_2^2 = -r^2$.

(2) *Hypersurfaces defined by $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^p \times \boldsymbol{S}^{n-p}(r) \to \boldsymbol{S}^{n+1}$ with*

$$\tilde{x}_1 = y_1/y_0\,, \quad \tilde{x}_2 = y_2/y_0\,, \quad y_0 \in \boldsymbol{R}^+\,, \quad y_1 \in \boldsymbol{R}^{p+1}\,,$$
$$y_2 \in \boldsymbol{R}^{n-p+1}\,, \quad 2 \le p \le n-1\,, \; r > 0\,.$$

Here $(y_0, y_1) : N^p \to \boldsymbol{H}^{p+1}(-r^{-2}) \hookrightarrow \boldsymbol{L}^{p+2}$ is an immersed umbilic-free hypersurface into the $(p+1)$-dimensional hyperbolic space of sectional curvature $-r^{-2}$ with constant mean curvature H_1 and constant scalar curvature $\tilde{R}_1$ such that $-y_0^2 + y_1^2 = -r^2$ and

$$H_1 = -\frac{n}{p}\lambda\,, \quad R_1 = -\frac{np(p-1)+(n-1)r^2}{nr^2} + n(n-1)\lambda^2\,,$$

whereas $y_2 : \boldsymbol{S}^{n-p}(r) \hookrightarrow \boldsymbol{R}^{n-p+1}$ is the standard embedding of the $(n-p)$-dimensional sphere of radius r.

REMARK 2.1. In the special case where $\lambda = 0$, Theorem 2.4 was first obtained by Li and Zhang [16, 17]. See also [10] for details of the description and calculations.

3. Möbius isoparametric hypersurfaces with $\gamma = 4$. In this section, we consider the case that $x : M^5 \to \boldsymbol{S}^6$ is a Möbius isoparametric hypersurfaces with $\gamma = 4$. According to [8], the Möbius second fundamental form $\mathbf{B}$ in this case is non-parallel.

For our choice of the local orthonormal basis $\{E_i\}_{1\le i\le 5}$, the fact that $\boldsymbol{\Psi}$ has constant eigenvalues is equivalent to the fact that the matrix (B_{ij}) has constant eigenvalues. From $\boldsymbol{\Phi} = 0$ and (2.15), we see that for all i, j

$$\sum_k (B_{ik}A_{kj} - A_{ik}B_{kj}) = 0\,. \tag{3.1}$$

This implies that we can choose $\{E_i\}$ to diagonalize (A_{ij}) and (B_{ij}) simultaneously. Let us write

$$(B_{ij}) = \mathrm{diag}(b_1, \dots, b_5)\,, \quad (A_{ij}) = \mathrm{diag}(a_1, \dots, a_5)\,, \tag{3.2}$$

where $\{b_i\}$ are all constants. From (2.19) we have

$$b_1 + b_2 + b_3 + b_4 + b_5 = 0\,, \quad b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 = 4/5\,. \tag{3.3}$$

Without loss of generality, from the assumption $\gamma = 4$, we can assume that $b_4 = b_5$ and b_1, b_2, b_3, b_4 are mutually distinct. Substituting $\Phi = 0$ into (2.14) and (2.16), we see that both $B_{ij,k}$ and $A_{ij,k}$ are totally symmetric tensors. As usual we define

$$\omega_{ij} = \sum_k \Gamma^i_{kj}\omega_k\,, \quad \Gamma^i_{kj} = -\Gamma^j_{ki}\,. \tag{3.4}$$

From the definition $\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{jk}\omega_{ki}$, and from (3.2), (3.4) and the assumption that all b_i are constant, we get

$$B_{ij,k} = (b_i - b_j)\Gamma^i_{kj} \quad \text{for all } i, j, k\,, \tag{3.5}$$

and thus

$$B_{ij,i} = B_{ii,j} = 0 \quad \text{for all } i, j\,; \quad B_{14,5} = B_{24,5} = B_{34,5} = 0\,. \tag{3.6}$$

First of all, as preliminary facts we derive the following Lemma 3.1 through Lemma 3.4.

LEMMA 3.1. *Under the above assumptions, there holds*

$$\frac{B_{12,4}B_{12,5}}{(b_1 - b_2)(b_4 - b_2)} = \frac{B_{13,4}B_{13,5}}{(b_1 - b_3)(b_3 - b_4)}\,, \tag{3.7}$$

$$\frac{B_{12,4}B_{12,5}}{(b_1 - b_4)(b_1 - b_2)} = \frac{B_{23,4}B_{23,5}}{(b_2 - b_3)(b_3 - b_4)}\,. \tag{3.8}$$

PROOF. Using (3.5), (3.6) and (2.21), we obtain

$$\begin{aligned}\sum_k B_{14,5k}\omega_k &= dB_{14,5} + \sum_k (B_{14,k}\omega_{k5} + B_{k4,5}\omega_{k1} + B_{1k,5}\omega_{k4})\\ &= B_{14,3}(\Gamma^3_{15}\omega_1 + \Gamma^3_{25}\omega_2) + B_{12,4}(\Gamma^2_{15}\omega_1 + \Gamma^2_{35}\omega_3)\\ &\quad + B_{12,5}(\Gamma^2_{14}\omega_1 + \Gamma^2_{34}\omega_3) + B_{13,5}(\Gamma^3_{14}\omega_1 + \Gamma^3_{24}\omega_2)\,,\end{aligned}$$

and therefore

$$B_{14,51} = \frac{2B_{13,5}B_{13,4}}{b_3 - b_4} + \frac{2B_{12,4}B_{12,5}}{b_2 - b_4}\,. \tag{3.9}$$

Analogously, we have

$$B_{11,45} = \frac{2B_{12,4}B_{12,5}}{b_2 - b_1} + \frac{2B_{13,4}B_{13,5}}{b_3 - b_1}\,. \tag{3.10}$$

On the other hand, (2.22) and (3.2) give that

$$B_{ij,kl} = B_{ij,lk} + \sum_m (B_{mj}R_{mikl} + B_{im}R_{mjkl}) = B_{ij,lk} + (b_i - b_j)R_{ijkl}\,. \tag{3.11}$$

Notice that, by (2.17) and (3.2), it holds that

$$R_{ijkl} = 0 \quad \text{if three of } \{i, j, k, l\} \text{ are either the same or distinct}\,. \tag{3.12}$$

From (3.11) and (3.12), we get

$$B_{ij,kl} = B_{ij,lk} \quad \text{if three of } \{i, j, k, l\} \text{ are distinct}\,. \tag{3.13}$$

Then (3.7) follows from (3.9), (3.10) and $B_{14,51} = B_{11,45}$. Similarly, from $B_{24,52} = B_{22,45}$ we easily obtain (3.8). □

LEMMA 3.2. *Denote by i, j, k the three distinct elements of $\{1, 2, 3\}$ with arbitrarily given order. Then we have*

$$R_{ijij} = \frac{2B^2_{12,3}}{(b_k - b_i)(b_k - b_j)} + \frac{2(B^2_{ij,4} + B^2_{ij,5})}{(b_4 - b_i)(b_4 - b_j)}\,, \tag{3.14}$$

$$R_{i4i4} = \frac{2B^2_{ij,4}}{(b_j - b_i)(b_j - b_4)} + \frac{2B^2_{ik,4}}{(b_k - b_i)(b_k - b_4)}\,, \tag{3.15}$$

$$R_{i5i5} = \frac{2B^2_{ij,5}}{(b_j - b_i)(b_j - b_5)} + \frac{2B^2_{ik,5}}{(b_k - b_i)(b_k - b_5)}\,. \tag{3.16}$$

PROOF. Using (3.5), (3.6) and (2.21), we obtain

$$\begin{aligned}\sum_l B_{ii,jl}\omega_l &= dB_{ii,j} + 2\sum_l B_{il,j}\omega_{li} + \sum_l B_{ii,l}\omega_{lj} = 2\sum_l B_{il,j}\omega_{li}\\ &= 2B_{ik,j}\omega_{ki} + 2B_{i4,j}\omega_{4i} + 2B_{i5,j}\omega_{5i}\,,\end{aligned} \tag{3.17}$$

and therefore

$$B_{ii,jj} = \frac{2B^2_{12,3}}{b_k - b_i} + \frac{2(B^2_{i4,j} + B^2_{i5,j})}{b_4 - b_i}\,. \tag{3.18}$$

Analogously, we have

$$B_{jj,ii} = \frac{2B^2_{ij,k}}{b_k - b_j} + \frac{2(B^2_{j4,i} + B^2_{j5,i})}{b_4 - b_j}\,. \tag{3.19}$$

From (3.11), we get

$$B_{ii,jj} = B_{ij,ij} = B_{ij,ji} + (b_i - b_j)R_{ijij} = B_{jj,ii} + (b_i - b_j)R_{ijij}\,. \tag{3.20}$$

Then (3.14) follows from (3.18) through (3.20). Similarly, we can obtain (3.15) and (3.16). □

LEMMA 3.3. *For any chosen $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $l, r \in \{4, 5\}$ with $l \neq r$, the assumption $B_{12,3}B_{ij,l} \neq 0$ implies that $B_{12,r} = B_{23,r} = B_{13,r} = 0$.*

PROOF. Without loss of generality, we show that the assumption $B_{12,3}B_{12,4} \neq 0$ implies that $B_{12,5} = B_{23,5} = B_{13,5} = 0$. Indeed, if otherwise, $B_{12,5} \neq 0$. Then from (3.4), (3.5), the definition of $A_{ij,k}$ and the totally symmetric of $A_{ij,k}$ and $B_{ij,k}$, we get

$$A_{12,3} = (a_1 - a_2)\Gamma^1_{32} = (a_1 - a_3)\Gamma^1_{23} = (a_2 - a_3)\Gamma^2_{13}\,, \tag{3.21}$$

$$B_{12,3} = (b_1 - b_2)\Gamma^1_{32} = (b_1 - b_3)\Gamma^1_{23} = (b_2 - b_3)\Gamma^2_{13}\,, \tag{3.22}$$

$$A_{12,4} = (a_1 - a_2)\Gamma^1_{42} = (a_1 - a_4)\Gamma^1_{24} = (a_2 - a_4)\Gamma^2_{14}\,, \tag{3.23}$$

$$B_{12,4} = (b_1 - b_2)\Gamma^1_{42} = (b_1 - b_4)\Gamma^1_{24} = (b_2 - b_4)\Gamma^2_{14}\,, \tag{3.24}$$

$$A_{12,5} = (a_1 - a_2)\Gamma^1_{52} = (a_1 - a_5)\Gamma^1_{25} = (a_2 - a_5)\Gamma^2_{15}\,, \tag{3.25}$$

$$B_{12,5} = (b_1 - b_2)\Gamma^1_{52} = (b_1 - b_5)\Gamma^1_{25} = (b_2 - b_5)\Gamma^2_{15}\,. \tag{3.26}$$

From (3.21) through (3.26), we derive

$$\frac{A_{12,3}}{B_{12,3}} = \frac{a_1 - a_2}{b_1 - b_2} = \frac{a_1 - a_3}{b_1 - b_3} = \frac{a_2 - a_3}{b_2 - b_3}\,,$$

$$\frac{A_{12,4}}{B_{12,4}} = \frac{a_1 - a_2}{b_1 - b_2} = \frac{a_1 - a_4}{b_1 - b_4} = \frac{a_2 - a_4}{b_2 - b_4}\,,$$

$$\frac{A_{12,5}}{B_{12,5}} = \frac{a_1 - a_2}{b_1 - b_2} = \frac{a_1 - a_5}{b_1 - b_5} = \frac{a_2 - a_5}{b_2 - b_5}\,,$$

and therefore there exists a function λ such that

$$\frac{a_1 - a_2}{b_1 - b_2} = \frac{a_1 - a_3}{b_1 - b_3} = \frac{a_2 - a_3}{b_2 - b_3} = \frac{a_1 - a_4}{b_1 - b_4} = \frac{a_2 - a_4}{b_2 - b_4} = \frac{a_1 - a_5}{b_1 - b_5} = \frac{a_2 - a_5}{b_2 - b_5} = -\lambda\,.$$

This implies the existence of another function μ such that

$$a_1 + \lambda b_1 = a_2 + \lambda b_2 = a_3 + \lambda b_3 = a_4 + \lambda b_4 = a_5 + \lambda b_5 = \mu. \tag{3.27}$$

As (3.27) says $\mathbf{A} + \lambda\mathbf{B} - \mu g = 0$, we can apply the result of Li and Wang [13] to conclude that λ and μ are constant and $x : M \to \boldsymbol{S}^6$ is locally Möbius equivalent to one of the following hypersurfaces:

(1) A hypersurface $\tilde{x} : \tilde{M} \to \boldsymbol{S}^6$ with constant mean curvature and constant scalar curvature.

(2) The image under σ of a hypersurface $\tilde{x} : \tilde{M} \to \boldsymbol{R}^6$ with constant mean curvature and constant scalar curvature.

(3) The image under τ of a hypersurface $\tilde{x} : \tilde{M} \to \boldsymbol{H}^6$ with constant mean curvature and constant scalar curvature.

Now, according to Propositions 3.1 and 3.2 of [11] and the fact that $\{b_i\}$ consists of constants, we see that the above $\tilde{x} : \tilde{M} \to \boldsymbol{S}^6$, $\tilde{x} : \tilde{M} \to \boldsymbol{R}^6$ and $\tilde{x} : \tilde{M} \to \boldsymbol{H}^6$, respectively, should all be Euclidean isoparametric hypersurfaces with four distinct principal curvatures. From the classical result that isoparametric hypersurfaces in $\boldsymbol{R}^6$ and $\boldsymbol{H}^6$ can have at most two distinct principal curvatures, we find that Cases (2) and (3) do not occur. Hence x is Möbius equivalent to an open subset of some isoparametric hypersurface in $\boldsymbol{S}^6$ with four distinct principal curvatures.

On the other hand, by the well-known fact for multiplicities of principal curvatures on isoparametric hypersurface in the sphere with four distinct principal curvatures (cf. also Cecil-Chi-Jensen [7]), we see that there is no Euclidean isoparametric hypersurface in $\boldsymbol{S}^6$ with four distinct principal curvatures. This is a contradiction and therefore we have $B_{12,5} = 0$.

Analogously, we can prove $B_{23,5} = B_{13,5} = 0$. □

LEMMA 3.4. $B_{12,4}B_{12,5} = B_{13,4}B_{13,5} = B_{23,4}B_{23,5} = 0.$

PROOF. Assume on the contrary that $B_{12,4}B_{12,5} \neq 0$. Then by Lemma 3.1, we have $B_{13,4}B_{13,5} \neq 0$ and $B_{23,4}B_{23,5} \neq 0$. Therefore (3.27) holds for some functions λ and μ. Now a similar argument as that in the proof of Lemma 3.3 will give a contradiction. □

From (3.6), we see that in the set $\{B_{ij,k}\}_{i\le j\le k}$, there are only seven elements, namely

$$\{B_{12,3},\ B_{12,4},\ B_{12,5},\ B_{23,4},\ B_{23,5},\ B_{13,4},\ B_{13,5}\}$$

can be probably nonzero. Now we separate our discussion into two cases:

Case I. $B_{12,3} \neq 0$; Case II. $B_{12,3} = 0$.

In Case I, according to Lemma 3.1, Lemma 3.3 and Lemma 3.4, the number α of nonzero elements in $\{B_{12,4}, B_{12,5}, B_{23,4}, B_{23,5}, B_{13,4}, B_{13,5}\}$ is at most three.

If $\alpha = 0$, we have one case: I-1.

If $\alpha = 1$, without loss of generality we may assume $B_{12,4} \neq 0$. Then we have one case: I-2.

If $\alpha = 2$ and $B_{12,4} \neq 0$, by Lemma 3.3, $B_{12,5} = B_{23,5} = B_{13,5} = 0$, so that exactly one of $\{B_{23,4}, B_{13,4}\}$ is nonzero. Without loss of generality we may assume $B_{23,4} \neq 0$, $B_{13,4} = 0$. So we have one case: I-3.

If $\alpha = 3$ and $B_{12,4}B_{23,4}B_{13,4} \neq 0$, then we are left I-4.

In summary, for the case I, it is sufficient to consider the following four independent cases:

I-1. $B_{12,3} \neq 0$ and $B_{12,4} = B_{12,5} = B_{23,4} = B_{23,5} = B_{13,4} = B_{13,5} = 0$.

I-2. $B_{12,3}B_{12,4} \neq 0$ and $B_{12,5} = B_{23,5} = B_{13,5} = B_{23,4} = B_{13,4} = 0$.

I-3. $B_{12,3}B_{12,4}B_{23,4} \neq 0$ and $B_{12,5} = B_{23,5} = B_{13,5} = B_{13,4} = 0$.

I-4. $B_{12,3}B_{12,4}B_{23,4}B_{13,4} \neq 0$ and $B_{12,5} = B_{23,5} = B_{13,5} = 0$.

For Case II, since **B** is assumed to be non-parallel, without loss of generality, we further assume $B_{12,4} \neq 0$. Then Lemma 3.4 implies that $B_{12,5} = 0$. Now, according to Lemma 3.1, Lemma 3.3 and Lemma 3.4, the number β of nonzero elements in $\{B_{23,4}, B_{23,5}, B_{13,4}, B_{13,5}\}$ is at most two.

If $\beta = 0$, we have one case: II-1.

If $\beta = 1$, the symmetry of indices 1 and 2 implies that we need only to consider two cases, i.e., II-2: $B_{23,4} \neq 0$ with $B_{23,5} = B_{13,4} = B_{13,5} = 0$ and II-3: $B_{23,5} \neq 0$ with $B_{23,4} = B_{13,4} = B_{13,5} = 0$.

If $\beta = 2$, the symmetry of indices 1 and 2 and Lemma 3.4 imply that we have to consider three cases, i.e., II-4: $B_{23,4}B_{13,5} \neq 0$ with $B_{23,5} = B_{13,4} = 0$; II-5: $B_{23,4}B_{13,4} \neq 0$ with $B_{23,5} = B_{13,5} = 0$; II-6: $B_{23,5}B_{13,5} \neq 0$ with $B_{23,4} = B_{13,4} = 0$. However, using the symmetry of indices 1, 2 and 3, and the symmetry of indices 4 and 5, we easily see that II-6 can be transformed into II-4.

In summary, in Case II, it is sufficient to consider the following five independent cases:

II-1. $B_{12,4} \neq 0$ and $B_{12,3} = B_{12,5} = B_{23,4} = B_{23,5} = B_{13,4} = B_{13,5} = 0$.

II-2. $B_{12,4}B_{23,4} \neq 0$ and $B_{12,3} = B_{12,5} = B_{23,5} = B_{13,4} = B_{13,5} = 0$.

II-3. $B_{12,4}B_{23,5} \neq 0$ and $B_{12,3} = B_{12,5} = B_{23,4} = B_{13,4} = B_{13,5} = 0$.

II-4. $B_{12,4}B_{23,4}B_{13,5} \neq 0$ and $B_{12,3} = B_{12,5} = B_{23,5} = B_{13,4} = 0$.

II-5. $B_{12,4}B_{23,4}B_{13,4} \neq 0$ and $B_{12,3} = B_{12,5} = B_{23,5} = B_{13,5} = 0$.

Now, we are ready to prove the following crucial propositions.

PROPOSITION 3.1. *The Cases* I-2, I-3 *and* II-1 *through* II-5 *do not occur.*

PROOF. We will check each of the cases one by one.

I-2: $B_{12,3}B_{12,4} \neq 0$ and $B_{12,5} = B_{23,5} = B_{13,5} = B_{23,4} = B_{13,4} = 0$.

From (3.15), (3.16) and (2.17) we have

$$R_{2424} = b_2b_4 + a_2 + a_4 = \frac{2B_{12,4}^2}{(b_1 - b_2)(b_1 - b_4)}\,, \tag{3.28}$$

$$R_{2525} = b_2b_5 + a_2 + a_5 = 0\,, \tag{3.29}$$

$$R_{3434} = b_3b_4 + a_3 + a_4 = 0\,, \tag{3.30}$$

$$R_{3535} = b_3b_5 + a_3 + a_5 = 0\,. \tag{3.31}$$

(3.30) and (3.31) imply that $a_4 = a_5$. Then (3.28) and (3.29) give

$$\frac{2B_{12,4}^2}{(b_1 - b_2)(b_1 - b_4)} = 0\,,$$

which is a contradiction. Thus case I-2 does not occur.

I-3: $B_{12,3}B_{12,4}B_{23,4} \neq 0$ and $B_{12,5} = B_{23,5} = B_{13,5} = B_{13,4} = 0$.

In this case, we have

$$\begin{aligned} &\omega_{21} = \Gamma_{31}^2\omega_3 + \Gamma_{41}^2\omega_4\,, \quad \omega_{23} = \Gamma_{13}^2\omega_1 + \Gamma_{43}^2\omega_4\,, \quad \omega_{24} = \Gamma_{14}^2\omega_1 + \Gamma_{34}^2\omega_3\,, \\ &\omega_{14} = \Gamma_{24}^1\omega_2\,, \quad \omega_{34} = \Gamma_{24}^3\omega_2\,, \quad \omega_{13} = \Gamma_{23}^1\omega_2\,, \quad \omega_{15} = \omega_{25} = \omega_{35} = 0\,. \end{aligned} \tag{3.32}$$

Then, by (3.2) and (3.32), we have the following:

$$\begin{aligned} -R_{1313}\omega_1 \wedge \omega_3 &= -\frac{1}{2}\sum_{k,l} R_{13kl}\omega_k \wedge \omega_l = d\omega_{13} - \sum_k \omega_{1k} \wedge \omega_{k3} \\ &= \frac{dB_{12,3}}{b_1 - b_3} \wedge \omega_2 + \Gamma_{23}^1(\omega_{21} \wedge \omega_1 + \omega_{23} \wedge \omega_3 + \omega_{24} \wedge \omega_4) - \omega_{12} \wedge \omega_{23} - \omega_{14} \wedge \omega_{43} \\ &= \frac{dB_{12,3}}{b_1 - b_3} \wedge \omega_2 + (\Gamma_{23}^1\Gamma_{31}^2 - \Gamma_{23}^1\Gamma_{13}^2 - \Gamma_{32}^1\Gamma_{13}^2)\omega_3 \wedge \omega_1 \\ &\quad + (\Gamma_{23}^1\Gamma_{41}^2 - \Gamma_{23}^1\Gamma_{14}^2 - \Gamma_{42}^1\Gamma_{13}^2)\omega_4 \wedge \omega_1 + (\Gamma_{23}^1\Gamma_{43}^2 - \Gamma_{23}^1\Gamma_{34}^2 + \Gamma_{32}^1\Gamma_{43}^2)\omega_4 \wedge \omega_3\,. \end{aligned}$$

Comparing both sides of the above equation, we obtain

$$\Gamma_{23}^1\Gamma_{41}^2 - \Gamma_{23}^1\Gamma_{14}^2 - \Gamma_{42}^1\Gamma_{13}^2 = 0\,, \quad \Gamma_{23}^1\Gamma_{43}^2 - \Gamma_{23}^1\Gamma_{34}^2 + \Gamma_{32}^1\Gamma_{43}^2 = 0\,,$$

or equivalently, by (3.5) and $B_{12,3}B_{12,4}B_{23,4} \neq 0$,

$$\frac{1}{(b_1 - b_3)(b_2 - b_1)} - \frac{1}{(b_1 - b_3)(b_2 - b_4)} - \frac{1}{(b_1 - b_2)(b_2 - b_3)} = 0\,, \tag{3.33}$$

$$\frac{1}{(b_1 - b_3)(b_2 - b_3)} - \frac{1}{(b_1 - b_3)(b_2 - b_4)} + \frac{1}{(b_1 - b_2)(b_2 - b_3)} = 0\,. \tag{3.34}$$

Then (3.33) and (3.34) give $3/(b_1 - b_2)(b_2 - b_3) = 0$, a contradiction. Thus case I-3 does not occur.

II-1: $B_{12,4} \neq 0$ and $B_{12,3} = B_{12,5} = B_{23,4} = B_{23,5} = B_{13,4} = B_{13,5} = 0$.

From (3.14), (3.16) and (2.17) we have

$$R_{1313} = b_1 b_3 + a_1 + a_3 = 0\,, \tag{3.35}$$

$$R_{1515} = b_1 b_5 + a_1 + a_5 = 0\,, \tag{3.36}$$

$$R_{2323} = b_2 b_3 + a_2 + a_3 = 0\,, \tag{3.37}$$

$$R_{2525} = b_2 b_5 + a_2 + a_5 = 0\,. \tag{3.38}$$

Then (3.35) + (3.38) − (3.36) − (3.37) gives

$$(b_3 - b_5)(b_1 - b_2) = 0\,, \tag{3.39}$$

which is a contradiction. Thus II-1 does not occur.

II-2: $B_{12,4}B_{23,4} \neq 0$ and $B_{12,3} = B_{12,5} = B_{23,5} = B_{13,4} = B_{13,5} = 0$.

From Lemma 3.2 and (2.17), we now have

$$\begin{aligned}
&R_{1212} = \frac{2B_{12,4}^2}{(b_4 - b_1)(b_4 - b_2)}\,, \quad R_{1414} = \frac{2B_{12,4}^2}{(b_2 - b_1)(b_2 - b_4)}\,,\\
&R_{2323} = \frac{2B_{23,4}^2}{(b_4 - b_2)(b_4 - b_3)}\,, \quad R_{3434} = \frac{2B_{23,4}^2}{(b_2 - b_3)(b_2 - b_4)}\,,\\
&R_{2424} = \frac{2B_{12,4}^2}{(b_1 - b_2)(b_1 - b_4)} + \frac{2B_{23,4}^2}{(b_3 - b_2)(b_3 - b_4)}\,,\\
&R_{1313} = R_{1515} = R_{2525} = R_{3535} = 0\,,\\
&R_{1414} - R_{1515} = R_{2424} - R_{2525} = R_{3434} - R_{3535} = a_4 - a_5\,.
\end{aligned} \tag{3.40}$$

Then we find from (3.40) that $R_{1414} = R_{2424} = R_{3434}$. This implies that

$$B_{12,4}^2(b_2 - b_3) = B_{23,4}^2(b_2 - b_1)\,, \tag{3.41}$$

$$\frac{B_{12,4}^2}{(b_1 - b_2)(b_1 - b_4)} + \frac{B_{23,4}^2}{(b_3 - b_2)(b_3 - b_4)} = \frac{B_{23,4}^2}{(b_2 - b_3)(b_2 - b_4)}\,. \tag{3.42}$$

As $B_{12,4}B_{23,4} \neq 0$, we can cancel $B_{12,4}$, $B_{23,4}$ from (3.41) and (3.42) to obtain

$$\begin{aligned}
0 &= (b_3 - b_4)(b_2 - b_4) + (b_2 + b_3 - 2b_4)(b_1 - b_4)\\
&= b_1 b_2 + b_2 b_3 + b_3 b_1 - 2b_4(b_1 + b_2 + b_3) + 3b_4^2\\
&= (1/2)(b_1 + b_2 + b_3)^2 - (1/2)(b_1^2 + b_2^2 + b_3^2) + 7b_4^2 = -2/5 + 10b_4^2\,,
\end{aligned} \tag{3.43}$$

where we have used (3.3) and $b_4 = b_5$. This gives

$$b_4^2 = 1/25\,. \tag{3.44}$$

From (2.17) and (3.40), we get

$$R_{2525} = b_2 b_5 + a_2 + a_5 = 0\,, \tag{3.45}$$

$$R_{3535} = b_3 b_5 + a_3 + a_5 = 0\,, \tag{3.46}$$

$$R_{1212} = R_{1212} - R_{1313} = b_1(b_2 - b_3) + a_2 - a_3 = \frac{2B_{12,4}^2}{(b_4 - b_1)(b_4 - b_2)}\,. \tag{3.47}$$

The algebraic summation (3.45)−(3.46)−(3.47) gives

$$B_{12,4}^2 = \frac{1}{2}(b_1 - b_4)^2(b_2 - b_3)(b_2 - b_4)\,. \tag{3.48}$$

Similarly, using (2.17), (3.40) and (3.45) through (3.47), we have

$$\begin{aligned} R_{1212} - R_{2323} - R_{1515} + R_{3535} &= (b_2 - b_4)(b_1 - b_3) \\ &= \frac{2B_{12,4}^2}{(b_4 - b_1)(b_4 - b_2)} - \frac{2B_{23,4}^2}{(b_4 - b_2)(b_4 - b_3)}\,. \end{aligned} \tag{3.49}$$

Putting (3.41) and (3.48) into (3.49), we obtain

$$(b_2 - b_4)(b_1 - b_3) = (b_1 - b_4)(b_2 - b_3) + \frac{(b_1 - b_4)^2(b_2 - b_3)^2}{(b_2 - b_1)(b_4 - b_3)}\,,$$

which can be written as

$$(b_2 - b_4)(b_1 - b_3)[(b_2 - b_1)(b_4 - b_3) - (b_1 - b_4)(b_2 - b_3)] = 0\,.$$

Thus we have

$$(b_1 - b_4)(b_2 - b_3) - (b_2 - b_1)(b_4 - b_3) = 0\,. \tag{3.50}$$

By solving the system of equations (3.50), (3.3) and (3.44), that is,

$$\begin{cases} b_1 + b_2 + b_3 = -2b_4\,, \quad b_4^2 = \dfrac{1}{25}\,, \\ b_1^2 + b_2^2 + b_3^2 = \dfrac{4}{5} - 2b_4^2\,, \\ (b_1 - b_4)(b_2 - b_3) - (b_2 - b_1)(b_4 - b_3) = 0\,, \end{cases}$$

we find that $(b_1,\ b_2,\ b_3,\ b_4)$ can only have the following six possibilities:

$$\left(\frac{1}{5},\ -\frac{4}{5},\ \frac{1}{5},\ \frac{1}{5}\right), \quad \left(-\frac{4}{5},\ \frac{1}{5},\ \frac{1}{5},\ \frac{1}{5}\right), \quad \left(\frac{1}{5},\ \frac{1}{5},\ -\frac{4}{5},\ \frac{1}{5}\right),$$
$$\left(-\frac{1}{5}, \frac{4}{5}, -\frac{1}{5}, -\frac{1}{5}\right), \quad \left(-\frac{1}{5}, -\frac{1}{5}, \frac{4}{5}, -\frac{1}{5}\right), \quad \left(\frac{4}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}\right).$$

This contradicts the assumption that $b_1,\ b_2,\ b_3,\ b_4$ are distinct. Thus II-2 does not occur.

II-3: $B_{12,4}B_{23,5} \neq 0$ and $B_{12,3} = B_{12,5} = B_{23,4} = B_{13,4} = B_{13,5} = 0$.

In this case, by (3.4) and (3.5), we have $\omega_{13} = \omega_{15} = \omega_{34} = 0$. Then, by (3.12), we have

$$\begin{aligned} -R_{3434}\omega_3 \wedge \omega_4 &= d\omega_{34} - \sum_k \omega_{3k} \wedge \omega_{k4} \\ &= -\omega_{32} \wedge \omega_{24} - \omega_{35} \wedge \omega_{54} = \Gamma_{53}^2\omega_5 \wedge (\Gamma_{14}^2\omega_1) - \Gamma_{25}^3\omega_2 \wedge \omega_{54}\,. \end{aligned}$$

Comparing both sides of the above equation, we obtain

$$\Gamma_{53}^{2}\Gamma_{14}^{2}=\frac{B_{23,5}B_{12,4}}{(b_2-b_3)(b_2-b_4)}=0\,,$$

which is a contradiction. Therefore II-3 does not occur.

II-4: $B_{12,4}B_{23,4}B_{13,5}\neq 0$ and $B_{12,3}=B_{12,5}=B_{23,5}=B_{13,4}=0$.

In this case, as in Lemma 3.3, we find that (3.27) holds for some functions λ and μ. Now, similar argument as in the proof of Lemma 3.3 will give a contradiction. Therefore II-4 does not occur.

II-5: $B_{12,4}B_{23,4}B_{13,4}\neq 0$ and $B_{12,3}=B_{12,5}=B_{23,5}=B_{13,5}=0$.

Now, we have $\omega_{12}=\Gamma_{42}^{1}\omega_4,\ \omega_{13}=\Gamma_{43}^{1}\omega_4,\ \omega_{32}=\Gamma_{42}^{3}\omega_4,\ \omega_{15}=\omega_{25}=\omega_{35}=0$. Then we have the following:

$$\begin{aligned}-R_{1212}\omega_1\wedge\omega_2&=d\omega_{12}-\sum_k\omega_{1k}\wedge\omega_{k2}\\&=\frac{dB_{12,4}}{b_1-b_2}\wedge\omega_4+\Gamma_{42}^{1}(\Gamma_{21}^{4}\omega_2+\Gamma_{31}^{4}\omega_3)\wedge\omega_1\\&\quad+\Gamma_{42}^{1}(\Gamma_{12}^{4}\omega_1+\Gamma_{32}^{4}\omega_3)\wedge\omega_2+\Gamma_{42}^{1}(\Gamma_{13}^{4}\omega_1+\Gamma_{23}^{4}\omega_2)\wedge\omega_3\\&\quad-(\Gamma_{24}^{1}\omega_2+\Gamma_{34}^{1}\omega_3)\wedge(\Gamma_{12}^{4}\omega_1+\Gamma_{32}^{4}\omega_3)\,.\end{aligned}$$

Comparing the coefficients of $\omega_1\wedge\omega_3$ and $\omega_2\wedge\omega_3$, respectively, on both sides of the above equation, we obtain

$$\Gamma_{42}^{1}\Gamma_{31}^{4}-\Gamma_{42}^{1}\Gamma_{13}^{4}-\Gamma_{34}^{1}\Gamma_{12}^{4}=0\,,\quad \Gamma_{42}^{1}\Gamma_{32}^{4}-\Gamma_{42}^{1}\Gamma_{23}^{4}+\Gamma_{24}^{1}\Gamma_{32}^{4}=0\,,$$

or equivalently, by (3.5) and $B_{12,4}B_{23,4}B_{13,4}\neq 0$,

$$\frac{1}{(b_1-b_2)(b_4-b_1)}-\frac{1}{(b_1-b_2)(b_4-b_3)}-\frac{1}{(b_1-b_4)(b_4-b_2)}=0\,,\tag{3.51}$$

$$\frac{1}{(b_1-b_2)(b_4-b_2)}-\frac{1}{(b_1-b_2)(b_4-b_3)}+\frac{1}{(b_1-b_4)(b_4-b_2)}=0\,.\tag{3.52}$$

Then (3.51) and (3.52) give the contradiction $3/(b_1-b_4)(b_4-b_2)=0$. Thus case II-5 does not occur.

We have completed the proof of Proposition 3.1.

PROPOSITION 3.2. *In both Cases* I-1 *and* I-4, *the tensor* $\mathbf{A}+b_4\mathbf{B}$ *has exactly two distinct constant eigenvalues*

PROOF. We will deal the following two cases separately.

Case I-1: $B_{12,3}\neq 0$ and $B_{12,4}=B_{12,5}=B_{23,4}=B_{23,5}=B_{13,4}=B_{13,5}=0$.

From (2.17) and Lemma 3.2, we have

$$R_{1515}=R_{1414}=b_1b_4+a_1+a_4=0\,,\tag{3.53}$$

$$R_{2525}=R_{2424}=b_2b_4+a_2+a_4=0\,,\tag{3.54}$$

$$R_{3535}=R_{3434}=b_3b_4+a_3+a_4=0\,,\quad a_4=a_5\,,\tag{3.55}$$

$$R_{1212} = \frac{2B_{12,3}^2}{(b_3 - b_1)(b_3 - b_2)} = b_1 b_2 + a_1 + a_2\,, \tag{3.56}$$

$$R_{1313} = \frac{2B_{12,3}^2}{(b_2 - b_1)(b_2 - b_3)} = b_1 b_3 + a_1 + a_3\,, \tag{3.57}$$

$$R_{2323} = \frac{2B_{12,3}^2}{(b_1 - b_2)(b_1 - b_3)} = b_2 b_3 + a_2 + a_3\,. \tag{3.58}$$

From these we get

$$R_{1414} + R_{2323} - R_{2424} - R_{1313} = (b_1 - b_2)(b_4 - b_3) = \frac{2B_{12,3}^2(b_1 + b_2 - 2b_3)}{(b_1 - b_2)(b_2 - b_3)(b_1 - b_3)}\,, \tag{3.59}$$

$$R_{1212} + R_{1313} - R_{2323} = b_1 b_2 + b_1 b_3 - b_2 b_3 + 2a_1 = \frac{4B_{12,3}^2}{(b_3 - b_1)(b_1 - b_2)}\,, \tag{3.60}$$

and thus $B_{12,3}$ is constant. Moreover, (3.59) and (3.60) give

$$a_1 = \frac{(b_1 - b_2)(b_2 - b_3)(b_3 - b_4)}{b_1 + b_2 - 2b_3} + \frac{1}{2}(b_2 b_3 - b_1 b_2 - b_1 b_3)\,, \tag{3.61}$$

which implies that a_1 is constant. It follows from (3.53) through (3.57) that the set $\{a_i\}_{2\le i\le 5}$ consists of constants. Now, from (3.53) through (3.55), we see that

$$a_1 + b_4 b_1 = a_2 + b_4 b_2 = a_3 + b_4 b_3 = -a_4\,. \tag{3.62}$$

On the other hand, we claim that

$$a_4 + b_4 b_4 = a_5 + b_4 b_5 = \text{constant} \neq -a_4\,. \tag{3.63}$$

Indeed, if $a_4 + b_4 b_4 = a_5 + b_4 b_5 = -a_4$, then $\mathbf{A} + b_4\mathbf{B}$ has only one constant eigenvalue. Then, according to [13], M is locally Möbius equivalent to an Euclidean isoparametric hypersurface in $\boldsymbol{S}^6$. This is impossible for $\gamma = 4$, cf. [7].

We have proved the assertion in the case I-1.

Case I-4: $B_{12,3}B_{12,4}B_{23,4}B_{13,4} \neq 0$ and $B_{12,5} = B_{23,5} = B_{13,5} = 0$.

In this case, $\omega_{15} = \omega_{25} = \omega_{35} = 0$. By (2.17), Lemma 3.2 and Lemma 3.3, we have

$$R_{1515} = b_5 b_1 + a_1 + a_5 = 0\,, \tag{3.64}$$

$$R_{2525} = b_5 b_2 + a_2 + a_5 = 0\,, \tag{3.65}$$

$$R_{3535} = b_5 b_3 + a_3 + a_5 = 0\,, \tag{3.66}$$

$$R_{1212} = \frac{2B_{12,3}^2}{(b_3 - b_1)(b_3 - b_2)} + \frac{2B_{12,4}^2}{(b_4 - b_1)(b_4 - b_2)} = b_1 b_2 + a_1 + a_2\,, \tag{3.67}$$

$$R_{1313} = \frac{2B_{12,3}^2}{(b_2 - b_1)(b_2 - b_3)} + \frac{2B_{13,4}^2}{(b_4 - b_1)(b_4 - b_3)} = b_1 b_3 + a_1 + a_3\,, \tag{3.68}$$

$$R_{2323} = \frac{2B_{12,3}^2}{(b_1 - b_2)(b_1 - b_3)} + \frac{2B_{23,4}^2}{(b_4 - b_2)(b_4 - b_3)} = b_2 b_3 + a_2 + a_3\,, \tag{3.69}$$

$$R_{1414} = \frac{2B_{12,4}^2}{(b_2 - b_1)(b_2 - b_4)} + \frac{2B_{13,4}^2}{(b_3 - b_1)(b_3 - b_4)} = b_1 b_4 + a_1 + a_4\,, \tag{3.70}$$

$$R_{2424} = \frac{2B_{12,4}^2}{(b_1 - b_2)(b_1 - b_4)} + \frac{2B_{23,4}^2}{(b_3 - b_2)(b_3 - b_4)} = b_2 b_4 + a_2 + a_4\,, \tag{3.71}$$

$$R_{3434} = \frac{2B_{23,4}^2}{(b_2 - b_3)(b_2 - b_4)} + \frac{2B_{13,4}^2}{(b_1 - b_3)(b_1 - b_4)} = b_3 b_4 + a_3 + a_4\,. \tag{3.72}$$

From (2.20) and (3.64) through (3.66), we see that

$$0 = d\omega_{15} - \sum_k \omega_{1k} \wedge \omega_{k5} = -\omega_{14} \wedge \omega_{45} = -(\Gamma_{24}^1 \omega_2 + \Gamma_{34}^1 \omega_3) \wedge \omega_{45}\,,$$

$$0 = d\omega_{25} - \sum_k \omega_{2k} \wedge \omega_{k5} = -\omega_{24} \wedge \omega_{45} = -(\Gamma_{14}^2 \omega_1 + \Gamma_{34}^2 \omega_3) \wedge \omega_{45}\,,$$

$$0 = d\omega_{35} - \sum_k \omega_{3k} \wedge \omega_{k5} = -\omega_{34} \wedge \omega_{45} = -(\Gamma_{14}^3 \omega_1 + \Gamma_{24}^3 \omega_2) \wedge \omega_{45}\,,$$

which imply $\omega_{45} = 0$. Then we have

$$R_{4545} = b_5 b_4 + a_4 + a_5 = 0\,. \tag{3.73}$$

We observe from (3.67) through (3.72) that

$$\begin{aligned} 0 &= R_{1212} + R_{1313} + R_{2323} + R_{1414} + R_{2424} + R_{3434} \\ &= b_1 b_2 + b_2 b_3 + b_1 b_3 + b_1 b_4 + b_2 b_4 + b_3 b_4 + 3(a_1 + a_2 + a_3 + a_4) \\ &= \frac{1}{2}[(b_1 + b_2 + b_3 + b_4)^2 - (b_1^2 + b_2^2 + b_3^2 + b_4^2)] \\ &\quad - 3b_5(b_1 + b_2 + b_3 + b_4) - 12a_5\,. \end{aligned} \tag{3.74}$$

Here in the last step we use $a_1 + a_2 + a_3 + a_4 = -b_5(b_1 + b_2 + b_3 + b_4) - 4a_5$, which is derived from (3.64) through (3.66) and (3.73).

From (3.3) and (3.74), we get $a_5 = b_5^2/3 - 1/30 =$ constant. It follows from (3.64) through (3.66) and (3.73) that each element of $\{a_i\}_{i=1}^5$ is constant. Moreover, we see that

$$a_1 + b_5 b_1 = a_2 + b_5 b_2 = a_3 + b_5 b_3 = a_4 + b_5 b_4 = -a_5 = \text{constant}\,. \tag{3.75}$$

On the other hand, we claim that

$$a_5 + b_5 b_5 = \text{constant} \neq -a_5\,. \tag{3.76}$$

Indeed, if $a_5 + b_5 b_5 = -a_5$, then $\mathbf{A} + b_4 \mathbf{B}$ has only one constant eigenvalue. Then, according to [13] again, M is locally Möbius equivalent to an Euclidean isoparametric hypersurface in $\boldsymbol{S}^6$ and this is impossible for $\gamma = 4$.

Now (3.75) and (3.76) say that $\mathbf{A} + b_4 \mathbf{B}$ has exactly two distinct constant eigenvalues. This gives the conclusion in Case I-4, and we have completed the proof of Proposition 3.2.

Finally, combining Proposition 3.1, Proposition 3.2 and Theorem 2.4, we obtain the following

THEOREM 3.1. *Let $x : M^5 \to \boldsymbol{S}^6$ be a Möbius isoparametric hypersurface with four distinct Möbius principal curvatures. Then there exists a constant λ such that the linear combination $\mathbf{A} + \lambda\mathbf{B}$ of the Blaschke tensor $\mathbf{A}$ and the Möbius second fundamental form $\mathbf{B}$ has exactly two distinct constant eigenvalues. Moreover, locally x can only be Möbius equivalent to one of the following families of hypersurfaces in $\boldsymbol{S}^6$:*

($\mathfrak{D}_1$) *Hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^5 = N^p \times \boldsymbol{H}^{5-p}(-r^{-2}) \to \boldsymbol{S}^6 \tag{3.77}$$

with $\tilde{x}_1 = y_1/y_0,\ \tilde{x}_2 = y_2/y_0,\ y_0 \in \boldsymbol{R}^+,\ y_1 \in \boldsymbol{R}^{p+2},\ y_2 \in \boldsymbol{R}^{5-p},\ 2 \le p \le 4,\ r > 0$. Here $y_1 : N^p \to \boldsymbol{S}^{p+1}(r) \hookrightarrow \boldsymbol{R}^{p+2}$ is an immersed umbilic free hypersurface with constant mean curvature H_1 and constant scalar curvature R_1 in the $(p+1)$-dimensional sphere of radius r such that

$$H_1 = -\frac{5}{p}\lambda\,, \quad R_1 = \frac{5p(p-1) - 4r^2}{5r^2} + 20\lambda^2\,, \tag{3.78}$$

and $(y_0, y_2) : \boldsymbol{H}^{5-p}(-r^{-2}) \hookrightarrow \boldsymbol{L}^{6-p}$ is the standard embedding of the hyperbolic space of sectional curvature $-r^{-2}$ into the $(6-p)$-dimensional Lorentz space with $-y_0^2 + y_2^2 = -r^2$.

($\mathfrak{D}_2$) *Hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^5 = N^p \times \boldsymbol{S}^{5-p}(r) \to \boldsymbol{S}^6 \tag{3.79}$$

with $\tilde{x}_1 = y_1/y_0,\ \tilde{x}_2 = y_2/y_0,\ y_0 \in \boldsymbol{R}^+,\ y_1 \in \boldsymbol{R}^{p+1},\ y_2 \in \boldsymbol{R}^{6-p},\ 2 \le p \le 4,\ r > 0$. Here $(y_0, y_1) : N^p \to \boldsymbol{H}^{p+1}(-r^{-2}) \hookrightarrow \boldsymbol{L}^{p+2}$, with $y_0 > 0$ and $-y_0^2 + y_1^2 = -r^2$, is an immersed umbilic free hypersurface into the $(p+1)$-dimensional hyperbolic space of sectional curvature $-r^{-2}$ with constant mean curvature H_1 and constant scalar curvature R_1 such that

$$H_1 = -\frac{5}{p}\lambda\,, \quad R_1 = -\frac{5p(p-1) + 4r^2}{5r^2} + 20\lambda^2\,, \tag{3.80}$$

and $y_2 : \boldsymbol{S}^{5-p}(r) \hookrightarrow \boldsymbol{R}^{6-p}$ is the standard embedding of $(5-p)$-dimensional sphere of radius r.

To settle the problem which hypersurfaces in ($\mathfrak{D}_1$) and ($\mathfrak{D}_2$) are Möbius isoparametric, we need to calculate their Möbius invariants. This will be done in Section 5.

4. Möbius isoparametric hypersurfaces with $\gamma = 5$. In this section, we assume that $x : M^5 \to \boldsymbol{S}^6$ is a Möbius isoparametric hypersurfaces with $\gamma = 5$. As stated in Introduction, any Möbius isoparametric hypersurfaces with $\gamma = 5$, if it exists, can not be compact.

We choose $\{E_i\}$ such that (3.1) through (3.5) hold. Since $b_1,\ b_2,\ b_3,\ b_4$ and b_5 are mutually distinct, we see from (3.5) that $B_{ii,j} = B_{ij,i} = 0$ hold for all $i,\ j$, and therefore we have

$$\Gamma_{ij}^i = -\Gamma_{ii}^j = 0\,, \quad i, j = 1,\ 2,\ 3,\ 4,\ 5\,.$$

LEMMA 4.1. *Let i, j, k, l, s denote the five elements of* $\{1, 2, 3, 4, 5\}$ *with order arbitrarily given. Then we have*

$$R_{ijij} = \frac{2B_{ij,k}^2}{(b_k - b_i)(b_k - b_j)} + \frac{2B_{ij,l}^2}{(b_l - b_i)(b_l - b_j)} + \frac{2B_{ij,s}^2}{(b_s - b_i)(b_s - b_j)} . \tag{4.1}$$

PROOF. This is similar to the proof of Lemma 3.2. □

For $\gamma = 5$, from [8] we know that M has non-parallel Möbius second fundamental form. Without loss of generality, hereafter we assume that $B_{12,3} \neq 0$ in this section.

LEMMA 4.2. $B_{12,4}B_{12,5} = B_{13,4}B_{13,5} = B_{23,4}B_{23,5} = 0$.

PROOF. Suppose on the contrary that $B_{12,3}B_{12,4}B_{12,5} \neq 0$. Then, similar to the proof of Lemma 3.3, we see that $\mathbf{A} + \lambda\mathbf{B} + \mu g = 0$ holds for some smooth function λ and μ. Then, according to the result of Li and Wang [13], we know that M is Möbius equivalent to an Euclidean isoparametric hypersurfaces in $\boldsymbol{S}^6$ with five distinct principal curvatures. This is impossible because the number of distinct principal curvature of any Euclidean isoparametric hypersurfaces of the sphere can only be one of $\{1, 2, 3, 4, 6\}$. Therefore, we have $B_{12,4}B_{12,5} = 0$. Analogously, we obtain $B_{13,4}B_{13,5} = B_{23,4}B_{23,5} = 0$. □

LEMMA 4.3. *There are no Möbius isoparametric hypersurfaces M^5 in $\boldsymbol{S}^6$ with five distinct principal curvatures, which satisfy* $B_{12,3} \neq 0$, $B_{12,4} = B_{12,5} = B_{13,4} = B_{13,5} = B_{23,4} = B_{23,5} = 0$.

PROOF. We first show that, under the assumption, at most one of $\{B_{14,5}, B_{24,5}, B_{34,5}\}$ is zero. Indeed, if $B_{24,5} = B_{34,5} = 0$, then using Lemma 4.1 and (2.17), we have

$$R_{2424} = b_2b_4 + a_2 + a_4 = 0\,, \quad R_{2525} = b_2b_5 + a_2 + a_5 = 0\,,$$
$$R_{3434} = b_3b_4 + a_3 + a_4 = 0\,, \quad R_{3535} = b_3b_5 + a_3 + a_5 = 0\,.$$

It follows that $0 = R_{2424} - R_{2525} + R_{3535} - R_{3434} = (b_2 - b_3)(b_4 - b_5)$. This contradicts to $\gamma = 5$.

We now assume $B_{14,5}B_{24,5} \neq 0$. From this and $B_{12,3} \neq 0$, a method similar to the proof of Lemma 3.3 implies the existence of λ and μ such that

$$a_1 + \lambda b_1 = a_2 + \lambda b_2 = a_4 + \lambda b_4 = a_5 + \lambda b_5\,, \qquad a_1 + \mu b_1 = a_2 + \mu b_2 = a_3 + \mu b_3\,.$$

This implies that $\lambda = \mu$ and all eigenvalues of $\mathbf{A} + \lambda\mathbf{B}$ are the same. Now the same argument as in the proof of Lemma 4.2 gives the assertion. □

By Lemma 4.3, in the remainder of this section we can assume $B_{12,3}B_{12,4} \neq 0$.

LEMMA 4.4. *If* $B_{12,3}B_{12,4} \neq 0$, *then* $B_{12,5} = B_{13,5} = B_{23,5} = B_{14,5} = B_{24,5} = B_{34,5} = 0$.

PROOF. If otherwise, as in the proof of Lemma 3.3 and Lemma 4.3, we easily find that $\mathbf{A} + \lambda\mathbf{B} + \mu g = 0$ holds for some smooth functions λ and μ. From this we get a contradiction. See the proof of Lemma 4.2. □

LEMMA 4.5. *If* $B_{12,3}B_{12,4} \neq 0$, *then* $a_1,\ a_2,\ a_3,\ a_4,\ a_5$ *are all constant, and the tensor* $\mathbf{A} + b_5\mathbf{B}$ *has exactly two distinct constant eigenvalues.*

PROOF. From Lemma 4.1 and Lemma 4.4, we have $R_{1515} = R_{2525} = R_{3535} = R_{4545} = 0$. This implies that

$$a_1 + b_5 b_1 = a_2 + b_5 b_2 = a_3 + b_5 b_3 = a_4 + b_5 b_4 = -a_5\,. \tag{4.2}$$

Now, by the same argument as in the proof of Proposition 3.2 for Case I-4, we conclude that a_5 and then $a_1,\ a_2,\ a_3,\ a_4$ are all constant.

By virtue of the fact that there are no Euclidean isoparametric hypersurfaces in $\boldsymbol{S}^6$ with five distinct principal curvatures, we can complete the remaining proof just copying word by word from the proof of Proposition 3.2, Case I-4. □

Combining the above Lemmas and Theorem 2.4, we obtain

THEOREM 4.1. *Let* $x : M^5 \to \boldsymbol{S}^6$ *be a Möbius isoparametric hypersurface with five distinct Möbius principal curvatures. Then there exists a constant* λ *such that the linear combination* $\mathbf{A} + \lambda\mathbf{B}$ *of the Blaschke tensor* $\mathbf{A}$ *and the Möbius second fundamental form* $\mathbf{B}$ *has exactly two distinct constant eigenvalues. Moreover, locally* x *is Möbius equivalent to one of the two families of hypersurfaces* $(\mathfrak{D}_1)$ *and* $(\mathfrak{D}_2)$ *as stated in Theorem* 3.1.

5. Möbius invariants of hypersurfaces in $(\mathfrak{D}_1)$ **and** $(\mathfrak{D}_2)$**.** The hypersurfaces in $(\mathfrak{D}_1)$ and $(\mathfrak{D}_2)$, as defined in Theorem 3.1, might be not Möbius isoparametric. In this section, using direct calculations, we will determine all those in $(\mathfrak{D}_1)$ and $(\mathfrak{D}_2)$ that are Möbius isoparametric. In [10], a similar work has been done for the special case that $H_1 = 0$.

EXAMPLE 5.1 (cf. [23]). Calculation for hypersurfaces in $(\mathfrak{D}_1)$.

For $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ as defined in (3.77), we have

$$d\tilde{x} = -\frac{dy_0}{y_0^2}(y_1, y_2) + \frac{1}{y_0}(dy_1, dy_2)\,, \tag{5.1}$$

and then its Euclidean induced metric is given by

$$\tilde{I} = d\tilde{x} \cdot d\tilde{x} = y_0^{-2}(-dy_0^2 + dy_1^2 + dy_2^2)|_{\tilde{M}^5}\,. \tag{5.2}$$

Let ξ_1 be the unit normal vector field of $y_1 : N^p \to \boldsymbol{S}^{p+1}(r) \hookrightarrow \boldsymbol{R}^{p+2}$. Then $\xi = (\xi_1, 0) \in \boldsymbol{R}^7$ is a unit normal vector field of $\tilde{x}$. Consequently, by (5.1), the (Euclidean) second fundamental form $\tilde{h}$ of $\tilde{x}$ is related to the (Euclidean) second fundamental form $\tilde{h}^*$ of y_1 by

$$\tilde{h} = -d\xi \cdot d\tilde{x} = -y_0^{-1}(d\xi_1 \cdot dy_1) = y_0^{-1}\tilde{h}^*. \tag{5.3}$$

Let $\{\tilde{E}_i\}_{1\le i\le p}$ and $\{\tilde{E}_i\}_{p+1\le i\le 5}$ be the local orthonormal basis on (N^p, dy_1^2) and $\boldsymbol{H}^{5-p}(-r^{-2})$, respectively, such that $\tilde{h}^*_{ij} = \tilde{h}^*(\tilde{E}_i, \tilde{E}_j) = \eta_i\delta_{ij}$ for $1 \le i, j \le p$. Then $\{\tilde{E}_i\}_{1\le i\le 5}$ form a local orthonormal basis on $\tilde{M}^5$ with respect to the metric $(-dy_0^2 + dy_1^2 + dy_2^2)|_{\tilde{M}^5} = y_0^2\tilde{I}$.

Denote $\tilde{e}_i = y_0\tilde{E}_i,\ 1 \le i \le 5$. Then $\{\tilde{e}_i\}_{1\le i\le 5}$ is a local orthonormal basis on $\tilde{M}^5$ with respect to the metric $\tilde{I}$. Thus we have

$$\text{(5.4)}\quad \begin{cases} \tilde{h}_{ij} = \tilde{h}(\tilde{e}_i, \tilde{e}_j) = y_0^2\tilde{h}(\tilde{E}_i, \tilde{E}_j) = y_0\tilde{h}^*(\tilde{E}_i, \tilde{E}_j) = y_0\tilde{h}^*_{ij} = y_0\eta_i\delta_{ij}\,, & 1 \le i, j \le p\,, \\ \tilde{h}_{ij} = 0,\ \ i > p \ \text{ or } \ j > p\,. \end{cases}$$

From (5.4) and the fact that y_1 is of constant mean curvature $H_1 = -5\lambda/p$, we see that the mean curvature of $\tilde{x}$ is $\tilde{H} = -\lambda y_0$. Therefore, by definition, the Möbius factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\text{(5.5)}\qquad \rho^2 = \frac{5}{4}\Big(\sum_{i,j=1}^{5}\tilde{h}_{ij}^2 - 5\tilde{H}^2\Big) = \frac{5}{4}y_0^2\Big(\sum_{i=1}^{p}\eta_i^2 - 5\lambda^2\Big) = y_0^2\,.$$

Here in the last equality, we make use of the fact $\sum_{i,j=1}^{p}(\tilde{h}^*_{ij})^2 = \sum_{i=1}^{p}\eta_i^2 = 5\lambda^2 + 4/5$, which is implied by (3.78) and the Gauss equation of y_1. Hence, the Möbius position vector of $\tilde{x}$ is $\tilde{Y} = \tilde{\rho}(1, \tilde{x}) = (y_0, y_1, y_2) \in \boldsymbol{L}^8$ and the Möbius metric of $\tilde{x}$ is

$$\text{(5.6)}\qquad \tilde{g} = \langle d\tilde{Y}, d\tilde{Y}\rangle = (-dy_0^2 + dy_1^2 + dy_2^2)|_{\tilde{M}^5} = y_0^2\tilde{I}\,.$$

Therefore, $\{\tilde{E}_i\}_{1\le i\le 5}$ is in fact a local orthonormal basis with respect to the Möbius metric $\tilde{g}$. Furthermore, the Möbius second fundamental form of $\tilde{x}$ is

$$\text{(5.7)}\qquad \tilde{\mathbf{B}} = \tilde{\rho}^{-1}\sum_{i,j=1}^{5}(\tilde{h}_{ij} - \tilde{H}\delta_{ij})\tilde{\omega}_i\tilde{\omega}_j = \sum_{i=1}^{p}\eta_i\tilde{\omega}_i^2 + \sum_{i=1}^{5}\lambda\tilde{\omega}_i^2\,,$$

where $\{\tilde{\omega}_i\}_{1\le i\le 5}$ is the dual basis of $\{\tilde{E}_i\}_{1\le i\le 5}$ on $\tilde{M}^5$. Note that (5.7) is equivalent to

$$\text{(5.8)}\qquad \tilde{B}_{ij} = (\eta_i + \lambda)\delta_{ij}\,,\quad 1 \le i, j \le p\,;\quad \tilde{B}_{ij} = \lambda\delta_{ij}\,,\quad i > p \ \text{ or } \ j > p\,.$$

Since (5.6) shows that $(\tilde{M}^5, \tilde{g})$ is the Riemannian *direct* product

$$(\tilde{M}^5, \tilde{g}) = (N^p, dy_1^2) \times \boldsymbol{H}^{5-p}(-r^{-2})\,,$$

using the Gauss equation, we can write down the Ricci tensor of $\tilde{g}$ with respect to $\{\tilde{E}_i\}_{1\le i\le 5}$ as follows

$$\text{(5.9)}\quad \begin{cases} \tilde{R}_{ij} = \Big(\dfrac{p-1}{r^2} - 5\lambda\eta_i - \eta_i\eta_j\Big)\delta_{ij}\,, & \text{if } 1 \le i, j \le p\,, \\ \tilde{R}_{ij} = -\dfrac{4-p}{r^2}\delta_{ij}\,, & \text{if } p+1 \le i, j \le 5\,, \\ \tilde{R}_{ij} = 0\,, & \text{for all other cases}\,. \end{cases}$$

This implies that the normalized scalar curvature $\tilde{R}$ of $\tilde{g}$ satisfies

$$\text{(5.10)}\qquad 20\tilde{R} = \frac{p(p-1) - (5-p)(4-p)}{r^2} + 20\lambda^2 - \frac{4}{5}\,.$$

Thus we have

$$\text{(5.11)}\qquad \frac{1}{10}(1 + 25\tilde{R}) = \frac{2p-5}{2r^2} + \frac{5\lambda^2}{2}\,.$$

From (2.18), (2.19) and (5.8) through (5.11), the Blaschke tensor $\tilde{\mathbf{A}} = \sum_{i,j=1}^{5} \tilde{A}_{ij}\tilde{\omega}_i\tilde{\omega}_j$ of $\tilde{x}$ can be easily calculated as follows:

$$
\begin{cases}
\tilde{A}_{ij} = \left(\dfrac{1}{2r^2} - \dfrac{\lambda^2}{2} - \lambda\eta_i\right)\delta_{ij}\,, & \text{if } 1 \le i, j \le p\,, \\
\tilde{A}_{ij} = -\left(\dfrac{1}{2r^2} + \dfrac{\lambda^2}{2}\right)\delta_{ij}, & \text{if } p+1 \le i, j \le 5\,, \\
\tilde{A}_{ij} = 0\,, & \text{for all other cases}\,.
\end{cases}
\tag{5.12}
$$

From (5.8) and (5.12), we see that $\mathbf{A}+\lambda\mathbf{B}$ has two constant eigenvalues, namely $\lambda^2/2+1/(2r^2)$ and $\lambda^2/2 - 1/(2r^2)$ with multiplicities p and $5-p$, respectively.

For the Möbius form $\tilde{\boldsymbol{\Phi}} = \sum_{i=1}^{5} \tilde{C}_i\tilde{\omega}_i$ of $\tilde{x}$, from (2.11), (5.4) and that $\tilde{H} = -\lambda y_0$, $\tilde{\rho} = y_0$, we see that

$$
\begin{aligned}
\tilde{C}_i &= -\tilde{\rho}^{-2}\Big[\tilde{e}_i(\tilde{H}) + \sum_{j=1}^{5}(\tilde{h}_{ij} - \tilde{H}\delta_{ij})\tilde{e}_j(\log\tilde{\rho})\Big] \\
&= \begin{cases}
-y_0^{-1}\eta_i\tilde{e}_i(\log y_0) = 0\,, & \text{if } 1 \le i \le p\,, \\
-\tilde{\rho}^{-2}[\tilde{e}_i(\tilde{H}) - \tilde{H}\tilde{e}_i(\log\tilde{\rho})] = 0\,, & \text{if } p+1 \le i \le 5\,.
\end{cases}
\end{aligned}
\tag{5.13}
$$

Therefore, we have $\tilde{\boldsymbol{\Phi}} = 0$.

As a summary of the above calculation, we can now prove the following

THEOREM 5.1. *A hypersurface $\tilde{x} : M^5 \to \boldsymbol{S}^6$ in $(\mathfrak{D}_1)$ is a Möbius isoparametric hypersurface with four or five distinct principal curvatures if and only if one of the following occurs*:

(1) $p = 3$ *and $y_1 : N^3 \to \boldsymbol{S}^4(r)$ is a non-minimal isoparametric hypersurface with three distinct principal curvatures.*

(2) $p = 4$ *and $y_1 : N^4 \to \boldsymbol{S}^5(r)$ is an Euclidean isoparametric hypersurface with four distinct principal curvatures*

PROOF. From (5.8) and (5.13) we see that $\tilde{x}$ in $(\mathfrak{D}_1)$ is Möbius isoparametric if and only if $\{\eta_i\}_{1\le i\le p}$ consists of constants. If this is the case, then $y_1 : N^p \to \boldsymbol{S}^{p+1}(r)$ is an Euclidean isoparametric hypersurface. Moreover, (5.8) shows that $\tilde{x} : M^5 \to \boldsymbol{S}^6$ can have at most $p+1$ distinct Möbius principal curvatures. As $p \le 4$, we see that p must be 3 or 4.

If $p = 3$, we see that it must be the case that $\gamma = 4$. Furthermore, we have $\eta_1\eta_2\eta_3 \ne 0$ and none of $\{\eta_1, \eta_2, \eta_3\}$ is zero. According to Cartan [3], isoparametric hypersurfaces in $\boldsymbol{S}^4(r)$ with three non-zero distinct principal curvatures do exist. Moreover, this is equivalent to the fact that the hypersurface is non-minimal. This shows that Case (1) occurs. According to (3.62), (3.63), (3.75), (4.2) and (5.8), we see that this corresponds to Case I-1.

If $p = 4$, from (5.8) we see that γ can be either 4 or 5. If $\gamma = 4$, then $\{0, \eta_1, \eta_2, \eta_3, \eta_4\}$ consists of four distinct constants. Since an Euclidean isoparametric hypersurface $y_1 : N^4 \to \boldsymbol{S}^5(r)$ can not have three distinct principal curvatures, we see that $\eta_1, \eta_2, \eta_3, \eta_4$ are mutually distinct and one of them is zero. According to Cartan [3], this does not occur. If $\gamma = 5$, then $0,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4$ are mutually distinct, and therefore $y_1 : N^4 \to \boldsymbol{S}^5(r)$ is an Euclidean isoparametric hypersurface with four distinct principal curvatures. According to Cartan [3], this does occur and we obtain (2). As a matter of fact, it corresponds to that implied by Lemma 4.5. □

EXAMPLE 5.2 (cf. [23]). Calculation for hypersurfaces in $(\mathfrak{D}_2)$.

For $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ as defined in (3.79), we have

$$d\tilde{x} = -\frac{dy_0}{y_0^2}(y_1, y_2) + \frac{1}{y_0}(dy_1, dy_2) \tag{5.14}$$

and then its Euclidean induced metric is given by

$$\tilde{I} = d\tilde{x}\cdot d\tilde{x} = y_0^{-2}(-dy_0^2 + dy_1^2 + dy_2^2)|_{\tilde{M}^5}\,. \tag{5.15}$$

Let (ξ_0, ξ_1) be the unit normal vector field of $\tilde{y} := (y_0, y_1) : N^p \to \boldsymbol{H}^{p+1}(-r^{-2}) \hookrightarrow \boldsymbol{L}^{p+2}$, where $\xi_0 \in \boldsymbol{R}^+,\ \xi_1 \in \boldsymbol{R}^{p+1}$. Then we can verify that $\xi = (\xi_1, 0) - \xi_0\tilde{x} \in \boldsymbol{R}^7$ is a unit normal vector field of $\tilde{x}$, and the Euclidean second fundamental form $\tilde{h}$ of $\tilde{x}$ is given by

$$\begin{aligned}\tilde{h} &= -d\xi\cdot d\tilde{x} = \xi_0 d\tilde{x}\cdot d\tilde{x} - (d\xi_1, 0)\cdot d\tilde{x} = -y_0^{-1}(-d\xi_0 dy_0 + d\xi_1\cdot dy_1) + \xi_0\tilde{I}\\ &= -y_0^{-1}\langle d(\xi_0,\xi_1), d(y_0,y_1)\rangle + \xi_0\tilde{I} = y_0^{-1}\tilde{h}^* + \xi_0\tilde{I}\,,\end{aligned} \tag{5.16}$$

where $\tilde{h}^*$ denotes the *Euclidean* second fundamental form of $\tilde{y} : N^p \to \boldsymbol{H}^{p+1}(-r^{-2})$. Note that in the third equality, we have used $d\xi_1\cdot y_1 = y_0 d\xi_0$, which is implied by

$$-\xi_0 y_0 + \xi_1\cdot y_1 = -\xi_0 dy_0 + \xi_1\cdot dy_1 = 0\,. \tag{5.17}$$

Let $\{\tilde{E}_i\}_{1\le i\le p}$ and $\{\tilde{E}_i\}_{p+1\le i\le 5}$ be the local orthonormal basis on $(N^p, d\tilde{y}^2)$ and $\boldsymbol{S}^{5-p}(r)$, respectively, such that $\tilde{h}^*_{ij} = \tilde{h}^*(\tilde{E}_i, \tilde{E}_j) = \eta_i\delta_{ij}$ for $1 \le i, j \le p$. Then $\{\tilde{E}_i\}_{1\le i\le 5}$ form a local orthonormal basis on $\tilde{M}^5$ with respect to the metric $(-dy_0^2 + dy_1^2 + dy_2^2)|_{\tilde{M}^5} = y_0^2\tilde{I}$.

Denote $\tilde{e}_i = y_0\tilde{E}_i,\ 1 \le i \le 5$. Then $\{\tilde{e}_i\}_{1\le i\le 5}$ is a local orthonormal basis on $\tilde{M}^5$ with respect to the metric $\tilde{I}$. Thus we have

$$\begin{aligned}&\tilde{h}_{ij} = \tilde{h}(\tilde{e}_i, \tilde{e}_j) = y_0^2\tilde{h}(\tilde{E}_i, \tilde{E}_j) = (y_0\eta_i + \xi_0)\delta_{ij}\,, &&\text{if } 1 \le i, j \le p\,,\\ &\tilde{h}_{ij} = \tilde{h}(\tilde{e}_i, \tilde{e}_j) = y_0^2\tilde{h}(\tilde{E}_i, \tilde{E}_j) = \xi_0\delta_{ij}\,, &&\text{if } i > p \text{ or } j > p\,.\end{aligned} \tag{5.18}$$

From (3.80) and (5.18), we see that the mean curvature of $\tilde{x} : \tilde{M}^5 \to \boldsymbol{S}^6$ is

$$\tilde{H} = \frac{1}{5}\sum_{i=1}^{5}\tilde{h}_{ii} = \frac{y_0}{5}\sum_{i=1}^{p}\tilde{h}^*_{ii} + \xi_0 = -\lambda y_0 + \xi_0\,. \tag{5.19}$$

Therefore, by definition, the Möbius factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\rho^2 = \frac{5}{4}\Big(\sum_{i,j=1}^{5}\tilde{h}_{ij}^2 - 5\tilde{H}^2\Big) = \frac{5}{4}y_0^2\Big(\sum_{i=1}^{p}\eta_i^2 - 5\lambda^2\Big) = y_0^2\,. \tag{5.20}$$

Here in the last equality, we use the Gauss equation of $\tilde{y}$ to obtain

$$\begin{aligned}\sum_{i=1}^{p}\eta_i^2 &= \sum_{i,j=1}^{p}(\tilde{h}^*_{ij})^2 = -\frac{p(p-1)}{r^2} + p^2H_1^2 - R_1 \\ &= -\frac{p(p-1)}{r^2} + 25\lambda^2 + \frac{p(p-1)}{r^2} + \frac{4}{5} - 20\lambda^2 \\ &= 5\lambda^2 + \frac{4}{5}\,.\end{aligned}$$

Hence, the Möbius position vector of $\tilde{x}$ is $\tilde{Y} = \tilde{\rho}(1, \tilde{x}) = (y_0, y_1, y_2) \in \boldsymbol{L}^8$ and the Möbius metric of $\tilde{x}$ is

$$\tilde{g} = \langle d\tilde{Y}, d\tilde{Y}\rangle = (-dy_0^2 + dy_1^2 + dy_2^2)\,|_{\tilde{M}^5} = y_0^2\tilde{I}\,. \tag{5.21}$$

Therefore, $\{\tilde{E}_i\}_{1\le i\le 5}$ is in fact a local orthonormal basis of the Möbius metric $\tilde{g}$. Furthermore, the Möbius second fundamental form of $\tilde{x}$ is

$$\tilde{\mathbf{B}} = \tilde{\rho}^{-1}\sum_{i,j=1}^{5}(\tilde{h}_{ij} - \tilde{H}\delta_{ij})\tilde{\omega}_i\tilde{\omega}_j = \sum_{i=1}^{p}\eta_i\tilde{\omega}_i^2 + \sum_{i=1}^{5}\lambda\tilde{\omega}_i^2\,. \tag{5.22}$$

Here $\{\tilde{\omega}_i\}_{1\le i\le 5}$ is the dual basis of $\{\tilde{E}_i\}_{1\le i\le 5}$ on $\tilde{M}^5$. (5.22) is equivalent to

$$\tilde{B}_{ij} = (\eta_i + \lambda)\delta_{ij}\,, \quad 1 \le i, j \le p\,; \quad \tilde{B}_{ij} = \lambda\delta_{ij}\,, \quad i > p \text{ or } j > p\,. \tag{5.23}$$

Since (5.21) shows that $(\tilde{M}^5, \tilde{g})$ is the Riemannian product $(\tilde{M}^5, \tilde{g}) = (N^p, dy_1^2) \times \boldsymbol{S}^{5-p}(r)$, by making use of the Gauss equation, the Ricci tensor of $\tilde{g}$ with respect to $\{\tilde{E}_i\}_{1\le i\le 5}$ can be written as follows:

$$\begin{cases} \tilde{R}_{ij} = \left(-\dfrac{p-1}{r^2} - 5\lambda\eta_i - \eta_i\eta_j\right)\delta_{ij}\,, & \text{if } 1 \le i, j \le p\,, \\ \tilde{R}_{ij} = \dfrac{4-p}{r^2}\delta_{ij}\,, & \text{if } p+1 \le i, j \le 5\,, \\ \tilde{R}_{ij} = 0\,, & \text{for all other cases}\,. \end{cases} \tag{5.24}$$

This implies that the normalized scalar curvature $\tilde{R}$ of $\tilde{g}$ satisfies

$$\frac{1}{10}(1 + 25\tilde{R}) = \frac{(5-p)(4-p) - p(p-1)}{8r^2} + \frac{5\lambda^2}{2}\,. \tag{5.25}$$

From (2.18), (2.19) and (5.23) through (5.25), the Blaschke tensor $\tilde{\mathbf{A}} = \sum_{i,j=1}^{5}\tilde{A}_{ij}\tilde{\omega}_i\tilde{\omega}_j$ of $\tilde{x}$ can be easily calculated as follows:

$$\begin{cases} \tilde{A}_{ij} = -\left(\dfrac{1}{2r^2} + \dfrac{\lambda^2}{2} + \lambda\eta_i\right)\delta_{ij}, & \text{if } 1 \le i, j \le p\,, \\ \tilde{A}_{ij} = \left(\dfrac{1}{2r^2} - \dfrac{\lambda^2}{2}\right)\delta_{ij}, & \text{if } p+1 \le i, j \le 5\,, \\ \tilde{A}_{ij} = 0\,, & \text{for all other cases}\,. \end{cases} \tag{5.26}$$

From (5.23) and (5.26), we see that $\mathbf{A} + \lambda\mathbf{B}$ has two constant eigenvalues, namely $\lambda^2/2 - 1/(2r^2)$ and $\lambda^2/2 + 1/(2r^2)$ with multiplicities p and $5 - p$, respectively.

For the Möbius form $\tilde{\boldsymbol{\Phi}} = \sum_{i=1}^{5} \tilde{C}_i \tilde{\omega}_i$ of $\tilde{x}$, (2.11) and (5.18) through (5.20) imply the following:

If $1 \le i \le p$, then from the definition $\tilde{e}_i\{(\xi_0, \xi_1)\} = -\eta_i \tilde{e}_i\{(y_0, y_1)\}$ we see that

$$\tilde{C}_i = -\tilde{\rho}^{-2}\Big[\tilde{e}_i(\tilde{H}) + \sum_{j=1}^{5}(\tilde{h}_{ij} - \tilde{H}\delta_{ij})\tilde{e}_j(\log\tilde{\rho})\Big] = -y_0^{-2}\big[\tilde{e}_i(\xi_0) + \eta_i \tilde{e}_i(y_0)\big] = 0\,. \tag{5.27}$$

If $p + 1 \le i \le 5$, then as $\tilde{e}_i(y_0) = \tilde{e}_i(\xi_0) = 0$, we see that

$$\begin{aligned} \tilde{C}_i &= -y_0^{-2}[\tilde{e}_i(-\lambda y_0 + \xi_0) + \xi_0 \tilde{e}_i(\log y_0) - (-\lambda y_0 + \xi_0)\tilde{e}_i(\log y_0)] \\ &= -y_0^{-2}\tilde{e}_i(\xi_0) = 0\,. \end{aligned} \tag{5.28}$$

Therefore, we have $\tilde{\boldsymbol{\Phi}} = 0$.

Now we can prove the following

THEOREM 5.2. *There are no Möbius isoparametric hypersurfaces with four or five distinct principal curvatures in* $(\mathfrak{D}_2)$.

PROOF. From (5.23), (5.27) and (5.28), we see that $\tilde{x}$ in $(\mathfrak{D}_2)$ is Möbius isoparametric if and only if $\{\eta_i\}_{1\le i\le p}$ consists of constants. If this is the case, then $(y_0, y_1) : N^p \to \boldsymbol{H}^{p+1}(-r^{-2})$ is an Euclidean isoparametric hypersurface. Moreover, (5.23) shows that $\tilde{x} : M^5 \to \boldsymbol{S}^6$ can have at most $p + 1$ distinct Möbius principal curvatures. Since $2 \le p \le 4$, to guarantee $\tilde{x} : M^5 \to \boldsymbol{S}^6$ possesses at least four distinct Möbius principal curvatures, p must be 3 or 4 and $(y_0, y_1) : N^p \to \boldsymbol{H}^{p+1}(-r^{-2})$ must have at least three distinct principal curvatures. This is impossible, because, according to Cartan [2], an isoparametric hypersurface in $\boldsymbol{H}^{n+1}$ can have at most two distinct principal curvatures, namely, it must be either totally umbilic or else an open subset of a standard product $\boldsymbol{S}^k \times \boldsymbol{H}^{n-k}$, $1 \le k \le n-1$. This completes the proof of Theorem 5.2.

6. Completion of the proof of the Classification Theorem. Let $x : M^5 \to \boldsymbol{S}^6$ be a Möbius isoparametric hypersurface with γ denoting the number of distinct Möbius principal curvatures. Then we have exactly four cases: $\gamma = 2, 3, 4, 5$. If $\gamma = 2$, Theorem 2.2 shows that the hypersurface must be locally Möbius equivalent to one of the hypersurfaces (1), (2) and (3), as stated in the Classification Theorem. If $\gamma = 3$, Theorem 2.3 shows that the hypersurface must be locally Möbius equivalent to one of the hypersurfaces (4) and (5), as stated in the Classification Theorem. For the remaining cases $\gamma = 4$ and $\gamma = 5$, we can apply Theorem 3.1, Theorem 4.1, Theorem 5.1 and Theorem 5.2 to conclude that the hypersurface must be locally Möbius equivalent to either the hypersurface (6) or (7), as stated in the Classification Theorem.

We have completed the proof of the Classification Theorem.

FINAL REMARKS. As a counterpart of the Cecil-Ryan conjecture for Dupin hypersurfaces which states that a compact embedded Dupin hypersurface in a space form is Lie equivalent to an isoparametric hypersurface, C. P. Wang, in a private communication, made a similar conjecture. Namely, every compact embedded Möbius isoparametric hypersurface in S^{n+1} is Möbius equivalent to an isoparametric hypersurface. It is worthwhile to note that all the accomplished classification of Möbius isoparametric hypersurfaces strengthen this conjecture.

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