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# ON THE CONVERSE THEOREM FOR INTEGRAL STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS

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**1. Introduction.** Recently, Chow and Yorke [1] have extended Vrkoč's result [2] on integrally asymptotic stability for ordinary differential equations. They have given substantially simpler proofs of Vrkoč's results based on the method used in [3] and shown that every integrally asymptotically stable system behaves nicely not only for perturbations integrable on $[0, \infty)$ but also for the larger class of interval bounded functions, i.e., the class of functions $p(t)$ such that

$$\sup_{t\geqq 0}\int_t^{t+1} |p(u)|\,du < \infty \ .$$

On the other hand, Kato and Yoshizawa [4] have extended Chow and Yorke's results to functional differential equations without using Liapunov's method. Here, we show that the converse theorem holds for integral stability of functional differential equations under suitable conditions.

**2. Preliminaries.** Let $I$ denote the interval $0 \leqq t < \infty$ and let $|x|$ be the Euclidean norm of $x \in R^m$. For a given $h > 0$, $C$ denotes the space of continuous functions mapping the interval $[-h, 0]$ into $R^m$ and for $\phi \in C$, $\|\phi\| = \sup_{-h\leqq\theta\leqq 0}|\phi(\theta)|$. Let $C_H$ be the set of $\phi \in C$ such that $\|\phi\| \leqq H$. Let $C_H(L)(0 < L < \infty)$ be the set of $\phi \in C_H$ such that $|\phi(\theta_1) - \phi(\theta_2)| \leqq L|\theta_1 - \theta_2|$ for all $\theta_1, \theta_2 \in [-h, 0]$ and $C_H(\infty)$ be $\bigcup_{L>0} C_H(L)$. For $\phi \in C_H(\infty)$, let $\|\phi\|_1$ be the norm defined by $\|\phi\|_1 = \|\phi\| + \int_{-h}^0 |\dot\phi(\theta)|\,d\theta$, where $\dot\phi(\theta)$ denotes the right-hand derivative of $\phi(u)$ at $u = \theta$ if it exists and 0 if it does not exist. For any continuous function $x(u)$ defined on an interval including $[t-h, t]$, the symbol $x_t$ will denote the restriction of $x(u)$ to the interval $[t-h, t]$, i.e., $x_t$ is an element of $C$ defined by $x_t(\theta) = x(t+\theta)$, $-h \leqq \theta \leqq 0$. Let $B_0$ be the set of measurable functions $p: I \to R^m$ such that $\operatorname{ess\,sup}_{u\in J}|p(u)| < \infty$ for any compact interval $J$ in $I$. Let $B$ be a normed vector space in $B_0$, and we denote the norm of $p$ by $\|p\|_B$.

Consider the functional differential equation

$$\dot{x}(t) = f(t, x_t) , \tag{1.1}$$

where $f(t, \phi)$ is an $m$-vector functional which is defined on $I \times C_H$. Corresponding to (1.1), consider a perturbed system

$$\dot{y}(t) = f(t, y_t) + p(t) , \tag{1.2}$$

where $p$ is an element of $B$. We make the following assumptions.

(H1) $f : I \times C_H \to R^m$ is continuous in $(t, \phi)$.

(H2) $| f(t, \phi) | \leqq l(t) \| \phi \|$ on $I \times C_H$, where $l(t)$ is continuous.

(H3) $\int_t^{t+h} l(u)du \leqq b$ for all $t \geqq 0$.

REMARK. (H1) can be replaced by the following more general assumption:

(H1*) $f(\cdot, \phi)$ is measurable for each $\phi$, $f(t, \cdot)$ is continuous for each $t$, and for any $\varepsilon > 0$, any compact set $S$ in $C_H$ and any compact interval $J$ in $I$, there exists a $\gamma(\varepsilon, S, J) > 0$ such that $\| \phi - \psi \| < \gamma(\varepsilon, S, J)$ implies $| f(s, \phi) - f(s, \psi) | < \varepsilon$ for all $s \in J$, $\phi \in S$ and $\psi \in C_H$.

In the following, we shall denote by $x(t, t_0, \phi_0)$ a solution of (1.1) through $(t_0, \phi_0)$ and similarly by $y(t, t_0, \phi_0)$ a solution of (1.2).

Let $0 < a < H$, $r > h$ and $L > a/(r-h)$. For each $(t, \phi)$ in $[r, \infty) \times C_a(\infty)$, $A_a(t, \phi, L)$ will denote the set of Lipschitz continuous functions $\xi: [-h, t] \to R^m$ such that

$$\xi_0 = 0,\ \xi_t = \phi,\ \xi_s \in C_a(L) \text{ for all } s \in [0, t-h] .$$

Let $V(t, \phi, L)$: $[r, \infty) \times C_a(\infty) \to R^1$ be a functional and $S_{(1.1)}(t, \phi)$ be the set of $x(s, t, \phi)$, and we define the functional

$$V'_{(1.1)}(t, \phi, L) = \sup_{x(s) \in S_{(1.1)}(t,\phi)} \overline{\lim_{\delta \to 0+}} \frac{1}{\delta} \{ V(t+\delta, x_{t+\delta}, L) - V(t, \phi, L) \} .$$

DEFINITION 1. The zero solution of (1.1) is stable under $B$ perturbations (hereafter called $S$ under $B$), if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $t_0 \geqq 0$, any $\phi_0 \in C_H$ and any $p \in B$, $\| \phi_0 \| < \delta(\varepsilon)$ and $\| p \|_B < \delta(\varepsilon)$ imply $| y(t, t_0, \phi_0) | < \varepsilon$ for all $t \geqq t_0$.

DEFINITION 2. The zero solution of (1.1) is attracting under $B$ perturbations ($A$ under $B$ for brevity), if there exists a $\delta_0 > 0$ and for any $\varepsilon > 0$, there exist a $T(\varepsilon) > 0$ and an $\eta(\varepsilon) > 0$ such that for any $t_0 \geqq 0$, any $\phi_0 \in C_H$ and any $p \in B$, $\| \phi_0 \| < \delta_0$ and $\| p \|_B < \eta(\varepsilon)$ imply $| y(t, t_0, \phi_0) | < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$.

DEFINITION 3. The zero solution of (1.1) is asymptotically stable under $B$ perturbations ($AS$ under $B$), if it is stable under $B$ perturbations and is attracting under $B$ perturbations.

DEFINITION 4. A function $p \in B_0$ is said to be interval bounded if

$$\sup_{t \geqq 0} \int_t^{t+1} |p(u)| du < \infty .$$

We shall denote the space of interval bounded functions in $B_0$ by $B_{IB}$ with norm $\|p\|_{IB} = \sup_{t \geqq 0} \int_t^{t+1} |p(u)| du$. Especially, when $B = B^1 = B_0 \cap L^1[0, \infty)$ and the zero solution is $S$ under $B$, we sometimes say the zero solution of (1.1) is integrally stable ($IS$).

LEMMA 1. *Let $B$ be either $B_{IB}$ or $B^1$. Then $S$ under $B$ is equivalent to $(\|\cdot\|_1, \|\cdot\|_1) - S$ under $B$ w.r.t. $C_H(\infty)$ on $[r, \infty)$. $A$ under $B$, $(\|\cdot\|_1, \|\cdot\|_1) - A$ under $B$ w.r.t. $C_H(\infty)$ on $[r, \infty)$ and $(\|\cdot\|, \|\cdot\|_1) - A$ under $B$ w.r.t. $C_H(\infty)$ on $[r, \infty)$ are equivalent. Moreover, if the zero solution of* (1.1) *is $A$ under $B$, it is $S$ under $B$.*

Here $(\|\cdot\|_1, \|\cdot\|_1) - S$ under $B$ w.r.t. $C_H(\infty)$ on $[r, \infty)$ means that $t_0 \geqq 0$, $\phi_0 \in C_H$, $\|\phi_0\|$ and $|y(t, t_0, \phi_0)|$ are replaced by $t_0 \geqq r$, $\phi_0 \in C_H(\infty)$, $\|\phi_0\|_1$ and $\|y_t(t_0, \phi_0)\|_1$ respectively in the above definition of $S$ under $B$.

It is not difficult to prove this lemma. It is sufficient to show that for any $\varepsilon > 0$, there exist a $\delta(\varepsilon) > 0$ and an $\eta(\varepsilon) > 0$ such that for any $t_0 \geqq 0$, any $\phi_0 \in C_H$ and any $p \in B_{IB}$, $\|\phi_0\| < \delta(\varepsilon)$ and $\|p\|_{IB} < \eta(\varepsilon)$ imply $\|y_{t_0+r+h}(t_0, \phi_0)\|_1 < \varepsilon$, and to show that for any $\varepsilon > 0$, there exist a $\delta(\varepsilon) > 0$ and an $\eta(\varepsilon) > 0$ such that for any $p \in B_{IB}$ and any $y(t, t_0, \phi_0)$, $|y(t, t_0, \phi_0)| < \delta(\varepsilon)$ for all $t \geqq t_0$ and $\|p\|_{IB} < \eta(\varepsilon)$ imply $\|y_t(t_0, \phi_0)\|_1 < \varepsilon$ for all $t \geqq t_0 + 2h$. To show this, notice that a solution $y(t, t_0, \phi_0)$ can be written as follows

$$y(t, t_0, \phi_0) = \begin{cases} \phi_0(t - t_0) , & t_0 - h \leqq t \leqq t_0 , \\ \phi_0(0) + \int_{t_0}^t f(u, y_u(t_0, \phi_0))du + \int_{t_0}^t p(u)du , & t > t_0 . \end{cases}$$

From this it follows that $\|y_t(t_0, \phi_0)\| \leqq \|\phi_0\| + \int_{t_0}^t l(u) \|y_u(t_0, \phi_0)\| du + \int_{t_0}^t |p(u)| du$, and by Gronwall's inequality we obtain

$$\|y_t(t_0, \phi_0)\| \leqq \left(\|\phi_0\| + \int_{t_0}^{t_1} |p(u)| du\right) \exp\left\{\int_{t_0}^t l(u)du\right\} , \quad t_0 \leqq t \leqq t_1 .$$

Moreover for $t_0 + h \leqq t \leqq t_1$, we have $y_t(t_0, \phi_0) \in C_H(\infty)$ and

$$\|y_t(t_0, \phi_0)\|_1 \leqq \left(\|\phi_0\| + \int_{t_0}^{t_1} |p(u)| du\right)\left(1 + \int_{t-h}^t l(u)du\right) \times \exp\left\{\int_{t_0}^t l(u)du\right\} + \int_{t-h}^t |p(u)| du .$$

for all $t \geqq t_0 + 2h$.

**3. Liapunov functionals.** Let $0 < a < H$, $r > h$ and $L > a/(r-h)$. For each $(t, \phi) \in [r, \infty) \times C_a(\infty)$, let $V(t, \phi, L)$ be defined by

$$V(t, \phi, L) = \inf_{\xi \in A_a(t,\phi,L)} \int_0^t e^{-\lambda(t-u)} |\dot{\xi}(u) - f(u, \xi_u)| \, du \,, \tag{3.1}$$

where $\lambda \geqq 0$ is a constant. In case $L_2 > L_1 > a/(r-h)$, we have $0 \leqq V(t, \phi, L_2) \leqq V(t, \phi, L_1)$, because $A_a(t, \phi, L_1)$ is contained in $A_a(t, \phi, L_2)$. Therefore $W(t, \phi)$ can be defined by

$$W(t, \phi) = \inf_{L > a/(r-h)} V(t, \phi, L) = \lim_{L \to \infty} V(t, \phi, L) \,. \tag{3.2}$$

On the other hand, $A_a(t, \phi, L)$ contains an element which attains the value of $V(t, \phi, L)$ since $A_a(t, \phi, L)$ is compact. Moreover $V(t, \phi, L)$ has the following properties.

LEMMA 2. *Let $t \geqq r$, $\phi \in C_a(\infty)$ and $p \in B_0$. Then we have the inequalities:*

$$0 \leqq V(t, \phi, L) \leqq (b + b' + 1) \|\phi\|_1 \,, \tag{3.3}$$

*where* $\sup_{t \geqq 0} \int_t^{t+r-h} l(u) du \leqq b'$. *For $t$ satisfying*

$$|p(t)| = \lim_{\delta \to 0+} \frac{1}{\delta} \int_t^{t+\delta} |p(u)| \, du \,,$$

$$V'_{(1.2)}(t, \phi, L) \leqq V'_{(1.1)}(t, \phi, L) + |p(t)| \,. \tag{3.4}$$

PROOF. Let $\xi \in A_a(t, \phi, L)$ be a function such that $\xi(u) = 0$ on $[-h, t-h-a/L]$ and the graph of $\xi$ on $[t-h-a/L, t-h]$ is a straight line between $(t-h-a/L, 0)$ and $(t-h, \phi(-h))$. Then we have

$$\begin{aligned}
0 &\leqq V(t, \phi, L) \\
&\leqq \int_0^t e^{-\lambda(t-u)} |\dot{\xi}(u) - f(u, \xi_u)| \, du \leqq \int_{t-h-a/L}^t |\dot{\xi}(u) - f(u, \xi_u)| \, du \\
&\leqq \int_{t-h-a/L}^{t-h} |\dot{\xi}(u) - f(u, \xi_u)| \, du + \int_{t-h}^t |\dot{\xi}(u) - f(u, \xi_u)| \, du \\
&\leqq |\phi(-h)| + |\phi(-h)| \int_{t-h-a/L}^{t-h} l(u) du + \int_{-h}^0 |\dot{\phi}(\theta)| \, d\theta + \|\phi\| \int_{t-h}^t l(u) du \\
&\leqq \|\phi\| \left(1 + \int_{t-h-a/L}^t l(u) du\right) + \int_{-h}^0 |\dot{\phi}(\theta)| \, d\theta \\
&\leqq \|\phi\|_1 + \|\phi\|(b + b') \leqq (b + b' + 1) \|\phi\|_1 \,,
\end{aligned}$$

where $b' > 0$ is a constant such that

$$\sup_{t\geq 0}\int_t^{t+a/L} l(u)du \leqq \sup_{t\geq 0}\int_t^{t+r-h} l(u)du \leqq b' .$$

Now, the assumption $p \in B_0$ implies $|p(t)| = \lim_{\delta\to 0+} 1/\delta \int_t^{t+\delta} p(u)\,|\,du$ a.e. in $t$. For such a $t$, we take $x = x(s) \in S_{(1.1)}(t, \phi)$ and $y = y(s) \in S_{(1.2)}(t, \phi)$, and we choose $0 < \delta < h$ such that $x$ and $y$ exist on $[t, t+\delta]$. Let $\xi^\delta \in A_a(t+\delta, x_{t+\delta}, L)$ be a function such that

$$V(t+\delta, x_{t+\delta}, L) = \int_0^{t+\delta} e^{-\lambda(t+\delta-u)}\,|\,\dot{\xi}^\delta(u) - f(u, \xi^\delta_u)\,|\,du .$$

Moreover, let $\eta^\delta$ be a function such that $\eta^\delta = \xi^\delta$ on $[-h, t]$ and $\eta^\delta(u) = y(u)$ on $[t, t+\delta]$. Then we obtain

$$V(t+\delta, y_{t+\delta}, L) \leqq \int_0^{t+\delta} e^{-\lambda(t+\delta-u)}\,|\,\dot{\eta}^\delta(u) - f(u, \eta^\delta_u)\,|\,du .$$

Now let $L' > \max(L, M)$, where $M = H\max_{t\leqq u\leqq t+h} l(u)$. From the assumption (H1) it follows that for any $\varepsilon > 0$, there exists a $\gamma(\varepsilon) > 0$ such that for any $0 \leqq s \leqq t+h$, and $\psi_1 \in C_H(L')$ and any $\psi_2 \in C_H$, $|f(s, \psi_1) - f(s, \psi_2)| < \varepsilon$ if $\|\psi_1 - \psi_2\| < \gamma(\varepsilon)$. Therefore, if $\delta_1$, $0 < \delta_1 < h$, satisfies $\delta_1 < \gamma(\varepsilon)/4M$ and if $\int_t^{t+\delta_1} |p(u)|\,du < \gamma(\varepsilon)/2$ and $0 < \delta < \delta_1$, we have $x_{t+\delta} \in C_H(L')$ and $\|x_{t+\delta} - y_{t+\delta}\| < \gamma(\varepsilon)$. Since we have

$$\begin{aligned}
V'_y(t, \phi, L) &= \overline{\lim_{\delta\to 0+}}\frac{1}{\delta}\{V(t+\delta, y_{t+\delta}, L) - V(t, \phi, L)\}\\
&\leqq \overline{\lim_{\delta\to 0+}}\frac{1}{\delta}\{V(t+\delta, x_{t+\delta}, L) - V(t, \phi, L)\}\\
&\quad + \overline{\lim_{\delta\to 0+}}\frac{1}{\delta}\{V(t+\delta, y_{t+\delta}, L) - V(t+\delta, x_{t+\delta}, L)\}\\
&\leqq V'_x(t, \phi, L) + \overline{\lim_{\delta\to 0+}}\frac{1}{\delta}\Big\{\int_0^{t+\delta} e^{-\lambda(t+\delta-u)}\,|\,\dot{\eta}^\delta(u) - f(u, \eta^\delta_u)\,|\,du\\
&\quad - \int_0^{t+\delta} e^{-\lambda(t+\delta-u)}\,|\,\dot{\xi}^\delta(u) - f(u, \xi^\delta_u)\,|\,du\Big\}\\
&\leqq V'_x(t, \phi, L) + \overline{\lim_{\delta\to 0+}}\frac{1}{\delta}\int_t^{t+\delta} e^{-\lambda(t+\delta-u)}\{|\,\dot{\eta}^\delta(u) - f(u, \eta^\delta_u)\,|\\
&\quad - |\,\dot{\xi}^\delta(u) - f(u, \xi^\delta_u)\,|\}\,du \leqq V'_x(t, \phi, L)\\
&\quad + \overline{\lim_{\delta\to 0+}}\frac{1}{\delta}\int_t^{t+\delta}\{|\,\dot{y}(u) - \dot{x}(u)\,| + |\,f(u, \eta^\delta_u) - f(u, \xi^\delta_u)\,|\}\,du\\
&\leqq V'_x(t, \phi, L) + \overline{\lim_{\delta\to 0+}}\frac{1}{\delta}\int_t^{t+\delta}|\,f(u, y_u) + p(u) - f(u, x_u)\,|\,du + \varepsilon\\
&\leqq V'_x(t, \phi, L) + |\,p(t)\,| + \varepsilon \leqq V'_{(1.1)}(t, \phi, L) + |\,p(t)\,| + \varepsilon ,
\end{aligned}$$

we see that $V'_{(1.2)}(t, \phi, L) \leqq V'_{(1.1)}(t, \phi, L) + |p(t)|$, because $\varepsilon > 0$ is arbitrary.

REMARK. If $f(t, \phi)$ satisfies $|f(t, \phi) - f(t, \psi)| \leqq l(t) \|\phi - \psi\|$, we can prove $|V(t, \phi, L) - V(t, \psi, L)| \leqq (b + 1)\|\phi - \psi\|_1$ for $\phi$ and $\psi$ such that $\phi(-h) = \psi(-h)$.

LEMMA 3. *Let $r > h$, $L > a/(r - h)$ and $0 < a_1 < a$. For $t \geqq r$ and $\phi \in C_{a_1}(\infty)$, let $x(s)$, $t - h \leqq s \leqq t + A$ $(A > 0)$, be a function such that $x_t = \phi$ and $x_s \in C_{a_1}(L_0)$ on $[t, t + A]$. If $L \geqq L_0$, $v(s) = V(s, x_s, L)$ is continuous on $[t, t + A]$. Especially, if $x(s) \in S_{(1.1)}(t, \phi)$, $v(s)$ is non-increasing and we have*

$$V'_{(1.1)}(s, x_s, L) \leqq -\lambda V(s, x_s, L) . \tag{3.5}$$

PROOF. Let $\alpha = a - a_1$ and let $\beta$ be a positive number such that $|x(s)|/s \leqq L - \beta$ on $[t - h, t + A]$. For $s \in [t, t + A]$, we take $\xi \in A_a(s, x_s, L)$. Let $\eta \in A_a(s, x_s, L)$ be a function such that the graph of $\eta$ on $[0, s - h]$ is a straight line between $(0, 0)$ and $(s - h, x(s - h))$. Then $|\eta(u)| \leqq a - \alpha = a_1$ and $|\dot{\eta}(u)| \leqq L - \beta$ for all $u \in [0, s]$. For $\xi$, $\eta$ and $q$ $(0 < q < 1)$, let $\xi^q = (1 - q)\xi + q\eta$. Then $\xi^q \in A_{a-q\alpha}(s, x_s, L - q\beta)$ and consequently $\xi^q \in A_a(s, x_s, L)$. Since $f(u, \psi)$ is uniformly continuous on $[0, t + A] \times C_a(L)$, for any $\varepsilon > 0$ there exists a $\gamma(\varepsilon) > 0$ such that for any $u_1, u_2 \in [0, t + A]$ and any $\psi_1, \psi_2 \in C_a(L)$, $|f(u_1, \psi_1) - f(u_2, \psi_2)| < \varepsilon/(16(t + A))$ if $|u_1 - u_2| + \|\psi_1 - \psi_2\| < \gamma(\varepsilon)$. For this $\gamma(\varepsilon)$, we choose a $q$ such that $2aq < \gamma(\varepsilon)$, $2Lq(t + A) < \varepsilon/16$ and $0 < q < 1$. Then for $0 \leqq u \leqq s$, we have

$$\|\xi_u - \xi^q_u\| \leqq q \sup_{0 \leqq \theta \leqq s} |\xi(\theta) - \eta(\theta)| \leqq 2aq < \gamma(\varepsilon) ,$$

and hence we obtain

$$\begin{aligned} &\left| \int_0^s e^{-\lambda(s-u)} |\dot{\xi}(u) - f(u, \xi_u)| du - \int_0^s e^{-\lambda(s-u)} |\dot{\xi}^q(u) - f(u, \xi^q_u)| du \right| \\ &\leqq \int_0^s |\dot{\xi}(u) - \dot{\xi}^q(u)| du + \int_0^s |f(u, \xi_u) - f(u, \xi^q_u)| du \\ &\leqq q \int_0^s |\dot{\xi}(u) - \dot{\eta}(u)| du + \frac{\varepsilon}{16} < \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8} . \end{aligned} \tag{3.6}$$

For $t \leqq s_1 < s_2 < t + A$, $\sigma = (s_1 - h)/(s_2 - h)$ and $\xi \in A_a(s_2, x_{s_2}, L)$, define $\zeta(u)$ by

$$\zeta(u) = \begin{cases} 0 & \text{if } -h \leqq u \leqq 0 \\ \xi^q(\sigma u) + \dfrac{u}{s_1 - h}(x(s_1 - h) - x(s_2 - h)) & \text{if } 0 \leqq u \leqq s_1 - h \\ x(u) & \text{if } s_1 - h \leqq u \leqq s_1 . \end{cases}$$

Let $0 < r_1 < q/L \min(\alpha, \beta(t-h)/2)$ and $|s_1 - s_2| < r_1$. Then for $0 \leqq u \leqq s_1$, we have

$$|\zeta(u)| \leqq |\xi^q(\sigma u)| + \frac{u}{s_1 - h}|x(s_1 - h) - x(s_2 - h)| \leqq a - q\alpha + Lr_1 \leqq a$$

and

$$|\dot{\zeta}(u)| \leqq \sigma|\dot{\xi}^q(\sigma u)| + \frac{1}{s_1 - h}|x(s_1 - h) - x(s_2 - h)| \leqq \sigma(L - q\beta) + \frac{Lr_1}{s_1 - h}$$
$$\leqq \left(1 + \frac{r_1}{s_1 - h}\right)(L - q\beta) + \frac{Lr_1}{s_1 - h} \leqq L ,$$

and hence $\zeta$ belongs to $A_a(s_1, x_{s_1}, L)$. On the other hand, let $0 < r_2 < 1/L \min(\varepsilon/16, \gamma(\varepsilon))$, $|s_1 - s_2| < r_2$ and $\chi(u) = \xi^q(\sigma u)$. Then for $0 \leqq u \leqq s_1 - h$, we obtain

$$\|\zeta_u - \chi_u\| = \sup_{-h \leqq \theta \leqq 0} |\zeta(u + \theta) - \chi(u + \theta)| \leqq L|s_1 - s_2| < Lr_2$$

and

$$|\dot{\zeta}(u) - \dot{\chi}(u)| \leqq \frac{1}{s_1 - h}|x(s_1 - h) - x(s_2 - h)| \leqq \frac{Lr_2}{s_1 - h} .$$

Thus we have

$$\left|\int_0^{s_1 - h} e^{-\lambda(s_1 - u)}|\dot{\zeta}(u) - f(u, \zeta_u)|\,du - \int_0^{s_1 - h} e^{-\lambda(s_1 - u)}|\dot{\chi}(u) - f(u, \chi_u)|\,du\right|$$
$$(3.7) \qquad \leqq \int_0^{s_1 - h} |\dot{\zeta}(u) - \dot{\chi}(u)|\,du + \int_0^{s_1 - h} |f(u, \zeta_u) - f(u, \chi_u)|\,du$$
$$\leqq Lr_2 + \frac{\varepsilon}{16} < \frac{\varepsilon}{8} .$$

Now let $M = a \max_{0 \leqq u \leqq t + A} l(u)$ and $r_3 > 0$ be a number such that

$$r_3 < \min\left(1, \frac{t - h}{Lh}\right)\gamma(\varepsilon) , \quad Lr_3 < \frac{\varepsilon}{8}$$

and

$$\sup_{|s_1 - s_2| \leqq r_3} \sup_{0 \leqq u \leqq t + A} \left|\frac{e^{-\lambda(s_1 - u/\sigma)}}{\sigma} - e^{-\lambda(s_2 - u)}\right| < \frac{\varepsilon}{8(L + M)(t + A)} .$$

If $|s_1 - s_2| \leqq r_3$, then for $0 \leqq u \leqq s_1 - h$, we have

$$\begin{aligned}\|\chi_u - \xi_{\sigma u}\| &= \sup_{-h\leqq\theta\leqq 0} |\chi(u+\theta) - \xi^q(\sigma u+\theta)| \\ &\leqq \sup_{-h\leqq\theta\leqq 0} |\xi^q(\sigma u+\sigma\theta) - \xi^q(\sigma u+\theta)| \\ &\leqq Lh(\sigma-1) \leqq Lh\frac{r}{t-h} < \gamma(\varepsilon)\,,\end{aligned}$$

and hence we obtain the following inequality.

$$\begin{aligned}\Big|\int_0^{s_1-h} e^{-\lambda(s_1-u)}|\dot{\chi}(u) - f(u,\chi_u)|\,du \\ -\int_0^{s_2-h} e^{-\lambda(s_2-u)}|\dot{\xi}^q(u) - f(u,\xi^q_u)|\,du\Big| < \frac{4\varepsilon}{8}\,,\end{aligned} \tag{3.8}$$

since

$$\begin{aligned}&\Big|\int_0^{s_1-h} e^{-\lambda(s_1-u)}|\dot{\chi}(u) - f(u,\chi_u)|\,du \\ &\qquad -\int_0^{s_2-h} e^{-\lambda(s_2-u)}|\dot{\xi}^q(u) - f(u,\xi^q_u)|\,du\Big| \\ &\leqq \Big|\int_0^{s_1-h} e^{-\lambda(s_1-u)}|\dot{\xi}^q(\sigma u) - f(u,\chi_u)|\,du \\ &\qquad -\int_0^{s_2-h} e^{-\lambda(s_2-u)}|\dot{\xi}^q(u) - f(u,\xi^q_u)|\,du\Big| \\ &\qquad +\int_0^{s_1-h}(\sigma-1)|\dot{\xi}^q(u)|\,du \\ &\leqq \Big|\int_0^{s_1-h} e^{-\lambda(s_1-u)}|\dot{\xi}^q(\sigma u) - f(u,\xi^q_{\sigma u})|\,du \\ &\qquad -\int_0^{s_2-h} e^{-\lambda(s_2-u)}|\dot{\xi}^q(u) - f(u,\xi^q_u)|\,du\Big| \\ &\qquad +\int_0^{s_1-h}|f(u,\chi_u) - f(u,\xi^q_{\sigma u})|\,du + Lr_3 \\ &\leqq \Big|\int_0^{s_1-h} e^{-\lambda(s_1-u)}|\dot{\xi}^q(\sigma u) - f(u,\xi^q_{\sigma u})|\,du \\ &\qquad -\int_0^{s_2-h} e^{-\lambda(s_2-u)}|\dot{\xi}^q(u) - f(u,\xi^q_u)|\,du\Big| \\ &\qquad +\int_0^{s_1-h}|f(u,\xi^q_{\sigma u}) - f(\sigma u,\xi^q_{\sigma u})|\,du + \frac{2\varepsilon}{8} \\ &\leqq \Big|\frac{1}{\sigma}\int_0^{s_2-h} e^{-\lambda(s_1-u/\sigma)}|\dot{\xi}^q(u) - f(u,\xi^q_u)|\,du \\ &\qquad -\int_0^{s_2-h} e^{-\lambda(s_2-u)}|\dot{\xi}^q(u) - f(u,\xi^q_u)|\,du\Big| + \frac{3\varepsilon}{8}\end{aligned}$$

$$\leqq \int_0^{s_2-h} \left| \frac{e^{-\lambda(s_1-u/\sigma)}}{\sigma} - e^{-\lambda(s_2-u)} \right| |\dot{\xi}^q(u) - f(u, \xi^q_u)| \, du$$

$$+ \frac{3\varepsilon}{8} < \frac{4\varepsilon}{8} .$$

Moreover, let $0 < r_4 < \varepsilon/8(L + M)$ and $|s_1 - s_2| < r_4$. Then we have

$$\int_{s_1-h}^{s_2-h} e^{-\lambda(s_1-u)} |\dot{\zeta}(u) - f(u, \zeta_u)| \, du < \frac{\varepsilon}{8} . \tag{3.9}$$

Finally, let $0 < 2Lr_5 < \gamma(\varepsilon)$, $(e^{\lambda r_5} - 1)(L + M)(t + A) < \varepsilon/16$ and $|s_1 - s_2| < r_5$. Then we obtain

$$\begin{aligned}
&\sup_{0 \leqq u \leqq s_1-h} |\zeta(u) - \xi^q(u)| \\
&= \max\left( \sup_{0 \leqq u \leqq s_1-h} |\zeta(u) - \xi^q(u)|, \sup_{s_1-h \leqq u \leqq s_2-h} |\zeta(u) - \xi^q(u)| \right) \\
&\leqq \max\left( \sup_{0 \leqq u \leqq s_1-h} |\zeta(u) - \chi(u)| + \sup_{0 \leqq u \leqq s_1-h} |\chi(u) - \xi^q(u)| , \right. \\
&\qquad \left. \sup_{s_1-h \leqq u \leqq s_2-h} |\zeta(u) - x(s_2 - h)| + \sup_{s_1-h \leqq u \leqq s_2-h} |\xi^q(u) - x(s_2 - h)| \right) \\
&\leqq \max\Big( |x(s_2 - h) - x(s_1 - h)| \\
&\qquad + \sup_{0 \leqq u \leqq s_1-h} |\xi^q(\sigma u) - \xi^q(u)|, L(s_2 - s_1) + L(s_2 - s_1)\Big) \\
&\leqq \max\left(L(s_2 - s_1) + L(\sigma - 1)(s_1 - h), 2Lr_5\right) \\
&\leqq \max\left(2Lr_5, 2Lr_5\right) = 2Lr_5 < \gamma(\varepsilon) .
\end{aligned}$$

From this it follows that

$$\begin{aligned}
&\left| \int_{s_2-h}^{s_1} e^{-\lambda(s_1-u)} |\dot{\zeta}(u) - f(u, \zeta_u)| \, du \right. \\
&\qquad \left. - \int_{s_2-h}^{s_1} e^{-\lambda(s_2-u)} |\dot{\xi}^q(u) - f(u, \xi^q_u)| \, du \right| \\
&\leqq \left| \int_{s_1-h}^{s_1} e^{-\lambda(s_1-u)} \{ |\dot{\zeta}(u) - f(u, \zeta_u)| - |\dot{\xi}^q(u) - f(u, \xi^q_u)| \} \, du \right| \\
&\qquad + (e^{\lambda r_5} - 1) \int_{s_2-h}^{s_1} (|\dot{\xi}^q(u)| + |f(u, \xi^q_u)|) \, du \\
&\leqq \int_{s_2-h}^{s_1} |f(u, \zeta_u) - f(u, \xi^q_u)| \, du \\
&\qquad + (e^{\lambda r_5} - 1)(L + M)(t + A) < \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8} ,
\end{aligned} \tag{3.10}$$

since $\dot{\zeta}(u) = \dot{\xi}^q(u)$ on $[s_1 - h, s_1]$. Now let $r_0 = \min(r_1, r_2, r_3, r_4, r_5)$ and let

$\xi \in A_a(s_2, x_{s_2}, L)$ be a function such that

$$v(s_2) = V(s_2, x_{s_2}, L) = \int_0^{s_2} e^{-\lambda(s_2-u)} |\dot{\xi}(u) - f(u, \xi_u)| du ,$$

where $0 < s_2 - s_1 < r_0$ and $s_1, s_2 \in [t, t+A]$. Then from (3.6) through (3.10), we obtain

$$\begin{aligned}
v(s_1) &\leqq \int_0^{s_1} e^{-\lambda(s_1-u)} |\dot{\zeta}(u) - f(u, \zeta_u)| du \\
&\leqq \int_0^{s_1-h} e^{-\lambda(s_1-u)} |\dot{\zeta}(u) - f(u, \zeta_u)| du \\
&\quad + \int_{s_1-h}^{s_1} e^{-\lambda(s_1-u)} |\dot{\zeta}(u) - f(u, \zeta_u)| du \\
&\leqq \int_0^{s_1-h} e^{-\lambda(s_1-u)} |\dot{\chi}(u) - f(u, \chi_u)| du + \frac{\varepsilon}{8} \\
&\quad + \int_{s_1-h}^{s_1} e^{-\lambda(s_1-u)} |\dot{\zeta}(u) - f(u, \zeta_u)| du \\
&\leqq \int_0^{s_2-h} e^{-\lambda(s_2-u)} |\dot{\xi}^q(u) - f(u, \xi^q_u)| du + \frac{5\varepsilon}{8} \\
&\quad + \int_{s_1-h}^{s_2-h} e^{-\lambda(s_1-u)} |\dot{\zeta}(u) - f(u, \zeta_u)| du \\
&\quad + \int_{s_2-h}^{s_1} e^{-\lambda(s_1-u)} |\dot{\zeta}(u) - f(u, \zeta_u)| du \\
&\leqq \int_0^{s_2-h} e^{-\lambda(s_2-u)} |\dot{\xi}^q(u) - f(u, \xi^q_u)| du \\
&\quad + \int_{s_2-h}^{s_1} e^{-\lambda(s_2-u)} |\dot{\xi}^q(u) - f(u, \xi^q_u)| du + \frac{7\varepsilon}{8} \\
&\leqq \int_0^{s_2} e^{-\lambda(s_2-u)} |\dot{\xi}^q(u) - f(u, \xi^q_u)| du + \frac{7\varepsilon}{8} \\
&< \int_0^{s_2} e^{-\lambda(s_2-u)} |\dot{\xi}(u) - f(u, \xi_u)| du + \varepsilon = v(s_2) + \varepsilon ,
\end{aligned}$$

that is, $v(s_1) - v(s_2) < \varepsilon$. On the other hand, let $\xi \in A_a(s_1, x_{s_1}, L)$ be a function such that

$$v(s_1) = V(s_1, x_{s_1}, L) = \int_0^{s_1} e^{-\lambda(s_1-u)} |\dot{\xi}(u) - f(u, \xi_u)| du ,$$

and extend $\xi$ so as to be $\xi(u) = x(u)$ on $[s_1, s_2]$. Let $0 < (L+M)r' < \varepsilon$ and $|s_1 - s_2| < r'$. Then we have

$$\begin{aligned} v(s_2) &\leqq \int_0^{s_2} e^{-\lambda(s_2-u)} \,|\, \dot{\xi}(u) - f(u, \xi_u) \,|\, du \\ &\leqq e^{-\lambda(s_2-s_1)} \int_0^{s_2} e^{-\lambda(s_1-u)} \,|\, \dot{\xi}(u) - f(u, \xi_u) \,|\, du \\ &\leqq \int_0^{s_1} e^{-\lambda(s_1-u)} \,|\, \dot{\xi}(u) - f(u, \xi_u) \,|\, du \\ &\quad + \int_{s_1}^{s_2} e^{-\lambda(s_1-u)} \,|\, \dot{\xi}(u) - f(u, \xi_u) \,|\, du \\ &\leqq v(s_1) + (L + M)r' < v(s_1) + \varepsilon \,, \end{aligned}$$

and thus $v(s)$ is continuous on $[t, t + A]$.

Especially, when $x(s) = x(s, t, \phi)$, we extend $\xi \in A_a(s, x_s, L)$ so as to be $\xi(u) = x(u)$ on $[s, s + \delta]$ for some $\delta > 0$. Then we obtain

$$\begin{aligned} &\int_0^{s+\delta} e^{-\lambda(s+\delta-u)} \,|\, \dot{\xi}(u) - f(u, \xi_u) \,|\, du \\ &\qquad \leqq e^{-\lambda\delta} \int_0^{s} e^{-\lambda(s-u)} \,|\, \dot{\xi}(u) - f(u, \xi_u) \,|\, du \,. \end{aligned}$$

Since this is true for all $\xi \in A_a(s, x_s, L)$, $v(s)$ is non-increasing. For $x^1 = x^1(u) \in S_{(1.1)}(s, x_s)$, we have

$$V(s + \delta, x^1_{s+\delta}, L) \leqq e^{-\lambda\delta} V(s, x_s, L) \,,$$

and hence

$$V'_{x^1}(s, x_s, L) \leqq -\lambda V(s, x_s, L) \,,$$

which implies (3.5).

4. **Theorems.** We are ready to prove our theorems.

THEOREM 1. *In order that the zero solution of* (1.1) *be integrally stable, it is necessary and sufficient that for some* $a$, $0 < a < H$, *and* $r > h$, *there exists a family of Liapunov functionals* $\{V(t, \phi, L)\}$, $L > a/(r - h)$, *defined on* $[r, \infty) \times C_a(\infty)$ *which satisfies the following conditions*:

( i ) *$V(t, \phi, L)$ is continuous along a curve which is $L_0$-Lipschitz continuous, where* $L \geqq L_0$.

(ii) $b(\|\phi\|_1) \leqq V(t, \phi, L) \leqq K \|\phi\|_1$ *on* $[r, \infty) \times C_a(\infty)$ ,

*where $b(s)$ is continuous, increasing and positive definite.*

(iii) $V'_{(1.2)}(t, \phi, L) \leqq V'_{(1.1)}(t, \phi, L) + \overline{\lim}_{\delta\to 0+} 1/\delta \int_t^{t+\delta} |\, p(u) \,|\, du$ *on* $[r, \infty) \times C_a(\infty)$, *where* $p \in B_0$.

(iv) $V'_{(1.1)}(t, \phi, L) \leqq 0$ *on* $[r, \infty) \times C_a(L)$.

PROOF. Assume that there exists a family of Liapunov functionals which satisfies the conditions in the theorem and the zero solution of (1.1) is not integrally stable. Then, since $IS$ is equivalent to $(\|\cdot\|_1, \|\cdot\|_1)-IS$ w.r.t. $C_H(\infty)$ on $[r, \infty)$ by Lemma 1, there exists an $\varepsilon_0 > 0$ and sequences $\{t_n\}$, $\{\tau_n\}$, $\{p_n(t)\}$ and $\{\phi_n\}$ such that

$$r \leqq t_n \leqq \tau_n\,, \quad p_n \in B^1\,, \quad \int_{t_n}^{\infty} |p_n(u)|\, du < \frac{1}{n}\,,$$

$$\phi_n \in C_H(\infty)\,, \quad \|\phi_n\|_1 < \frac{1}{n} \quad \text{and} \quad \|y^n_{\tau_n}(t_n, \phi_n)\|_1 \geqq \varepsilon_0\,,$$

where $y^n = y^n(s, t_n, \phi_n)$ is a solution of (1.2) with $p(t) = p_n(t)$. Choose an $n$ so large that $1/n < a$ and $(K+1)/n < b(\varepsilon_0)$. Let $\phi_n \in C_a(L_n)$, $M_n = a\max_{t_n \leqq u \leqq \tau_n} l(u)$ and $P_n = \operatorname{ess\,sup}_{t_n \leqq u \leqq \tau_n} |p_n(u)|$. For an $L$ such that $L > \max(L_n, M_n + P_n, a/(r-h))$, consider $V(t, \phi, L)$. By (iii) and (iv), we have $V'_{y^n}(s, y^n_s, L) \leqq \overline{\lim}_{\delta \to 0+} 1/\delta \int_t^{t+\delta} |p(u)|\, du$, because $V'_{y^n}(s, y^n_s, L) \leqq V'_{(1.2)}(s, y^n_s, L)$. Hence, for $t_n \leqq s \leqq \tau_n$, we obtain

$$V(s, y^n_s(t_n, \phi_n), L) \leqq V(t_n, y^n_{t_n}(t_n, \phi_n), L) + \int_{t_n}^{s} |p_n(u)|\, du\,.$$

Setting $s = \tau_n$, we have

$$V(\tau_n, y^n_{\tau_n}(t_n, \phi_n), L) \leqq K\|\phi_n\|_1 + \frac{1}{n} < \frac{K+1}{n} < b(\varepsilon_0)\,.$$

On the other hand, by $\|y^n_{\tau_n}(t_n, \phi_n)\|_1 \geqq \varepsilon_0$ and (i), we obtain

$$V(\tau_n, y^n_{\tau_n}(t_n, \phi_n), L) \geqq b(\|y^n_{\tau_n}(t_n, \phi_n)\|_1) \geqq b(\varepsilon_0)\,,$$

which is a contradiction. Thus the zero solution of (1.1) is integrally stable.

Now for an $a$, $0 < a < H$, $r > h$ and $\lambda = 0$, define $V(t, \phi, L)$ by (3.1), where $L > a/(r-h)$. By Lemma 3, $V(t, \phi, L)$ is continuous along a curve which is $L_0$-Lipschitz continuous, where $L \geqq L_0$, and $V(t, \phi, L)$ satisfies (iv). From Lemma 2 it follows that $V(t, \phi, L)$ satisfies $V(t, \phi, L) \leqq K\|\phi\|_1$ and (iii). Thus it is sufficient to prove that $W(t, \phi)$ is positive definite if the zero solution of (1.1) is $IS$. Suppose not. Then there exists an $\varepsilon_0 > 0$, sequences $\{t_n\}$ and $\{\phi_n\}$ such that $t_n \geqq r$, $\phi_n \in C_a(\infty)$, $\|\phi_n\|_1 \geqq \varepsilon_0$ and

$$W(t_n, \phi_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty\,.$$

Let $\delta(\varepsilon_0)$ be the number in the definition of $(\|\cdot\|_1, \|\cdot\|_1)-IS$ w.r.t. $C_H(\infty)$. Choose an $n$ so large that $W(t_n, \phi_n) < \delta(\varepsilon_0)$. Then for sufficiently large $L > a/(r-h)$, we have $V(t_n, \phi_n, L) < \delta(\varepsilon_0)$. Now let $\xi \in A_a(t_n, \phi_n, L)$ be a function such that

$$\int_0^{t_n} |\dot{\xi}(u) - f(u, \xi_u)|\, du < \delta(\varepsilon_0)$$

and define $p(t)$ by

$$p(t) = \begin{cases} \dot{\xi}(t) - f(t, \xi_t) & \text{for} \quad t \in [0, t_n] \\ 0 & \text{for} \quad t \in (t_n, \infty) . \end{cases}$$

Then $\xi(t)$ is a solution of $\dot{x}(t) = f(t, x_t) + p(t)$ through $(0, 0)$ on the interval $0 \leqq t \leqq t_n$, but $\|\xi_{t_n}\|_1 = \|\phi_n\|_1 \geqq \varepsilon_0$. This contradicts the definition of $(\|\cdot\|_1, \|\cdot\|_1) - IS$ w.r.t. $C_H(\infty)$. This proves the theorem.

THEOREM 2. *In order that the zero solution of* (1.1) *be integrally attracting, it is necessary and sufficient that for some* $a$, $0 < a < H$, *and* $r > h$, *there exists a family of Liapunov functionals* $\{V(t, \phi, L)\}$, $L > a/(r-h)$, *defined on* $[r, \infty) \times C_a(\infty)$ *which satisfies conditions* (i), (ii) *and* (iii) *in Theorem* 1, *and*

(iv)′ $V'_{(1.1)}(t, \phi, L) \leqq -V(t, \phi, L)$ *on* $[r, \infty) \times C_a(\infty)$ .

PROOF. Assume that there exists a family of Liapunov functionals which satisfies the conditions in the theorem and the zero solution of (1.1) is not integrally attracting. Then, since $IA$ is equivalent to $(\|\cdot\|_1, \|\cdot\|_1) - IA$ w.r.t. $C_H(\infty)$ on $[r, \infty)$ by Lemma 1, for any $\delta > 0$ such that $\delta K < b(a)$, there exists an $\varepsilon_0 > 0$, sequences $\{t_n\}$, $\{\tau_n\}$, $\{p_n(t)\}$ and $\{\phi_n\}$ such that $t_n \geqq r$, $\tau_n \geqq t_n + n$, $p_n \in B^1$, $\int_{t_n}^{\infty} |p_n(u)|\, du < 1/n$, $\phi_n \in C_H(\infty)$, $\|\phi_n\|_1 < \delta$ and $\|y^n_{\tau_n}(t_n, \phi_n)\|_1 \geqq \varepsilon_0$, where $y^n(s, t_n, \phi_n)$ is a solution of (1.2) with $p(t) = p_n(t)$. Now choose an $n$ so that $\delta K + 1/n < b(a)$ and $\delta K e^{-n} + 1/n < b(\varepsilon_0)$. Let $\phi_n \in C_a(L_n)$, $M_n = a \max_{t_n \leqq u \leqq \tau_n} l(u)$ and

$$P_n = \operatorname*{ess\,sup}_{t_n \leqq u \leqq \tau_n} |p_n(u)| .$$

For an $L$ such that $L > \max(L_n, M_n + P_n, a/(r-h))$, consider $V(t, \phi, L)$. From (iii) and (iv), it follows that for $t_n \leqq s \leqq \tau_n$,

$$V(s, y^n_s(t_n, \phi_n), L) \leqq V(t_n, y^n_{t_n}(t_n, \phi_n), L) e^{-(s-t_n)} + \int_{t_n}^{s} |p_n(u)|\, du .$$

Setting $s = \tau_n$, we have

$$V(\tau_n, y^n_{\tau_n}(t_n, \phi_n), L) \leqq \delta K e^{-n} + \frac{1}{n} < b(\varepsilon_0) .$$

But, by $\|y^n_{\tau_n}(t_n, \phi_n)\|_1 \geqq \varepsilon_0$ and (i), we have

$$V(\tau_n, y^n_{\tau_n}(t_n, \phi_n), L) \geqq b(\|y^n_{\tau_n}(t_n, \phi_n)\|_1) \geqq b(\varepsilon_0) ,$$

which is a contradiction. Thus the zero solution of (1.1) is $IA$.

Now assume that the zero solution of (1.1) is $IA$. Then by Lemma 1, the zero solution of (1.1) is $IS$ and $IA$ is equivalent to $(\|\cdot\|, \|\cdot\|_1) - IA$ w.r.t. $C_H(\infty)$ on $[r, \infty)$. Let $\delta_0$ correspond to the $\delta_0$ in the definition of $(\|\cdot\|, \|\cdot\|_1) - IA$ w.r.t. $C_H(\infty)$ on $[r, \infty)$. For $\delta_0^*$ such that $0 < \delta_0^* < \delta_0$, let $a = \delta_0^*$ and let $r > h$. For $\lambda = 1$, define $V(t, \phi, L)$ by (3.1), where $L > a/(r-h)$. It is sufficient to prove the positive definiteness of $W(t, \phi)$. Suppose not. Then there exists an $\varepsilon_0 > 0$ and sequences $\{t_n\}$ and $\{\phi_n\}$ such that $t_n \geqq r$, $\phi_n \in C_a(\infty)$, $\|\phi_n\|_1 \geqq \varepsilon_0$ and

$$W(t_n, \phi_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

If $\{t_n\}$ is bounded, we have a contradiction in a similar way to the proof of Theorem 1. Now we consider the case where

$$t_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty .$$

Choose an $n$ such that

$$t_n > T(\varepsilon_0) + r + 1\ , \quad W(t_n, \phi_n) < \eta(\varepsilon_0)e^{-(T(\varepsilon_0)+1)}\ ,$$

where $T(\varepsilon_0)$ and $\eta(\varepsilon_0)$ are numbers corresponding to those in the definition of $(\|\cdot\|, \|\cdot\|_1) - IA$ w.r.t. $C_H(\infty)$ on $[r, \infty)$. Then for sufficiently large $L > a/(r-h)$, we have $V(t_n, \phi_n, L) < \eta(\varepsilon_0)e^{-(T(\varepsilon_0)+1)}$. Moreover, let $\xi \in A_a(t_n, \phi_n, L)$ be a function such that

$$\int_0^{t_n} e^{-(t_n-u)}\,|\dot{\xi}(u) - f(u, \xi_u)|\,du < \eta(\varepsilon_0)e^{-(T(\varepsilon_0)+1)}\ ,$$

and set $t_n - (T(\varepsilon_0)+1) = t_0$. Then $t_0 \geqq r$ and $t_n > t_0 + T(\varepsilon_0)$. Then clearly

$$\begin{aligned} &e^{-(T(\varepsilon_0)+1)}\int_{t_0}^{t_n} |\dot{\xi}(u) - f(u, \xi_u)|\,du \\ &= \int_{t_0}^{t_n} e^{-(t_n-t_0)}\,|\dot{\xi}(u) - f(u, \xi_u)|\,du \\ &\leqq \int_{t_0}^{t_n} e^{-(t_n-u)}\,|\dot{\xi}(u) - f(u, \xi_u)|\,du < \eta(\varepsilon_0)e^{-(T(\varepsilon_0)+1)}\ , \end{aligned}$$

and hence, we have

$$\int_{t_0}^{t_n} |\dot{\xi}(u) - f(u, \xi_u)|\,du < \eta(\varepsilon_0)\ .$$

Define $p(t)$ by

$$p(t) = \begin{cases} \dot{\xi}(t) - f(t, \xi_t) & \text{for} \quad t \in [0, t_n] \\ 0 & \text{for} \quad t \in (t_n, \infty)\ , \end{cases}$$

Then $\xi(t)$ is a solution of $\dot{x}(t) = f(t, x_t) + p(t)$ on $t_0 \leqq t \leqq t_n$ such that

$\|\xi_{t_0}\| < \delta_0$. However $\|\xi_{t_n}\|_1 \geqq \varepsilon_0$, which contradicts the definition of $(\|\cdot\|, \|\cdot\|_1) - IA$ w.r.t. $C_H(\infty)$ on $[r, \infty)$ since $t_n > t_0 + T(\varepsilon_0)$. This proves the theorem.

Now we shall show the equivalence between $IA$ and $A$ under $B_{IB}$.

THEOREM 3. *If the zero solution of* (1.1) *is integrally attracting, then it is attracting under* $B_{IB}$ *perturbations.*

PROOF. Assume that the zero solution of (1.1) is integrally attracting. First we prove that the zero solution of (1.1) is $S$ under $B_{IB}$. By Theorem 2, there exists a family of Liapunov functionals $\{V(t, \phi, L)\}$, $L > a/(r-h)$, defined on $[r, \infty) \times C_a(\infty)$ which satisfies (i), (ii), (iii) and (iv)$'$. Suppose that the zero solution of (1.1) is not $S$ under $B_{IB}$. Then by Lemma 1, it is not $(\|\cdot\|_1, \|\cdot\|_1) - S$ under $B_{IB}$ w.r.t. $C_H(\infty)$ on $[r, \infty)$. Therefore there exists an $\varepsilon_0$, $0 < \varepsilon_0 < a$, and for any $\delta > 0$, there exist $p(t)$, $t_0$, $t_1$, $\phi_0$ and $y = y(s, t_0, \phi_0)$ such that $p \in B_{IB}$, $\|p\|_{IB} < \delta$, $r \leqq t_0 \leqq t_1$, $\phi_0 \in C_a(\infty)$, $\|\phi_0\|_1 < \delta$, $\|y_{t_1}(t_0, \phi_0)\|_1 \geqq \varepsilon_0$ and $|y(t, t_0, \phi_0)| \leqq a$ for $t \in [t_0, t_1]$. We take a $k$, $0 < k < 1/K$, such that $Kb(k\varepsilon_0) < b(\varepsilon_0)$. We may assume $\delta$ is so small that

$$\delta < b\left(\frac{b(k\varepsilon_0)}{K}\right) \leqq b(k\varepsilon_0)\,, \quad Kb(k\varepsilon_0) + \delta < b(\varepsilon_0)\,.$$

Since $\|\phi_0\|_1 < \delta < b((b(k\varepsilon_0))/K) \leqq b(k\varepsilon_0) \leqq Kk\varepsilon_0 < \varepsilon_0$, there is some $t_2 \in (t_0, t_1)$ such that

$$\|y_{t_2}(t_0, \phi_0)\|_1 = b\left(\frac{b(k\varepsilon_0)}{K}\right), \ \|y_t(t_0, \phi_0)\|_1 > b\left(\frac{b(k\varepsilon_0)}{K}\right) \text{ for } t \in (t_2, t_1)\,.$$

Let $y_{t_2}(t_0, \phi_0) \in C_a(L_1)$, $M = a \max_{t_1 \leqq u \leqq t_2} l(u)$ and $P = \operatorname{ess\,sup}_{t_1 \leqq u \leqq t_2} |p(u)|$. Consider $V(t, \phi, L)$, where $L > \max(L_1, M + P, a/(r-h))$. From (ii), (iii) and (iv)$'$, it follows that

$$\begin{aligned} V'_y(t, y_t(t_0, \phi_0), L) &\leqq V'_{(1.2)}(t, y_t(t_0, \phi_0), L) \\ &\leqq -V(t, y_t(t_0, \phi_0), L) + q(t) \leqq -b(\|y_t(t_0, \phi_0)\|_1) + q(t) \\ &\leqq -b\left(\frac{b(k\varepsilon_0)}{K}\right) + q(t) < -\delta + q(t) \end{aligned}$$

where $q(t) = \overline{\lim}_{\delta \to 0+} 1/\delta \int_t^{t+\delta} |p(u)|\, du$. Now integrating from $t_2$ to $t_1$, we have

$$V(t_1, y_{t_1}(t_0, \phi_0), L) - V(t_2, y_{t_2}(t_0, \phi_0), L) \leqq -\delta(t_1 - t_2) + \int_{t_2}^{t_1} |p(u)|\, du$$

and thus

$$\begin{aligned} b(\varepsilon_0) &\leqq b(\| y_{t_1}(t_0, \phi_0) \|_1) \leqq V(t_1, y_{t_1}(t_0, \phi_0), L) \\ &\leqq V(t_2, y_{t_2}(t_0, \phi_0), L) + \delta \leqq K \| y_{t_2}(t_0, \phi_0) \|_1 + \delta \\ &\leqq Kb\Big(\frac{b(k\varepsilon_0)}{K}\Big) + \delta \leqq Kb(k\varepsilon_0) + \delta \,, \end{aligned}$$

which contradicts the choice of $\delta$. Thus the zero solution of (1.1) is $S$ under $B_{IB}$.

Next, we shall prove that the zero solution of (1.1) is $A$ under $B_{IB}$. By Lemma 1, it is sufficient to prove the $(\|\cdot\|_1, \|\cdot\|_1)$ – Attraction under $B_{IB}$ perturbations w.r.t. $C_H(\infty)$ on $[r, \infty)$. By the above-mentioned, there exist two increasing functions $\delta = \delta(\varepsilon)$ and $\eta = \eta(\varepsilon)$ on $[0, a]$ such that

$$\| p \|_{IB} < \eta \text{ implies } \| y_t(t_0, \phi_0) \|_1 < \varepsilon \tag{4.1}$$

for all $p \in B_{IB}$, $\phi_0 \in C_a(\infty)$, $\|\phi_0\|_1 < \delta$ and $t \geqq t_0 \geqq r$. Let $\delta_0 = \delta(a)$. Let $\varepsilon > 0$ be given. We claim that

$$\bar{\eta}(\varepsilon) = \min\Big(\eta(\varepsilon), \frac{1}{2}b(\delta(\varepsilon))\Big) \text{ and } T(\varepsilon) = \frac{Ka + \frac{1}{2}b(\delta(\varepsilon))}{\frac{1}{2}b(\delta(\varepsilon))}$$

are the required numbers in the definition of $(\|\cdot\|_1, \|\cdot\|_1) - A$ under $B_{IB}$ w.r.t. $C_H(\infty)$ on $[r, \infty)$. All we have to show is that there exists $t^* \in (t_0, t_0 + T(\varepsilon))$ such that $\| y_{t^*}(t_0, \phi_0) \|_1 < \delta(\varepsilon)$, where $\phi_0 \in C_a(\infty)$ and $\|\phi_0\|_1 < \delta_0 = \delta(a)$, because we have

$$\| y_t(t_0, \phi_0) \|_1 < \varepsilon \text{ for all } t \geqq t_0 + T(\varepsilon) \geqq t^*$$

by (4.1) since $\bar{\eta}(\varepsilon) \leqq \eta(\varepsilon)$.

Now suppose that there does not exist such a $t^*$. Then

$$\delta(\varepsilon) \leqq \| y_t(t_0, \phi_0) \|_1 \leqq a \text{ for all } t \in [t_0, t_0 + T(\varepsilon)] \,.$$

In a way similar to the previous proof, we obtain

$$\begin{aligned} 0 < b(\delta(\varepsilon)) &\leqq V(t_0 + T, y_{t_0+T}(t_0, \phi_0), L) \\ &\leqq V(t_0, y_{t_0}(t_0, \phi_0), L) - b(\delta(\varepsilon))T + (T+1)\bar{\eta} \\ &\leqq K\|\phi_0\|_1 - (b(\delta(\varepsilon)) - \bar{\eta})T + \bar{\eta} \\ &\leqq Ka - \Big(b(\delta(\varepsilon)) - \frac{1}{2}b(\delta(\varepsilon))\Big)T + \frac{1}{2}b(\delta(\varepsilon)) = Ka - Ka = 0 \,. \end{aligned}$$

This contradiction shows that the zero solution of (1.1) is $A$ under $B_{IB}$.

REMARK. If the zero solution of (1.1) is $IA$, then it is totally asymptotically stable (TAS), too. Here, the zero solution of (1.1) is TAS if it is $S$ under $B_T$ and $A$ under $B_T$.

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