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# THE GAUSS MAP OF KAEHLER IMMERSIONS

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**1. Introduction.** It is a central idea in the study of immersions of Riemannian manifolds that the associated Gauss map clarifies the relationship among geometric objects under consideration. Fundamental results in this aspect which are closely related to our study are the following.

First, in the classical theory of surfaces, with an oriented 2-manifold $M$ in a Euclidean 3-space there is associated the spherical Gauss map, $M \rightarrow S^2$, which assigns to a point $p$ of $M$ the well-defined unit normal vector at $p$, identified with a point of the unit sphere $S^2$ by parallel displacement. It is well-known that the Jacobian coincides with the second fundamental form of $M$ [2].

More generally, with a Riemannian $n$-manifold $M$ immersed in a simply connected complete $N$-space of constant curvature, Obata [4] associates the (generalized) Gauss map, which assigns to each point $p$ of $M$ the totally geodesic $n$-subspace tangent to $M$ at $p$. By the Gauss map in this sense is given a geometric interpretation of the third fundamental form of the immersion.

The purpose of this note is first to define the Gauss map à la Obata for a holomorphic immersion of a Kaehlerian $n$-manifold $M$ into a simply connected complete Kaehlerian $N$-space $V$ of constant holomorphic sectional curvature, and then to obtain a relationship among the Ricci form of $M$, the fundamental 2-form of $M$ and the third fundamental form of the immersion (Theorem 4.1). The Gauss map in our generalized sense is a mapping: $M \rightarrow Q$, where $Q$ stands for the space, which has a natural complex structure and a quadratic differential form (see § 3), of all the totally geodesic complex $n$-subspaces in $V$, and will be proved to be anti-holomorphic (Theorem 3.2). As an application of Theorem 4.1, we obtain a characterization of Einstein submanifolds in terms of the Gauss map (Theorem 4.2). A new interpretation of theorems of Smyth [6] and Ogiue [5] will also be given from the Gauss map viewpoint (Theorems 4.3 and 4.4).

It should be remarked that, in the corresponding case, the Gauss map in this paper is essentially the same one as that of Nomizu-Smyth

[3] defined for a complex hypersurface $M$ of the complex $(n+1)$-space $C^{n+1}$, which is a mapping of $M$ into the complex projective $n$-space $P^n(C)$ and relates the Kaehlerian connections of $M$ and $P^n(C)$.

**2. Preliminaries.** We will summerize some of the basic formulas of Kaehlerian geometry, to begin with. For details, see [1, 2].

In order to avoid repetition, it will be agreed that our indices have the following ranges throughout this paper:

$$0 \leqq A, B, C, \cdots \leqq N,$$
$$1 \leqq \alpha, \beta, \gamma, \cdots \leqq N,$$
$$1 \leqq i, j, k, \cdots \leqq n,$$
$$n+1 \leqq r, s, t, \cdots \leqq N.$$

Let $V$ be a Kaehlerian $N$-manifold with metric $g$. Then $g$ defines a Hermitian scalar product on each tangent space of $V$ and a connection of type (1, 0) under whose parallelism the scalar product is preserved. More precisely, let $(x, e_1, \cdots, e_N)$ be a field of unitary frames, defined for $x$ in a neighborhood of $V$. Its dual coframe field consists of $N$ complex-valued linear differential forms $\theta^\alpha$ of type (1, 0) such that $g$ can be locally written as

$$g = 2\sum_\alpha \theta^\alpha \otimes \bar{\theta}^\alpha .$$

Then the connection forms $\theta^\alpha_\beta$ are characterized by the conditions

$$(1) \qquad \theta^\alpha_\beta + \bar{\theta}^\beta_\alpha = 0 , \qquad d\theta^\alpha = -\sum_\beta \theta^\alpha_\beta \wedge \theta^\beta .$$

The curvature forms $\Theta^\alpha_\beta$ of $V$ are defined by

$$(2) \qquad d\theta^\alpha_\beta = -\sum_\gamma \theta^\alpha_\gamma \wedge \theta^\gamma_\beta + \Theta^\alpha_\beta ,$$

and thus we have

$$\Theta^\alpha_\beta = -\bar{\Theta}^\beta_\alpha = \sum_{\gamma,\delta} R^\alpha_{\beta\gamma\bar{\delta}}\theta^\gamma \wedge \bar{\theta}^\delta ,$$

where $R^\alpha_{\beta\gamma\bar{\delta}}$ are components of the curvature tensor of $V$. $V$ is of constant holomorphic sectional curvature $c$ if and only if

$$(3) \qquad \Theta^\alpha_\beta = (c/2)(\theta^\alpha \wedge \bar{\theta}^\beta + \delta_{\alpha\beta}\sum_\gamma \theta^\gamma \wedge \bar{\theta}^\gamma) .$$

The system of equations in (1) and (2) are called the structure equations of $V$.

The fundamental 2-form $\tilde{\Phi}$ and the Ricci form $\tilde{\Psi}$ of $V$ are defined respectively by

$$\tilde{\Phi} = -2i \sum_{\alpha} \theta^{\alpha} \wedge \bar{\theta}^{\alpha} ,$$

$$\tilde{\Psi} = -2i \sum_{\alpha} \Theta_{\alpha}^{\alpha} = -2i \sum_{\alpha,\beta} S_{\alpha\bar{\beta}} \theta^{\alpha} \wedge \bar{\theta}^{\beta} ,$$

where $S_{\alpha\bar{\beta}} = \sum_{\gamma} R^{\gamma}_{\alpha\gamma\bar{\beta}}$ are components of the Ricci tensor of $V$. $V$ is called *Einsteinian* if $\tilde{\Psi}$ is proportional to $\tilde{\Phi}$ with constant factor, i.e. $\tilde{\Psi} = k\tilde{\Phi}$ for a constant $k$.

Now, let $V$ denote one of the following simply connected complete Kaehlerian $N$-manifolds:

(i) $P^N(C)$, a complex projective $N$-space.

(ii) $C^N$, a complex $N$-space.

(iii) $H^N(C)$, a complex hyperbolic $N$-space.

The bundle $F(V)$ of the unitary frames on $V$ can be identified with the group $G(N)$ which is one of the following according to the type of $V$:

(i) $U(N+1)$, the group of all linear isometries of $C^{N+1}$ equiped with the standard Hermitian metric: $F(z, w) = \sum_A z^A \bar{w}^A$.

(ii) $E(N)$, the group consisting of all transformations $y \to u(y) + x$, $u \in U(N)$, $x \in C^N$, of $C^N$.

(iii) $U(1, N)$, the group of all linear isometries of $C^{N+1}$ equiped with the indefinite Hermitian metric: $F(z, w) = -z^0\bar{w}^0 + \sum_\alpha z^\alpha \bar{w}^\alpha$.

In fact, fixing a point $p^0$ of $V$ and a unitary frame $b^0 = (p^0, e_1^0, \cdots, e_N^0)$ at $p^0$, there is one and only one transformation $h$ in $G(N)$ which sends $b^0$ into a frame $b = (p, e_1, \cdots, e_N)$ at a point $p$ of $V$, and the correspondence $b \leftrightarrow h$ is the desired identification. The isotropy subgroup $K(N)$ at $p^0$ is $U(1) \times U(N)$ in the cases (i) and (iii), and $U(N)$ in the case (ii). Obviously $V$ is the homogeneous space $G(N)/K(N)$.

Let $\varphi_B^A$ be the Maurer-Cartan forms on $G(N)$. Then $\varphi_B^A$ satisfy the following algebraic relations:

$$\varphi_0^0 = \varepsilon^2 \varphi_0^0 , \quad \varepsilon\varphi_0^\alpha + \bar{\varphi}_\alpha^0 = 0 , \quad \varphi_\beta^\alpha + \bar{\varphi}_\alpha^\beta = 0 ,$$

where from now on $\varepsilon$ takes the value

$$\varepsilon = \begin{cases} 1 & \text{if} \quad G(N) = U(N+1) , \quad V = P^N(C) , \\ 0 & \text{if} \quad G(N) = E(N) , \quad V = C^N , \\ -1 & \text{if} \quad G(N) = U(1, N) , \quad V = H^N(C) . \end{cases}$$

$\varphi_B^A$ also satisfy the structure equations:

$$d\varphi_B^A = -\sum_C \varphi_C^A \wedge \varphi_B^C . \tag{4}$$

On putting

$$\theta^\alpha = \varphi_0^\alpha ,$$
$$\theta_\beta^\alpha = \varphi_\beta^\alpha - \delta_{\alpha\beta}\varphi_0^0 ,$$

the Kaehler metric $d\sigma^2$ on $V$ is given by

$$d\sigma^2 = 2\sum_\alpha \theta^\alpha\bar{\theta}^\alpha ,$$

and (4) becomes

$$(5) \qquad d\theta^\alpha = -\sum_\beta \theta_\beta^\alpha \wedge \theta^\beta ,$$
$$d\theta_\beta^\alpha = -\sum_\gamma \theta_\gamma^\alpha \wedge \theta_\beta^\gamma + \varepsilon(\theta^\alpha \wedge \bar{\theta}^\beta + \delta_{\alpha\beta}\sum_\gamma \theta^\gamma \wedge \bar{\theta}^\gamma) ,$$

which are the structure equations of $V$. From (5), the curvature form $\Theta_\beta^\alpha$ of $V$ is given by

$$\Theta_\beta^\alpha = \varepsilon(\theta^\alpha \wedge \bar{\theta}^\beta + \delta_{\alpha\beta}\sum_\gamma \theta^\gamma \wedge \bar{\theta}^\gamma) ,$$

which shows by (3) that $V$ is of constant holomorphic sectional curvature $2\varepsilon$.

Throughout the rest of this note $V$ always denotes one of the above-mentioned simply connected complete Kaehlerian $N$-manifolds. Let $M$ be a Kaehlerian $n$-manifold isometrically immersed into the space $V$ by a holomorphic mapping $x: M \to V$, $F(M)$ denote the bundle of unitary frames on $M$, and $B$ be the set of elements $b = (p, e_1, \cdots, e_N)$ such that $(p, e_1, \cdots, e_n) \in F(M)$ and $(x(p), dx(e_1), \cdots, dx(e_n), e_{n+1}, \cdots, e_N) \in F(V)$. $B$ becomes naturally a differentiable manifold and $\psi: B \to M$, $\psi(p, e_1, \cdots, e_N) = p$, can be viewed as a principal bundle with the fibre $U(n) \times U(N-n)$. The natural immersion $\tilde{x}: B \to F(V) = G(N)$ is defined by $\tilde{x}(b) = (x(p), dx(e_1), \cdots, dx(e_n), e_{n+1}, \cdots, e_N)$, which is nothing but the natural identification of an element of $B$ with a frame of $F(V)$.

Let $\omega^\alpha, \omega_\beta^\alpha$ be the 1-forms on $B$ induced from $\theta^\alpha, \theta_\beta^\alpha$ by the map $\tilde{x}$: $\omega^\alpha = \tilde{x}^*\theta^\alpha$, $\omega_\beta^\alpha = \tilde{x}^*\theta_\beta^\alpha$. Then we have

$$(6) \qquad \omega^r = 0 ,$$

and the Kaehler metric $ds^2$ on $M$ is given by

$$ds^2 = 2\sum_i \omega^i\bar{\omega}^i .$$

From (5) and (6), we obtain

$$d\omega^i = -\sum_k \omega_k^i \wedge \omega^k ,$$
$$d\omega_j^i = -\sum_k \omega_k^i \wedge \omega_j^k - \sum_r \omega_r^i \wedge \omega_j^r + \varepsilon(\omega^i \wedge \bar{\omega}^j + \delta_{ij}\sum_k \omega^k \wedge \bar{\omega}^k) ,$$

where the second one is called the Gauss equation of the immersion $x$.

The curvature form $\Omega^i_j$ of $M$ can then be written as

$$\begin{aligned}\Omega^i_j &= d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j \\ &= -\sum_r \omega^r_j \wedge \bar{\omega}^r_i + \varepsilon(\omega^i \wedge \bar{\omega}^j + \delta_{ij} \sum_k \omega^k \wedge \bar{\omega}^k) .\end{aligned}$$

It follows that

$$\Psi - \varepsilon(n+1)\Phi + III = 0 , \tag{7}$$

where $\Psi$ denotes the Ricci form of $M$, $\Phi$ the fundamental 2-form of $M$, and we have put

$$III = -2i \sum_{i,r} \omega^r_i \wedge \bar{\omega}^r_i .$$

Finally note that the vanishing of all $\omega^r_i$ defines a *totally geodesic* immersion $x$.

**3. The Gauss map.** Let $Q$ be the set of all the totally geodesic complex $n$-subspaces in $V$. Then the group $G(N)$ acts on $Q$ transitively. Take a point $p$ in $Q$. Then the isotropy subgroup at $p$ is identified with $G(n) \times U(N-n)$, where $G(n)$ is viewed as acting on the totally geodesic complex $n$-subspace $V_0$ representing the point $p$ in $Q$ and $U(N-n)$ on the totally geodesic complex $(N-n)$-subspace orthogonal to $V_0$ at the point of intersection which is kept fixed. Therefore $Q$ is identified with a homogeneous space

$$Q = G(N)/G(n) \times U(N-n) .$$

By using the Maurer-Cartan forms $\varphi^A_B$ of $G(N)$ we introduce a quadratic differential form $d\Sigma^2$ and the associated 2-form $\Xi$ on $Q$ respectively by

$$\begin{aligned} d\Sigma^2 &= 2 \sum_r \varepsilon \varphi^r_0 \bar{\varphi}^r_0 + 2 \sum_{i,r} \varphi^r_i \bar{\varphi}^r_i , \\ \Xi &= -2i \sum_r \varepsilon \varphi^r_0 \wedge \bar{\varphi}^r_0 - 2i \sum_{i,r} \varphi^r_i \wedge \bar{\varphi}^r_i ,\end{aligned}$$

which are obviously invariant under the action of $G(N)$. Furthermore we introduce an invariant complex structure $J$ on $Q$ given by

$$J\varphi^r_0 = -i\varphi^r_0 , \quad J\varphi^r_i = -i\varphi^r_i ,$$

i.e. the $\varphi^r_0$, $\varphi^r_i$ are 1-forms of type (0, 1) on $Q$.

The structure $(d\Sigma^2, J)$ on $Q$ is natural in the following sense:

In the case $G(N) = U(N+1)$, $Q$ is the complex Grassmann manifold $G_{n+1,N+1}(C)$ of the complex $(n+1)$-subspaces through the origin in the complex $(N+1)$-space and $(d\Sigma^2, J)$ is the standard Kaehlerian structure on it with respect to which $Q$ is a Hermitian symmetric space. If, in particular, $n = N-1$, then $Q$ is nothing but the complex projective

$N$-space $P^N(C)$ with the Fubini-Study metric of constant holomorphic sectional curvature 2.

In the case $G(N) = U(1, N)$, $(d\Sigma^2, J)$ is the standard pseudo-Riemannian Kaehlerian structure with respect to which $Q$ is a pseudo-Riemannian Hermitian symmetric space.

In the case $G(N) = E(N)$, $d\Sigma^2$ is obviously degenerate. However, if we consider the natural projection of $Q$ onto the complex Grassmann manifold $G_{n,N}(C)$, obtained by identifying the parallel planes, $(d\Sigma^2, J)$ coincides with the structure induced from the standard Kaehlerian structure on $G_{n,N}(C)$ by the projection.

DEFINITION. With an immersion $x: M \to V$ we associate the (generalized) *Gauss map* $f: M \to Q$, where $f(p)$, $p \in M$, is the totally geodesic complex $n$-subspace tangent to $x(M)$ at $x(p)$.

We consider the following diagram of mappings:

$$\begin{array}{ccl} B & \xrightarrow{\tilde{x}} & F(V) = G(N) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & Q = G(N)/G(n) \times U(N-n)\,, \end{array} \tag{8}$$

where $\pi$ is the natural projection and $\tilde{x}$ is the natural identification of a frame in $B$ with an element of $G(N)$ mentioned in § 2. The diagram (8) is clearly commutative.

It can be easily seen from (8) that the form $f^*\Xi$ induced from $\Xi$ on $Q$ by the Gauss map $f$ coincides with $III$:

$$III = f^*\Xi = -2i \sum_{i,r} \omega_i^r \wedge \bar{\omega}_i^r\,, \tag{9}$$

since we observe that

$$\begin{aligned} f^*\varphi_0^r &= \omega^r\,, \\ f^*\varphi_i^r &= \omega_i^r\,, \end{aligned} \tag{10}$$

and $\omega^r = 0$ by (6). We call $III$ the *third fundamental form* of the immersion $x$.

PROPOSITION 3.1. *The Gauss map $f$ is a constant map if and only if the immersion $x$ is totally geodesic.*

In fact, from (9) the Gauss map $f$ is a constant map if and only if $III$ vanishes identically, i.e. $\omega_i^r = 0$ identically. Moreover, we have

THEOREM 3.2. *The Gauss map $f$ is an anti-holomorphic mapping.*

It suffices to note that $\omega_i^r$ are 1-forms of type (1, 0) on $M$. Then the second equation of (10) shows that $f$ is anti-holomorphic.

**4. Results.** First, we state the relation (7) as

THEOREM 4.1. *Suppose that a Kaehlerian n-manifold $M$ is holomorphically and isometrically immersed into a simply connected complete Kaehlerian $N$-space $V$ of constant holomorphic sectional curvature $2\varepsilon$. Then the relation* (7) *holds among the Ricci form $\Psi$ on $M$, the fundamental 2-form $\Phi$ of $M$ and the third fundamental form $III$ of the immersion.*

Note that from (7) $\Psi$ is proportional to $\Phi$ if and only if $III$ is. Thus we have

THEOREM 4.2. *Let $M$ and $V$ be as above. Then $M$ is an Einstein manifold if and only if the Gauss map $f$ is a homothety or a constant map.*

If, in particular, $M$ is a complex hypersurface of $V$, i.e.$n = N - 1$, then the case in which the Gauss map is homothetic is very limited. For example, let $V$ be a complex $(n + 1)$-space $C^{n+1}$. Then the scalar curvature $S = 2\sum_i S_{i\bar{i}}$ of $M$ is non-positive. On the other hand, the Gauss map can be viewed, by projecting $Q$ onto $G_{n,n+1}(C) = P^n(C)$, as a mapping of $M$ into a complex projective $n$-space $P^n(C)$, which has a positive scalar curvature. Hence there exists no homothety between $M$ and $Q$, since every homothety preserves the sign of the scalar curvature. More precisely, we obtain

THEOREM 4.3. *Let $M$ be a complex hypersurface immersed into a simply connected complete Kaehlerian $(n + 1)$-space $V$ of constant holomorphic sectional curvature $2\varepsilon$. If the Gauss map $f$ is a homothety, then $V$ must be the complex projective $(n + 1)$-space $P^{n+1}(C)$ and $f$ is an isometry of $M$ into $Q = P^{n+1}(C)$.*

It should be remarked that, on account of Proposition 3.1 and Theorem 4.2, this theorem is equivalent to the classification theorem of Smyth [1, 6] for complex Einstein hypersurfaces of $V$, which states that such an $M$ is totally geodesic or else $\varepsilon > 0$ and $M$ is locally holomorphically isometric to the complex hyperquadric $Q^n(C)$ in $P^{n+1}(C)$. In fact, we have only to note here that for $Q^n(C)$ in $P^{n+1}(C)$, the connection forms $\omega_i^{n+1}$ coincide with $\omega^i$ under a suitable change of the frame field.

In the light of this example, it may be said that it is of particular interest to find sufficient conditions for the Gauss map to reduce to an isometry. To close the note, we give a new interpretation of a theorem of Ogiue [5] from this point of view. Namely,

THEOREM 4.4 *Let $M$ be a compact complex hypersurface imbedded into the complex projective $(n+1)$-space $P^{n+1}(C)$ of constant holomorphic sectional curvature 2. If every holomorphic sectional curvature of $M$ is positive, then the Gauss map $f$ is an isometry or a constant map.*

Of course, in Theorem 4.4 (Theorem 4.3), if the Gauss map $f$ is an isometry, then $M$ is (locally) a complex hyperquadric $Q^n(C)$ in $P^{n+1}(C)$, and the image of $f$ is also (in) a complex hyperquadric $Q^n(C)$ of $Q = P^{n+1}(C)$.

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