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THE MAIN COMPONENT OF THE TORIC HILBERT SCHEME

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Abstract. Let $\boldsymbol{X}$ be an affine toric variety with big torus $\boldsymbol{T} \subset \boldsymbol{X}$ and let $T \subset \boldsymbol{T}$ be a subtorus. The general T-orbit closures in $\boldsymbol{X}$ and their flat limits are parametrized by the main component H_0 of the toric Hilbert scheme. Further, the quotient torus $\boldsymbol{T}/T$ acts on H_0 with a dense orbit. We describe the fan of this toric variety; this leads us to an integral analogue of the fiber polytope of Billera and Sturmfels. We also describe the relation of H_0 to the main component of the inverse limit of GIT quotients of $\boldsymbol{X}$ by T.

1. Introduction. The multigraded Hilbert scheme parametrizes, in a technical sense specified below, all homogeneous ideals in a polynomial algebra (or, more generally, in an arbitrary finitely generated algebra) having a fixed Hilbert function with respect to a grading by an abelian group. In [8] it was shown that the multigraded Hilbert scheme always exists as a quasiprojective scheme.

We consider the following case. Let $\boldsymbol{X}$ be an affine toric (not necessarily normal) variety with big torus $\boldsymbol{T} \subset \boldsymbol{X}$ and let $T \subset \boldsymbol{T}$ be a subtorus acting on $\boldsymbol{X}$ by the restriction of the action of $\boldsymbol{T}$. This defines a grading of the algebra of regular functions $k[\boldsymbol{X}]$ by the group of characters of T. Denote by $H_{\boldsymbol{X},T}$ the toric Hilbert scheme, that is, the multigraded Hilbert scheme parametrizing those T-invariant ideals in $k[\boldsymbol{X}]$ having the same Hilbert function as the toric T-variety $X = \overline{Tx}$, where $x \in \boldsymbol{X}$ lies in the open $\boldsymbol{T}$-orbit [11]. There is a canonical irreducible component H_0 of $H_{\boldsymbol{X},T}$ parametrizing general T-orbit closures in $\boldsymbol{X}$ and their flat limits (Proposition 3.6(2)). This component contains an open orbit for a natural action of $\boldsymbol{T}/T$ on $H_{\boldsymbol{X},T}$. The main result of this work is a description of the fan of this toric variety (Theorem 4.5). Also, we compare the fan of H_0 with the fan of the toric Chow quotient.

The Chow quotient of a *projective* toric variety was considered in [10]. In particular, in this paper there is a description of its fan. Namely, recall that the fan of a projective toric variety is the normal fan of a convex polytope P in the space generated by the lattice of characters of $\boldsymbol{T}$. Let Q be the projection of this polytope on the space $\mathcal{X}(T)_{\boldsymbol{R}}$ generated by the lattice of characters of the subtorus T. Then the fan of the Chow quotient is the normal fan to the fiber polytope $F(P, Q)$ [3], which , in a well-defined sense, is the average over all fibers of the projection of P on Q. More generally, the fiber fan for a projection of an arbitrary

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polyhedron was defined in [4]. In this paper some results of [10] were generalized on the case of a variety that is projective over some affine variety.

In our affine setting, we show that the fan corresponding to the toric variety H_0 is the normal fan to the average over all "integral" fibers of the corresponding cone projection. Here by an integral fiber we mean the polyhedron generated by all integral points of a fiber of the projection. Thus this object can be regarded as an integral analogue of the fiber fan. If $\boldsymbol{X}$ is a finite-dimensional T-module and the grading of $k[\boldsymbol{X}]$ by the weights of T is positive, then the fan of H_0 coincides with the normal fan to the state polytope of Sturmfels (see [12, Theorem 2.5]).

In the last section we consider the toric Chow morphism from the Hilbert scheme to the inverse limit of GIT quotients $\boldsymbol{X}/_{\chi}T$. This morphism was constructed in [8, Section 5] in the case when $\boldsymbol{X}$ is a finite-dimensional T-module. We generalize this to the case of a normal affine toric $\boldsymbol{T}$-variety $\boldsymbol{X}$ (Theorem 5.4).

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2. Terminology and notation. We consider the category of schemes over an algebraically closed field k. A *variety* is a separated integral scheme of finite type. Recall that any scheme Z is characterized by its *functor of points* from the category of k-algebras to the category of sets:

$$\underline{Z} : \underline{k\text{-}Alg} \to \underline{Set}\,, \quad \underline{Z}(R) := \mathrm{Mor}(\mathrm{Spec}\, R, Z)\,,$$

where $\mathrm{Mor}(\mathrm{Spec}\, R, Z)$ is the set of morphisms of schemes over k from $\mathrm{Spec}\, R$ to Z (we denote the functor of points of a scheme by the corresponding underlined letter). Our main reference on schemes is [5]. We denote by $\mathcal{O}_Z$ the structure sheaf of Z, and if Z is affine, then $k[Z]$ denotes the algebra of sections of $\mathcal{O}_Z$ over Z. We denote by $\boldsymbol{A}^n$ the affine space $\mathrm{Spec}\, k[x_1, \dots, x_n]$.

An n-dimensional *torus* T is an algebraic group isomorphic to the direct product of n copies of the multiplicative group $\boldsymbol{G}_m$ of the field k. For the lattices of characters and one-parameter subgroups of T, we use the notations $\mathcal{X}(T) = \mathrm{Hom}(T, \boldsymbol{G}_m)$ and $\Lambda(T) = \mathrm{Hom}(\boldsymbol{G}_m, T)$. We denote by $\langle \cdot, \cdot \rangle$ the natural pairing between $\mathcal{X}(T)$ and $\Lambda(T)$. For a lattice $\mathcal{X}$, let $\mathcal{X}_{\boldsymbol{R}} = \mathcal{X} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. If $\Sigma \subset \mathcal{X}$ is a monoid, then $\mathrm{cone}(\Sigma)$ denotes the cone in $\mathcal{X}_{\boldsymbol{R}}$ generated by Σ. For subsets D_1, D_2 of a vector space, we denote by $D_1 + D_2$ the Minkowski sum.

By a *toric variety* under a torus T we mean a variety X such that T is embedded as an open subset into X, the action of T on itself by multiplication extends to an action on X, and X admits an open covering by affine T-invariant charts. We do not require X to be normal.

We denote by $\mathcal{C}_X$ the associated fan of a toric variety X, so the cones of $\mathcal{C}_X$ lie in $\Lambda(T)_{\boldsymbol{R}}$ (see [7, Sec. 1.4]). The T-orbits on X are in order-reversing one-to-one correspondence with the cones of $\mathcal{C}_X$. If $\sigma(Y)$ is the cone in $\mathcal{C}_X$ corresponding to a T-orbit Y, then a one-parameter subgroup $\lambda \in \Lambda(T)$ lies in the interior of $\sigma(Y)$ if and only if $\lim_{s\to 0} \lambda(s)$ exists and lies in Y. A toric variety is determined by its fan up to normalization.

3. Definitions and background on multigraded Hilbert schemes. Let $\boldsymbol{X}$ be an affine variety over k with an action of a torus T, so its algebra of regular functions $S := k[\boldsymbol{X}]$ is graded by the group $\mathcal{X}(T)$ of characters of T:

$$S = \bigoplus_{\chi \in \mathcal{X}(T)} S_\chi ,$$

where S_χ is the subspace of T-semiinvariant functions of weight χ. Let

$$\Sigma := \{\chi \in \mathcal{X}(T) \ ; \ S_\chi \neq 0\} .$$

This is a finitely generated monoid. Conversely, if S is a finitely generated commutative k-algebra without zero divizors graded by $\mathcal{X}(T)$, then we have a T-action on the affine variety $\boldsymbol{X} = \operatorname{Spec} S$.

DEFINITION 3.1. The grading of S by $\mathcal{X}(T)$ is *positive* if $k[\boldsymbol{X}]_0 = k$ and cone(Σ) is strictly convex.

Notice that in the case of a positive grading there exists a unique minimal system of generators of Σ. The following definition was introduced in [8].

DEFINITION 3.2. Given a function $h : \mathcal{X}(T) \to \boldsymbol{N}$, the *Hilbert functor* is the covariant functor $\underline{H^h_{\boldsymbol{X},T}}$ from the category of k-algebras to the category of sets assigning to any k-algebra R the set of all T-invariant ideals $I \subseteq R \otimes_k S$ such that $(R \otimes_k S_\chi)/I_\chi$ is a locally free R-module of rank $h(\chi)$ for any $\chi \in \mathcal{X}(T)$.

Remark that we can also view $\underline{H^h_{\boldsymbol{X},T}}(R)$ as a set of closed T-invariant subschemes $Y \subset \operatorname{Spec} R \times \boldsymbol{X}$ such that the projection $Y \to \operatorname{Spec} R$ is flat.

In [8, Theorem 1.1] it was proved that there exists a quasiprojective scheme $H^h_{\boldsymbol{X},T}$ which represents this functor in the case when $\boldsymbol{X}$ is a finite-dimensional T-module V. In the case of an arbitrary $\boldsymbol{X}$ there exists a T-equivariant closed immersion $\boldsymbol{X} \hookrightarrow V$, where V is a finite-dimensional T-module. Then the Hilbert functor $\underline{H^h_{\boldsymbol{X},T}}$ is represented by a closed subscheme of $H^h_{V,T}$ (see [1, Lemma 1.6]). Namely, for an algebra R the subset $\underline{H^h_{\boldsymbol{X},T}}(R) \subset \underline{H^h_{V,T}}(R)$ consists of those ideals $I \subset R \otimes_k k[V]$ that $I \in \underline{H^h_{V,T}}(R)$ and $R \otimes_k I_{\boldsymbol{X}} \subset I$.

Recall that the *universal family* is the closed subscheme $\boldsymbol{W}_{\boldsymbol{X},T}$ of $H_{\boldsymbol{X},T} \times \boldsymbol{X}$ corresponding to the identity map $\{\mathrm{Id} : H_{\boldsymbol{X},T} \to H_{\boldsymbol{X},T}\} \in \underline{H_{\boldsymbol{X},T}}(H_{\boldsymbol{X},T}) := \operatorname{Mor}(H_{\boldsymbol{X},T}, H_{\boldsymbol{X},T})$. For any $Y \in \underline{H_{\boldsymbol{X},T}}(R)$ (so Y is a closed subscheme in $\operatorname{Spec}(R \otimes_k S)$) we have $Y = \boldsymbol{W}_{\boldsymbol{X},T} \times_{H_{\boldsymbol{X},T}} \operatorname{Spec} R$. In fact, the k-rational points of $\boldsymbol{W}_{\boldsymbol{X},T}$ are those pairs (y, Y), where $Y \in \underline{H_{\boldsymbol{X},T}}(k)$ and $y \in \underline{Y}(k)$.

If V is a finite dimensional T-module such that $k[V]^T = k$, then $H^h_{V,T}$ is projective (see [8, Corollary 1.2]) The following lemma generalizes this statement.

LEMMA 3.3. *Assume that* $h(0) = 1$. *Then the morphism*

$$p : H^h_{X,T} \longrightarrow X//T := \operatorname{Spec} k[X]^T$$

which assigns to any element $I \in \underline{H^h_{X,T}}(R)$ *the morphism* $k[X]^T \to (R \otimes k[X]^T)/I^T \simeq R$, *is projective.*

PROOF. Since we know that $H^h_{X,T}$ quasiprojective, it is sufficient to check that the valuative criterion of properness for p is satisfied. Let S be the spectrum of a discrete valuation ring R with generic point η and closed point s. We have to show that any morphism $\phi_\eta : \eta \to H^h_{X,T}$ such that the composition $p \circ \phi : \eta \to X//T$ extends to a morphism $S \to X//T$, extends to a unique morphism $S \to H^h_{X,T}$. Consider $Y_\eta = \eta \times_{H^h_{X,T}} W^h_{X,T} \subset \eta \times X$. By [9, Prop. 9.7], a closed subscheme $Y \subset S \times X$ such that $Y \times_S \eta = Y_\eta$, is flat over S if and only if Y is the closure of Y_η in $S \times X$. It follows that the desired extension $Y \in \underline{H^h_{X,T}}(S)$ is unique. For the existence, we consider $Y := \overline{Y_\eta} \subset S \times X$. It remains to show that the fiber Y_s is non-empty (then, by flatness, it has the Hilbert function h). Indeed, we have the following commutative diagram:

$$\begin{array}{ccc} Y = \overline{Y_\eta} \subset & S \times X \\ \downarrow & \downarrow \\ S = \bar{\eta} \ \subset & S \times X//T \,, \end{array}$$

where the morphism $Y \to S$ is the quotient by T, so it is surjective. $\square$

We prove the following lemma to treat one particular case of the Hilbert scheme which we shall need later (see the corollary below).

LEMMA 3.4. *Let P be an N-graded algebra*: $P = \bigoplus_{r \geq 0} P_r$, *and*

$(*)$ *there exists r_0 such that $P_{r+1} = P_1 P_r$ for any $r \geq r_0$.*

Consider the Hilbert scheme H_P of the graded algebra P for the Hilbert function

$$h(r) := \begin{cases} 1 & \text{if } r \geq 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let R be an algebra and $Y = \operatorname{Spec}(R \otimes_k P/I) \in \underline{H_P}(R)$. *Then the projection* $Y \to \operatorname{Spec} R$ *is a locally trivial bundle with fiber A^1.*

PROOF. (1) Consider the open subscheme in $\operatorname{Spec} P$ that is the complement to the subscheme defined by the ideal $\bigoplus_{r>0} P_r$:

$$(\operatorname{Spec} P)_0 = \left\{ p \in \operatorname{Spec} P \ ; \ p \not\supseteq \left(\bigoplus_{r>0} P_r \right) \right\}$$

and the natural morphism

$$\psi : (\operatorname{Spec} P)_0 \to \operatorname{Proj} P\,.$$

Locally ψ is given by the embeddings of algebras $(P_f)_0 \subset P_f$, where $f \in P$ is homogeneous, $\deg f > 0$ (it is clear that the corresponding morphisms of affine schemes satisfy the compatibility conditions). Note that ψ is a locally trivial bundle with fiber $\boldsymbol{G}_m$. Indeed, condition $(*)$ implies that Proj P is covered by open affine subschemes $\operatorname{Spec}(P_h)_0$, where $h \in P_1$, and for any $h \in P_1$ we have $P_h = (P_h)_0[h, h^{-1}]$.

(2) Consider $Y_0 = Y \cap (\operatorname{Spec} R \times (\operatorname{Spec} P)_0)$. We have the morphisms

$$Y_0 \xrightarrow{\rho} \operatorname{Proj}(R \otimes_k P/I) \xrightarrow{\delta} \operatorname{Spec} R\,.$$

Since $R \otimes_k P/I$ satisfies condition $(*)$, by (1), it follows that ρ is a locally trivial bundle with fiber $\boldsymbol{G}_m$. So δ is an isomorphism. Consider the following morphism from Spec R to Proj P:

$$\operatorname{Spec} R \cong \operatorname{Proj}(R \otimes_k P/I) \subset \operatorname{Spec} R \times \operatorname{Proj} P \xrightarrow{p} \operatorname{Proj} P\,,$$

where p is the projection.

(a) Note that $Y_0 = (\operatorname{Spec} P)_0 \times_{\operatorname{Proj} P} \operatorname{Spec} R$. Indeed, locally we have

$$R \otimes_k P_f/I_f \simeq P_f \otimes_{(P_f)_0} \left(R \otimes_k (P_f)_0/(I_f)_0\right)\,,$$

where $f \in P$ is homogeneous of positive degree.

(b) Consider $Y' = Y_0 \times_{\boldsymbol{G}_m} \boldsymbol{A}^1$. Here $Y_0 \times_{\boldsymbol{G}_m} \boldsymbol{A}^1$ denotes the categorical quotient $(Y_0 \times \boldsymbol{A}^1)//\boldsymbol{G}_m$, where $\boldsymbol{G}_m$ acts on $\boldsymbol{A}^1$ as follows: $t \cdot s = t^{-1}s$, $t \in \boldsymbol{G}_m, s \in \boldsymbol{A}^1$. Then Y' is a locally trivial bundle over Spec R with fiber $\boldsymbol{A}^1$ and we have the natural morphism $\eta : Y' \to Y$, which is locally given by the homomorphisms

$$R \otimes_k P/I \to \bigoplus_{r \geq 0} (R \otimes_k P_f/I_f)_r\,,$$

where $f \in P$ is homogeneous of positive degree. So we have a commutative diagram:

$$\begin{tikzcd} Y' \arrow[rr, "\eta"] \arrow[dr, "\alpha'"'] & & Y \arrow[dl, "\alpha"] \\ & \operatorname{Spec} R. & \end{tikzcd}$$

Note that for any $r \geq 0$, the corresponding homomorphism $\alpha_*(\mathcal{O}_Y)_r \to \alpha'_*(\mathcal{O}'_Y)_r$ is a surjective homomorphism of locally free sheaves of R-modules of rank 1 and, consequently, is an isomorphism. Thus η is an isomorphism. □

The statement of the following corollary was given in [8, Section 5] with a proof for algebras generated by elements of degree 1.

COROLLARY 3.5. *With the notation of the previous lemma, the Hilbert scheme H_P is isomorphic to* Proj P.

PROOF. We shall show that Proj P represents the Hilbert functor $\underline{H_P}$. For this we prove that the tautological bundle over Proj P is the universal family, i.e., we are going to prove the universal property for $E := (\operatorname{Spec} P)_0 \times_{\boldsymbol{G}_m} \boldsymbol{A}^1$. Let $Y = \operatorname{Spec}(R \otimes_k P/I) \in \underline{H_P}(R)$. We have to show that $Y = E \times_{\operatorname{Proj} P} \operatorname{Spec} R$. Indeed, we have $Y = Y_0 \times_{\boldsymbol{G}_m} \boldsymbol{A}^1 = ((\operatorname{Spec} P)_0 \times_{\operatorname{Proj} P} \operatorname{Spec} R) \times_{\boldsymbol{G}_m} \boldsymbol{A}^1 = E \times_{\operatorname{Proj} P} \operatorname{Spec} R$. □

Let us return to the case of an affine toric $\boldsymbol{T}$-variety $\boldsymbol{X}$. We have

$$S = k[\boldsymbol{X}] = \bigoplus_{\nu \in \Omega} S_\nu \,,$$

where $\Omega \subset \mathcal{X}(\boldsymbol{T})$ is a finitely generated monoid and S_ν is the subspace of $\boldsymbol{T}$-semiinvariant functions of weight ν ($\dim S_\nu = 1$). Let $T \subset \boldsymbol{T}$ be a subtorus. We have a surjective linear map $\pi : \mathcal{X}(\boldsymbol{T}) \to \mathcal{X}(T)$ given by the restriction. The action of T on $\boldsymbol{X}$ arising from the action of $\boldsymbol{T}$ gives a grading

$$S = \bigoplus_{\chi \in \Sigma} S_\chi \,,$$

where $\Sigma = \pi(\Omega)$. We shall consider the following Hilbert function:

$$h(\chi) := \begin{cases} 1 & \text{if } \chi \in \Sigma \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Let $H_{\boldsymbol{X},T}$ be the corresponding Hilbert scheme (we shall also denote it by $H_{S,T}$). Note that all the ideals $I \in \underline{H_{V,T}}(k)$ are binomial (see [6, Proposition 1.11]). If $x \in \boldsymbol{X}$ lies in the open $\boldsymbol{T}$-orbit, then we have the point $X := \overline{T \cdot x} \in \underline{H_{\boldsymbol{X},T}}(k)$.

The group $\underline{\boldsymbol{T}}(R)$ acts on $\underline{H_{\boldsymbol{X},T}}(R)$ in the natural way. Namely, we have an action of $\underline{\boldsymbol{T}}(R)$ on $R \otimes_k S$: for $f \in R \otimes_k S_\nu$, where $\nu \in \Omega$, and $t \in \underline{\boldsymbol{T}}(R)$ let $t \cdot f = \nu(t) f$. Hence for $I \in \underline{H_{\boldsymbol{X},T}}(R)$ let $t \cdot I = \{t \cdot f \,;\ f \in I\}$. These actions commute with base extensions, thus we have an action of $\boldsymbol{T}$ on $H_{\boldsymbol{X},T}$. Since T acts trivially, this yields an action of the torus $\boldsymbol{T}/T$. The universal family $\boldsymbol{W}_{\boldsymbol{X},T}$ is invariant under the diagonal action of $\boldsymbol{T}$ on $H_{\boldsymbol{X},T} \times \boldsymbol{X}$.

Let H_0 be the toric orbit closure $\overline{\boldsymbol{T} \cdot X} \subset H_{\boldsymbol{X},T}$, and denote by $\boldsymbol{W}_0$ its preimage under the projection

$$p : \boldsymbol{W}_{\boldsymbol{X},T} \to H_{\boldsymbol{X},T}$$

(we consider H_0 and $\boldsymbol{W}_0$ with their structure of reduced schemes).

PROPOSITION 3.6. (1) *The stabilizer of X under the action of $\boldsymbol{T}$ on H_0 is T. Moreover, H_0 is a toric variety under the torus $\boldsymbol{T}/T$.*

(2) *The orbit $\boldsymbol{T} \cdot X$ is open in $H_{\boldsymbol{X},T}$. Consequently, H_0 is an irreducible component of $H_{\boldsymbol{X},T}$.*

(3) *$\boldsymbol{W}_0$ is a toric variety under the torus $\boldsymbol{T}$ (and, consequently, $\boldsymbol{W}_0$ is an irreducible component of $\boldsymbol{W}_{X,T}$).*

PROOF. (1) If $t \cdot X = X$ for $t \in \boldsymbol{T}$, then $t \cdot x \in T \cdot x$ and $t \in T$. So we have only to show that $H_{\boldsymbol{X},T}$ admits an open covering by affine T-invariant charts. Indeed, let $\chi \in \Sigma$. Then for any $I \in \underline{H_{\boldsymbol{X},T}}(R)$ the locally trivial R-module $(R \otimes k[\boldsymbol{X}]_\chi)/I_\chi$ defines a morphism from $\operatorname{Spec} R$ to the projectivisation $\boldsymbol{P}(k[\boldsymbol{X}]_\chi^*) = \operatorname{Proj}(\operatorname{Sym}(k[\boldsymbol{X}]_\chi))$, where $\operatorname{Sym}(k[\boldsymbol{X}]_\chi)$ denotes the symmetric algebra. These maps commute with base changes and, consequently, define a morphism $p_\chi : H_{\boldsymbol{X},T} \to \boldsymbol{P}(k[\boldsymbol{X}]_\chi^*)$. Note that p_χ is $\boldsymbol{T}$-equivariant (the action of $\boldsymbol{T}$ on $\boldsymbol{P}(k[\boldsymbol{X}]_\chi^*)$ is induced by the linear action of $\boldsymbol{T}$ on $k[\boldsymbol{X}]_\chi^*$). By [8, Proposition 3.2, Corollary 3.4], it follows that there exists a finite set of characters $\chi_1, \ldots, \chi_r \in \Sigma$ such that the morphism

$$p \times p_{\chi_1} \times \cdots \times p_{\chi_r} : H_{\boldsymbol{X},T} \to \boldsymbol{X}/\!/T \times \boldsymbol{P}(k[\boldsymbol{X}]^*_{\chi_1}) \times \cdots \times \boldsymbol{P}(k[\boldsymbol{X}]^*_{\chi_r})$$

is injective. Since the morphism p is projective (Lemma 3.3), it follows that $p \times p_{\chi_1} \times \cdots \times p_{\chi_r}$ is a closed embedding. Since any $\boldsymbol{P}(k[\boldsymbol{X}]^*_{\chi_i})$ admits an open covering by $\boldsymbol{T}$-invariant affine charts, it follows that $H_{\boldsymbol{X},T}$ does.

(2) We shall prove that $\boldsymbol{T} \cdot X$ is open in $H_{\boldsymbol{X},T}$. Since the stabilizer of X in $\boldsymbol{T}$ is T, it suffices to prove that $\dim T_X H_{\boldsymbol{X},T} \leq \dim \boldsymbol{T} \cdot X = \dim \boldsymbol{T} - \dim T$, where $T_X H_{\boldsymbol{X},T}$ denotes the tangent space to $H_{\boldsymbol{X},T}$ at X. By [8, Prop. 1.6], we have

$$T_X H_{\boldsymbol{X},T} = \mathrm{Hom}_{k[\boldsymbol{X}]}(I_X, k[X])_0 .$$

This vector space is isomorphic to

$$\mathrm{Hom}_{k[\boldsymbol{T}]}(I_T, k[T])_0 = \mathrm{Hom}_{k[\boldsymbol{T}]}(I_T/I_T^2, k[T])_0 ,$$

where I_T is the ideal of functions in $k[\boldsymbol{T}]$ vanishing on T. Indeed, since $(I_X)_\chi \subset k[\boldsymbol{X}]_\chi (I_T)_0$, for any $\phi \in \mathrm{Hom}_{k[\boldsymbol{T}]}(I_T, k[T])_0$ we have $\phi(I_X) \subset k[\boldsymbol{X}]\phi((I_T)_0) = k[\boldsymbol{X}]$. Conversely, $I_T = k[\boldsymbol{T}]I_X$, so any $\phi \in \mathrm{Hom}_{k[\boldsymbol{X}]}(I_X, k[X])_0$ can be extended to a homomorphism of $k[\boldsymbol{T}]$-modules from I_T to $k[T]$.

Further, we can choose coordinates on $\boldsymbol{T}$ such that

$$k[\boldsymbol{T}] = k[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}, s_1, s_1^{-1}, ..., s_r, s_r^{-1}] ,$$

where $r = \dim \boldsymbol{T} - \dim T$, and the ideal I_T is generated by $s_i - 1$ for $i = 1, \ldots, r$. The linear space I_T is spanned by the elements $t_1^{a_1} \cdots t_n^{a_n} s_1^{b_1} \cdots s_m^{b_m} (s_i - 1)$, where $a_i, b_j \in \boldsymbol{Z}$, and the projections of the elements $t_1^{a_1} \cdots t_n^{a_n} (s_i - 1)$ span the linear space I_T/I_T^2 (since $s_i(s_j - 1) = (s_j - 1) + (s_i - 1)(s_j - 1)$ and $s_i^{-1}(s_j - 1) = (s_j - 1) - s_i^{-1}(s_i - 1)(s_j - 1)$). Hence a homomorphism of $k[\boldsymbol{T}]$-modules from I_T to $k[T]$ is uniquely determined by the images of $s_i - 1$. Thus the dimension of the vector space of such homomorphisms of degree zero is not greater than r.

(3) Consider the restriction p_0 of p to $\boldsymbol{W}_0$:

$$p_0 : \boldsymbol{W}_0 \to H_0 .$$

This is a flat morphism. By Lemma 3.7 below and [9, Corollary 9.6], the dimension of any irreducible component Z of $\boldsymbol{W}_0$ is equal to $\dim \boldsymbol{T}$. This implies that $p_0(Z) = H_0$ and $Z \subset \overline{p^{-1}(\boldsymbol{T} \cdot X)}$. Thus $\boldsymbol{W}_0 = \overline{p^{-1}(\boldsymbol{T} \cdot X)} = \overline{\boldsymbol{T} \cdot (x, X)}$ is irreducible and $\boldsymbol{T} \cdot (x, X) \subset \boldsymbol{W}_0$ is dense and, consequently, open. Since $\boldsymbol{W}_0$ is a closed subscheme in $H_0 \times \boldsymbol{X}$, it follows that $\boldsymbol{W}_0$ admits an open covering by affine $\boldsymbol{T}$-invariant charts. □

LEMMA 3.7. *For any point $Y \in H_{\boldsymbol{X},T}$, the dimension of any irreducible component of its fiber $p^{-1}(Y)$ equals* $\dim T$.

PROOF. We denote by $k(Y)$ the residue field of $Y \in H_{\boldsymbol{X},T}$. Then we have

$$p^{-1}(Y) = \mathrm{Spec}\, k(Y) \times_{H_{\boldsymbol{X},T}} \boldsymbol{W}_{\boldsymbol{X},T} = \mathrm{Spec}\, L ,$$

where L is a coherent sheaf of Σ-graded $k(Y)$-algebras:

$$L = \bigoplus_{\chi \in \Sigma} L_\chi ,$$

and $L_\chi := k(Y) \otimes_{\mathcal{O}_{H_{X,T}}} (\mathcal{O}_{\boldsymbol{W}_{X,T}})_\chi$ is isomorphic to $k(Y)$. Let

$$\Sigma_{\text{red}} := \{\chi \in \Sigma \; ; \; L_\chi \text{ is not nilpotent}\} .$$

Note that $\operatorname{cone}(\Sigma_{\text{red}}) = \operatorname{cone}(\Sigma)$. Every point $Y \in H_{\boldsymbol{X},T}$ gives us a subdivision of $\operatorname{cone}(\Sigma)$ into subcones, namely two points $\chi, \chi' \in \Sigma_{\text{red}}$ lie in the same cone if and only if $L_\chi L_{\chi'} \neq 0$. The irreducible components Z of $p^{-1}(Y)$ correspond to the maximal cones C of this subdivision:

$$Z = \operatorname{Spec}\left(\bigoplus_{\chi \in \Sigma_{\text{red}} \cap C} L_\chi \right) .$$

Note that $\Sigma_{\text{red}} \cap C$ is a monoid. It suffices to prove that the dimension of Z is equal to $\dim T$. We can extend the action of T on Z to an action of the torus $T \times \operatorname{Spec} k(Y)$ (over the field $k(Y)$). Thus Z is a toric variety under the torus $T \times \operatorname{Spec} k(Y)$ and $\dim Z = \dim C = \dim T$. □

4. Fan of a toric Hilbert scheme. Our aim is to describe the fans of the toric varieties H_0 and $\boldsymbol{W}_0$.

Let us fix the notations. Recall that $\boldsymbol{X}$ is an affine toric variety under an action of a torus $\boldsymbol{T}$:

$$\boldsymbol{X} = \overline{\boldsymbol{T} \cdot x_0} ,$$

$T \subset \boldsymbol{T}$ is a subtorus, and

$$\pi : \mathcal{X}(\boldsymbol{T}) \to \mathcal{X}(T)$$

is the restriction map. Fix isomorphisms $\boldsymbol{T} \simeq \boldsymbol{G}_m^n$, $T \simeq \boldsymbol{G}_m^r$, this gives us a basis in $k[\boldsymbol{T}]$ and $k[T]$:

$$k[\boldsymbol{T}] = \bigoplus_{\nu \in \mathcal{X}(\boldsymbol{T})} kt^\nu , \quad k[T] = \bigoplus_{\chi \in \mathcal{X}(T)} kt^\chi .$$

We denote by X the T-orbit closure $\overline{T \cdot x_0}$ and $I_X \subset k[\boldsymbol{X}]$ denotes the corresponding ideal. Also, we have

$$S = k[\boldsymbol{X}] = \bigoplus_{\nu \in \Omega} kt^\nu , \quad k[X] = \bigoplus_{\chi \in \Sigma} kt^\chi .$$

The restriction homomorphisme $k[\boldsymbol{X}] \to k[X]$ is given by $t^\nu \to t^{\pi(\nu)}$ and its kernel I_X is generated by all the binomials of the form $t^{\nu_1} - t^{\nu_2}$ such that $\pi(\nu_1) = \pi(\nu_2)$ (see [12, Lemma 4.1]).

Let us recall some definitions concerning convex polyhedra. They are taken from [12], which we shall use as a general reference on convex polyhedra.

DEFINITION 4.1. Let P be a convex polyhedron in a vector space V. For any face F of P the *normal cone* $N_F(P)$ is the following cone in the dual vector space V^*:

$$N_F(P) := \{l \in V^* \; ; \; l(v - v') \geq 0 \text{ for all } v \in P, v' \in F\}.$$

The *normal fan* $N(P)$ of P is the fan whose cones are normal cones to the faces of P.

DEFINITION 4.2. The *recession cone* of a polyhedron $P \subset V$ is the set of those vectors $v \in V$ such that $u + v \in P$ for any $u \in P$.

DEFINITION 4.3. A fan $\mathcal{C}_1$ is a *refinement* of a fan $\mathcal{C}_2$ if any cone of $\mathcal{C}_1$ is contained in some cone of $\mathcal{C}_2$.

DEFINITION 4.4. We say that two polyhedra $P_1, P_2 \subset \mathcal{X}(\boldsymbol{T})_{\boldsymbol{R}}$ are *equivalent* if they have the same normal fan.

We fix an open $\boldsymbol{T}$-equivariant embedding of $\boldsymbol{T}/T$ (resp. of $\boldsymbol{T}$) in H_0 (resp. in W_0) such that the image of eT (resp. of e) is X (resp. (X, x_0)), where e is the unit in $\boldsymbol{T}$.

THEOREM 4.5. (1) *The fan $\mathcal{C}_{H_0} \subset \Lambda(\boldsymbol{T})_{\boldsymbol{R}}$ of the toric $\boldsymbol{T}/T$-variety H_0 is the coarsest common refinement of the normal fans $\mathcal{C}_\chi$ of the polyhedra*

$$P_\chi := \mathrm{conv}(\pi^{-1}(\chi) \cap \Omega) \subset \mathcal{X}(\boldsymbol{T})_{\boldsymbol{R}},$$

where $\chi \in \Sigma$.

(2) *The fan of $\boldsymbol{W}_0$ is the coarsest common refinement of the fans $\mathcal{C}_{H_0}$ and $N(\mathrm{cone}(\Omega))$.*

REMARK 4.6. We can consider fans in $\Lambda(\boldsymbol{T}/T)_{\boldsymbol{R}}$ as fans in $\Lambda(\boldsymbol{T})_{\boldsymbol{R}}$ whose cones contain $\Lambda(T)_{\boldsymbol{R}}$. In particular, we view the fan of the toric $\boldsymbol{T}/T$-variety H_0 as a fan in $\Lambda(\boldsymbol{T})_{\boldsymbol{R}}$.

PROOF. Let us recall that a one-parameter subgroup $\lambda \in \Lambda(\boldsymbol{T})_{\boldsymbol{R}}$ belongs to the support of the fan C_{H_0} if and only if there exists a limit of $X \in H_0$ under λ. Further, one-parameter subgroups $\lambda, \lambda' \in \Lambda(\boldsymbol{T})_{\boldsymbol{R}}$ lie in the interior of the same cone of C_{H_0} if and only if they define the same limit of $X \in H_0$.

We shall calculate the limit of X under a one-parameter subgroup $\lambda \in \Lambda(\boldsymbol{T})$. Consider the closed embedding

$$\boldsymbol{G}_m \times X \subset \boldsymbol{G}_m \times \boldsymbol{X},$$

$$(s, x) \to (s, \lambda(s) \cdot x).$$

Let Ξ be the closure of the image of this embedding in $\boldsymbol{A}^1 \times \boldsymbol{X}$ (so Ξ is a variety). Since the projection $p_{\boldsymbol{A}^1} : \Xi \to \boldsymbol{A}^1$ is a flat morphism, we have a morphism $\boldsymbol{A}^1 \to H_{\boldsymbol{X},T}$ such that $\Xi = \boldsymbol{W}_{\boldsymbol{X},T} \times_{H_{\boldsymbol{X},T}} \boldsymbol{A}^1$. Thus the limit X_λ of X under λ is equal to the fiber of $p_{\boldsymbol{A}^1}$ over 0 if this fiber is non-empty and the limit does not exist otherwise. Consider the commutative diagram:

$$\begin{array}{ccc} \Xi & \supset & \boldsymbol{G}_m \times X \\ \cap & & \cap \\ \boldsymbol{A}^1 \times \boldsymbol{X} & \supset & \boldsymbol{G}_m \times \boldsymbol{X}. \end{array}$$

We have the corresponding homomorphisms of algebras:

$$\begin{array}{ccc} k[\Xi] & \hookrightarrow & k[\boldsymbol{G}_m \times X] \\ \uparrow & & \uparrow \\ k[\boldsymbol{A}^1 \times \boldsymbol{X}] & \hookrightarrow & k[\boldsymbol{G}_m \times \boldsymbol{X}], \end{array}$$

where the vertical maps are surjective.

Denote by s the coordinate in $\boldsymbol{A}^1$. Then the homomorphism $k[\boldsymbol{G}_m \times \boldsymbol{X}] \twoheadrightarrow k[\boldsymbol{G}_m \times X]$ is given by $s \to s$ and $t^\nu \to s^{\langle \lambda, \nu \rangle} t^{\pi(\nu)}$. Thus, the vector subspace $k[\Xi] \subset k[\boldsymbol{G}_m \times X]$ is generated by the elements of the form $s^m t^{\pi(\nu)}$, where $m \geq \langle \lambda, \nu \rangle$, $\nu \in \Omega$. The fiber $p_{\boldsymbol{A}^1}^{-1}(0)$ is empty if and only if the ideal $sk[\Xi]$ contains 1. Thus λ belongs to the support of the fan $\mathcal{C}_{H_0}$ if and only if $\langle \lambda, \nu \rangle \leq 0$ for any $\nu \in \pi^{-1}(0) \cap \Omega$. Since any $k[\boldsymbol{X}]_\chi$ is a finitely generated $k[\boldsymbol{X}]_0$-module, this is equivalent to say that λ attains its minimum on $\pi^{-1}(\chi) \cap \Omega$ for any $\chi \in \Sigma$. In this case

$$k[X_\lambda] = \bigoplus_{\chi \in \Sigma} k s^{n_\lambda(\chi)} t^\chi ,$$

where

$$n_\lambda(\chi) := \min_{\nu \in \pi^{-1}(\chi) \cap \Omega} \langle \lambda, \nu \rangle .$$

The product $s^{n_\lambda(\chi_1)} t^{\chi_1} s^{n_\lambda(\chi_2)} t^{\chi_2} = s^{n_\lambda(\chi_1) + n_\lambda(\chi_2)} t^{\chi_1 + \chi_2}$ equals zero if and only if $n_\lambda(\chi_1) + n_\lambda(\chi_2) > n_\lambda(\chi_1 + \chi_2)$. The embedding of X_λ in $\boldsymbol{X}$ is given by the homomorphism of algebras $k[\boldsymbol{X}] \to k[X_\lambda]$, where t^ν, $\nu \in \Omega$, maps to $s^{n_\lambda(\pi(\nu))} t^{\pi(\nu)}$ if $\langle \lambda, \nu \rangle = n_\lambda(\chi)$, and to 0 otherwise. We denote by I_λ the kernel of this homomorphism. Hence we see that one-parameter subgroups λ_1 and λ_2 define the same limit if and only if $I_{\lambda_1} = I_{\lambda_2}$. This holds if and only if λ_1 and λ_2 attain the minimum over $\pi^{-1}(\chi) \cap \Omega$ at the same point for any $\chi \in \Sigma$ or, equivalently, λ_1 and λ_2 lie in the interior of the same cone of $N(P_\chi)$ for any $\chi \in \Sigma$.

(2) Since $\boldsymbol{W}_0 = \overline{\boldsymbol{T} \cdot (X, x)} \subset H_0 \times \boldsymbol{X}$, the second statement is evident. □

The statement below follows directly from the description of the limit of X under λ in the proof of the theorem.

REMARK 4.7. Let $\prec_\lambda$ be the preorder on $\mathcal{X}(\boldsymbol{T})$ such that $\nu_1 \prec_\lambda \nu_2$ if $\langle \lambda, \nu_1 \rangle \leq \langle \lambda, \nu_2 \rangle$. For any $f = \sum f_{\nu_i}$, $f_{\nu_i} \in k[\boldsymbol{X}]_{\nu_i}$, denote by $\mathrm{in}_\lambda(f)$ the sum of f_{ν_i}'s where ν_i is maximal with respect to $\prec_\lambda$. Then the limit of $I_X \in H_0$ under λ exists if and only if $\langle \lambda, \nu \rangle \geq 0$ for any $\nu \in \pi^{-1}(0) \cap \Omega$. In this case the limit is the ideal $\mathrm{in}_\lambda(I_X)$ generated by all $\mathrm{in}_\lambda(f)$, $f \in I_X$.

EXAMPLE 4.8. Let $\boldsymbol{X} = \boldsymbol{A}^n$, $\boldsymbol{T} = \boldsymbol{G}_m^n$ act on $\boldsymbol{A}^n$ by rescaling of coordinates, $T = \boldsymbol{G}_m$, and let the $\mathcal{X}(T)$-grading of $k[x_1, \ldots, x_n]$ be positive.

(1) Consider the case $n = 3$. It was proved by Arnold, Korkina, Post, Roelfols (see, for example, [12, Theorem 10.2]), that any ideal $I \in \underline{H_{A^n,T}}(k)$ is of the form $t \cdot in_\lambda(I_X)$ for some $t \in \boldsymbol{G}_m^n$ and $\lambda \in \Lambda(\boldsymbol{G}_m^n)$. This means that in this case the toric Hilbert scheme is irreducible.

(2) Let $n = 4$ and $\chi_1 = 1$, $\chi_2 = 3$, $\chi_3 = 4$, $\chi_4 = 7$. Then the toric Hilbert scheme is reducible. Moreover, in $H_{A^n,T}$ there are infinitely many orbits of $\boldsymbol{G}_m^n$ (see [12, Theorem 10.4]).

PROPOSITION 4.9. (1) *The support of any $\mathcal{C}_\chi$ is the cone generated by those one-parameter subgroups λ that $\langle\lambda, \nu\rangle \geq 0$ for any $\nu \in \pi^{-1}(0) \cap \Omega$. In particular, the grading of S by $\mathcal{X}(T)$ is positive if and only if this support is the whole space $\Lambda(\boldsymbol{T})_{\boldsymbol{R}}$, i.e., any polyhedron P_χ is a polytope. This holds if and only if H_0 is projective.*

(2) *There are only finitely many non-equivalent polyhedra P_χ for $\chi \in \Sigma$. Hence $\mathcal{C}_{H_0}$ is the normal fan of the Minkowski sum of representatives of the equivalence classes* (*we denote this sum by P_{H_0}*).

PROOF. (1) First note that P_0 is a cone and its normal cone $\mathcal{C}_0$ is generated by those one-parameter subgroups λ that $\langle\lambda, \nu\rangle \geq \langle\lambda, 0\rangle = 0$ for any $\nu \in \pi^{-1}(0) \cap \Omega$. Further, note that the recession cone of any P_χ is P_0. Indeed, S_χ is a finitely generated S_0-module. Let $\mu_1, \ldots, \mu_d \in \mathcal{X}(\boldsymbol{T})$ be the weights of a set of $\boldsymbol{T}$-semiinvariant generators. Then

$$P_\chi = \operatorname{conv}\left(\bigcup_{i=1}^{d}(\mu_i + P_0)\right) = \operatorname{conv}(\mu_1, \ldots, \mu_d) + P_0\,.$$

It follows that the support of $\mathcal{C}_\chi$ is $\mathcal{C}_0$.

If the support of $\mathcal{C}_{H_0}$ is not $\Lambda(\boldsymbol{T})_{\boldsymbol{R}}$, then H_0 is not complete and, consequently, is not projective. Conversely, if the grading is positive, then the Hilbert scheme $H_{\boldsymbol{X},T}$ is projective, and H_0 is projective.

(2) There are only finitely many fans $\mathcal{C}$ such that $\mathcal{C}_{H_0}$ is a refinement of $\mathcal{C}$ and the supports of $\mathcal{C}$ and $\mathcal{C}_{H_0}$ coincide. □

REMARK 4.10. By [12, Theorem 7.15], it follows that in the case when $\boldsymbol{X} = \boldsymbol{A}^n$, $\boldsymbol{T} = \boldsymbol{G}_m^n$ acts by rescaling of coordinates, and the $\mathcal{X}(T)$-grading of $k[\boldsymbol{X}]$ is positive, the polytope P_{H_0} is equivalent to the Minkowski sum of P_χ corresponding to the weights χ of the elements of the universal Gröbner basis of $I_{\boldsymbol{X}}$.

Let $\boldsymbol{X}$ be normal. Now we are going to give a precise description of those characters $\chi \in \Sigma$ having equivalent polyhedra P_χ. Recall that we have a homomorphism of lattices $\pi : \mathcal{X}(\boldsymbol{T}) \to \mathcal{X}(T)$, a finitely generated monoid $\Omega \subset \mathcal{X}(\boldsymbol{T})$ such that $\Omega = \operatorname{cone}(\Omega) \cap \mathcal{X}(\boldsymbol{T})$, and we put $\Sigma = \pi(\Omega)$. To any point $\chi \in \Sigma$ we associate the polyhedron

$$P_\chi = \operatorname{conv}(\pi^{-1}(\chi) \cap \Omega) \subset \mathcal{X}(\boldsymbol{T})_{\boldsymbol{R}}\,.$$

Two points $\chi, \chi' \in \Sigma$ are said to be *equivalent* if the corresponding polyhedra P_χ and $P_{\chi'}$ are equivalent. The question is to describe equivalence classes constructively.

Denote by $\pi_{\boldsymbol{R}}$ the linear map induced by π:

$$\pi_{\boldsymbol{R}} : \mathcal{X}(\boldsymbol{T})_{\boldsymbol{R}} \to \mathcal{X}(T)_{\boldsymbol{R}}\,.$$

Let $\mathcal{C}_\chi^{\boldsymbol{R}}$ denote the normal fan to the polyhedron

$$P_\chi^{\boldsymbol{R}} := \pi_{\boldsymbol{R}}^{-1}(\chi) \cap \operatorname{cone}(\Omega)\,.$$

DEFINITION 4.11. (See [4].) The *cell decomposition of* cone(Σ) *induced by* $\pi_{\boldsymbol{R}}$ is the subdivision of cone(Σ) into the following set of cones: the characters χ and χ' lie in the interior of the same cone of this decomposition if and only if the set of those faces of $\Omega_{\boldsymbol{R}_+}$ whose images under $\pi_{\boldsymbol{R}}$ contain χ coincides with the set of such faces for χ'.

REMARK 4.12. Note that the cell decomposition of $\operatorname{cone}(\Sigma)$ induced by $\pi_{\boldsymbol{R}}$ coincides with the subdivision by GIT-cones ([2, Section 2]).

Note that if χ lies in the interior of a cone σ of the cell decomposition and $\chi' \in \sigma$, then $\mathcal{C}_\chi^{\boldsymbol{R}}$ refines $\mathcal{C}_{\chi'}^{\boldsymbol{R}}$. In particular, the polyhedra $P_\chi^{\boldsymbol{R}}$ corresponding to interior points χ of σ are equivalent. Let $P_{\boldsymbol{R}}$ denote the Minkowski sum of $P_\chi^{\boldsymbol{R}}$ for representatives of interior points for all cones of the cell decomposition and let $\mathcal{C}_{\boldsymbol{R}}$ denote the normal fan to $P_{\boldsymbol{R}}$ (note that in the Minkowski sum it suffices to take representatives of interior points for the maximal cones of the cell decomposition).

REMARK 4.13. In [4] the fan $\mathcal{C}_{\boldsymbol{R}}$ is called the *fiber fan* by analogy with the normal fan of the fiber polytope for a linear projection of polytopes (see [3]).

DEFINITION 4.14. (See [8, Definition 5.4].) A character $\chi \in \Sigma$ is *integral* if the inclusion of the convex polyhedra $P_\chi \subseteq P_\chi^{\boldsymbol{R}}$ is an equality.

We shall denote by $\Sigma_{\boldsymbol{X}}^{\mathrm{int}}$ the set of integral characters. The following proposition gives us an algorithm for computing the fan of H_0.

PROPOSITION 4.15. *For any cone σ of the cell decomposition of* $\operatorname{cone}(\Sigma)$ *induced by π let $\mu_1, \dots, \mu_r$ be generators of the monoid $\sigma \cap \Sigma$ and let $c_1, \dots, c_r \in \boldsymbol{N}$ be such that $c_i\mu_i$ are integral, $i = 1, \dots, r$. Then, the polyhedra P_χ, where $\chi = \sum_{i=1}^r d_i\mu_i$ and $0 < d_i < l(\sigma)c_i$, form representatives of all equivalence classes of points in σ up to Minkowski sum with $P_{\boldsymbol{R}}$. Here $l(\sigma)$ is the number of vertices of $P_\chi^{\boldsymbol{R}}$ for χ lying in the interior of σ.*

Hence P_{H_0} is the Minkowski sum of such representatives for all (*maximal*) *cones σ of the cell decomposition of* $\operatorname{cone}(\Sigma)$ *induced by $\pi_{\boldsymbol{R}}$.*

PROOF. Consider a point χ lying in the interior of σ and the corresponding polyhedron $P_\chi^{\boldsymbol{R}}$. For any vertex v of $P_\chi^{\boldsymbol{R}}$ there exists a unique minimal face F of $\operatorname{cone}(\Omega)$ such that $F \cap P_\chi^{\boldsymbol{R}} = \{v\}$ (indeed, since $P_\chi^{\boldsymbol{R}}$ is the intersection of $\operatorname{cone}(\Omega)$ with the affine subspace $\pi_{\boldsymbol{R}}^{-1}(\chi)$, it follows that any face of $P_\chi^{\boldsymbol{R}}$ is the intersection of $\pi_{\boldsymbol{R}}^{-1}(\chi)$ with some face of $\operatorname{cone}(\Omega)$). Let $v_1^\chi, \dots, v_{l(\sigma)}^\chi \in \mathcal{X}(\boldsymbol{T})_{\boldsymbol{R}}$ be the vertices of $P_\chi^{\boldsymbol{R}}$ and let $F_1^\sigma, \dots, F_{l(\sigma)}^\sigma$ be the corresponding faces (the set of such faces does not depend on a point χ in the interior of σ). Note also that the intersection $F_i \cap P_\chi^{\boldsymbol{R}}$ is a vertex of $P_\chi^{\boldsymbol{R}}$ for any $\chi \in \sigma$. Just as above, we denote this vertex by v_i^χ. For two vectors $u, u' \in \mathcal{X}(\boldsymbol{T})_{\boldsymbol{R}}$ we say $u \prec u'$ if $u' - u \in \operatorname{cone}(\Omega)$.

Let us show that if $\chi = \sum_{i=1}^r d_i\mu_i$ lies in the interior of σ and there exists i such that $d_i \geq c_i l(\sigma)$, then

$$(**) \quad P_\chi = P_{\chi - c_i\mu_i} + P_{c_i\mu_i}\,.$$

Indeed, the inclusion $P_{\chi - c_i\mu_i} + P_{c_i\mu_i} \subseteq P_\chi$ is evident. For the converse, it is sufficient to show that $P_\chi \cap \Omega \subset P_{\chi - c_i\mu_i} + P_{c_i\mu_i}$. Note that $P_\chi \cap \Omega = P_\chi^{\boldsymbol{R}} \cap \Omega$. Denote by D_χ the convex hull of the v_i^χ, $i = 1, \dots, l(\sigma)$. By Proposition 4.9 (1), $P_\chi^{\boldsymbol{R}} = D_\chi + P_0$. Then for any $v \in P_\chi \cap \Omega$ we have $v = u + v_0$ for some $v_0 \in P_0, u \in D_\chi$ and $u = \sum_{j=1}^{l(\sigma)} q_j v_j^\chi$ for some $q_j \geq 0$ such that $\sum_{j=1}^{l(\sigma)} q_j = 1$. There exists j such that $q_j \geq 1/l(\sigma)$. Hence $v \succ q_j v_j^\chi = q_j(v_j^{\chi - c_i\mu_i} + v_j^{c_i\mu_i}) \succ v_j^{c_i\mu_i}$. Thus $v - v_j^{c_i\mu_i} \in \mathrm{cone}(\Omega) \cap \mathcal{X}(\boldsymbol{T}) = \Omega$.

In particular, this implies that for any χ in the interior of σ the polyhedron $P_\chi + P_{\boldsymbol{R}}$ is equivalent to $P_{\chi'} + P_{\boldsymbol{R}}$ for some $\chi' = \sum_{i=1}^r d_i\mu_i$ such that $d_i < c_i l(\sigma)$ for any i. The second statement of the proposition is evident. □

COROLLARY 4.16. *With the preceding notation, if $\chi = \sum_{i=1}^r d_i\mu_i$ lies in the interior of σ and there exists i such that $d_i \geq c_i l(\sigma)$, then P_χ is equivalent to $P_{\chi+c_i\mu_i}$.*

PROOF. By $(**)$, it follows that $P_{\chi+c_i\mu_i} = P_{\chi-c_i\mu_i} + 2P_{c_i\mu_i}$ is equivalent to P_χ. □

EXAMPLE 4.17. Let $\boldsymbol{X} = \boldsymbol{A}^n$, $\boldsymbol{T} = \boldsymbol{G}_m^n$ act be rescaling of coordinates, and let $T = \boldsymbol{G}_m$ act on $\boldsymbol{A}^n$ with characters $\chi_1, \ldots, \chi_n \in \boldsymbol{Z}$. Then $\Omega \subset \boldsymbol{Z}^n$ is the set of vectors with integral non-positive coordinates, and $\Sigma \subset \boldsymbol{Z}$ is the monoid generated by $-\chi_i$. Moreover, $\Sigma = (\Sigma \cap \boldsymbol{Z}_+) \cup (\Sigma \cap \boldsymbol{Z}_-)$ is the subdivision of Σ induced by π. Let n_+ and n_- be the numbers of positive and negative χ_i respectively. A number $\chi \in \boldsymbol{Z}_+$ (resp. $\boldsymbol{Z}_-$) is integral (in the sense of Definition 4.14) if and only if χ is divisible by any $\chi_i < 0$ (resp. > 0). Let χ_+ (resp. χ_-) be the least common (positive) multiple of all positive (resp. negative) χ_i. Then P_{H_0} is the Minkowski sum of polyhedra P_χ for $-n_+\chi_+ < \chi < n_-\chi_-$.

5. Toric Chow morphism. We are going to describe the toric Chow morphism from the Hilbert scheme to the inverse limit of GIT quotients $\boldsymbol{X}/\!_\chi T$. In [8, Section 5] the toric Chow morphism was constructed in the case when $\boldsymbol{X} = \boldsymbol{A}^n$ is a T-module. We generalize this to the case of a normal affine toric $\boldsymbol{T}$-variety $\boldsymbol{X}$.

In this section we fix a $\boldsymbol{T}$-equivariant closed embedding $\boldsymbol{X} \hookrightarrow V$, where V is a finite-dimensional $\boldsymbol{T}$-module such that $\boldsymbol{X}$ is not contained in a proper $\boldsymbol{T}$-submodule. We use the notations of the previous sections. Let

$$S^{(\chi)} := \bigoplus_{r=0}^{\infty} S_{r\chi}\,,$$

and let

$$\boldsymbol{X}/\!_\chi T := \mathrm{Proj}\ S^{(\chi)}$$

be the GIT quotient. In particular, $\boldsymbol{X}/_0 T = \boldsymbol{X}/\!/T = \mathrm{Spec}(S_0)$. Notice also that $\boldsymbol{X}/\!_\chi T = \boldsymbol{X}_\chi^{ss}/\!/T$, where

$$\boldsymbol{X}_\chi^{ss} := \{x \in \boldsymbol{X}\,;\, f(x) \neq 0 \text{ for some homogeneous } f \in S^{(\chi)}\}\,.$$

If χ lies in the interior of $\mathrm{cone}(\Sigma)$, then $\boldsymbol{X}/\!_\chi T$ is a normal toric $\boldsymbol{T}/T$-variety whose fan is $\mathcal{C}_\chi^{\boldsymbol{R}}$, the normal fan to the polyhedron $P_\chi^{\boldsymbol{R}}$.

It is easy to see that for any $\chi_1, \chi_2 \in \Sigma$, the inclusion $\boldsymbol{X}_{\chi_1}^{ss} \subseteq \boldsymbol{X}_{\chi_2}^{ss}$ holds if and only if χ_1 belongs to the cone of the cell decomposition of $\mathrm{cone}(\Sigma)$ induced by π (see Definition 4.11) containing χ_2 in its interior. We consider the morphisms between GIT-quotients $\boldsymbol{X}/\!\!/_{\chi}T$ induced by inclusions between $\boldsymbol{X}^{ss}_{\chi}$, where $\chi \in \Sigma$. So the GIT-quotients $\boldsymbol{X}/\!\!/_{\chi}T$ form a finite inverse system with $\boldsymbol{X}/\!\!/T$ sitting at the end. Consider the inverse limit

$$\boldsymbol{X}/\!\!/_{C}T := \varprojlim\{\boldsymbol{X}/\!\!/_{\chi}T;\ \chi \text{ lies in the interior of } \Sigma\}\,.$$

It is a closed subscheme in the product $\boldsymbol{X}/\!\!/_{\chi_1}T \times \cdots \times \boldsymbol{X}/\!\!/_{\chi_r}T$, where $\chi_1, \ldots, \chi_r$ are representatives of interior points of all maximal cones of the cell decomposition of $\mathrm{cone}(\Sigma)$ induced by π. Note also that $\boldsymbol{X}/\!\!/_{C}T$ is a closed subscheme in $V/\!\!/_{C}T$.

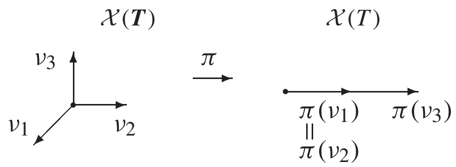
DEFINITION 5.1. The *main component* $(\boldsymbol{X}/\!\!/_{C}T)_0$ of the inverse limit $\boldsymbol{X}/\!\!/_{C}T$ is the closure of the image of the map $\boldsymbol{T} \to \boldsymbol{X}/\!\!/_{C}T$ induced by the maps $\boldsymbol{T} \to \boldsymbol{X}/\!\!/_{\chi}T$, where χ lies in the interior of Σ.

By [4, Proposition 3.8], it follows that the main component $(\boldsymbol{X}/\!\!/_{C}T)_0$ is an irreducible component of $\boldsymbol{X}/\!\!/_{C}T$ which satisfies the following universal property: given a $\boldsymbol{T}/T$-variety Y containing an irreducible component Y_0 such that Y_0 is a toric $\boldsymbol{T}/T$-variety, and given $\boldsymbol{T}/T$-equivariant morphisms $\phi_{\chi} : Y \to \boldsymbol{X}/\!\!/_{\chi}T$, where χ lies in the interior of Σ, such that the ϕ_{χ} induce birational morphisms $Y_0 \to \boldsymbol{X}/\!\!/_{\chi}T$ and the ϕ_{χ} are compatible with the morphisms of the inverse system (so the ϕ_{χ} give a morphism $\phi : Y \to \boldsymbol{X}/\!\!/_{C}T$); then restricting the morphism ϕ to Y_0 we have a birational morphism of toric $\boldsymbol{T}/T$-varieties $Y_0 \to (\boldsymbol{X}/\!\!/_{C}T)_0$.

REMARK 5.2. By [4, Proposition 3.10], it follows that the fan of $(\boldsymbol{X}/\!\!/_{C}T)_0$ is $\mathcal{C}_{\boldsymbol{R}}$, the maximal common refinement of all the normal fans to the polyhedra $P^{\boldsymbol{R}}_{\chi}$, $\chi \in \Sigma$. Since every character $\chi \in \Sigma$ has some integral positive multiple $c\chi \in \Sigma^{\mathrm{int}}_{\boldsymbol{X}}$ ($c \in \boldsymbol{N}$), the fan $\mathcal{C}_{H_0}$ is a refinement of the fan $\mathcal{C}_{\boldsymbol{R}}$.

The following example shows that $\mathcal{C}_{H_0}$ and $\mathcal{C}_{\boldsymbol{R}}$ do not always coincide.

EXAMPLE 5.3. Let $\boldsymbol{X} = \boldsymbol{A}^3$, $\boldsymbol{T} = \boldsymbol{G}_m^3$ act by rescaling of coordinates, and let $T = \boldsymbol{G}_m$ act by $t(x_1, x_2, x_3) = (tx_1, tx_2, t^2x_3)$.



The Hilbert scheme $H_{\boldsymbol{A}^3,T}$ is the closed subscheme in $\boldsymbol{P}^1 \times \boldsymbol{P}^3$ defined by the equations $z_1w_3 - z_2w_1 = 0$ and $z_1w_2 - z_2w_3 = 0$ (where z_1, z_2 and w_1, w_2, w_3, w_4 are homogeneous coordinates in $\boldsymbol{P}^1$ and $\boldsymbol{P}^3$ respectively). The integral (in the sense of Definition 4.14) degrees are even. The fan $\mathcal{C}_{H_0}$ consists of the following cones:

$$\boldsymbol{R}_+(e_1 + e_2) + \boldsymbol{R}_+e_2\,,$$

$$\boldsymbol{R}_+(e_1 + e_2) + \boldsymbol{R}_+(-e_2)\,,$$

$$\boldsymbol{R}_+(e_2 - e_1) + \boldsymbol{R}_+e_2\,,$$

$$\boldsymbol{R}_+(e_2 - e_1) + \boldsymbol{R}_+(-e_2)\,,$$

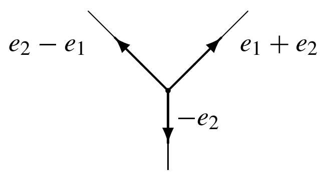
where $e_1 = \nu_1^* + \nu_3^*$, $e_2 = -\nu_3^*$ is a basis of $\Lambda(\boldsymbol{T}/T)$. The inverse limit of GIT-quotients is $\boldsymbol{A}^3/\!\!/_{C} T = \operatorname{Proj}\ k[x_1, x_2, x_3]$ (where $k[x_1, x_2, x_3]$ is graded by the weights of T), and its fan $\mathcal{C}_{\boldsymbol{R}}$ consists of the following cones:

$$\boldsymbol{R}_+(e_1 + e_2) + \boldsymbol{R}_+(-e_2)\,,$$

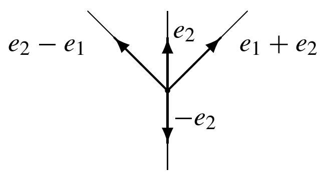
$$\boldsymbol{R}_+(e_2 - e_1) + \boldsymbol{R}_+(-e_2)\,,$$

$$\boldsymbol{R}_+(e_1 + e_2) + \boldsymbol{R}_+(e_2 - e_1)\,.$$

$\mathcal{C}_{\boldsymbol{R}}$



$\mathcal{C}_{H_0}$



By the statement $(**)$ from the proof of Proposition 4.15, it follows that if a character $\chi \in \Sigma$ is integral, then there exists r_0 such that $S_{(r+1)\chi} = S_\chi S_{r\chi}$ for all $r \geq r_0$. Thus Corollary 3.5 implies that

$$H_{S^{(\chi)},T} = \operatorname{Proj} S^{(\chi)} = \boldsymbol{X}/\!\!/_{\chi} T\,,$$

for any $\chi \in \Sigma_{\boldsymbol{X}}^{\mathrm{int}}$.

For any subset $D \subset \Sigma$ we can consider the restriction of the Hilbert scheme $H_{\boldsymbol{X},T}$ on degrees D, that is, the quasiprojective scheme $H_{\boldsymbol{X},T}^D$ representing the covariant functor

$$\underline{H_{\boldsymbol{X},T}^D} : \underline{k - Alg} \to \underline{Set}$$

such that $\underline{H_{\boldsymbol{X},T}^D}(R)$ is the set of families $\{L_\chi\}_{\chi \in D}$, where $L_\chi \subset R \otimes_k S_\chi$ is an R-submodule, such that $(R \otimes_k S_\chi)/L_\chi$ is a locally free R-module of rank 1 and $f L_{\chi_2} \subset L_{\chi_1}$ for any $\chi_1, \chi_2 \in D$ and any $f \in S_{\chi_1 - \chi_2}$ (see [8, Section 2]). In particular, $H_{\boldsymbol{X},T}^{\Sigma} = H_{\boldsymbol{X},T}$ and $H_{S^{(\chi)},T} = H_{\boldsymbol{X},T}^{D^\chi}$, where $D^\chi := \{c\chi\ ;\ c \in \boldsymbol{Z}_+\}$. Note also that $H_{\boldsymbol{X},T}^D$ is a closed subscheme of $H_{V,T}^D$. For any $D \subset \Sigma$ we have a degree restriction morphism $H_{\boldsymbol{X},T} \to H_{\boldsymbol{X},T}^D$. In particular, we have canonical morphisms

$$\phi_{\boldsymbol{X}}^\chi : H_{\boldsymbol{X},T} \to \boldsymbol{X}/\!\!/_{\chi} T\,.$$

The following theorem was proved in [8, Theorem 5.6] for the case when $\boldsymbol{X}$ is a finite-dimensional T-module.

THEOREM 5.4. *Let $H_{\boldsymbol{X},T}^{\mathrm{int}} := H_{\boldsymbol{X},T}^{\Sigma_{\boldsymbol{X}}^{\mathrm{int}}}$ be the toric Hilbert scheme restricted to the set of integral degrees. Then there is a canonical morphism*

$$\phi_{\boldsymbol{X}}^{\mathrm{int}} : H_{\boldsymbol{X},T}^{\mathrm{int}} \to \boldsymbol{X}/\!\!/_{C} T$$

which induces an isomorphism of the corresponding reduced schemes. In particular, composing ϕ_X^{int} with the degree restriction morphism, we obtain a canonical Chow morphism from the toric Hilbert scheme to the inverse limit of the GIT quotients

$$\phi_X : H_{X,T} \to X/_C T .$$

PROOF. As in [8, Lemma 5.7], we see that the morphisms ϕ_X^χ satisfy the compatibility conditions for $\chi \in \Sigma_X^{\mathrm{int}}$ and, consequently, give a canonical morphism

$$H_{X,T} \to H_{X,T}^{\mathrm{int}} \xrightarrow{\phi_X^{\mathrm{int}}} X/_C T .$$

Further, note that for any algebra R the morphism

$$\underline{\phi_X^{\mathrm{int}}}(R) : \underline{H_{X,T}^{\mathrm{int}}}(R) \to \underline{X/_C T}(R)$$

is injective (since $H_{S^{(\chi)},T} = X/\!\!/_\chi T$, we view any element of $\underline{X/_C T}(R)$ as a family of R-submodules $\{I_\chi \subset R \otimes S_\chi\}_{\chi \in \Sigma_X^{\mathrm{int}}}$ such that $(R \otimes S_\chi)/I_\chi$ is a locally free R-module of rank 1, $I^{(\chi)} := \bigoplus_{n \geq 0} I_{n\chi}$ is an ideal in $R \otimes S^{(\chi)}$, so it defines a point of $\underline{H_{S^{(\chi)},T}}(R)_{\chi \in \Sigma_X^{\mathrm{int}}}$, and these points satisfy the compatibility conditions of the direct system). Hence to prove that ϕ_X^{int} induces an isomorphism of the reduced schemes, it suffices to show that $\underline{\phi_X^{\mathrm{int}}}(R)$ is surjective for any reduced R.

Note that ϕ_X^χ coincides with the restriction of ϕ_V^χ to $H_{X,T} \subset H_{V,T}$ for any $\chi \in \Sigma_V^{\mathrm{int}} \subset \Sigma_X^{\mathrm{int}}$. By [8, Theorem 5.6], the map $\underline{\phi_V^{\mathrm{int}}}(R)$ is surjective for any reduced R, and it follows that any element $\{I_\chi\}_{\chi \in \Sigma_X^{\mathrm{int}}}$ in $\underline{X/_C T}(R) \subset \underline{V/_C T}(R)$ gives an element $\{I_\chi\}_{\chi \in \Sigma_V^{\mathrm{int}}}$ in $\underline{H_{V,T}^{\mathrm{int}}}(R)$, i.e., $fI_{\chi_2} \subset I_{\chi_1}$ for any $\chi_1, \chi_2 \in \Sigma_V^{\mathrm{int}}$ and any $f \in S_{\chi_1 - \chi_2}$. We have to prove that this condition holds for any $\chi_1, \chi_2 \in \Sigma_X^{\mathrm{int}}$. There exists $c \in N$ such that $c\chi_1, c\chi_2 \in \Sigma_V^{\mathrm{int}}$. For any $f' \in I_{\chi_2}$ we have $f^c(f')^c \in I_{c\chi_1}$. By Lemma 3.4, we see that the projection of $\mathrm{Spec}((R \otimes_k S^{(\chi_1)})/I^{(\chi_1)})$ to $\mathrm{Spec}\, R$ is a locally trivial bundle with fiber A^1. Consequently, $(R \otimes_k S^{(\chi_1)})/I^{(\chi_1)}$ is reduced and $ff' \in I^{(\chi_1)}$. □

REMARK 5.5. Note that restricting ϕ_X to the main component H_0, we obtain a birational morphism of toric T/T-varieties from H_0 to $(X/_C T)_0$.

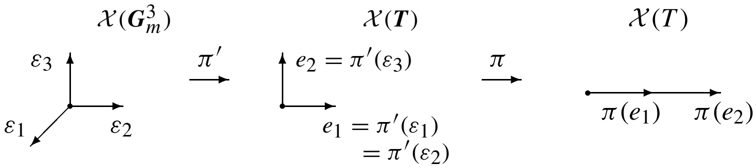
EXAMPLE 5.6. Let $V = A^3$ where G_m^3 and $T = G_m$ act as in Example 5.3, and let $T = G_m^2$ be embedded in G_m^3 by $(t_1, t_2) \to (t_1, t_1, t_2)$. Consider the variety $X = \overline{T \cdot (1,1,1)} = \mathrm{Spec}\, S$, where $S = k[x_1, x_2, x_3]/I_X$ and $I_X = (x_1 - x_2)$. So $H_{X,T}$ is defined in $H_{A^3,T}$ by the equation $z_1 = z_2$. We have the homomorphisms of groups of characters

$$Z^3 = \mathcal{X}(G_m^3) \xrightarrow{\pi'} Z^2 = \mathcal{X}(T) \xrightarrow{\pi} Z = \mathcal{X}(T)$$

and of monoids

$$\Omega_{A^3} \to \Omega \to \Sigma ,$$

where Ω_{A^3} is the monoid in $\mathcal{X}(G_m^3)$ generated by characters with negative coordinates.



Note that $\Sigma_{\boldsymbol{X}}^{\text{int}} = \Sigma_{\boldsymbol{A}^3}^{\text{int}}$ is the set of even numbers. The scheme $H_{\boldsymbol{A}^3,T}^{\text{int}}$ is the closed subscheme in $\boldsymbol{P}^3$ defined by the equation $w_3^2 = w_1 w_2$, and $H_{\boldsymbol{X},T}^{\text{int}}$ is defined by the equations $w_1 = w_2 = w_3$. The isomorphism

$$\phi_{\boldsymbol{X}}^{\text{int}} : H_{\boldsymbol{X},T}^{\text{int}} \to \boldsymbol{X} /\!\!/_C T = \operatorname{Proj} S$$

is the restriction of the isomorphism

$$\phi_{\boldsymbol{A}^3}^{\text{int}} : H_{\boldsymbol{A}^3,T}^{\text{int}} \to \boldsymbol{A}^3 /\!\!/_C T = \operatorname{Proj} k[x_1, x_2, x_3]\,,$$

where the inverse isomorphism is given by

$$(\phi_{\boldsymbol{A}^3}^{\text{int}})^{-1}(x_1 : x_2 : x_3) = (x_1^2 : x_2^2 : x_1 x_2 : x_3)\,.$$

Concerning the morphism $\phi_{\boldsymbol{A}^3} : H_{\boldsymbol{A}^3,T} \to \boldsymbol{A}^3 /\!\!/_C T$, note that $\phi_{\boldsymbol{A}^3}^{-1}(\boldsymbol{X} /\!\!/_C T)$ is not contained in $H_{\boldsymbol{X},T}$. Indeed, consider the ideal $I = (x_1, x_2^2) \in \underline{H_{\boldsymbol{A}^3,T}}(k)$. We have $(I_{\boldsymbol{X}})_r \subset I_r$ for any even r, so $\phi_{\boldsymbol{A}^3}(I) \in \underline{\boldsymbol{X} /\!\!/_C T}(k)$, but $I \notin \underline{H_{\boldsymbol{X},T}}(k)$.

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