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ON STABILITY OF HOLOMORPHIC MAPS OF COMPACT COMPLEX MANIFOLDS

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Introduction. Let V and W be connected compact complex manifolds. According to Douady [2], the set $H(V, W)$ of all holomorphic maps of V into W admits an analytic space*) structure whose underlying topology is the compact-open topology. For $f \in H(V, W)$, the Zariski tangent space $T_f H(V, W)$ to $H(V, W)$ at f is canonically isomorphic to a subspace of $H^0(V, f^*TW)$, the zero-th cohomology group of (the sheaf of holomorphic sections of) the pull back f^*TW of the holomorphic tangent bundle TW of W over f. (See §1.)

Now, we say that $f \in H(V, W)$ is *stable* if and only if there is an open neighbourhood U of f in $H(V, W)$ such that, for each $g \in U$, there are automorphisms (holomorphic isomorphisms) a of V and b of W with $g = bfa$.

We also say that a map $f \in H(V, W)$ is *infinitesimally stable* if and only if

$$f^*H^0(W, TW) + f_*H^0(V, TV) = H^0(V, f^*TW)\ ,$$

where

$$f^*\colon H^0(W, TW) \to H^0(V, f^*TW)\ ,$$
$$f_*\colon H^0(V, TV) \to H^0(V, f^*TW)$$

be the induced linear maps defined by $f^*(\eta) = \eta f$, for $\eta \in H^0(W, TW)$ and $f_*(\xi) = (df)(\xi)$, for $\xi \in H^0(V, TV)$.

The purpose of this paper is to prove the following theorem. (cf., Mather [5]).

THEOREM 1. *A holomorphic map f of V into W is infinitesimally stable if and only if* (1) *f is stable and* (2) *the Zariski tangent space $T_f H(V, W)$ to $H(V, W)$ at f is isomorphic to $H^0(V, f^*TW)$.*

As an example, let V be a compact Riemann surface and let $\boldsymbol{P}^1$ be the complex projective line. Then $H(V, \boldsymbol{P}^1)$ is the set of all algebraic

*) By *an analytic space*, we mean a reduced, Hausdorff, complex analytic space.

functions on V. We prove:

THEOREM 2. *Any algebraic function of order* 2 *on* V *is stable.*

1. Infinitesimal displacement maps. Let V, W and $H(V, W)$ be as above. The map

$$F: (P, f) \in V \times H(V, W) \to f(P) \in W$$

is holomorphic. Following Kodaira [4], we define a linear map

$$\sigma_f: T_f H(V, W) \to H^0(V, f^* TW)$$

by $\sigma_f(\partial/\partial s) = (\partial F/\partial s)_f$, where s is a coordinate in an ambient space of $H(V, W)$ around f. We call σ_f *the infinitesimal displacement map at* f.

The map σ_f is injective. In fact, as was shown in [6], (considering the graph Γ_f of f), the analytic space $H(V, W)$ is locally (around f) identified with

$$S = \{\xi \in B \mid K(\xi) = 0\} ,$$

where B is an open neighbourhood of 0 in $H^0(V, f^* TW)$ and K is a holomorphic map of B into $H^1(V, f^* TW)$, the first cohomology group of (the sheaf holomorphic sections of) $f^* TW$. Moreover, it is easy to see that σ_f is identified with the inclusion map $T_0 S \subset H^0(V, f^* TW)$.

Note that the automorphism groups $\mathrm{Aut}(V)$ of V and $\mathrm{Aut}(W)$ of W are open (and closed*)) in $H(V, V)$ and in $H(W, W)$, respectively. We denote by e and e' the identities of $\mathrm{Aut}(V)$ and $\mathrm{Aut}(W)$, respectively. Then the infinitesimal displacement maps

$$\sigma_e: T_e \mathrm{Aut}(V) \to H^0(V, TV) ,$$
$$\sigma_{e'}: T_{e'} \mathrm{Aut}(W) \to H^0(W, TW)$$

are linear isomorphisms. In fact, each $\xi \in H^0(V, TV)$(resp. $\eta \in H^0(W, TW)$) defines the one-parameter group $\exp t\xi$, $t \in \boldsymbol{C}$, (resp. $\exp t\eta$, $t \in \boldsymbol{C}$) of automorphisms of V (resp. W). We have then

$$\left.\frac{d \exp t\xi}{dt}\right|_{t=0} = \xi \qquad \left(\text{resp. } \left.\frac{d \exp t\eta}{dt}\right|_{t=0} = \eta\right) .$$

Now, for $f \in H(V, W)$, we define a holomorphic map

$$A_f: \mathrm{Aut}(W) \times \mathrm{Aut}(V) \to H(V, W)$$

by $A_f(b, a) = bfa$, for $(b, a) \in \mathrm{Aut}(W) \times \mathrm{Aut}(V)$.

LEMMA 1. *The following diagram is commutative:*

*) Using Hurwitz's theorem, we can easily show that $\mathrm{Aut}(V)$ is closed in $H(V, V)$.

$$\begin{array}{ccc} T_{e'}\operatorname{Aut}(W) \times T_e \operatorname{Aut}(V) & \xrightarrow{(dA_f)_{(e',e)}} & T_f H(V, W) \\ \sigma_{e'} \times \sigma_e \Big\downarrow & & \sigma_f \Big\downarrow \\ H^0(W, TW) \times H^0(V, TV) & \xrightarrow{f^* + f_*} & H^0(V, f^*TW)\,, \end{array}$$

where $(f^* + f_*)(\eta, \xi) = f^*\eta + f_*\xi$, *for* $(\eta, \xi) \in H^0(W, TW) \times H^0(V, TV)$.

PROOF. Let $\{U_i\}_{i\in I}$ and $\{U_i'\}_{i\in I}$ be finite open coverings of V such that, for each $i \in I$,

(1) $U_i \Subset U_i'$ (i.e., the closure $\bar{U}_i$ is compact and is contained in U_i'),

(2) there is on U_i' a coordinate system

$$z_i = (z_i^1, \cdots, z_i^d)\ .$$

Let $\{W_i\}_{i\in I}$ and $\{W_i'\}_{i\in I}$ be finite open coverings of $f(V)$ such that, for each $i \in I$,

(3) $f(U_i') \Subset W_i$,

(4) $W_i \Subset W_i'$,

(5) there is on W_i' a coordinate system

$$w_i = (w_i^1, \cdots, w_i^r)\ .$$

The map f is expressed by the equations

$$w_i = f_i(z_i),\ z_i \in U_i',\ i \in I\ ,$$

where f_i is a vector valued holomorphic function on U_i'.

Let ε be a small positive number. We denote by B_ε the ε-disc in C with the center 0. For $\xi \in H^0(V, TV)$ and $\eta \in H^0(W, TW)$, the one-parameter families $\exp t\xi$, $t \in B_\varepsilon$, and $\exp s\eta$, $s \in B_\varepsilon$, of automorphisms of V and W, respectively, are expressed as

$$(z_i, t) \in U_i \times B_\varepsilon \to a_i(z_i, t) \in U_i'\ ,$$
$$(w_i, s) \in W_i \times B_\varepsilon \to b_i(w_i, s) \in W_i'\ ,$$

respectively. Then the map $\exp s\eta\, f \exp t\xi$, $(s, t) \in B_\varepsilon \times B_\varepsilon$, is expressed by the equations:

$$w_i = b_i(f_i(a_i(z_i, t)), s)\ , \quad \text{for} \quad (z_i, s, t) \in U_i \times B_\varepsilon \times B_\varepsilon\ .$$

Hence we have

(6) $(\partial w_i/\partial t)_{t=0} = (\partial f_i/\partial z_i)_{z_i}(\partial a_i/\partial t)_{t=0} = (\partial f_i/\partial z_i)_{z_i}\xi_i(z_i)$,

(7) $(\partial w_i/\partial s)_{s=0} = \eta_i(f_i(z_i))$,

where $\xi = \{\xi_i(z_i)\}_{i\in I}$ and $\eta = \{\eta_i(w_i)\}_{i\in I}$.

Now, (6) and (7) imply Lemma 1. q.e.d.

2. Proof of Theorem 1. Assume that f is infinitesimally stable.

Then, by Lemma 1, σ_f is a linear isomorphism and $(dA_f)_{(e',e)}$ is surjective. Since $\mathrm{Aut}(W) \times \mathrm{Aut}(V)$ is non-singular, this implies that $H(V, W)$ is non-singular at f and A_f is a local submersion around (e', e). Hence f is stable and $T_f H(V, W)$ is isomorphic to $H^0(V, f^* TW)$.

Conversely, assume that f is stable and $T_f H(V, W)$ is isomorphic to $H^0(V, f^* TW)$. Since f is stable, $H(V, W)$ is non-singular at f. Assume that f is not infinitesimally stable. Then, by Lemma 1, $(dA_f)_{(e',e)}$ is not surjective. Let M be a complex submanifold through f of an open neighbourhood of f in $H(V, W)$ such that

$$(dA_f)_{(e',e)}(T_{e'} \mathrm{Aut}(W) \times T_e \mathrm{Aut}(V)) \oplus T_f M = T_f H(V, W) .$$

Then $\dim M > 0$.

LEMMA 2. *If M is sufficiently small, then $M \cap A_f$ $(\mathrm{Aut}(W) \times \mathrm{Aut}(V))$ is at most countable.*

PROOF OF LEMMA 2. (cf., Chevalley [1], p. 95). We put

$$G = \mathrm{Aut}(W) \times \mathrm{Aut}(V) ,$$
$$I_f = \{(b, a) \in G \mid bfa = f\} .$$

Then I_f is a closed submanifold of G through (e', e).

Let L be a closed submanifold of a small open neighbourhood of (e', e) in G passing (e', e) such that

$$T_{(e',e)} I_f \oplus T_{(e',e)} L = T_{(e',e)} G .$$

We define a holomorphic map

$$A: G \times H(V, W) \rightarrow H(V, W)$$

by $A((b, a), g) = bga$, for $((b, a), g) \in G \times H(V, W)$. Then

$$(dA)_{((e',e),f)}: T_{(e',e)} L \oplus T_f M \rightarrow T_f H(V, W)$$

is a linear isomorphism. Hence, we may assume that

$$A: L \times M \rightarrow U$$

is a holomorphic isomorphism, where U is an open neighbourhood of f in $H(V, W)$.

For $m \in M$, we put

$$S_m = \{bma \mid (b, a) \in L\} .$$

Then it is easy to see that, for any $m \in M$, either $S_m \cap A_f(G)$ is empty or $S_m \subset A_f(G)$. It is also easy to see that, for any compact subset K of G,

$$\{m \in M \mid S_m \cap A_f(K) \text{ is not empty}\}$$

is a finite set.

Now, G satisfies the second axiom of countability. Hence G is written as

$$G = \bigcup_{n=1}^{\infty} K_n ,$$

where each K_n is a compact subset of G. Then

$$A_f(G) = \bigcup_{n=1}^{\infty} A_f(K_n) .$$

Hence, noting that $m \in S_m$ for $m \in M$, we have

$$\begin{aligned} M \cap A_f(G) &= \left\{m \in M \mid S_m \cap \left(\bigcup_{n=1}^{\infty} A_f(K_n)\right) \text{ is not empty}\right\} \\ &= \{m \in M \mid S_m \cap A_f(K_n) \text{ is not empty for some } n\} \\ &= \bigcup_{n=1}^{\infty} M_n , \end{aligned}$$

where $M_n = \{m \in M \mid S_m \cap A_f(K_n) \text{ is not empty}\}$ is a finite set, for each $n = 1, 2, \cdots$. Thus $M \cap A_f(G)$ is at most countable. q.e.d. of Lemma 2

Now, by Lemma 2, for any open neighbourhood U of f in $H(V, W)$,

$$U \cap M - A_f(\mathrm{Aut}(W) \times \mathrm{Aut}(V))$$

is not empty. This shows that f is not stable, a contradiction.

This completes the proof of Theorem 1.

3. Proof of Theorem 2. Let V be a compact Riemann surface of genus g. Let $\boldsymbol{P}^1$ be the complex projective line. Then $H(V, \boldsymbol{P}^1)$ is the set of all algebraic functions on V.

LEMMA 3. *Let f be an algebraic function on V. Then*

$$f^* T\boldsymbol{P}^1 = [2D_\infty] ,$$

where D_∞ is the polar divisor of f and $[2D_\infty]$ is the line bundle determined by the divisor $2D_\infty$. Moreover, if f is of order $n \geqq g$, then

$$\dim H^0(V, f^* T\boldsymbol{P}^1) = 2n + 1 - g .$$

PROOF. Let (z_0, z_1) be the standard homogeneous coordinate on $\boldsymbol{P}^1$. Then we have easily $T\boldsymbol{P}^1 = [2(\infty)]$, where (∞) is the divisor of the point $\infty = (0, 1) \in \boldsymbol{P}^1$. Hence

$$f^* T\boldsymbol{P}^1 = f^*[2(\infty)] = [2D_\infty] .$$

If f is of order $n \geqq g$, then

$$\deg(2D_\infty) = 2n > 2g - 2 .$$

Hence, by Riemann-Roch Theorem,

$$\dim H^0(V, f^* TP^1) = \dim H^0(V, [2D_\infty]) = 2n + 1 - g .$$

q.e.d.

Now, we prove Theorem 2 by dividing into three cases.

Case 1: $g \geqq 2$. Let f be an algebraic function of order 2 on V. In this case, V is a hyperelliptic Riemann surface. f is a two sheeted ramified covering of V onto P^1 with $(2g+2)$-branch points. Let P be one of the branch points. Let b be an automorphism of P^1 mapping $f(P)$ to $\infty = (0, 1)$. Then P is the only pole of order 2 of the function bf. By Lemma 3 above and by Theorem 14, [3],

$$\dim H^0(V, (bf)^* TP^1) = \dim H^0(V, [4P])$$
$$= 4 + 1 - \{\text{the number of the gaps at } P \leqq 4\} .$$

By Theorem 25, [3], P is a Weierstrass point of V. Since the Weierstrass gap sequence at a Weierstrass point is $1, 3, 5, \cdots, 2g-1$, we have

(1) $$\dim H^0(V, (bf)^* TP^1) = 4 + 1 - 2 = 3.$$

Note that

(2) $$\dim H^0(P^1, TP^1) = \dim \mathrm{Aut}(P^1) = 3 ,$$

(3) $$\dim H^0(V, TV) = \dim \mathrm{Aut}(V) = 0 .$$

We put $\hat{f} = bf$. In order to prove that f is stable, it suffices to prove that $\hat{f}$ is stable. By (1), (2), (3) and by Theorem 1, it then suffices to prove that

$$\hat{f}^*: H^0(P^1, TP^1) \to H^0(V, \hat{f}^* TP^1)$$

is injective.

Note that an element of $H^0(P^1, TP^1)$ is written as

$$X = (p\xi^2 + q\xi + r)\partial/\partial\xi \quad \text{on} \quad P^1 - \infty ,$$

where $\xi = z_1/z_0$ is the inhomogeneous coordinate on $P^1 - \infty$, $\infty = (0, 1)$, and p, q and r are constants.

The universal covering space of V is the unit disc $D = \{z \in C \mid |z| < 1\}$. Let $\pi: D \to V$ be the covering map. We put $\tilde{f} = \hat{f}\pi$. Then $\tilde{f}$ is a meromorphic function on D.

Now, assume that $\hat{f}^* X = 0$. Then

$$0 = \pi^* \hat{f}^* X = \tilde{f}^* X = p\tilde{f}(z)^2 + q\tilde{f}(z) + r ,$$

on $D - \tilde{f}^{-1}(\infty)$. This implies that $p = q = r = 0$. Hence, $\hat{f}^*$ is injective.

Case 2: $g = 1$. In this case, V is a complex 1-torus. Let f be an elliptic function of order 2 on V. By Lemma 3, $\dim H^0(V, f^* TP^1) = 4$. Note that

$$\dim H^0(V, TV) = \dim \operatorname{Aut}(V) = 1 .$$

Hence, in order to prove Theorem 2 in this case, using Theorem 1, it suffices to prove that

$$f^* + f_* : H^0(\boldsymbol{P}^1, T\boldsymbol{P}^1) \times H^0(V, TV) \to H^0(V, f^* T\boldsymbol{P}^1)$$

is injective.

The universal covering space of V is $\boldsymbol{C}$. Let $\pi: \boldsymbol{C} \to V$ be the covering map. We denote by z a coordinate on V induced by π. Then an element of $H^0(V, TV)$ is written as

$$Y = s\partial/\partial z ,$$

where s is a constant.

Now, for $X = (p\xi^2 + q\xi + r)\partial/\partial\xi \in H^0(\boldsymbol{P}, T\boldsymbol{P}^1)$, assume that $f^* X + f_* Y = 0$. Then

$$pf(z)^2 + qf(z) + r + sf'(z) = 0 ,$$

on $V - f^{-1}(\infty)$. This implies that $p = q = r = s = 0$. Hence $f^* + f_*$ is injective.

REMARK. We can show that, for any elliptic function f of order 2 on V, there are $b \in \operatorname{Aut}(\boldsymbol{P}^1)$ and $a \in \operatorname{Aut}(V)$ with $f = b\wp a$, where $\wp$ is Weierstrassian $\wp$-function on V. (See [7].)

Case 3: $g = 0$. In this case, $V = \boldsymbol{P}^1$. Let f be a rational function of order 2. Then f is a two sheeted ramified covering of $\boldsymbol{P}^1$ onto $\boldsymbol{P}^1$ with two branch points P and Q. Let a be an automorphism of $\boldsymbol{P}^1$ mapping $0 = (1, 0)$ to P and $\infty = (0, 1)$ to Q. Let b_1 be an automorphism of $\boldsymbol{P}^1$ mapping $f(P)$ to 0 and $f(Q)$ to ∞. Then

$$b_1 f a(\xi) = p\xi^2 , \quad \text{for} \quad \xi = z_1/z_0 \in \boldsymbol{P}^1 - \infty ,$$

where p is a non-zero constant. Let b_2 be the automorphism of $\boldsymbol{P}^1$ defined by $b_2(\xi) = (1/p)\xi$, for $\xi = z_1/z_0 \in \boldsymbol{P}^1 - \infty$. Put $b = b_2 b_1$. Then

$$bfa(\xi) = \xi^2 , \quad \text{for} \quad \xi = z_1/z_0 \in \boldsymbol{P}^1 - \infty .$$

This shows that f is stable.

This completes the proof of Theorem 2.

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