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WEAK GEOMETRIC STRUCTURES ON SUBMANIFOLDS OF AFFINE SPACES

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Abstract. A few affine invariant structures depending only on the second fundamental form relative to arbitrary transversal bundles on submanifolds of the standard affine spaces are introduced. A notion of "local strong convexity" is proposed for arbitrary codimensional submanifolds. In the case of n-dimensional submanifolds of $2n$-dimensional real affine spaces, complex structures on the ambient spaces are used as a tool for studying real affine invariants.

1. Introduction. In the present paper we study affine invariants on submanifolds of affine spaces, defined by using only the second fundamental form determined by any transversal bundle. Therefore we call the structures weak. For instance, the local strong convexity of hypersurfaces is such an invariant. In the first section we propose a notion of local strong convexity for arbitrary codimensional submanifolds. The notion we propose extends the one known in the theory of hypersurfaces of affine spaces. Our considerations cover the Riemannian case. We give four equivalent conditions which characterize the local strong convexity in the language of affine and metric geometries. According to this definition the 2-dimensional Clifford torus in the Euclidean 4-space is locally strongly convex.

We define the cone of the second fundamental form as the set $\{h(X, X)\ ;\ X \in T_xM, X \neq 0\} \subset \mathcal{N}_x$, where $\mathcal{N}$ is a transversal bundle (arbitrary chosen) for a given immersion. Whether the cone fills the whole space $\mathcal{N}_x$ or the origin of $\mathcal{N}_x$ belongs to the cone or it is symmetric relative to its vertex are well-defined affine invariants of a given immersion. In the case of 2-dimensional Riemannian surfaces in the Euclidean space $\boldsymbol{R}^N$ the cone of the second fundamental form is generated by the ellipse of curvature. Here, in the general setting, we also have ellipses depending on chosen bases of vector planes contained in the tangent spaces. Some properties of such ellipses are well-defined from the affine viewpoint. For instance, whether the ellipse reduces to a segment of a line passing through the origin of the vector space $\mathcal{N}_x$ or the origin lies in the interior of the domain surrounded by the ellipse are such properties. Of course, being a circle, which is interesting from the metric geometry viewpoint, is not an affine invariant.

We pay a special attention to n-dimensional submanifolds of affine spaces $\boldsymbol{R}^{2n}$. Every such submanifold can be locally viewed as a purely real (in other words affine Lagrangian) submanifold relative to some complex structure on $\boldsymbol{R}^{2n}$. Since the normal bundle for a purely real submanifold is isomorphic to the tangent bundle, it follows that affine invariants defined

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by the normal bundle can be transported to the tangent bundle. This makes the study easier. For instance, the second fundamental form (having values in some transversal vector bundle) becomes now a $(1,2)$ tensor field on a manifold and by fixing one vector we get an endomorphism of a tangent space. We can now take into account invariants of endomorphisms and define well-defined affine properties of a submanifold. Some of such properties are studied in Section 2. In particular, we obtain an n-linear symmetric form, whose proportionality class is an affine invariant. In the case of 2-dimensional submanifolds of $\boldsymbol{R}^4$ we get the same conformal class (possibly degenerate) which was discovered in the classical affine geometry of surfaces in $\boldsymbol{R}^4$. In this paper we study the properties of this conformal class which have not been studied in the classical case. In particular, we give a local classification of surfaces in $\boldsymbol{R}^4$ for which the conformal structure vanishes. We also give a description of compact orientable surfaces in $\boldsymbol{R}^4$ whose rank of the conformal structure is constant.

The properties considered in the paper can also be used to answer some questions of the following type. Having an immersion $f : M \to \boldsymbol{R}^N$, is it possible to chose such a homogeneous geometric structure on $\boldsymbol{R}^N$ relative to which f has "good" properties? For instance, let f be an immersion of a two-dimensional manifold M into $\boldsymbol{R}^4$. Is it possible to find a complex structure on $\boldsymbol{R}^4$ relative to which f (assume additionally, real analycity) becomes a holomorphic curve, or, is there a Kaehler structure on $\boldsymbol{R}^4$ relative to which f is Lagrangian in the metric sense?

2. Locally strongly convex submanifolds of $\boldsymbol{R}^N$. Assume that $f : M \to \boldsymbol{R}^N$ is an immersion of an n-dimensional connected manifold M into the affine space $\boldsymbol{R}^N$. Let $k = N - n$. The affine structure on $\boldsymbol{R}^N$ is given by the standard connection $\tilde{\nabla}$. If $\mathcal{N}$ is a transversal vector bundle for f, then we can write the formulas of Gauss and Weingarten as

$$\tilde{\nabla}_X f_* Y = f_*(\nabla_X Y) + h(X, Y)\,, \tag{1}$$

$$\tilde{\nabla}_X \xi = -f_*(S_\xi X) + \nabla'_X \xi\,, \tag{2}$$

where $h(X, Y)$ and $\nabla'_X \xi$ have values in $\mathcal{N}$ for any vector fields X, Y on M and a section ξ of $\mathcal{N}$. h is an $\mathcal{N}$-valued symmetric 2-form on M, S_ξ is a $(1,1)$-tensor field on a domain of ξ, ∇ is a torsion-free connection on M, and ∇' is a connection on the bundle $\mathcal{N}$. h is called the second fundamental form of f, ∇ the induced connection and ∇' the normal connection. If a transversal bundle $\mathcal{N}$ is fixed, we call it the normal bundle.

Let $\mathcal{N}$ be a fixed transversal bundle for f. For a fixed point $x_0 \in M$ consider the subset

$$\tilde{\mathcal{H}}_{x_0} = \{h(X, X)\,;\ X \in T_{x_0}M,\, X \neq 0\} \tag{3}$$

of the space $\mathcal{N}_{x_0}$. In general, it is a cone with vertex at the origin of the vector space N_{x_0}. The vertex might belong to the cone or not. The cone may be symmetric relative to the vertex or not. It may reduce to a vector subspace or a half-subspace. All those properties are affine invariants of a submanifold at a fixed point x_0 independent of a choice of a transversal bundle. We shall call $\tilde{\mathcal{H}}_{x_0}$ the cone of the second fundamental form. For instance, if $\boldsymbol{R}^N = \boldsymbol{C}^m$ and we have a complex submanifold of $\boldsymbol{C}^m$, then it can be equipped with a complex transversal bundle. In such a case $h(\mathrm{i}X, \mathrm{i}X) = -h(X, X)$, where i is the imaginary unit in $\boldsymbol{C}$ acting on $\boldsymbol{C}^m$ in the standard way. It means that the cone of the second fundamental form is symmetric relative to the vertex. Therefore, if we have an even-dimensional submanifold of $\boldsymbol{R}^{2m}$ and its cone of the second fundamental form (for a transversal bundle) at a point x_0 is not symmetric relative to the vertex, then the submanifold cannot be complex (even locally) relative to any complex structure on $\boldsymbol{R}^{2m}$ (parallel relative to the standard connection $\tilde{\nabla}$).

The cone of the second fundamental form is generated by the set

$$\tilde{\mathcal{E}}_{x_0} = \{h(X, X)\ ;\ X \in S\}\,, \tag{4}$$

where S is an ellipsoid centered at 0 in the tangent space $T_{x_0}M$. This set essentially depends on a choice of an ellipsoid S.

If $f : M \to \boldsymbol{R}^N$ is a Riemannian surface, $\boldsymbol{R}^N$ being regarded as a Euclidean space, M is 2-dimensional, $\mathcal{N}$ is a metric normal bundle and S is the unit circle relative to the induced scalar product on $T_{x_0}M$, then $\tilde{\mathcal{E}}_{x_0}$ is called the ellipse of curvature at x_0. In Riemannian geometry it is interesting (for example) when the ellipse is a circle. Here we study affine properties of the sets $\tilde{\mathcal{H}}_{x_0}$ and $\tilde{\mathcal{E}}_{x_0}$.

In general, if $f : M \to \boldsymbol{R}^N$ is an immersion equipped with any transversal bundle $\mathcal{N}$ and Π is a vector plane of the tangent space $T_{x_0}M$, then the mapping $\Pi : X \to h(X, X) \in \mathcal{N}_{x_0}$ sends every ellipse centered at the origin into an ellipse. More precisely, if X, Y is a basis of Π and $W = \cos\theta\, X + \sin\theta\, Y$ (i.e., X, Y are half-axes of an ellipse in Π), then

$$\begin{aligned} h(W, W) &= \frac{1}{2}(h(X, X) + h(Y, Y)) \\ &\quad + \sin 2\theta \cdot h(X, Y) + \cos 2\theta \cdot \frac{1}{2}(h(X, X) - h(Y, Y))\,. \end{aligned} \tag{5}$$

The affine shape of the ellipse (for instance, whether it reduces to a point or a line segment) is independent of a choice of $\mathcal{N}$, but depends on X, Y.

We shall consider the following properties of an immersion f at a fixed point $x_0 \in M$:

I. $h(X, X) \neq 0$ for all $X \in T_{x_0}M$, $X \neq 0$ (equivalently, $0 \notin \tilde{\mathcal{H}}_{x_0}$).

II. There is a vector hyperplane Ω_{x_0} of $\mathcal{N}_{x_0}$ such that all vectors $h(X, X)$ for $X \in T_{x_0}M$, $X \neq 0$, lie on one side of Ω_{x_0} (i.e., in one open half-space of $\mathcal{N}_{x_0}$ determined by Ω_{x_0}).

III. For an arbitrary positive definite scalar product G on $\boldsymbol{R}^N$ there is a point o of the affine space $\boldsymbol{R}^N$ and a positive real number r such that $f(x_0) \in S^{N-1}$, $f_*(T_{x_0}M) \subset T_{f(x_0)}S^{N-1}$ and $f(\mathcal{U}) \subset \mathcal{B}^{N-1}$ for some neighborhood $\mathcal{U}$ of x_0, where S^{N-1} is a sphere (relative to G) of radius r centered at o and $\mathcal{B}^{N-1}$ is the closed ball surrounded by S^{N-1}.

IV. For an arbitrary positive definite scalar product G on $\boldsymbol{R}^N$ there is a basic point o of the affine space $\boldsymbol{R}^N$ such that the function

$$\Phi : M \ni x \to G(\overrightarrow{of(x)}, \overrightarrow{of(x)}) \in \boldsymbol{R} \tag{6}$$

attains its local maximum at x_0.

V. There is an affine hyperplane $f(x_0) + \mathcal{V}$ of the affine space $\boldsymbol{R}^N$ which supports f at x_0, that is, $f_*(T_{x_0}M) \subset \mathcal{V}$ and there is a neighborhood $\mathcal{U}$ of x_0 in M such that $f(\mathcal{U} \setminus \{x_0\})$ lies in one of the open half-spaces determined by the affine hyperplane $f(x_0) + \mathcal{V}$.

We shall first make some basic observations. The first two properties clearly do not depend on the choice of a transversal bundle $\mathcal{N}$. Therefore they are affine invariants. II is stronger than I. In order to check I or II it suffices to find the set $\tilde{\mathcal{E}}_{x_0}$.

The first property holding at every point of M is equivalent to the following one:

I$'$. For every regular curve γ in M, $f \circ \gamma$ is biregular in $\boldsymbol{R}^N$ (i.e., $(f \circ \gamma)'(t)$ and $(f \circ \gamma)''(t)$ are linearly independent for every t). Indeed, for a curve γ in M we have

$$(f \circ \gamma)'' = f_*(\nabla_{\gamma'}\gamma') + h(\gamma', \gamma'). \tag{7}$$

If I is satisfied for every point of M, then for a regular curve γ in M the vector $(f \circ \gamma)''(t)$ is linearly independent of $(f \circ \gamma)'(t)$ for any parameter t. Conversely, if $\gamma(t)$ is a geodesic (relative to the induced connection ∇) such that $\gamma(0) = x$ and $\gamma'(0) = X \neq 0$ and $(f \circ \gamma)''(0)$ is linearly independent of $(f \circ \gamma)'(0)$, then, since $f_*(\nabla_{\gamma'}\gamma')$ is parallel to $f_*(\gamma')$, we have $h(\gamma'(0), \gamma'(0)) \neq 0$.

It is clear that if a submanifold is a curve, then its biregularity implies II. It is also known that if we have a biregular curve in the Euclidean space $\boldsymbol{R}^3$ (in this case it means regular and of nowhere-vanishing curvature), then in a neighborhood of a fixed point the curve lies on one side of the rectifying plane.

We shall now prove the following

THEOREM 2.1. *The properties* II, III *and* IV *are equivalent. Moreover*, II *implies* V *and* V *together with* I *imply* II.

PROOF. For a fixed scalar product G the properties III and IV are obviously equivalent.

We shall now prove that II implies IV. Let $\mathcal{N}$ be a metric normal bundle relative to a positive definite scalar product G on $\boldsymbol{R}^N$. Take the affine straight line L passing through $f(x_0)$ and perpendicular to the vector hyperplane $\mathcal{V} = f_*(T_{x_0}M) + \Omega_{x_0}$ of $\boldsymbol{R}^N$. The vector half-space $\mathcal{N}_{x_0}$ determined by Ω_{x_0} which contains all $h(X, X)$ for $X \in T_{x_0}M$, $X \neq 0$, determines an open half-line of this straight line. Let o' be any (for a moment) point of this half-line. For each $X \in T_{x_0}M$, $X \neq 0$, we have $G(h(X, X), \overrightarrow{f(x_0)o'}) > 0$. Take o' such that $\xi = \overrightarrow{f(x_0)o'}$ is unit relative to G. Consider the function

$$S^{n-1}(1) \ni X \to G(h(X, X), \xi) \in \boldsymbol{R}^+,$$

where $S^{n-1}(1)$ is the unit sphere centered at $0 \in T_{x_0}M$ relative to the induced metric. The above function attains its minimum, say m, which is positive. Take $r > 1/m$ and a point o on the line L such that $\overrightarrow{f(x_0)o} = r\xi$. We now have

$$G(h(X, X), \overrightarrow{f(x_0)o}) = rG(h(X, X), \xi) > 1 \tag{8}$$

for $X \in S^{n-1}(1)$. Consider the function Φ given by (6). Since $G(f_*(T_{x_0}M), \overrightarrow{of(x_0)}) = 0$, we have $d_{x_0}\Phi = 0$. Differentiating Φ twice at x_0, we obtain

$$(d^2_{x_0}\Phi)(X, Y) = -G(h(X, Y), \overrightarrow{f(x_o)o}) + G(f_*X, f_*Y)\,. \tag{9}$$

By (8) the form $d^2_{x_0}\Phi$ is negative definite, which implies that Φ attains a local maximum at x_0.

Assume now IV. We shall prove II. We have the function Φ given by (6) which attains a local maximum at x_0. Let $X \in T_{x_0}M$, $X \neq 0$, be arbitrary. Let $\gamma(t)$ be a curve such that $\gamma(0) = x_0$ and $\gamma'(0) = X$. Then $G(f_*(T_{x_0}M), \overrightarrow{of(x_0)}) = 0$ and

$$G((f \circ \gamma)''(0), \overrightarrow{of(x_0)}) + G((f \circ \gamma)'(0), (f \circ \gamma)'(0)) \leq 0\,.$$

Since the second component of this sum is positive, we have

$$G((f \circ \gamma)''(0), \overrightarrow{of(x_0)}) < 0\,. \tag{10}$$

Take the metric normal bundle $\mathcal{N}$ (relative to G) for f. Then $\xi = \overrightarrow{of(x_0)} \in \mathcal{N}_{x_0}$. Take the orthogonal complement Ω_{x_0} to ξ in $\mathcal{N}_{x_0}$. By (10) and (7) we see that all $h(X, X)$ (for $X \neq 0$) are non-zero and lie on one side of Ω_{x_0}.

In order to prove that III implies V it suffices to observe that the affine hyperplane $f(x_0) + T_{f(x_0)}S^{N-1}$ supports f at x_0.

We shall now prove that V together with I imply II. We have a supporting hyperplane $f(x_0) + \mathcal{V}$ and its side as in V. Let ξ be any vector transversal to $\mathcal{V}$ determining this side. Take any algebraic complement Ω_{x_0} to $f_*(T_{x_0}M)$ in $\mathcal{V}$.

The constant vector bundle $\mathcal{N} = \Omega_{x_0} + \boldsymbol{R}\xi$ is transversal to f in a neighborhood of x_0 and it will be the normal bundle we choose.

Let $\gamma(t)$ be a naturally parametrized geodesic relative to the connection induced by $\mathcal{N}$ such that $\gamma(0) = x_0$ and $\gamma'(0) = X \in T_{x_0}M$, $X \neq 0$. We have

$$(f \circ \gamma)''(t) = h(\gamma'(t), \gamma'(t)) \in \mathcal{N} \tag{11}$$

for all t. We know that $(f \circ \gamma)(t) \in \mathcal{A}^+ = f(x_0) + \mathcal{V} + \boldsymbol{R}^+\xi$ for sufficiently small $t \neq 0$. For such a parameter t we have

$$\mathcal{A}^+ \ni (f \circ \gamma)(t) = (f \circ \gamma)(0) + t(f \circ \gamma)'(0) + \frac{t^2}{2}(f \circ \gamma)''(\theta t)$$

for some $0 < \theta < 1$. It follows that

$$h(\gamma'(\theta t), \gamma'(\theta t)) \in \mathcal{N} \cap (\mathcal{V} + \boldsymbol{R}^+\xi) = \Omega + \boldsymbol{R}^+\xi\,,$$

which implies that

$$h(\gamma'(0), \gamma'(0)) = \lim_{t \to 0} h(\gamma'(\theta t), \gamma'(\theta t)) \in \Omega_{x_0} + (\boldsymbol{R}^+ \cup \{0\})\xi\,.$$

If additionally I is satisfied, then we get II. □

We propose to call a submanifold $f : M \to \boldsymbol{R}^N$ satisfying II (equivalently III, IV or V) at $x_0 \in M$ locally strongly convex at x_0. Such a notion coincides with the one known in the theory of hypersurfaces. By a locally strongly convex submanifold we shall mean a submanifold which is locally strongly convex at each point of M.

3. n-dimensional submanifolds of $\boldsymbol{R}^{2n}$. We shall start with the following algebraic lemma

LEMMA 3.1. *Let Q be a $(1,2)$-tensor on an n-dimensional vector space V (over a field $\mathcal{F}$ of characteristic 0). There is a unique symmetric n-linear form $\mathcal{L}$ on V such that*

$$\det Q_X = \mathcal{L}(X, \dots, X) \tag{12}$$

for every $X \in V$.

PROOF. Let $E_1, \dots, E_n$ be a fixed basis of V. The linear map Q_X will be identified with its matrix relative to the fixed basis. Let Q_j denote the linear mapping sending $X \in V$ into the j-th column of Q_X. Then $Q_j(X) = Q_X(E_j)$. The mapping

$$V^n \ni (X_1, \dots, X_n) \to (Q_1(X_1), \dots, Q_n(X_n)) \in (\mathcal{F}^n)^n$$

is linear. Hence

$$\tilde{\mathcal{L}} : (X_1, \dots, X_n) \to \det(Q_1(X_1), \dots, Q_n(X_n)) \in \mathcal{F}$$

is n-linear. We have $\tilde{\mathcal{L}}(X, \dots, X) = \det Q_X$. By symmetrizing $\tilde{\mathcal{L}}$, we obtain a desired mapping $\mathcal{L}$. Such an $\mathcal{L}$ is unique. Indeed, it suffices to observe that if $\mathcal{L}$ is n-linear symmetric and $\mathcal{L}(X, \dots, X) = 0$, then $\mathcal{L} = 0$. If $X_1, \dots, X_n$ are arbitrary, then the polynomial $(t_1, \dots, t_n) \to \mathcal{L}(X, \dots, X)$, with $X = t_1X_1 + \cdots + t_nX_n$, vanishes identically. In particular, all its coefficients are zero. Looking at the coefficients of $t_1, \dots, t_n$, using the fact that $\mathcal{F}$ has characteristic zero and $\mathcal{L}$ is totally symmetric, we deduce that $\mathcal{L}$ vanishes identically. □

In the above lemma the symmetry of Q is not needed but for an anti-symmetric Q the form $\mathcal{L}$ vanishes.

By a complex structure J on $\boldsymbol{R}^{2n}$ we mean a complex structure parallel relative to the connection $\tilde{\nabla}$, that is, it is a complex structure on the vector (or affine) space $\boldsymbol{R}^{2n}$.

If a complex structure J on $\boldsymbol{R}^{2n}$ is fixed and $f : M \to \boldsymbol{R}^{2n}$ is an immersion of an n-dimensional manifold M, then f is called *purely real* (or *affine Lagrangian*) if the bundle $Jf_*(TM)$ is transversal to $f_*(TM)$. The bundle $Jf_*(TM)$ is the normal bundle for a purely real submanifold. As in the general case we have a Gauss formula, which in this case reeds as follows:

$$\tilde{\nabla}_X f_*Y = f_*(\nabla_X Y) + Jf_*Q(X, Y)\,, \tag{13}$$

where Q is a symmetric $(1,2)$-tensor field on M. Q is called the second fundamental tensor of f. The Weingarten formula reduces to the Gauss formula, because J gives an isomorphism between the tangent and the normal bundle and $\tilde{\nabla} J = 0$. For a fixed vector $X \in T_xM$ the tensor Q defines the endomorphism $Q_X : T_xM \to T_xM$ given by $Q_XY = Q(X, Y)$.

Recall that a purely real immersion $f : M \to \boldsymbol{R}^{2n}$ is called totally real (or Lagrangian) if $\boldsymbol{R}^{2n}$ is regarded as a Kaehler manifold (with $\tilde{\nabla}$ as a Kaehler connection and a Kaehler metric G on $\boldsymbol{R}^{2n}$) and if the bundle $Jf_*(TM)$ is G-orthogonal to $f_*(TM)$. In such a case the second fundamental tensor satisfies also the following condition

$$G(Q(X, Y), Z) = G(Q(X, Z), Y)\,, \tag{14}$$

where G is here the induced metric tensor on M.

Assume now we have an immersion $f : M \to \boldsymbol{R}^{2n}$ of an n-dimensional manifold M. Let x_0 be a fixed point of M. Denote by V the vector space $f_*(T_{x_o}M)$. There is a complex structure J on $\boldsymbol{R}^{2n}$ relative to which the immersion is purely real in a neighborhood of x_0. Namely, any complex structure J on $\boldsymbol{R}^{2n}$ relative to which the vector space V is purely real (that is, $JV \oplus V = \boldsymbol{R}^{2n}$) is good. If $\{e_1, \dots, e_n\}$ is any basis of V, then $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ is a basis of $\boldsymbol{R}^{2n}$. If $\hat{J}$ is another complex structure on $\boldsymbol{R}^{2n}$ relative to which f is purely real around x_0, then $\hat{J}$ can be expressed as PJP^{-1}, where P is a linear transformation of $\boldsymbol{R}^{2n} = V \oplus JV$ whose matrix relative to the basis $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ is given by

$$\begin{bmatrix} I & \mathcal{B} \\ 0 & \mathcal{C} \end{bmatrix}, \tag{15}$$

where I is the identity $n \times n$ matrix and $\det \mathcal{C} \neq 0$. Let $\hat{\nabla}$ and $\hat{Q}$ denote the induced connection and the second fundamental tensor for f relative to the complex structure $\hat{J}$. We have

$$\begin{aligned}(\tilde{\nabla}_X f_* Y)_{x_0} &= f_*(\nabla_X Y)_{x_0} + Jf_* Q_{x_0}(X, Y)\,,\\ (\tilde{\nabla}_X f_* Y)_{x_0} &= f_*(\hat{\nabla}_X Y)_{x_0} + \hat{J} f_* \hat{Q}_{x_0}(X, Y)\,.\end{aligned}$$

On the other hand

$$\hat{J} f_* \hat{Q}(X, Y) = \mathcal{B} f_* \hat{Q}(X, Y) + J\mathcal{C} f_* \hat{Q}(X, Y)\,,$$

where $\mathcal{B}$ and $\mathcal{C}$ are regarded here as endomorphisms of V given by the matrices $\mathcal{B}, \mathcal{C}$ relative to the basis $\{e_1, \dots, e_n\}$. We obtain the following relations between ∇, $\hat{\nabla}$, Q and $\hat{Q}$.

$$\begin{aligned}Q(X, Y) &= \mathcal{C}'\hat{Q}(X, Y)\,,\\ \nabla_X Y &= \hat{\nabla}_X Y + \mathcal{B}'\hat{Q}(X, Y)\,,\end{aligned} \tag{16}$$

where $\mathcal{C}' = f_*^{-1} \circ \mathcal{C} \circ f_* : T_{x_0}M \to T_{x_0}M$ and $\mathcal{B}' = f_*^{-1} \circ \mathcal{B} \circ f_* : T_{x_0}M \to T_{x_0}M$. It follows, in particular, that the property $\det Q_X = 0$ (or $\neq 0$) for a fixed $X \in T_{x_0}M$ is a property invariant, not only under the action of the general complex affine group $GA(\boldsymbol{C}, n)$ but is also under the action of the general real affine group $GA(\boldsymbol{R}, 2n)$. More precisely, if we have an immersion $f : M \to \boldsymbol{R}^{2n}$, where M is n-dimensional, and a fixed point x_0, we choose any complex structure on $\boldsymbol{R}^{2n}$ relative to which f is purely real around x_0. This gives Q. Some properties of Q do not depend on a choice of a complex structure. We shall say that the vanishing of $\det Q_X$ is a real affine invariant, not only a complex affine invariant. We shall collect some point-wise real affine invariants, that is, properties of the given immersion f at a fixed point x_0 which are invariant relative to the group $GA(\boldsymbol{R}, 2n)$. Among such properties are the following:

P1. $Q_X = 0$,
P2. $\det Q_X = 0$,
P3. $Q(X, X) = 0$,
P4. the mapping $\mathcal{Q} : T_{x_0}M \ni X \to Q_X \in \mathrm{HOM}\,(T_{x_0}M)$ is a monomorphism,
P5. $\mathrm{rk}\, Q_X = r$,
P6. $\dim \mathrm{im}\, Q_{x_o} = r$,

where X is a non-zero vector of $T_{x_0}M$ in P1, P2, P3, P5. In P6

$$\mathrm{im}\, Q_{x_0} = \mathrm{span}\{Q(X, Y)\,;\, X, Y \in T_{x_0}M\}\,.$$

Note that $\mathrm{im}\, Q_{x_0}$ is not a well-defined real affine invariant. In general, it essentially depends on the complex structure J. Therefore, if a complex structure is used as a tool in the study of an n-dimensional submanifold of $\boldsymbol{R}^{2n}$, then $\mathrm{im}\, Q$ is, in general, not a well-defined distribution, even if $\dim \mathrm{im}\, Q$ is constant on M.

One can find other invariants. Since $\det Q_X = \det \mathcal{C} \cdot \det \hat{Q}_X$, we have

$$\mathcal{L} = \det \mathcal{C} \cdot \hat{\mathcal{L}}\,,$$

where $\hat{\mathcal{L}}$ is determined by $\hat{J}$. The algebraic type of the symmetric form $\mathcal{L}$ is a real invariant.

If a complex structure on $\boldsymbol{R}^{2n}$ is fixed, then $\boldsymbol{R}^{2n}$ becomes $\boldsymbol{C}^n$. An immersion $f : M \to \boldsymbol{C}^n$ is purely real if for every frame $X_1, \ldots, X_n$ on M we have $\det_{\boldsymbol{C}}(f_*X_1, \ldots, f_*X_n) \neq 0$. On an oriented purely real submanifold one defines a volume form ν by the formula

$$|\nu(X_1, \ldots, X_n)| = |\det_{\boldsymbol{C}}(f_*X_1, \ldots, f_*X_n)|\,. \tag{17}$$

The volume form essentially depends on the complex structure. Indeed, let $\hat{J}$ be another complex structure on $\boldsymbol{R}^{2n}$ relative to which f is purely real around x_0. We have the following formulas for ν and $\hat{\nu}$, where $\hat{\nu}$ is determined by $\hat{J}$, at x_0:

$$\begin{aligned}(\hat{\nu}(X_1, \ldots, X_n))^2 &= \det_{\boldsymbol{R}}(f_*X_1, \ldots, f_*X_n, \hat{J}f_*X_1, \ldots, \hat{J}f_*X_n)\\ &= \det_{\boldsymbol{R}}(PP^{-1}f_*X_1, \ldots, PP^{-1}f_*X_n, PJP^{-1}f_*X_1, \ldots, PJP^{-1}f_*X_n)\\ &= \det P \cdot \det_{\boldsymbol{R}}(P^{-1}f_*X_1, \ldots, P^{-1}f_*X_n, JP^{-1}f_*X_1, \ldots, JP^{-1}f_*X_n)\\ &= \det \mathcal{C} \cdot (\nu(X_1, \ldots, X_n))^2\,,\end{aligned}$$

where $\hat{J} = PJP^{-1}$ and P is given by (15). In the above computations we used the fact that P (and P^{-1}) restricted to $f_*(T_{x_0}M)$ is the identity. We have obtained

$$\hat{\nu} = \sqrt{|\det \mathcal{C}|}\;\nu \tag{18}$$

at x_0.

If $n = 2$, then $\mathcal{L}$ is a bilinear symmetric form, which will be denoted by g. Define a function $\mathcal{K}$ by

$$\mathcal{K} = \det_\nu g\,. \tag{19}$$

This function is a complex affine invariant. If we change a complex structure as above, then

$$\hat{\mathcal{K}} = \det_{\hat{\nu}}\hat{g} = (\det \mathcal{C})^{-3}\mathcal{K}\,.$$

Hence, if $\mathcal{K}$ is nowhere vanishing (which is a real affine invariant and, in the classical theory of surfaces in $\boldsymbol{R}^4$, is called the non-degeneracy of an immersion), then the metric (possibly indefinite) tensor field

$$\tilde{g} = \mathcal{K}^{-1/3} g \tag{20}$$

is a real affine invariant defined globally on M. In the theory of surfaces of $\boldsymbol{R}^4$ it is called the affine metric, see for instance [1] and references given there. In particular, if $f : M \to R^4$ is a surface, then one defines a (0, 2) symmetric tensor of weight 2 by the formula:

$$g_u(Z, W) = \frac{1}{2}\{\det(X, Y, \tilde{\nabla}_Z X, \tilde{\nabla}_W Y) + \det(X, Y, \tilde{\nabla}_W X, \tilde{\nabla}_Z Y)\}\,, \tag{21}$$

where $u = (X, Y)$ is a local frame on M. One easily sees that g_u determines the same conformal class as g constructed above.

Assume that f is holomorphic relative to some complex structure J on $\boldsymbol{R}^4$ around a point $x_0 \in M$. Take $u = (X, JX)$. By putting $Z = X,\ W = X$, then $Z = JX,\ W = JX$, and then $Z = X,\ W = JX$ in (21), one sees that g_u is definite or zero. It means that g_{x_0} is definite or zero. Therefore, if g_{x_0} is not definite and is not zero, then the surface cannot be holomorphic around x_0 relative to any complex structure J on M.

4. Surfaces in R^4. Assume that $n = 2$ and J is a fixed complex structure on $\boldsymbol{R}^4$ making this space a complex space $\boldsymbol{C}^2$. Let $f : M \to \boldsymbol{C}^2$ be a purely real immersion. For a basis $\{X, Y\}$ of $T_x M$ the endomorphisms Q_X, Q_Y can be expressed by their matrices relative to the basis $\{X, Y\}$:

$$Q_X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad Q_Y = \begin{bmatrix} c & e \\ d & k \end{bmatrix}. \tag{22}$$

We have $g(X, X) = \det Q_X$, $g(Y, Y) = \det Q_Y$ and

$$g(X, Y) = \frac{1}{2}\det \begin{bmatrix} a & e \\ b & k \end{bmatrix}. \tag{23}$$

The conformal class of g is a real affine invariant of the immersion f. In particular, the rank of g_{x_0} (also definitness or indefinitness of g_{x_0}) is a real invariant of the immersion f at a point x_0. The same deals with the nullity space of g_{x_0}. In order to construct the conformal class we can use any complex structure relative to which f is purely real around x_0.

We shall now study the properties introduced in Section 2 in the case of surfaces in $\boldsymbol{R}^4$. In the case where M is n-dimensional and $f : M \to \boldsymbol{C}^n$ is a purely real immersion, the conditions I and II from Section 2 can be read as follows:

I. $Q(X, X) \neq 0$ for all $X \in T_{x_0}M$, $X \neq 0$.

II. There is a vector hyperplane Ω_{x_0} of $T_{x_0}M$ such that all vectors $Q(X, X)$, $X \in T_{x_0}M$, $X \neq 0$, lie in one open half-space of $T_{x_0}M$ determined by Ω_{x_0}.

We have the following

LEMMA 4.1. *Let $f : M \to \boldsymbol{R}^4$ be an immersion.*

(a) *If f does not satisfy* I *at x_0, then* $\operatorname{rk} g_{x_0} \leq 1$.

(b) *If* $\operatorname{rk} g_{x_0} = 1$, *then* f *does not satisfy* I *at* x_0.
(c) *If* g_{x_0} *is non-degenerate definite*, *then* f *satisfies* I *but does not satisfy* II *at* x_0.
(d) *If* g_{x_0} *is non-degenerate indefinite*, *then* f *satisfies* II *at* x_0.

PROOF. (a) Let $X \in T_{x_0}M$, $X \neq 0$, be such that $Q(X, X) = 0$. For any $Y \in T_{x_0}M$ we have

$$Q_X = \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix}, \quad Q_Y = \begin{bmatrix} c & e \\ d & k \end{bmatrix}. \tag{24}$$

It follows that $g(X, X) = \det Q_X = 0$ and

$$g(X, Y) = \frac{1}{2}\det\begin{bmatrix} 0 & e \\ 0 & k \end{bmatrix} = 0\,.$$

(b) Let $X \in T_{x_0}M$, $X \neq 0$, spans the nullity space of g_{x_0}, that is, $g(X, Y) = 0$ for every vector $Y \in T_{x_0}M$. Suppose that $Q(X, X) \neq 0$. Since $\det Q_X = g(X, X) = 0$, there is Y linearly independent of X such that $Q(X, Y) = 0$. But then $\det Q_Y = 0$, i.e., $g_{x_0} = 0$. This contradiction shows that $Q(X, X)$ must be zero.

(c) There exists a basis $\{X, Y\}$ of $T_{x_0}M$ such that $g(X, X)g(Y, Y) > 0$ and $g(X, Y) = 0$. Then

$$Q_X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad Q_Y = \begin{bmatrix} c & \lambda a \\ d & \lambda b \end{bmatrix} \tag{25}$$

and $ad - bc \neq 0$. Moreover, $\det Q_Y = -\lambda(ad - bc)$, which implies that $\lambda < 0$. The vectors $Q(X, X)$ and $Q(Y, Y)$ are linearly dependent. The vectors $Q(X, X)$ and $Q(X, Y)$ are linearly independent. If $W = w_1X + w_2Y$, then

$$Q(W, W) = (w_1^2 + \lambda w_2^2)Q(X, X) + 2w_1w_2Q(X, Y)\,.$$

It follows that $Q(W, W) = 0$ if and only if $W = 0$. We obtain I.

We have already observed that $Q(Y, Y) = \lambda Q(X, X)$, where $\lambda < 0$. Hence the vectors $Q(X, X)$ and $Q(Y, Y)$ cannot lie on one side of any vector line in the vector plane $T_{x_0}M$.

(d) Let X, Y span the asymptotic directions of g_{x_0}. In particular, $\det Q_X = \det Q_Y = 0$. Suppose that $X \in \ker Q_X$. Then

$$Q_X = \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix}, \quad Q_Y = \begin{bmatrix} c & e \\ d & k \end{bmatrix}. \tag{26}$$

It follows that $g(X, Y) = 0$, which is a contradiction. Thus $Q(X, X) \neq 0$. Let $Y' \in \ker Q_X$. Then Y' is linearly independent of X and

$$Q_X = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \quad Q_{Y'} = \begin{bmatrix} 0 & e \\ 0 & k \end{bmatrix}. \tag{27}$$

Thus $\det Q'_Y = 0$. It follows that Y' is proportional to Y. We can assume that $Y = Y'$. Since $g(X, Y) \neq 0$, $\det\begin{bmatrix} a & e \\ b & k \end{bmatrix} \neq 0$. It follows that $Q(X, X)$ and $Q(Y, Y)$ are linearly independent. If $W = w_1X + w_2Y$, then

$$Q(W, W) = w_1^2Q(X, X) + w_2^2Q(Y, Y)\,.$$

This implies that all vectors $Q(W, W)$, $W \neq 0$, lie on one side of a vector line in $T_{x_0}M$. □

REMARK 4.2. If $g_{x_0} = 0$ and $Q_{x_0} \neq 0$, then I (or II) may be satisfied or may be not. The examples will be provided after Theorem 5.4. Here we only indicate two possibilities. The following situations can happen

$$Q_X = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \quad Q_Y = \begin{bmatrix} 0 & \lambda a \\ 0 & \lambda b \end{bmatrix},$$

where $(a, b) \neq 0$ for some basis $\{X, Y\}$ of $T_{x_0}M$. If $\lambda > 0$, then II is satisfied. If $\lambda < 0$, then I is not satisfied.

By using the proof of Lemma 4.1, we also obtain

PROPOSITION 4.3. *Let $f : M \to \boldsymbol{R}^4$ be an immersion of a 2-dimensional manifold. If g is non-degenerate indefinite on M, then M admits a globally defined nowhere vanishing vector field.*

PROOF. Take any positive definite metric tensor field G on M. Let X, Y be unit (relative to G) vector fields (possibly local) spanning the asymptotic distributions of g. Then for a fixed G the vector field $Z = Q(X, X) + Q(Y, Y)$ is uniquely defined. Therefore it is globally defined. Since $Q(X, X)$ and $Q(Y, Y)$ are linearly independent, as it was observed in the proof of Lemma 4.1, the vector field Z nowhere vanishes on M. □

EXAMPLE 4.4. Take an immersion into $\boldsymbol{C}^2$ defined on $\boldsymbol{R}^2$, equipped with the canonical coordinates (u, v):

$$f(u, v) = e^{iv} L(u)\,, \tag{28}$$

where $L(u)$ is a curve in $\boldsymbol{C}^2$ such that $\det_{\boldsymbol{C}}(L(u), L'(u)) \neq 0$ for all u. Then $\det_{\boldsymbol{C}}(f_u, f_v) = ie^{2iv}\det_C(L(u), L'(u)) \neq 0$ and consequently the immersion is purely real. If we set

$$L(u) = (\sinh u + i \cosh u, \cosh u + i \sinh u)\,, \tag{29}$$

then the corresponding second fundamental tensor Q is given by

$$Q(X, X) = -Y\,, \quad Q(X, Y) = X\,, \quad Q(Y, Y) = Y\,,$$

where $X = \partial_u$ and $Y = \partial_v$. Hence g is positive definite on the whole $\boldsymbol{R}^2$. If we set

$$L(u) = (\cos u + i \sin u, \sin u + i \cos u)\,,$$

then

$$Q(X, X) = Y, \quad Q(X, Y) = X, \quad Q(Y, Y) = Y\,.$$

In this case g is non-degenerate indefinite.

EXAMPLE 4.5. Let $\boldsymbol{C}^2 = \boldsymbol{C}_1 \times \boldsymbol{C}_2$, where $\boldsymbol{C}_1 = \boldsymbol{C} = \boldsymbol{C}_2$. The two copies of $\boldsymbol{C}$ in $\boldsymbol{C}^2$ are just numbered. Let $p_2(v)$ be an arc-length parametrized curve in the Euclidean space $\boldsymbol{C}_2$.

We also assume that the curvature of the curve p_2 vanishes nowhere. Take the curve $q = p_2'$ but in $\boldsymbol{C}_1$. Consider the surface

$$f(u, v) = (p_1(v), p_2(v)) + u(q(v), 0)\,, \tag{30}$$

where $p_1(v)$ is an arbitrary curve in $\boldsymbol{C}_1$. The second fundamental tensor Q is given by the formulas

$$Q(X, X) = 0\,, \quad Q(X, Y) = cX\,, \quad Q(Y, Y) = eX + cY\,,$$

where c is a nowhere-vanishing function and $X = \partial_u$, $Y = \partial_v$. Here $\operatorname{rk} g = 1$. Other properties of such surfaces are studied in [4].

The second fundamental form Q gives the following simple criterion excluding some surfaces from being metric totally real relative to a Kaehler (definite or indefinite) structure on $\boldsymbol{R}^4$.

PROPOSITION 4.6. *If* $\operatorname{rk} Q_X = 1$ *for every* $X \in T_{x_0}M$, $X \neq 0$, *then there is no Kaehler* (*definite or indefinite*) *structure on* $\boldsymbol{R}^4$ *relative to which* f *is totally real in a neighborhood of* x_0.

PROOF. Assume that $f : M \to \boldsymbol{C}^2$ is a totally real immersion relative to a Kaehler positive definite matric G in $\boldsymbol{C}^2$. If $\{X, Y\}$ is a G-orthonormal basis of a tangent space $T_{x_0}M$, then the matrices (22) of Q_X and Q_Y are symmetric. Take X belonging to $\ker \tau$, where $\tau(Z) = \operatorname{tr} Q_Z$. Then

$$Q_X = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}. \tag{31}$$

It follows that $\operatorname{rk} Q_X = 2$ or 0.

Consider now the indefinite case. Suppose that f is totally real around x_0 relative to an indefinite Kaehler metric G. The metric G is of type $(+, +, -, -)$ and the induced metric tensor field G on M is of type $(+, -)$. Let $X, Y \in T_{x_0}M$ span the asymptotic directions of g_{x_0}. By (14) we have

$$Q_X = \begin{bmatrix} a & c \\ b & a \end{bmatrix}, \quad Q_Y = \begin{bmatrix} c & e \\ a & c \end{bmatrix}.$$

If $a \neq 0$, then the vector X can be chosen such that $a = 1$. Similarly, if $c \neq 0$, then Y can be chosen such that $c = 1$. Hence there are the following possibilities:

$$Q_X = \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}, \quad Q_Y = \begin{bmatrix} 1 & e \\ 1 & 1 \end{bmatrix}, \tag{32}$$

or

$$Q_X = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}, \quad Q_Y = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix}, \tag{33}$$

or

$$Q_X = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \quad Q_Y = \begin{bmatrix} 0 & e \\ 1 & 0 \end{bmatrix}, \tag{34}$$

or

$$Q_X = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}, \quad Q_Y = \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix}. \tag{35}$$

In the case described by (32), in order that $\operatorname{rk} Q_X = 1 = \operatorname{rk} Q_Y$, the numbers e and b must be equal to 1. Then $Q_{X-Y} = 0$, which is a contradiction. In (33) we have $\operatorname{rk} Q_Y = 2$. In (34) we have $\operatorname{rk} Q_X = 2$. In (35), if $\operatorname{rk} Q_X = \operatorname{rk} Q_Y = 1$, then $e \neq 0$ and $b \neq 0$ and consequently $\operatorname{rk} Q_{X+Y} = 2$. In all cases we have got contradictions. □

We shall now consider affine properties of the sets $\tilde{\mathcal{H}}_{x_0}$ and $\tilde{\mathcal{E}}_{x_0}$ for surfaces in $\boldsymbol{R}^4$. It suffices to assume that the immersion f is purely real relative to some complex structure J on $\boldsymbol{R}^4$ around x_o and observe the sets

$$\mathcal{E}_{x_0} = \{Q(X, X)\ ;\ X \in S\}\,, \tag{36}$$

$$\mathcal{H}_{x_0} = \{Q(X, X)\ ;\ X \neq 0, X \in T_{x_0}M\}\,, \tag{37}$$

where S is an ellipse in $T_{x_0}M$ centered at the origin. The set $\mathcal{H}_{x_0}$ will be called the cone of the second fundamental tensor.

We shall find the above sets depending on various types of the symmetric bilinear form g_{x_0}. Assume first that g_{x_0} is definite. Let X, Y be an orthonormal basis of $T_{x_0}M$ relative to g_{x_0}. Then, as in the proof of Lemma 4.1, we get

$$Q_X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad Q_Y = \begin{bmatrix} c & -a \\ d & -b \end{bmatrix}. \tag{38}$$

It follows that $Q(X, X)+Q(Y, Y) = 0$, that is, the ellipse $\mathcal{E}_{x_0}$ is centered at 0. Moreover, the vectors $Q(X, Y)$ and $Q(X, X)-Q(Y, Y)$ are linearly independent. Hence the ellipse does not reduce to a line segment. It means that the cone of the second fundamental tensor $\mathcal{H}_{x_0}$ is equal to $T_{x_0}M \setminus \{0\}$. If we take another basis of $T_{x_0}M$ and construct the ellipse $\mathcal{E}_{x_0}$, then its center is not necessary the origin of $T_{x_0}M$, but the origin belongs to the open domain surrounded by $\mathcal{E}_{x_0}$ (because the set $\mathcal{H}_{x_0}$ must be $T_{x_0}M \setminus \{0\}$).

Assume now that g_{x_0} is non-degenerate indefinite. Take a basis $\{X, Y\}$ of $T_{x_0}M$ as in the proof of Lemma 4.1. Then $Q(X, Y) = 0$ and the vectors $Q(X, X)+Q(Y, Y)$ and $Q(X, X)-Q(Y, Y)$ are linearly independent. It means that the ellipse $\mathcal{E}_{x_0}$ reduces to a line segment and the line does not pass through $0 \in T_{x_0}M$. The origin of $T_{x_0}M$ (the vertex of the cone $\mathcal{H}_{x_0}$) does not belong to the cone, as it was proved in Lemma 4.1. The cone is not symmetric relative to the vertex.

Consider the case where $\operatorname{rk} g_{x_0} = 1$. Using the proof of Lemma 4.1, we know that there is a basis $\{X, Y\}$ of $T_{x_0}M$ such that $Q(X, X) = 0$. Then $Q(X, X) + Q(Y, Y)$ and $Q(X, X) - Q(Y, Y)$ are opposite non-zero vectors and the axis $Q(X, Y)$ is also non-zero. It means that the cone of the second fundamental tensor contains $0 \in T_{x_0}M$ and does not reduce to a half-line and is not the whole $T_{x_0}M$. The cone is not symmetric relative to the vertex.

As it was observed in Section 2, a surface whose cone $\mathcal{H}_{x_0}$of the second fundamental tensor is not symmetric relative to its vertex cannot be complex (around x_0) relative to any complex structure on $\boldsymbol{R}^4$.

The case $g_{x_0} = 0$ will be considered in the next section.

5. Classification results. Recall that an immersion f into a space with a connection is called full if its codimension cannot be reduced, that is, there is no proper totally geodesic submanifold of the ambient space containing the image of f.

We have the following “reduction lemma” for submanifolds of the affine space $\boldsymbol{R}^N$.

LEMMA 5.1. *Let $\mathcal{N}$ be a transversal bundle for an immersion $f : M \to \boldsymbol{R}^N$. Suppose that* $\dim M = n$, *M is connected and there is a subbundle $\mathcal{N}'$ with* $\dim \mathcal{N}' = r < N - n$, *of $\mathcal{N}$ such that $h(X, Y) \in \mathcal{N}'$ for all $X, Y \in TM$, and $\mathcal{N}'$ is ∇'-parallel. Then there is an $(n + r)$-dimensional affine subspace $\mathcal{A}$ of $\boldsymbol{R}^N$ such that $f(M) \subset \mathcal{A}$.*

PROOF. Take the vector bundle

$$M \ni x \to \mathcal{V}_x = f_*(T_x M) + \mathcal{N}'_x \, .$$

By formulas (1) and (2) and the assumptions of the lemma, the bundle is parallel relative to the standard connection $\tilde{\nabla}$ on $\boldsymbol{R}^N$. Hence $\mathcal{V}_x = \mathcal{V}$ for some $(n + r)$-dimensional vector subspace $\mathcal{V}$ of $\boldsymbol{R}^N$ for all $x \in M$. It now suffices to observe that if $\gamma(t)$ is an arbitrary curve on M, then

$$(f \circ \gamma)(t) \in f(\gamma(0)) + \mathcal{V} \, .$$

This easily follows from the Taylor formula

$$(f \circ \gamma)(t) = (f \circ \gamma)(0) + t(f \circ \gamma)'(\theta t) \, . \qquad \square$$

We have an analogous result as in the classical theory of surfaces in $\boldsymbol{R}^3$:

LEMMA 5.2. *Let $f : U \ni (u, v) \to f(u, v) = p(v) + uq(v) \in \boldsymbol{R}^N$ be an immersion of an open connected domain $U \subset \boldsymbol{R}^2$ and $f_{uv} = \psi f_v$ for some function ψ satisfying the equation $\psi_u + \psi^2 = 0$ on U.*

(a) *If $\psi = 0$ on U, then f is a piece of a cylinder.*
(b) *If $\psi \neq 0$ and $\psi_v = 0$ on U, then f is a piece of a cone.*
(c) *If $\psi \neq 0$ and $\psi_v \neq 0$ on U, then f is a piece of a tangential developable.*

PROOF. In order to prove (a) it suffices to observe that if $\psi = 0$, then $q' = f_{uv} = 0$.

Assume now that $\psi \neq 0$ on U. Since $\psi_u + \psi^2 = 0$, we have

$$\psi(u, v) = \frac{1}{u + \mu(v)} \tag{39}$$

for some function μ. Therefore, using also the equality $f_{uv} = \psi f_v$, we obtain

$$p' = \mu q' \, . \tag{40}$$

Since f is an immersion, $\mu + u$ does not vanish and the vectors q and q' are linearly independent at each point.

We shall now prove (b). In this case we have (39), where μ is constant. By (40) we get $p = q_0 + \mu q$, where q_0 is a fixed point of $\boldsymbol{R}^N$. We have

$$f(u, v) = q_0 + (u + \mu)q(v)\,, \tag{41}$$

that is, f is a piece of a cone.

In the case of (c), $\mu'(v) \neq 0$. If we set $\tilde{p} = -\mu q + p$, we get

$$p + uq = \tilde{p} - \left(\frac{u + \mu}{\mu'}\right)\tilde{p}'\,. \tag{42}$$

This finishes the proof of the lemma.

Before stating the next theorem, we shall introduce the following notation. If ∇ is a torsion-free connection on a 2-dimensional manifold M, (u, v) is a coordinate system on M and $X = \partial_u$, $Y = \partial_v$, then we set

$$\nabla_X X = AX + BY\,, \quad \nabla_X Y = CX + DY\,, \quad \nabla_Y Y = EX + FY\,. \tag{43}$$

If Ric is the Ricci tensor of ∇, then

$$\begin{aligned} \mathrm{Ric}\,(X, X) &= B_v - D_u + D(A - D) + B(F - C)\,, \\ \mathrm{Ric}\,(X, Y) &= D_v - F_u + CD - BE\,, \\ \mathrm{Ric}\,(Y, X) &= C_u - A_v + CD - BE\,, \\ \mathrm{Ric}\,(Y, Y) &= E_u - C_v + E(A - D) + C(F - C)\,. \end{aligned} \tag{44}$$

Recall also that if $f : M \to \boldsymbol{C}^n$ is a purely real immersion, then the following fundamental equations of Gauss and Codazzi are satisfied ([2]):

$$R(Z, W) = Q_Z Q_W - Q_W Q_Z\,, \tag{45}$$

$$(\nabla_W Q)(Z, V) = (\nabla_Z Q)(W, V) \tag{46}$$

for all W, V, Z, where R is the curvature tensor of the induced connection ∇.

We can now prove the following

THEOREM 5.3. *Let $f : M \to \boldsymbol{R}^4$ be an immersion of a 2-dimensional connected manifold M and $g = 0$ identically on M. If $Q : T_xM \ni W \to Q_W \in \mathrm{HOM}\,(T_xM)$ is a monomorphism for every $x \in M$, then $f(M)$ is contained in some 3-dimensional affine subspace of $\boldsymbol{R}^4$. If for each $x \in M$ the mapping Q is not a monomorphism, then there is a dense open subset M' of M such that for each $x \in M'$, there is a neighborhood U of x such that f restricted to U is either a piece of a cylinder, a piece of a cone, or a piece of a tangential developable.*

PROOF. For a fixed point x_0 of M we choose a complex structure on $\boldsymbol{R}^4$ relative to which f is purely real in a neighborhood U_0 of x_0. Considerations will be now carried on around x_0, that is, in a sufficiently small neighborhood of x_0.

Observe first that $\mathrm{rk}\,Q \leq 1$. Since $g = 0$, the endomorphism Q_Z is singular for every $Z \in T_xM$. Let $0 \neq X \in T_xM$ be arbitrary. Let $0 \neq Y \in \ker Q_X$. Assume first that Y is linearly independent of X. The matrices of Q_X and Q_Y relative to the basis $\{X, Y\}$ are given by

$$Q_X = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \quad Q_Y = \begin{bmatrix} 0 & e \\ 0 & k \end{bmatrix}. \tag{47}$$

Since $\det Q_{X+Y} = 0$, the vectors (a, b) and (e, k) are proportional. It follows that rk $Q \leq 1$. If $X \in \ker Q_X$, then

$$Q_X = \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix}, \quad Q_Y = \begin{bmatrix} c & e \\ d & k \end{bmatrix}. \tag{48}$$

Since $\det Q_Y = 0$, the vectors (c, d) and (e, k) are proportional. Again, it follows that rk $Q \leq 1$.

Assume first that $\mathcal{Q} : TM \ni Z \to Q_Z \in \mathrm{HOM}(TM)$ is a monomorphism at each x. Thus rk $Q = 1$ and im Q is a 1-dimensional distribution around x. Recall that the distribution depends essentially on a complex structure chosen on $\boldsymbol{R}^4$ and, in general, it is not globally defined on M.

Let Y be a vector field spanning im Q. If (u, v) is any coordinate system around x such that $\partial_v = Y$, then we have

$$Q_X = \begin{bmatrix} 0 & 0 \\ b & d \end{bmatrix}, \quad Q_Y = \begin{bmatrix} 0 & 0 \\ d & k \end{bmatrix}, \tag{49}$$

where $X = \partial_u$.

We shall now use the Codazzi equation. We then have

$$\begin{aligned}
(\nabla_X Q)(Y, X) &= \nabla_X(dY) - Q(\nabla_X Y, X) - Q(Y, \nabla_X X) = dCX + L_1 Y \\
(\nabla_Y Q)(X, X) &= \nabla_Y(bY) - 2Q(\nabla_Y X, X) = bEX + L_2 Y\,, \\
(\nabla_Y Q)(X, Y) &= \nabla_Y(dY) - Q(\nabla_Y X, Y) - Q(X, \nabla_Y Y) = dEX + L_3 Y\,, \\
(\nabla_X Q)(Y, Y) &= \nabla_X(kY) - 2Q(\nabla_X Y, Y) = kCX + L_4 Y
\end{aligned}$$

for some functions L_1, L_2, L_3, L_4. Hence

$$dC = bE\,, \quad dE = kC\,. \tag{50}$$

We claim that the distribution im Q is ∇-parallel, that is, $E = C = 0$ at each point. Indeed, if $k = 0$ at x, then, since $Q_Y \neq 0$, we have $d \neq 0$. By (50) we get $E = C = 0$ at x. If $k \neq 0$ at x (and automatically around x), then there is a vector field X, linearly independent of Y, around x spanning $\ker Q_Y$. We can now choose a coordinate system in such a way that ∂_u spans $\ker Q_Y$ and Y spans im Q. The formulas (50) are still valid. Then $d = 0$, $k \neq 0$, $b \neq 0$. By (50) $E = C = 0$.

By the reduction lemma we know that for each point x of M there is an affine 3-dimensional subspace $\mathcal{A}$ of $\boldsymbol{R}^4$ and a neighborhood U of x such that $f(U) \subset \mathcal{A}$. Moreover the immersion $f_{|U} : U \to \mathcal{A}$ is full. It follows that the image of the whole (connected) M is contained in a 3-dimensional affine subspace of $\boldsymbol{R}^4$.

Assume now that the mapping $\mathcal{Q} : TM \ni Z \to Q_Z \in \mathrm{HOM}\,(TM)$ is not a monomorphism, although it is not trivial at each point. Then $\ker \mathcal{Q}$ is a 1-dimensional distribution. By the Gauss equation the induced connection is flat.

For any coordinate system (u, v) such that $X = \partial_u$ spans $\ker \mathcal{Q}$, we have

$$Q_X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_Y = \begin{bmatrix} 0 & e \\ 0 & k \end{bmatrix}, \tag{51}$$

where $Y = \partial_v$. Moreover

$$\begin{aligned}
&(\nabla_X Q)(Y, X) = -Q(Y, \nabla_X X) = -BQ(Y, Y)\,,\\
&(\nabla_Y Q)(X, X) = 0\,,\\
&(\nabla_X Q)(Y, Y) = (Xe)X + eAX + eBY + (Xk)Y + k(CX + DY) - 2D(eX + kY)\,,\\
&(\nabla_Y Q)(X, Y) = -D(eX + kY)\,.
\end{aligned}$$

By the Codazzi equation we have

$$B = 0\,, \quad Xe + eA + kC - De = 0\,, \quad Xk = 0\,. \tag{52}$$

In particular, the distribution $\ker \mathcal{Q}$ is totally geodesic relative to the connection ∇.

Define subsets U_1 and U_2 of U_0 by

$$U_1 = \{x \in U_0\,;\ \ker \mathcal{Q} = \operatorname{im} Q\}\,, \quad U_2 = U_0 \setminus U_1\,.$$

Consider the open set U_2. If $x \in U_2$, then $\ker \mathcal{Q}$ and $\operatorname{im} Q$ are 1-dimensional complementary distributions around x. Take a coordinate system (u, v) around x such that $X = \partial_u$ and $Y = \partial_v$ span $\ker \mathcal{Q}$ and $\operatorname{im} Q$, respectively. Then $e = 0$ and $k \neq 0$ in (51). Hence, by (52), $C = 0$, which, together with the fact that $B = 0$, means that the distribution $\ker \mathcal{Q}$ is ∇-parallel. Moreover, since ∇ is flat, by Formulas (44) we have the equality $A_v = 0$. It means that we can change the coordinate u in such a way that $\nabla_{\partial_u}\partial_u = 0$. Then, using again (44), we get the equality $D_u + D^2 = 0$. In the new coordinate system we have $f_{uu} = 0$ and $f_{uv} = Df_v$, where $D_u^2 + D^2 = 0$. We can now use Lemma 5.2.

Consider now the set $\operatorname{int} U_1$. If $x \in \operatorname{int} U_1$, then around x we have one totally geodesic 1-dimensional distribution $\ker \mathcal{Q}$. Take a vector $Y'_x \notin \ker \mathcal{Q}_x$ and extend it to a ∇-parallel vector field Y' around x, which is possible because ∇ is flat. The distribution spanned by Y' is ∇-parallel and complementary to $\ker \mathcal{Q}$ around x. Take a coordinate system (u, v) around x such that $X = \partial_u$ spans $\ker \mathcal{Q}$ and $Y = \partial_v$ is parallel to Y' at each point around x. We then have

$$\nabla_X X = AX\,, \quad \nabla_X Y = DY\,, \quad \nabla_Y Y = FY\,.$$

Since ∇ is flat, by (44), we obtain $A_v = 0$. Hence, by changing the coordinate u, we may assume $A = 0$ and, again by (44), we can choose a coordinate system (u, v) around x such that $f_{uu} = 0$, $f_{uv} = D\partial_v$, where $D_u + D^2 = 0$. We now apply Lemma 5.2.

Around any point of $\operatorname{int}\{Q = 0\}$ the immersion f can be regarded as a piece of a cylinder. The proof of the theorem is completed.

THEOREM 5.4. *If M is a compact 2-dimensional manifold and $f : M \to \boldsymbol{R}^4$ is an immersion, then there is a point $x_0 \in M$ such that g_{x_0} is either non-degenerate indefinite and* $\operatorname{rk} Q = 2$ *or* $g_{x_0} = 0$ *and* $\operatorname{rk} Q_{x_0} = 1$.

PROOF. Take any basic point o of $\boldsymbol{R}^4$ and any positive definite scalar product G on $\boldsymbol{R}^4$. The function $\Phi : M \ni x \to G(\overrightarrow{of(x)}, \overrightarrow{of(x)})$ attains its maximum at a point $x_0 \in M$. Therefore, f satisfies IV in Section 2. Consequently, by Theorem 2.1, it satisfies II at x_0. By Lemma 4.1 we know that g_{x_0} is non-degenerate indefinite or $g_{x_0} = 0$. In the first case, by the proof of Lemma 4.1, we know that the rank of g_{x_0} is 2. In the last case, by the proof of Theorem 5.3, we know that $\operatorname{rk} Q_{x_0} \leq 1$. Since II is satisfied, $\operatorname{rk} Q_{x_0}$ cannot be 0. Hence $\operatorname{rk} Q_{x_0} = 1$. □

We shall now look at the sets $\mathcal{H}_{x_0}$ and $\mathcal{E}_{x_0}$ in the case where $g_{x_0} = 0$ and $Q_{x_0} \neq 0$. Since $\operatorname{rk} Q_{x_0} = 1$, the vectors $Q(X, X) + Q(Y, Y)$, $Q(X, X) - Q(Y, Y)$ and $Q(X, Y)$ are all proportional to each other (for any basis $\{X, Y\}$ of $T_{x_0}M$). It follows that the ellipse $\mathcal{E}_{x_0}$ is a line segment and the line passes through the origin of the vector space $T_{x_0}M$. If the codimension of f around x_0 can be reduced and f, as a surface in a 3-dimensional affine subspace of $\boldsymbol{R}^4$, is locally strongly convex at x_0, then $\mathcal{H}_{x_0}$ is a half-line without the vertex (the origin of $T_{x_0}M$). If f is ruled around x_0, then the origin of $T_{x_0}M$ belongs to $\mathcal{H}_{x_0}$. In this case $\mathcal{H}_{x_0}$ might be the whole line (for instance, for special affine Lagrangian surfaces) or a half-line including the vertex. Examples of ruled special affine Lagrangian surfaces are given in [3].

It should be remarked that throughout the paper we have intensively used the fact that any n-dimensional submanifold of $\boldsymbol{R}^{2n}$ can be locally regarded as a purely real submanifold of $\boldsymbol{C}^n$. It is not true in the global setting. There exist topological obstructions for embedded submanifolds. There also exist n-dimensional immersed submanifolds of $\boldsymbol{R}^{2n}$ which cannot be globally purely real relative to any complex structure on $\boldsymbol{R}^{2n}$. We shall now consider this problem in case of surfaces of $\boldsymbol{R}^4$.

LEMMA 5.5. *Let M be a compact n-dimensional manifold and $f : M \to \boldsymbol{C}^n$ be a purely real immersion. The image of M cannot be contained in any* $(2n-1)$*-dimensional real affine subspace of* $\boldsymbol{C}^n$.

PROOF. Suppose that $\mathcal{A}$ is a $(2n-1)$-dimensional real affine subspace of $\boldsymbol{C}^n$ in which $f(M)$ is contained. Let V be the direction of $\mathcal{A}$. The space $W = V \cap JV$ is a $(2n-2)$-dimensional J-invariant vector subspace of $\boldsymbol{C}^n$. Let G be the standard scalar product on $\boldsymbol{C}^n$ and $\mathbf{e}$ be a non-zero vector of V perpendicular to W. Consider the function

$$M \ni x \to G(\overrightarrow{of(x)}, \mathbf{e}) \in \boldsymbol{R}\,,$$

where $o \in \boldsymbol{C}^n$ is any basic point of the affine space $\boldsymbol{C}^n$. If x_0 is a point of M, where the function attains its extremum, then $f_*(T_{x_0}M)$ is orthogonal to $\mathbf{e}$, that is, $f_*(T_{x_0}M) \subset W$. Moreover $J(f_*(T_{x_0}M)) \subset W$ because W is J-invariant. Hence

$$f_*(T_{x_0}M) + Jf_*(T_{x_0}M) \subset W\,,$$

which is impossible for a purely real immersion. □

Using now Lemma 5.5 and Theorems 5.3, 5.4, we obtain

THEOREM 5.6. *Let M be a 2-dimensional compact orientable manifold and $f : M \to \boldsymbol{R}^4$ be an immersion. If* $\operatorname{rk} g$ *is constant on M, then* $\operatorname{rk} g = 2$ *and g is non-degenerate indefinite at each point of M or $g = 0$ on M. In the first case M is a topological torus of genus 1, the immersion f is full in $\boldsymbol{R}^4$ and locally strongly convex at each point of M. In the second case, if f is locally strongly convex at each point of M, then M is a topological sphere, $f(M)$ is contained in a 3-dimensional affine subspace $\mathcal{A}$ of $\boldsymbol{R}^4$ and $f : M \to \mathcal{A}$ is an ovaloid. If f is purely real relative to some complex structure on $\boldsymbol{R}^4$, then the first case holds.*

PROOF. By Theorem 5.4 we know that there is a point $x_0 \in M$ such that g_{x_0} is either non-degenerate indefinite or $g_{x_0} = 0$ on $T_{x_0}M$. Since $\operatorname{rk} g$ (which is a real affine invariant) is constant on M, either g is non-degenerate indefinite at each point of M or $g = 0$ on M. In the first case, by Proposition 4.3 we know that the Euler characteristic of M is 0. Hence M must be a topological torus. Moreover, since in this case $\operatorname{rk} Q = 2$, the immersion must be full. In the second case it suffices to observe that a locally strongly convex surface is such that Q is a bundle monomorphism. By Theorem 5.3 we know that f is a locally strongly convex surface in a 3-dimensional affine subspace of $\boldsymbol{R}^4$.

The last statement is a consequence of Lemma 5.5. □

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