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CONFORMALLY FLAT RIEMANNIAN MANIFOLDS ADMITTING A TRANSITIVE GROUP OF ISOMETRIES II

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1. Introduction. Let M be an n-dimensional conformally flat Riemannian manifold which admits a transitive group of isometries. In [1], the author proved that M is isometric to one of the homogeneous Riemannian manifolds of following types:

(I) $$S^n(K)/\Gamma\ ,\quad E^n/\Gamma\ ,\quad H^n(-K)\ ,$$

(II) $$(S^r(K)/\Gamma)\times H^{n-r}(-K)\ ,\quad 2\leqq r\leqq n-2\ ,$$

(III) $$(E^1/\Gamma)\times H^{n-1}(-K)\ ,$$

(IV) $$(S^{n-1}(K)\times E^1)/\Gamma\ ,$$

where $S^m(K)$, E^m and $H^m(-K)$ denote a Euclidean m-sphere of radius $K^{-1/2}$, a Euclidean m-space and a hyperbolic m-space of curvature $-K$ respectively. N/Γ denotes a quotient space, where Γ is a group of isometries of N acting freely and properly discontinuously. And $\times$ denotes a Riemannian product.

J. A. Wolf (see [2]) classified the homogeneous Riemannian manifolds of the forms $S^n(K)/\Gamma$ and E^n/Γ. Thus, the problem left to us is to determine the groups Γ appearing in (IV). In [1], the author proved a few theorems about the structure of Γ. Making use of the theorem, we shall classify the manifolds of type IV completely.

2. Structure of Γ. We consider $S^{n-1}(K)$ as the set of vectors of norm $K^{-1/2}$ in a Euclidean vector space $\boldsymbol{R}^n$. Then, the group of all isometries of $S^{n-1}(K)$ is the orthogonal group $O(n)$. Let $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ denote the algebra of real quaternions and the multicative group of unit quaternions respectively. $\boldsymbol{Q}'=\{a=a_1+a_2i+a_3j+a_4k;\ \sum_{s=1}^4 a_s^2=1,\ a_s\in\boldsymbol{R}\}$ has an $SO(4l)$-representation ρ defined by

$$\rho\colon a\longrightarrow\overbrace{\begin{pmatrix}A & & \\ & \ddots & \\ & & A\end{pmatrix}}^{l}\ ,\quad \text{where}\quad A=\begin{pmatrix}a_1 & -a_2 & -a_3 & -a_4\\ a_2 & a_1 & -a_4 & a_3\\ a_3 & a_4 & a_1 & -a_2\\ a_4 & -a_3 & a_2 & a_1\end{pmatrix}.$$

The subgroup $C' = \{b = a_1 + a_2 i;\, a_1^2 + a_2^2 = 1,\, a_s \in \boldsymbol{R}\}$ has an $SO(2l)$-representation σ defined by

$$\sigma\colon b \longrightarrow \overbrace{\begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix}}^{l}, \quad \text{where} \quad B = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}.$$

The subgroup $\{\pm 1\}$ has an $O(n)$-representation ε given by

$$\varepsilon\colon \pm 1 \longrightarrow \pm \overbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}^{n}.$$

Then, ρ, σ and ε are faithful. Hereafter, we identify $\boldsymbol{Q}'$, $\boldsymbol{C}'$ and $\{\pm 1\}$ with the closed subgroups $\rho(\boldsymbol{Q}')$, $\sigma(\boldsymbol{C}')$ and $\varepsilon(\{\pm 1\})$ of $SO(4l)$, $SO(2l)$ and $O(n)$ respectively.

On the other hand, the additive group $\boldsymbol{R}$ is considered as the group of all parallel translations of E^1. $\boldsymbol{R}$ is a normal subgroup of the group $E(1)$ of all isometries of E^1.

THEOREM 2.1 (see [1]). (i) $(S^{n-1}(K) \times E^1)/\Gamma$ *is a homogeneous Riemannian manifold if and only if* Γ *is a discrete subgroup of* $\boldsymbol{Q}' \times \boldsymbol{R}$, $\boldsymbol{C}' \times \boldsymbol{R}$ *or* $\{\pm 1\} \times \boldsymbol{R}$*). (ii) *Let* H *be one of* $\boldsymbol{Q}'$, $\boldsymbol{C}'$ *and* $\{\pm 1\}$. *Then, a discrete subgroup* Γ *of* $H \times \boldsymbol{R}$ *is of the form*

(a) $\Gamma_1 \times \{0\}$ *or*

(b) *semi-direct product group* $\langle (x, y) \rangle \cdot (\Gamma_1 \times \{0\})$,

where Γ_1 *is a finite subgroup of* H, x *an element of the normalizer* $N(\Gamma_1)$ *of* Γ_1 *in* H *and* y *a positive real number.*

3. Finite subgroups of $\boldsymbol{Q}'$ (see Wolf [2]). Let $\boldsymbol{R}^3$ be viewed as the space of pure imaginary quaternions with basis $\{i, j, k\}$. Then, we have a map $\pi\colon \boldsymbol{Q}' \longrightarrow SO(3)$ defined by $\pi(x)(x') = xx'x^{-1}$. π is a two to one ($\pi(x) = \pi(-x)$) homomorphism of $\boldsymbol{Q}'$ onto $SO(3)$, and is a differential covering of $SO(3)$ by S^3. Thus, $\boldsymbol{Q}'$ is the universal covering group of $SO(3)$.

Every finite subgroup of $SO(3)$ appears as a group of symmetries of a regular polygon or a regular polyhedron in $\boldsymbol{R}^3$. And these groups are given in terms of generators and relations as follows:

*) For our purpose, we identify two subgroups which are conjugate in $O(n) \times E(1)$.

	order	generators	relations
Z_m	m	A	$A^m = 1$
D_m	$2m$	A, B	$A^m = B^2 = 1,\ BAB^{-1} = A^{-1}$
T	12	A, P, Q	$A^3 = P^2 = Q^2 = 1,\ PQ = QP,$ $APA^{-1} = Q,\ AQA^{-1} = PQ$
O	24	A, P, Q, R	$A^3 = P^2 = Q^2 = 1,\ PQ = QP,$ $APA^{-1} = Q,\ AQA^{-1} = PQ,$ $RAR^{-1} = A^{-1},\ RPR^{-1} = QP,\ RQR^{-1} = Q^{-1}$
I	60	A, B, C	$A^3 = B^2 = C^5 = ABC = 1$

D_m, T, O and I are called the dihedral, tetrahedral, octahedral and icosahedral groups respectively. T, I and O are isomorphic to the alternating groups A_4, A_5 and the symmetric group S_4 respectively. The binary dihedral, binary tetrahedral, binary octahedral and binary icosahedral groups are defined by

$$D_m^* = \pi^{-1}(D_m),\ T^* = \pi^{-1}(T),\ O^* = \pi^{-1}(O) \quad \text{and} \quad I^* = \pi^{-1}(I)\ .$$

Wolf proved

THEOREM 3.1. *Every finite subgroup of Q' is a cyclic, binary dihedral, binary tetrahedral, binary octahedral, or binary icosahedral group. If two finite subgroups of Q' are isomorphic then they are conjugate in Q'. A finite subgroup of Q' is contained in a complex subfield of Q if and only if it is cyclic, contained in the real subfield if and only if it is cyclic of order* 1 *or* 2.

For example, we can choose generators of these groups as follows:

	order	generators	relations
Z_m	m	$a = \cos(2\pi/m) + i\sin(2\pi/m)$	$a^m = 1$
D_m^*	$4m$	$a = \cos(\pi/m) + i\sin(\pi/m),\ j$	$a^m = j^2 = -1,\ jaj^{-1} = a^{-1}$
T^*	24	$a = (1/2)(1 + i + j + k),\ i,\ j$	$a^3 = i^2 = j^2 = -1,\ ij = -ji,$ $aia^{-1} = j,\ aja^{-1} = ij$
O^*	48	$a = (1/2)(1 + i + j + k),\ i,\ j,$ $b = (1/\sqrt{2})(i - k)$	$a^3 = i^2 = j^2 = -1,\ ij = -ji,$ $aia^{-1} = j,\ aja^{-1} = ij,$ $bib^{-1} = ji,\ bab^{-1} = a^{-1},\ bjb^{-1} = j^{-1}$
I^*	120	$a = (1/2)(1 + i + j + k),$ $b = \frac{\sqrt{5}-1}{4}i + \frac{\sqrt{5}+1}{4}j + \frac{1}{2}k,$ $c = \frac{\sqrt{5}+1}{4} + \frac{\sqrt{5}-1}{4}j + \frac{1}{2}k$	$a^3 = b^2 = c^5 = abc = -1$ $(a^{-1}bacba = -i,\ bacb = -j)$

Hereafter, we restrict the meaning of the notations $\boldsymbol{Z}_m$, $\boldsymbol{D}_m^*$, $\boldsymbol{T}^*$, $\boldsymbol{O}^*$ and $\boldsymbol{I}^*$ to those of the above table.

LEMMA 3.1. *For a subgroup G of $\boldsymbol{Q}'$, we denote by $N(G)$ the normalizer of G in $\boldsymbol{Q}'$. Then*

(i) $N(\boldsymbol{Z}_1) = N(\boldsymbol{Z}_2) = \boldsymbol{Q}'$,

(ii) $N(\boldsymbol{Z}_m) = \boldsymbol{C}' \cup \boldsymbol{C}'j \quad (m \geqq 3)$,

(iii) $N(\boldsymbol{D}_m^*) = \boldsymbol{D}_{2m}^* \quad (m \geqq 3)$,

(iv) $N(\boldsymbol{D}_2^*) = N(\boldsymbol{T}^*) = N(\boldsymbol{O}^*) = \boldsymbol{O}^*, \quad N(\boldsymbol{I}^*) = \boldsymbol{I}^*$.

PROOF. (i) It is obvious. (ii) Let $d = a_1 + a_2 i$ and $x = x_1 + x_2 i + x_3 j + x_4 k$ be elements of $\boldsymbol{Q}'$. Then we have

$$xdx^{-1} = a_1 + a_2(x_1^2 + x_2^2 - x_3^2 - x_4^2)i + 2a_2(x_1x_4 + x_2x_3)j + 2a_2(x_2x_4 - x_1x_3)k \,. \tag{3.1}$$

Thus, if $x \in N(\boldsymbol{Z}_m)$ and $d = a_1 + a_2 i \in \boldsymbol{Z}_m$ $(a_2 \neq 0)$, then $x_1x_4 + x_2x_3 = x_2x_4 - x_1x_3 = 0$, which implies $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$, that is, $x \in \boldsymbol{C}' \cup \boldsymbol{C}'j$. Conversely, if $d \in \boldsymbol{Z}_m$ and $x \in \boldsymbol{C}' \cup \boldsymbol{C}'j$, then $xdx^{-1} = d^{\pm 1} \in \boldsymbol{Z}_m$, that is, $x \in N(\boldsymbol{Z}_m)$. (iii) First, we note that $\boldsymbol{D}_m^* = \boldsymbol{Z}_{2m} \cup \boldsymbol{Z}_{2m}j$. If $x \in N(\boldsymbol{D}_m^*)$ and $d = a_1 + a_2 i \in \boldsymbol{D}_m^*$ $(a_1a_2 \neq 0)$, then, by (3.1), $xdx^{-1} \in \boldsymbol{Z}_{2m}$, that is, $x \in \boldsymbol{C}' \cup \boldsymbol{C}'j$. But, if $x \in \boldsymbol{C}' \cap N(\boldsymbol{D}_m^*)$ and $zj \in \boldsymbol{C}'j \cap N(\boldsymbol{D}_m^*)$, then $xjx^{-1} = x^2 j \in \boldsymbol{D}_m^*$ and $(zj)j(zj)^{-1} = z^2 j \in \boldsymbol{D}_m^*$, that is, $x \in \boldsymbol{Z}_{4m}$ and $zj \in \boldsymbol{Z}_{4m}j$ respectively. Thus, $N(\boldsymbol{D}_m^*) \subset \boldsymbol{D}_{2m}^*$. $N(\boldsymbol{D}_m^*) \supset \boldsymbol{D}_{2m}^*$ is clear. (iv) First, we note that

(1) $\boldsymbol{D}_4^* \supset \boldsymbol{D}_2^*$ (normal subgroup) ,

(2) $\boldsymbol{D}_{2m}^* \supset \boldsymbol{D}_2^*$ (not normal subgroup) , $(m \geqq 3)$,

(3) $\boldsymbol{T}^* \supset \boldsymbol{D}_2^*$ (normal subgroup) ,

(4) $\boldsymbol{O}^* \supset \boldsymbol{D}_2^*$ (normal subgroup) ,

(5) $\boldsymbol{I}^* \supset \boldsymbol{D}_2^*$ (not normal subgroup) ,

(6) $\boldsymbol{O}^* \supset \boldsymbol{T}^*$ (normal subgroup) ,

(7) $\boldsymbol{I}^* \supset \boldsymbol{T}^*$ (not normal subgroup) and

(8) $\boldsymbol{O}^* \supset \boldsymbol{D}_4^*$.

(1) ~ (8) are obvious from the last table and the fact that $\pi(\boldsymbol{I}^*) \cong A_5$ is a simple group. Next, we prove that $N(\boldsymbol{D}_2^*)$, $N(\boldsymbol{T}^*)$, $N(\boldsymbol{O}^*)$ and $N(\boldsymbol{I}^*)$ are finite groups. Let $x = x_1 + x_2 i + x_3 j + x_4 k$ be an element of $\boldsymbol{Q}'$. Then,

$$xix^{-1} = (x_1^2 + x_2^2 - x_3^2 - x_4^2)i + 2(x_1x_4 + x_2x_3)j + 2(x_2x_4 - x_1x_3)k$$

and

$$xjx^{-1} = 2(x_2x_3 - x_1x_4)i + (x_1^2 - x_2^2 + x_3^2 - x_4^2)j + 2(x_1x_2 + x_3x_4)k .$$

We put

$$f_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2,\ f_2 = x_1^2 + x_2^2 - x_3^2 - x_4^2,\ f_3 = 2(x_1x_4 + x_2x_3) ,$$

$f_4 = 2(x_2x_4 - x_1x_3)$, $f_5 = 2(x_2x_3 - x_1x_4)$, $f_6 = x_1^2 - x_2^2 + x_3^2 - x_4^2$ and $f_7 = 2(x_1x_2 + x_3x_4)$. Then, a map $f: \boldsymbol{R}^4 \rightarrow \boldsymbol{R}^7$ is defined by $f(x_1, \cdots, x_4) = (f_1, \cdots, f_7)$. Then, it is easy to show that rank $df_x = 4$ for every $x \in \boldsymbol{R}^4 - \{0\}$. That is, for each point $v \in \boldsymbol{R}^7$, $f^{-1}(v) \cap S^3(1)$ is a finite set. Since $i, j \in \boldsymbol{D}_2^*$, the above fact shows that $N(\boldsymbol{D}_2^*)$, $N(\boldsymbol{T}^*)$, $N(\boldsymbol{O}^*)$ and $N(\boldsymbol{I}^*)$ are finite set. Now the lemma is evident by (1) $\sim$ (8) and Theorem 3.1.

4. **Condition for Γ and Γ' to be conjugate.** Let Γ and Γ' be two groups appearing in Theorem 2.1, that is,

$$\Gamma = \langle (x, y)\rangle \cdot (\Gamma_1 \times \{0\}) , \quad \Gamma' = \langle (x', y')\rangle \cdot (\Gamma_1' \times \{0\}) .$$

LEMMA 4.1. *Γ and Γ' are conjugate in $H \times \boldsymbol{R}$ if and only if* (i) *$y = y'$ and there exists an element t of H satisfying* (ii) *$t\Gamma_1t^{-1} = \Gamma_1'$ and* (iii) *$x^{-1}t^{-1}x't \in \Gamma_1$.*

PROOF. Let (t, u) be an element of $H \times \boldsymbol{R}$ satisfying $(t, u)\Gamma(t^{-1}, -u) = \Gamma'$. Then, we have

$$\bigcup_{k\in \boldsymbol{Z}} \bigcup_{s_i \in \Gamma_1} (tx^k s_i t^{-1}, ky) = \bigcup_{k \in \boldsymbol{Z}} \bigcup_{s_j' \in \Gamma_1'} ((x')^k s_j', ky') .$$

In particular, we have (i), (ii) and (iii). Conversely, let t be an element of H satisfying (ii) and (iii). We show that $(t, 0)\Gamma(t^{-1}, 0) = \Gamma'$. First, (iii) implies $x^{-k}t^{-1}(x')^kt \in \Gamma_1$ for every $k \in \boldsymbol{Z}$. In fact, by the induction, we have $x^{-k}t^{-1}(x')^kt \in \Gamma_1$ for $k \geqq 0$. And $x^kt^{-1}(x')^{-k}t = x^k(x^{-k}t^{-1}(x')^kt)^{-1}x^{-k} \in \Gamma_1$. Then, we have $(x')^k\Gamma_1' = tx^k\Gamma_1t^{-1}$ by (ii), and hence $(t, 0)\Gamma(t^{-1}, 0) = \Gamma'$ by (i). q.e.d.

Now, we shall classify the groups of the form $\Gamma = \langle (x, y)\rangle\ (\Gamma_1 \times \{0\})$ up to the conjugate classes in $H \times \boldsymbol{R}$. By Lemma 4.1, we may assume that Γ_1 is $\boldsymbol{Z}_m$, $\boldsymbol{D}_m^*$, $\boldsymbol{T}^*$, $\boldsymbol{O}^*$ or $\boldsymbol{I}^*$ of the table in Section 3. And $\Gamma = \langle (x, y)\rangle \cdot (\Gamma_1 \times \{0\})$ and $\Gamma' = \langle (x', y)\rangle \cdot (\Gamma_1 \times \{0\})$ are conjugate in $H \times \boldsymbol{R}$ if and only if there exists $t \in N(\Gamma_1)$ satisfying $x^{-1}t^{-1}x't \in \Gamma_1$. This assertion is detailed as follows:

(I) $H = \boldsymbol{Q}'$:

(a) $\Gamma_1 = \boldsymbol{Z}_1$: $\Gamma \sim \Gamma'$*) if and only if there exists $t \in \boldsymbol{Q}'$ satisfying $x' = txt^{-1}$. But it is easy to see that $\boldsymbol{Q}' = \{tC't^{-1}; t \in \boldsymbol{Q}'\}$. Thus, we may

*) $\Gamma \sim \Gamma'$ means that Γ and Γ' are conjugate in $H \times \boldsymbol{R}$.

assume $x, x' \in C'$. If $txt^{-1} \in C'$ $(x \neq \pm 1)$, then $t \in C' \cup C'j$ and $txt^{-1} = x$ or x^{-1}. That is, $\Gamma \sim \Gamma'$ if and only if $x' = x$ or x^{-1}.

(b) $\Gamma_1 = \boldsymbol{Z}_2$: In the same manner as (a), we may assume $x, x' \in C'$, and $\Gamma \sim \Gamma'$ if and only if $x' = x, x^{-1}, -x$ or $-x^{-1}$.

(c) $\Gamma_1 = \boldsymbol{Z}_m (m \geqq 3)$: $\Gamma \sim \Gamma'$ if and only if there exists $t \in C' \cup C'j$ satisfying $x^{-1}t^{-1}x't \in \boldsymbol{Z}_m$. Then we have (1) $\sim$ (3):

(1) $x, x' \in C'$: $\Gamma \sim \Gamma'$ if and only if $x^{-1}x' \in \boldsymbol{Z}_m$ or $x^{-1}(x')^{-1} \in \boldsymbol{Z}_m$.

(2) $x, x' \in C'j$: $\Gamma \sim \Gamma'$, since $x^{-1}tx't = 1 \in \boldsymbol{Z}_m$ for $t \in C'$ satisfying $t^2 = x'x^{-1}$.

(3) $x \in C'j$, $x' \in C'$: $\Gamma \not\sim \Gamma'$, since $tC't^{-1} \subset C'$ and $tC'jt^{-1} \subset C'j$ for every $t \in C' \cup C'j$.

(d) $\Gamma_1 = \boldsymbol{D}_2^*$: $\Gamma \sim \Gamma'$ if and only if there exists $t \in \boldsymbol{O}^*$ satisfying $x^{-1}t^{-1}x't \in \boldsymbol{D}_2^*$. We note that $\boldsymbol{O}^* = \boldsymbol{D}_2^* \cup (a\boldsymbol{D}_2^* \cup a^2\boldsymbol{D}_2^*) \cup (b\boldsymbol{D}_2^* \cup ba\boldsymbol{D}_2^* \cup ba^2\boldsymbol{D}_2^*)$, where $a = (1/2)(1+i+j+k)$ and $b(1/\sqrt{2})(i-k)$. Then we have (1) $\sim$ (6):

(1) $x, x' \in s\boldsymbol{D}_2^*$ $(s \in \boldsymbol{O}^*)$: $\Gamma \sim \Gamma'$, since $x^{-1}x' \in \boldsymbol{D}_2^*$.

(2) $x \in a\boldsymbol{D}_2^*$, $x' \in a^2\boldsymbol{D}_2^*$: $\Gamma \sim \Gamma'$, since $x^{-1}bx'b^{-1} \in \boldsymbol{D}_2^*$.

(3) $x \in b\boldsymbol{D}_2^*$, $x' \in ba\boldsymbol{D}_2^*$: $\Gamma \sim \Gamma'$, since $x^{-1}a^{-1}x'a \in \boldsymbol{D}_2^*$.

(4) $x \in b\boldsymbol{D}_2^*$, $x' \in ba^2\boldsymbol{D}_2^*$: $\Gamma \sim \Gamma'$, since $x^{-1}a^{-2}x'a^2 \in \boldsymbol{D}_2^*$.

(5) $x \in a\boldsymbol{D}_2^*$ (or $x \in b\boldsymbol{D}_2^*$), $x' \in \boldsymbol{D}_2^*$: $\Gamma \not\sim \Gamma'$.

(6) $x \in b\boldsymbol{D}_2^*$, $x' \in a\boldsymbol{D}_2^*$: $\Gamma \not\sim \Gamma'$.

(e) $\Gamma_1 = \boldsymbol{D}_m^*$ $(m \geqq 3)$: $\Gamma \sim \Gamma'$ if and only if there exists $t \in \boldsymbol{D}_{2m}^*$ satisfying $x^{-1}t^{-1}x't \in \boldsymbol{D}_m^*$. We note that $\boldsymbol{D}_{2m}^* = \boldsymbol{D}_m^* \cup s\boldsymbol{D}_m^*$, where $s \in \boldsymbol{D}_{2m}^* - \boldsymbol{D}_m^*$. Then we have (1) $\sim$ (3):

(1) $x, x' \in \boldsymbol{D}_m^*$: $\Gamma \sim \Gamma'$.

(2) $x \in \boldsymbol{D}_{2m}^* - \boldsymbol{D}_m^*$, $x' \in \boldsymbol{D}_m^*$: $\Gamma \not\sim \Gamma'$.

(3) $x, x' \in \boldsymbol{D}_{2m}^* - \boldsymbol{D}_m^*$: $\Gamma \sim \Gamma'$.

(f) $\Gamma_1 = \boldsymbol{T}^*$: $\Gamma \sim \Gamma'$ if and only if there exists $t \in \boldsymbol{O}^*$ satisfying $x^{-1}t^{-1}x't \in \boldsymbol{T}^*$. We note that $\boldsymbol{O}^* = \boldsymbol{T}^* \cup b\boldsymbol{T}^*$. Then we have (1) $\sim$ (3):

(1) $x, x' \in \boldsymbol{T}^*$: $\Gamma \sim \Gamma'$.

(2) $x \in \boldsymbol{O}^* - \boldsymbol{T}^*$, $x' \in \boldsymbol{T}^*$: $\Gamma \not\sim \Gamma'$.

(3) $x, x' \in \boldsymbol{O}^* - \boldsymbol{T}^*$: $\Gamma \sim \Gamma'$.

(g) $\Gamma_1 = \boldsymbol{O}^*$: $\Gamma \sim \Gamma'$.

(h) $\Gamma_1 = \boldsymbol{I}^*$: $\Gamma \sim \Gamma'$.

(II) $H = C'$, $\Gamma_1 = \boldsymbol{Z}_m$ $(m = 1, 2, 3, \cdots)$: $\Gamma \sim \Gamma'$ if and only if $x^{-1}x' \in \boldsymbol{Z}_m$.

(III) $H = \{\pm 1\}$, $\Gamma_1 = \boldsymbol{Z}_m$ $(m = 1, 2)$: $\Gamma \sim \Gamma'$ if and only if $x^{-1}x' \in \boldsymbol{Z}_m$.

Next, we check the condition that $\Gamma = \langle (x, y) \rangle \cdot (\Gamma_1 \times \{0\})$ and $\Gamma' = \langle (x', y) \rangle \cdot (\Gamma_1 \times \{0\})$ are conjugate in $I(S^{n-1}(K) \times E^1) = O(n) \times E(1)$. It is easy to see that Γ and Γ' are conjugate in $O(n) \times E(1)$ if and only if there exists an element A of the normalizer of Γ_1 in $O(n)$ satisfying $x^{-1}A^{-1}x'A \in \Gamma_1$ or $xA^{-1}x'A \in \Gamma_1$. Then, it is easy to check that, if Γ and

Γ' are not conjugate in $\boldsymbol{Q}' \times \boldsymbol{R}$, then they are not conjugate in $O(4l) \times E(1)$. Γ and Γ' are conjugate in $O(4l+2) \times E(1)$ if and only if $x^{-1}x' \in \Gamma_1$ or $xx' \in \Gamma_1$. Γ and Γ' are conjugate in $O(2l+1) \times E(1)$ if and only if $x^{-1}x' \in \Gamma_1$ or $xx' \in \Gamma_1$.

5. Classification of $(S^{n-1}(K) \times E^1)/\Gamma$. Summing up the results of Section 4, we have the classification of the manifolds of the form $(S^{n-1}(K) \times E^1)/\Gamma$:

(I) $(S^{4l-1}(K) \times E^1)/\Gamma$:

(i) $\Gamma = \langle (x, y) \rangle$, where $x = \cos\theta + i \sin\theta$ $(0 \leqq \theta \leqq \pi)$.

(ii) $\Gamma = \langle (x, y) \rangle \cdot (\boldsymbol{Z}_2 \times \{0\})$, where $x = \cos\theta + i\sin\theta$ $(0 \leqq \theta \leqq \pi/2)$.

(iii) $\Gamma = \langle (x, y) \rangle \cdot (\boldsymbol{Z}_m \times \{0\})$ $(m = 3, 4, \cdots)$, where

$$x = \cos\theta + i \sin\theta \ (0 \leqq \theta \leqq \pi/m)$$

or j.

(iv) $\Gamma = \langle (x, y) \rangle \cdot (\boldsymbol{D}_2^* \times \{0\})$, where $x = 1$, $(1/2)(1 + i + j + k)$ or $(1/\sqrt{2})(i - k)$.

(v) $\Gamma = \langle (x, y) \rangle \cdot (\boldsymbol{D}_m^* \times \{0\})$ $(m = 3, 4, \cdots)$, where $x = 1$ or $\cos(\pi/2m) + i \sin(\pi/2m)$.

(vi) $\Gamma = \langle (x, y) \rangle \cdot (\boldsymbol{T}^* \times \{0\})$, where $x = 1$ or $(1/\sqrt{2})(i - k)$.

(vii) $\Gamma = \langle (1, y) \rangle \cdot (\boldsymbol{O}^* \times \{0\})$.

(viii) $\Gamma = \langle (1, y) \rangle \cdot (\boldsymbol{I}^* \times \{0\})$.

(ix) $\Gamma = \Gamma_1 \times \{0\}$, where $\Gamma_1 = \boldsymbol{Z}_m$ $(m = 1, 2, 3, \cdots)$, $\boldsymbol{D}_m^*$ $(m = 2, 3, \cdots)$, $\boldsymbol{T}^*$, $\boldsymbol{O}^*$ or $\boldsymbol{I}^*$.

(II) $(S^{4l+1}(K) \times E^1)/\Gamma$:

(i) $\Gamma = \langle (x, y) \rangle \cdot (\boldsymbol{Z}_m \times \{0\})$ $(m = 1, 2, 3, \cdots)$, where

$$x = \cos\theta + i \sin\theta \quad (0 \leqq \theta \leqq \pi/m) .$$

(ii) $\Gamma = \boldsymbol{Z}_m \times \{0\}$ $(m = 1, 2, 3, \cdots)$

(III) $(S^{2l}(K) \times E^1)/\Gamma$:

(i) $\Gamma = \langle (1, y) \rangle$, $\langle (-1, y) \rangle$ or $\langle (1, y) \rangle \cdot (\boldsymbol{Z}_2 \times \{0\})$.

(ii) $\Gamma = \boldsymbol{Z}_1 \times \{0\}$ or $\boldsymbol{Z}_2 \times \{0\}$.

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