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THE INJECTIVE RADIUS OF NON-COMPACT 3-DIMENSIONAL RIEMANNIAN MANIFOLDS

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In this note all Riemannian manifolds which we deal are connected and complete. For a point $p \in M$, $T_p(M)$ be the tangent space of M at p and $\exp_p$: $T_p(M) \rightarrow M$ be the exponential mapping of M. d denotes the metric distance of M induced from the Riemannian metric of M. All geodesics are parametrized by the arclength. As is well known, the function $i: M \rightarrow R \cup \{\infty\}$ defined by $i(p): = d(p, C(p))$ is continuous where $C(p)$ denotes the cut locus of p in M. $i(p)$ is called the injective radius of $\exp_p$. With respect to the estimation of the injective radius many results are known when M is compact. Let M be a non-compact Riemannian manifold. Then in [5], Toponogov asserted the following:

FACT. 1) if the sectional curvature K_σ satisfies $0 < K_\sigma \leqq \lambda$ for all tangent plane σ, then $i(q) \geqq \pi/\sqrt{\lambda}$ for all $q \in M$. 2) if $0 \leqq K_\sigma \leqq \lambda$, then there exists a positive constant L such that $i(q) \geqq L$ for all $q \in M$.

In [4], the author gave an anothor proof of assertion 1) and showed that the estimation of 1) is still true for a 2-dimensional simply connected Riemannian manifold M which satisfies $0 \leqq K \leqq \lambda$, where K is the Gaussian curvature of M. In this note, we show that the estimation 1) is still true for a 3-dimensional simply connected non-compact Riemannian manifold which satisfies $0 \leqq K_\sigma \leqq \lambda$. To prove this fact, we use the following facts which are proved by Cheeger and Gromoll in [2]. For a Riemannian manifold M, a subset A of M will be called totally convex if for any points $p, q \in A$ and any geodesic $c: [0, \beta] \rightarrow M$ from p to q, we have $c([0, \beta]) \subset A$. Let $A \subset M$ be a closed totally convex set, then A is an imbedded k-dimensional topological submanifold of M with totally geodesic interior and possibly non-smooth boundary which might be empty, see [2, Th. 1.6 pp 418]. Now, we assume that M is non-compact and its sectional curvature satisfies $0 \leqq K_\sigma$. Then, for a point $p \in M$, there exists a family of compact totally convex subsets $\{C_t\}_{t \geqq 0}$ such that

(1) $t_2 \geqq t_1$ implies $C_{t_2} \supset C_{t_1}$ and $C_{t_1} = \{q \in C_{t_2}: d(q, \partial C_{t_2}) \geqq t_2 - t_1\}$ in particular, $\partial C_{t_1} = \{q \in C_{t_2}: d(q, \partial C_{t_2}) = t_2 - t_1\}$,

(2) $\bigcup_{t \geqq 0} C_t = M$,

(3) $p \in C_0$ and if $\partial C_0 \neq \emptyset$, then $p \in \partial C_0$,

see [2; Prop. 1.3 pp 416]. Let C be a closed totally convex set. We set

$$C^a := \{q \in C : d(q, \partial C) \geqq a\}$$
$$C^{\max} := \bigcap_{C^a \neq \emptyset} C^a .$$

Then, for any $a \geqq 0$, C^a is totally convex and there exists $a_0 \geqq 0$ such that $C^{\max} = C^{a_0}$. Furthermore $\dim C^{\max} < \dim C$, see [2; Th. 1.9 pp 420]. For a family of totally convex sets $\{C_t\}_{t \geqq 0}$ as is mensioned above, if $\partial C_0 \neq \emptyset$, we set $C(1) := C_0$ and $C(2) := C(1)^{\max}$. Inductively, if $\partial C(i) \neq \emptyset$, we set $C(i+1) := C(i)^{\max}$ for $i = 1, 2, \cdots$. As is easily seen, we get the integer $k > 0$ such that $\partial C(k) = \emptyset$. We call $C(k)$ a soul of M and denote it by S. In the case $\dim C_0 = \dim M$, instead of $\{C_t\}_{t \geqq 0}$, we use a following family of totally convex sets $\{\tilde{C}_t\}_{t \geqq 0}$. Let $C_0^{a_0} = C_0^{\max}$. We set $\tilde{C}_0 := C_0^{a_0}$ and

$$\tilde{C}_t := \begin{cases} C_{t-a_0} & \text{if} \quad t \geqq a_0 \\ C_0^{a_0-t} & \text{if} \quad a_0 \geqq t \geqq 0 . \end{cases}$$

Then, thus obtained family $\{\tilde{C}_t\}_{t \geqq 0}$ also satisfies the property (1) and (2) for $\{C_t\}_{t \geqq 0}$. We do not use the property (3), so without confusion, we may denote again $\{\tilde{C}_t\}_{t \geqq 0}$ by $\{C_t\}_{t \geqq 0}$. Under this new index, $\dim C_t = \dim M$ for $t > 0$ and $\dim C_0 < \dim M$. And we also obtain a decreasing sequence of totally convex sets such that $C_0 = C(1), \cdots, C(k) = S$. Our assertion is:

THEOREM. *Let M be a simply connected 3-dimensional non-compact Riemannian manifold which satisfies $0 \leqq K_\sigma \leqq \lambda$, then*

$$i(q) \geqq \frac{\pi}{\sqrt{\lambda}} \quad \textit{for all} \quad q \in M .$$

For the moment, we assume that M is homeomorphic to E^3 and have the sectional curvature $0 \leqq K_\sigma \leqq \lambda$, where E^3 is a 3-dimentional Euclidean space. Let S be a soul of M. Then by [2; Th. 2.2 pp 423], S is a point set $\{s\}$, $s \in M$.

LEMMA 1. *For any soul $S = \{s\}$ of M, $i(s) \geqq \pi/\sqrt{\lambda}$.*

PROOF. If $i(s) < \pi/\sqrt{\lambda}$, then by the Theorem of Morse-Schoenberg and Lemma 2 [3; pp 226], there exists a geodesic loop $\gamma: [0, 2i(s)] \to M$ such that $\gamma(0) = \gamma(2i(s)) = s$. Then $\gamma([0, 2i(s)]) \subset \{s\}$, because $\{s\}$ is totally convex. This is a contradiction. q.e.d.

Let $p \in M$ be any point and $\{C_t\}_{t \geqq 0}$ be the family of the totally convex sets constructed from p. Under this situation, we have:

LEMMA 2. *For any point* $q \in C_0$, $i(q) \geqq \pi/\sqrt{\lambda}$.

PROOF. Assume that there exists a point $q_0^* \in C_0$ such that $i(q_0^*) < \pi/\sqrt{\lambda}$. Then by Lemma 1, $\partial C_0 \neq \emptyset$. Let $q_0 \in C_0$ be a point such that $i(q_0) = \min\{i(q): q \in C_0\}$. Then $i(q_0) \leqq i(q_0^*) < \pi/\sqrt{\lambda}$. Set $A: = \{q \in C_0: i(q) = i(q_0)\}$. Then by the compactness of A, there exists a point $q_1 \in A$ such that $d(q_1, \partial C_0) = \max\{d(q, \partial C_0): q \in A\}$. Set $t_1: = d(q_1, \partial C_0)$. Then $q_1 \in \partial C_0^{t_1}$ and $i(q_1) \leqq i(q_0^*) < \pi/\sqrt{\lambda}$. Then by the Theorem of Morse-Schoenberg and Lemma 2 [3], there exists a geodesic loop $\gamma_1: [0, 2i(q_1)] \rightarrow M$ such that $\gamma_1(0) = \gamma_1(2i(q_1)) = q_1$. Since $C_0^{t_1}$ is totally convex, we see $\gamma_1([0, 2i(q_1)]) \subset C_0^{t_1}$. Hence, by the choice of the point q_1, $i(\gamma_1(i(q_1))) = i(q_1)$. And again by Lemma 2 [3], γ_1 must be a closed geodesic. We also see $\gamma_1([0, 2i(q_1)]) \subset A$. And by the choice of the point q_1, we get $\gamma_1([0, 2i(q_1)]) \subset \partial C_0^{t_1}$. So $\gamma_1([0, 2i(q_1)]) = \partial C_0^{t_1}$, because $\dim C_0 \leqq 2$ and hence $\dim \partial C_0^{t_1} = 1$. By the choice of t_1 and continuity of the function i, we can choose t_2^* such that $t_1 < t_2^*$ and $\pi/\sqrt{\lambda} > \min\{i(q): q \in C_0^{t_2^*}\} > i(q_1)$. Let $q_2^* \in C_0^{t_2^*}$ be a point such that $i(q_2^*) = \min\{i(q): q \in C_0^{t_2^*}\}$ and $q_2 \in C_0^{t_2^*}$ be a point such that $d(q_2, \partial C_0) = \max\{d(q, \partial C_0): q \in C_0^{t_2^*}$ and $i(q) = i(q_2^*)\}$. Then $i(q_1) < i(q_2) < \pi/\sqrt{\lambda}$. By the same reason for q_1, there exists a closed geodesic γ_2: $[0, 2i(q_2)] \rightarrow M$ such that $\gamma_2(0) = \gamma_2(2i(q_2)) = q_2$. Set $t_2: = d(q_2, \partial C_0)$. Then we also have $\gamma_2([0, 2i(q_2)]) = \partial C_0^{t_2}$. Since $C_0^{t_1}$ and $C_0^{t_2}$ are homeomorphic to a 2-dimensional disk, by applying the Theorem of Gauss-Bonnet, we get

$$\iint_{C_0^{t_1}} K dv = \iint_{C_0^{t_2}} K dv = 2\pi ,$$

where K (resp. dv) is the Gaussian curvature (resp. the area element) of the totally geodesic surface $C_0^{t_1}$ of M and its totally geodesic surface $C_0^{t_2}$ having the boundary $\partial C_0^{t_1}$, $\partial C_0^{t_2}$. This equation means $K \equiv 0$ on $C_0^{t_1} - C_0^{t_2}$. That is $L(\gamma_1) = L(\gamma_2)$, where L denotes the length of a curve. Namely $2i(q_1) = 2i(q_2)$. This is a contradiction. q.e.d.

PROOF OF THE THEOREM. By the classification in [2; Th. 8.1 pp 438], M must be isometric to $\tilde{M} \times E^1$ or M is homeomorphic to E^3 where E^1 is a 1-dimensional Euclidean space and $\tilde{M}$ is homeomorphic to 2-dimensional sphere S^2. If M is isometric to $\tilde{M} \times E^1$, by using a result of [4], it is easily seen that our assertion is true. So we may assume that M is homeomorphic to E^3. We assume that there exists a point $q_0^* \in M$ such that $i(q_0^*) < \pi/\sqrt{\lambda}$ and derive a contradiction. Let p be a point of M. And $\{C_t\}_{t \geqq 0}$ be the familly of totally convex sets constructed from a point p. By Lemma 2, $q_0^* \notin C_0$. Choose a number $t_0 > 0$ such that $q_0^* \in C_{t_0}$. Let $q_0 \in C_{t_0}$ be a point such that $i(q_0) = \min\{i(q): q \in C_{t_0}\}$. Then $i(q_0) \leqq i(q_0^*) < \pi/\sqrt{\lambda}$. We set $A_1: = \{q \in C_{t_0}: i(q) = i(q_0)\}$. Then by Lemma 2, $A_1 \cap C_0 = \emptyset$. Since A_1 is compact, there exists a point $q_1 \in A_1$ such that $d(q_1, \partial C_{t_0}) = \max\{d(q, \partial C_{t_0}): q \in A_1\}$. Set $t_1: = d(q_1, \partial C_{t_0})$. Then $t_1 < t_0$ by Lemma 2. As is in the proof of Lemma 2, there exists a closed geodesic $\gamma_1: [0, 2i(q_1)] \to M$ such that $\gamma_1(0) = \gamma_1(2i(q_1)) = q_1$ and $\gamma_1([0, 2i(q_1)]) \subset \partial C_{t_0-t_1}$. By the choice of t_1 and the continuity of i, we can choose t_2^* such that $t_1 < t_2^* < t_0$ and $\pi/\sqrt{\lambda} > \min\{i(q): q \in C_{t_0-t_2^*}\} > i(q_1)$. $q_2^* \in C_{t_0-t_2^*}$ be a point such that $i(q_2^*) = \min\{i(q): q \in C_{t_0-t_2^*}\}$. Set $A_2: = \{q \in C_{t_0-t_2^*}: i(q) = i(q_2^*)\}$. Let $q_2 \in A_2$ be a point such that $d(q_2, \partial C_{t_0}) = \max\{d(q, \partial C_{t_0}): q \in A_2\}$. Set $t_2: = d(q_2, \partial C_{t_0})$. Then $t_2 < t_0$ by Lemma 2. And by the same reason for q_1, there exists a closed geodesic $\gamma_2: [0, 2i(q_2)] \to M$ such that $\gamma_2(0) = \gamma_2(2i(q_2)) = q_2$ and $\gamma_2([0, 2i(q_2)] \subset \partial C_{t_0-t_2}$. Continuing this operation, we obtain sequences $\{q_n\}$, $\{t_n\}$ and a family of closed geodesics $\gamma_n: [0, 2i(q_n)] \to M$ which satisfy the following conditions:

(1) $i(q_1) < i(q_2) < \cdots < i(q_n) < i(q_{n+1}) < \cdots < \pi/\sqrt{\lambda}$,

(2) $t_n: = d(q_n, \partial C_{t_0}), t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots < t_0$,

(3) $\gamma_n(0) = \gamma_n(2i(q_n)) = q_n, \gamma_n([0, 2i(q_n)]) \subset \partial C_{t_0-t_n}$.

For the sake of convenience, we extend the domain of γ_n as $\gamma_n: (-\infty, \infty) \to M$. We fix n and $\tilde{t} > t_0$. Then by [2; Th. 1.10 pp 420], the function $\psi: (-\infty, \infty) \to R$ defined by $\psi(u): = d(\gamma_n(u), \partial C_{\tilde{t}})$ is concave, i.e. for $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta = 1$, it holds $\psi(\alpha u_1 + \beta u_2) \geqq \alpha\psi(u_1) + \beta\psi(u_2)$. Since ψ is bounded, $\psi \equiv$ constant, say, $l > 0$. Let $c_{\tilde{t}}: [0, l] \to M$ be a minimal geodesic from $\gamma_n(0)$ to $\partial C_{\tilde{t}}$, then $\measuredangle(\dot{\gamma}_n(0), \dot{c}_{\tilde{t}}(0)) = \pi/2$ where $\measuredangle(v, w)$ denotes the angle between the vectors v and w. For if $\measuredangle(\dot{\gamma}_n(0), \dot{c}_{\tilde{t}}(0)) < \pi/2$, we can find $\tilde{u} > 0$ such that $d(\gamma_n(\tilde{u}), \partial C_{\tilde{t}}) < d(\gamma_n(0), \partial C_{\tilde{t}})$. That is $l = \psi(\tilde{u}) \leqq d(\gamma_n(\tilde{u}), \partial C_{\tilde{t}}) < d(\gamma_n(0), \partial C_{\tilde{t}}) = l$. This is a contradiction. Let X be the vector field along $c_{\tilde{t}}$ obtained by the parallel translation of $\dot{\gamma}_n(0)$. We define a differentiable mapping $V: [0, l] \times [0, \varepsilon] \to M$ by $V(s, u): = \exp_{c_{\tilde{t}}(s)} uX(s)$ where ε is a positive number. Set $V_u(s): = V(s,u)$. Then, by the convexity of $C_{\tilde{t}}$, $V_u(l) \notin \operatorname{int} C_{\tilde{t}}$ for $u \in [0, \varepsilon]$, see [1: Lemma 1.7 pp 419]. On the other hand, by the comparison theorem of Berger, if we put $\varepsilon_0: = \min\{\pi/(2\sqrt{\lambda}), \varepsilon\}$, then $L(V_u) \leqq L(V_0) = l$ for all $u \in [0, \varepsilon_0]$ and equality holding for some $u_0 \in (0, \varepsilon_0]$ if and only if $V|[0, l] \times [0, u_0]$ is a flat totally geodesic surface of M, see [1, Th. 1 pp 701]. Since we have seen $\psi \equiv l$ and $V_u(l) \notin \operatorname{int} C_{\tilde{t}}$, we get $l \leqq L(V_u) \leqq L(V_0) = l$ for all $u \in [0, \varepsilon_0]$. So $V|[0, l] \times [0, \varepsilon_0]$ defines a flat totally geodesic surface of M. Without confusion, $V: [0, l] \times [0, \varepsilon_0] \to M$ denotes the restriction $V|[0, l] \times [0, \varepsilon_0]$. We extend the surface $V: [0, l] \times [0, \varepsilon_0] \to M$ as $V: [0, l] \times [0, 2\varepsilon_0] \to M$ defining $V(s, u): = \exp_{V_{\varepsilon_0}(s)} u V_*(\partial/\partial u)_{|s,\varepsilon_0}$ for $u \in [\varepsilon_0, 2\varepsilon_0]$. Then we can also see that $V: [0, l] \times [0, 2\varepsilon_0] \to M$ is a flat totally geodesic surface of

M because $V|[0, l] \times [\varepsilon_0, 2\varepsilon_0]$ satisfies the same condition for V: $[0, l] \times [0, \varepsilon_0] \to M$. We extend V: $[0, l] \times [0, 2\varepsilon_0] \to M$ as V: $[0, l] \times [0, 3\varepsilon_0] \to M$ defining $V(s, u)$: $= \exp_{V_{2\varepsilon_0}(s)} u V_*(\partial/\partial u)_{|s,2\varepsilon_0}$ for $u \in [2\varepsilon_0, 3\varepsilon_0]$. V: $[0, l] \times [0, 3\varepsilon_0] \to M$ is also a flat totally geodesic surface of M. Continuing this method, finally we get an immersed flat totally geodesic surface V: $[0, l] \times (-\infty, \infty) \to M$ which is given by $V(s, u)$: $= \exp_{c_{\tilde{t}}(s)} uX(s)$. Set $Y(u)$: $= V_*(\partial/\partial s)_{|0,u}$. Then Y is a parallel vector field along γ_n.

ASSERTION 1. $Y(0) = Y(2i(q_n))$.

PROOF. We assume $Y(0) \neq Y(2i(q_n))$ and derive a contradiction. Set C_{q_n}: $= \{v \in T_{q_n}(M)$: $\exp_{q_n} u(v/\|v\|) \in C_{t_0-t_n}$ for some $u > 0\} \cup \{0\}$, T_{q_n}: $= \{v \in T_{q_n}(M)$: $\langle v, \dot{\gamma}_n(0)\rangle = 0$ and $\|v\| = 1\}$ and $C^*_{q_n}$: $= T_{q_n} \cap C_{q_n}$. C_{q_n} is called the tangent cone of $C_{t_0-t_n}$ at q_n, see [2]. Since $\dim C_{t_0-t_n} = 3$, T_{q_n} is isometric to the unit circle $S^1 = [0, 2\pi]$ and $C^*_{q_n}$ is the minor subarc of length $\alpha \in (0, \pi]$ by the convexity of $C_{t_0-t_n}$. Let φ: $[0, 2\pi](=S^1) \to T_{q_n}$ be the isometry such that $\varphi([0, \alpha]) = \bar{C}^*_{q_n}$ where closure is taken in T_{q_n}. Since $V_{2mi(q_n)}$ is a minimal geodesic from q_n to $\partial C_{\tilde{t}}$, we can easily see that $Y(2mi(q_n)) \in \varphi([\alpha + \pi/2, 2\pi - \pi/2])$ for $m = 0, 1, 2, \cdots$. Let $Y(0) = \varphi(\beta)$ for $\beta \in [\alpha + \pi/2, 2\pi - \pi/2]$. Then, by the assumption, without loss of generality, we can assume that $Y(2i(q_n)) = \varphi(\beta + \omega)$, where $\beta + \omega \langle 2\pi - \pi/2$ and $\omega > 0$. And it follows from the linearlity of the parallel displacement that $Y(2mi(q_n)) = \varphi(\beta + m\omega)$ and $\beta + m\omega < 2\pi - \pi/2$, because $\omega < \pi$ and $Y(2mi(q_n))$ is the parallel translation of $Y(2(m-1)i(q_n))$ along γ_n for $m = 1, 2, 3, \cdots$. But this is impossible. q.e.d.

From this assertion, we see the image of surface V is isometric to $[0, l] \times S^1(i(q_n)/\pi)$ where $S^1(r)$ denotes a circle of radius r. Let $\{\tilde{t}_k\}$ be a sequence such that $\tilde{t}_k \uparrow \infty$ and $\tilde{t}_1 > t_0$. For each $\tilde{t}_k$, let $c_{\tilde{t}_k}$: $[0, l_k] \to M$ be a minimal geodesic from $\gamma_n(0)$ to $\partial C_{\tilde{t}_k}$ where $l_k = d(\gamma_n(0), \partial C_{\tilde{t}_k})$. Since $\tilde{t} > t_0$ is any number, we can apply the above argument for each $\tilde{t}_k$ and we have a flat totally geodesic surface of M whose image is isometric to $[0, l_k] \times S^1(i(q_n)/\pi)$. We can choose subsequence $\{\tilde{t}_{k_j}\}$ of $\{\tilde{t}_k\}$ such that $\dot{c}_{\tilde{t}_{k_j}}(0) \to \dot{c}_n(0)$, $\dot{c}_n(0) \in T_{q_n}(M)$. Let P_n be the vector field along γ_n obtained by the parallel translation of $\dot{c}_n(0)$. Then by the construction, we can easily see that the surface given by the map V_n: $[0, \infty) \times (-\infty, \infty) \to M$ defined by $V_n(s, u)$: $= \exp_{\gamma_n(u)} sP_n(u)$ is an immersed flat totally surface of M and its image is isometric to $[0, \infty) \times S^1(i(q_n)/\pi)$. We denote the image of this surface by F_n. Now by the compactness of C_{t_0}, we can choose a subsequence $\{n_j\}$ of $\{n\}$ such that $\dot{\gamma}_{n_j}(0) \to \dot{\gamma}_\infty(0)$ and $\dot{c}_{n_j}(0) \to \dot{c}_\infty(0)$ where $\dot{\gamma}_\infty(0)$ and $\dot{c}_\infty(0) \in T_{q_\infty}(M)$, q_∞: $= \lim_{j\to\infty} q_{n_j} \in M$. Then the vector field P_∞ along the closed geodesic $\gamma_\infty(t)$: $= \exp t\dot{\gamma}_\infty(0)$ obtained by the parallel translation of $\dot{c}_\infty(0)$ satisfies $P_\infty(0) = P_\infty(2i(q_\infty))$ from the construction. And the surface given by the map $V_\infty\colon [0, \infty) \times (-\infty, \infty) \to M$ such that $V_\infty(s, u) := \exp_{\gamma_\infty(u)} sP_\infty(u)$ is an immersed flat totally geodesic surface of M whose image is isometric to $[0, \infty) \times S^1(i(q_\infty)/\pi)$. We also denote the image of this surface by F_∞. Hereafter, for the convenience, the sequence $\{m\}$ denotes the sequence $\{n_j\}$.

ASSERTION 2. *$F_m \cap F_\infty = \varnothing$ for all m.*

PROOF. We assume that there exists m_0 such that $F_{m_0} \cap F_\infty \neq \varnothing$ and derive a contradiction. Let ∂F_m and ∂F_∞ denote the image of the closed geodesic γ_m and γ_∞ respectively. And let $\operatorname{int} F_m := F_m - \partial F_m$, $\operatorname{int} F_\infty := F_\infty - \partial F_\infty$. We can consider two cases $\operatorname{int} F_{m_0} \cap \operatorname{int} F_\infty \neq \varnothing$ or $\partial F_m \cap \operatorname{int} F_\infty \neq \varnothing$ because $\partial F_\infty \subset \partial C_{t_0 - t_\infty}$ and $F_{m_0} \cap C_{t_0 - t_\infty} = \varnothing$ where $t_\infty := d(q_\infty, \partial C_{t_0})$. Suppose there exists a point $q \in \operatorname{int} F_{m_0} \cap \operatorname{int} F_\infty$. Since $\dim M = 3$ and $\dim F_{m_0} = \dim F_\infty = 2$, there exists a vector $v \in T_q(F_{m_0}) \cap T_q(F_\infty)$ such that $\|v\| = 1$. Let $c\colon (-\infty, \infty) \to M$ be the geodesic defined by $c(t) := \exp tv$. Then there exists a subarc of the geodesic c which is a geodesic in F_{m_0} and F_∞ because F_{m_0} and F_∞ are totally geodesic surface of M. We assert that $c((-\infty, \infty)) \cap \partial F_{m_0} \neq \varnothing$. For, if $c((-\infty, \infty)) \cap \partial F_{m_0} = \varnothing$, then as is easily seen, c is a closed geodesic in F_{m_0} and F_∞, since F_{m_0} and F_∞ are isometric to the half cylinder $[0, \infty) \times S^1(i(q_{m_0})/\pi)$ and $[0, \infty) \times S^1(i(q_\infty)/\pi)$ respectively. The fundamental period of a closed geodesic in F_{m_0} and F_∞ are $2i(q_{m_0})$ and $2i(q_\infty)$ respectively. So above fact means $2i(q_{m_0}) = 2i(q_\infty)$. This is a contradiction. So the assumption $\operatorname{int} F_{m_0} \cap \operatorname{int} F_\infty \neq \varnothing$ derives $\partial F_{m_0} \cap \operatorname{int} F_\infty \neq \varnothing$. Next we suppose $\partial F_{m_0} \cap \operatorname{int} F_\infty \neq \varnothing$. For each $u \in [0, 2i(q_{m_0})]$, let $c_u\colon [0, \infty) \to M$ be the geodesic defined by $c_u(s) := \exp_{\gamma_{m_0}(u)}(-s)P_{m_0}(u)$. Then for each $u \in [0, 2i(q_{m_0})]$, we will show $c_u([0, \infty)) \cap \operatorname{int} C_{t_0 - t_{m_0}} = \varnothing$. For, if some $u_0 \in [0, 2i(q_{m_0})]$ and $s_0 \in (0, \infty)$, $c_{u_0}(s_0) \in \operatorname{int} C_{t_0 - t_{m_0}}$, then by the total convexity of $C_{t_0 - t_{m_0}}$, $c_{u_0}((0, s_0]) \subset \operatorname{int} C_{t_0 - t_{m_0}}$. We define a differentiable mapping $V\colon [0, 2i(q_{m_0})] \times [0, \beta] \to M$ by $V(u, s) := c_u(s)$. We put $V_s(u) := V(u, s)$. Then by the comparison theorem of Berger, there exists an $\varepsilon_0 > 0$ depending on λ such that for all $0 \leqq s \leqq \varepsilon_0$, $L(V_s) \leqq L(V_0)$, see [1: Th. 1 pp 701]. By the assumption $V_{s_0}(u_0)(= c_{u_0}(s_0)) \in \operatorname{int} C_{t_0 - t_{m_0}}$ and by the choice of t_{m_0}, for all s, $0 < s \leqq s_0$, we get $i(V_s(u_0)) > i(V_0(u_0))$. Namely, for all $0 < s \leqq \min\{\varepsilon_0, s_0\}$, $i(V_s(u_0)) > i(V_0(u_0)) = (1/2)L(V_0) \geqq (1/2)L(V_s)$. Then by using the same technique which is used by W. Klingenberg to get the estimation of the injective radius of a certain compact Riemannian manifold, see [3: Th. pp 227], we can easily get a contradiction. For a point $\gamma_{m_0}(u^*) \in \partial F_{m_0} \cap \operatorname{int} F_\infty$, there exists uniquely $\tilde{u} \in [0, 2i(q_\infty(0))]$ and $\tilde{s}$ such that $\gamma_{m_0}(u^*) = \exp_{\gamma_\infty(\tilde{u})} \tilde{s} P_\infty(\tilde{u})$. From the construction of F_∞, the geodesic $\alpha\colon [0, \infty) \to M$ defined by $\alpha(s) := \exp_{\gamma_\infty(\tilde{u})} sP(\tilde{u})$ is a shortest connection from $\gamma_\infty(\tilde{u})$ to ∂C_t for each $t \geqq t_0$. And above fact shows that $\dot{\alpha}(\tilde{s}) \neq P_{m_0}(u^*)$ because $c_{u^*}([0, \infty)) \cap \operatorname{int} C_{t_0 - t_{m_0}} = \emptyset$ and $\gamma_\infty(\tilde{u}) \in \operatorname{int} C_{t_0 - t_{m_0}}$. The geodesic $\beta\colon [0, \infty) \to M$ defined by $\beta(s) := \exp_{\gamma_m(u^*)} sP_m(u^*)$ is also a minimal geodesic from $\gamma_m(u^*)$ to ∂C_t for each $t \geqq t_0$. In particular $\alpha | [0, \tilde{s} + t_m]$(resp. $\beta | [0, t_m]$) is a minimal connection from $\gamma_\infty(\tilde{u})$ (resp. $\gamma_m(u^*)$) to ∂C_{t_0} where $t_m = d(q_m, \partial C_{t_0})$. So, from the triangle inequality of distance, we can easily see $\dot{\alpha}(\tilde{s}) = \dot{\beta}(0) = P_m(u^*)$ and we get a contradiction.

q.e.d.

Now, since $F_m \to F_\infty$ as $m \to \infty$, we can easily find numbers m^* and s^* such that, for each minimal geodesic from a point of the set $\{s^*\} \times S^1(i(q_\infty)/\pi) \subset F_\infty$ to F_{m^*}, its end point lies in $\operatorname{int} F_{m^*}$. We consider that $\{s^*\} \times S^1(i(q_\infty)/\pi)$ is the image of the closed geodesic $\gamma_\infty\colon (-\infty, \infty) \to M$. F_{m^*} can be considered locally as a boundary of some convex set because F_{m^*} is a totally geodesic surface of M. And by the proof of Th. 1.10 [2: pp 420], the function $\varphi\colon (-\infty, \infty) \to R$ defined by $\varphi(s) := d(\gamma_\infty(s), F_{m^*})$ is concave. So φ must be constant $a > 0$, because φ is bounded. Let $c\colon [0, a] \to M$ be a minimal geodesic from $\gamma_\infty(0)$ to F_{m^*}. Then $\dot{c}(0) \perp F_{m^*}$ and $\dot{c}(0) \perp \dot{\gamma}_\infty(0)$. Let Z be the vector field along γ_∞ obtained by the parallel translation of $\dot{c}(0)$. Then, by the same argument just we have used to prove the fact $Y(0) = Y(2i(q_n))$, we can easily see $Z(0) = Z(2i(q_\infty))$ and the differentiable mapping $\tau\colon [0, a] \times [0, 2i(q_\infty)]$ defined by $\tau(u, s) := \exp_{\gamma_\infty(s)} uZ(s)$ gives a flat totally geodesic surface of M which is isometric to $[0, a] \times S^1(i(q_\infty)/\pi)$. Therefore we get $L(\tau_0) = L(\tau_a)$ where $\tau_u(s) := \tau(u, s)$. On the other hand τ_0 and τ_a are closed geodesics in F_∞ and F_{m^*} respectively. So $L(\tau_0) > L(\tau_a)$. This is a contradiction. q.e.d.

REMARK. For $\dim M = n \geqq 4$, this Theorem is not valid. Because M. Berger gave an example that on S^3, there exists a Riemannian metric g_0 such that $0 < K_\sigma \leqq \lambda$ satisfying $i(q_0) < \pi/\sqrt{\lambda}$ for some point $q_0 \in S^3$. Hence, for a simply connected non-compact Riemannian manifold $M := (S^3, g_0) \times E^1$ which satisfies $0 \leqq K_\sigma \leqq \lambda$, there exists a point $q \in M$ such that $i(q) < \pi/\sqrt{\lambda}$. So it might be significant to assume that M is homeomorphic to E^n for the proof of our assertion in the case $n \geqq 4$.

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