Tôhoku Math. Journ.
27 (1975), 381-388.

ON MARCINKIEWICZ INTEGRAL

SHIGEKI YANO

(Received April 25, 1974)

1. Introduction. Let P be a closed set in R^n and $\delta(x) = \delta_P(x)$ denote the distance of the point x from P. Let λ be a positive number and $f \in L^p(R^n)$, $1 \leqq p \leqq \infty$. We shall call the integral

$$J_\lambda(x) = J_\lambda(x; f) = \int_{R^n} \frac{\delta^\lambda(y) f(y)}{|x-y|^{n+\lambda}} dy \tag{1.1}$$

to be *the Marcinkiewicz "distance function" integral* of f.

Concerning this integral, following results are known:

If $f \in L^1(R^n)$, *then the integral* (1.1) *converges almost everywhere in* P. *In particular, if* P *is bounded and is contained in a finite cube* Q, *then*

$$\int_Q \frac{\delta^\lambda(y)}{|x-y|^{n+\lambda}} dy \tag{1.2}$$

is finite almost everywhere in P.

On the other hand, *if* $|\complement P| < \infty$[1], *then*

$$\int_{R^n} \frac{\delta^\lambda(y)}{|x-y|^{n+\lambda}} dy \tag{1.3}$$

is almost everywhere finite in P. For these results we refer the reader to Zygmund [7] and Stein [6; Chapter I].

The integral of the form (1.1) diverges in general outside P, so some variants are introduced, namely

$$H_\lambda(x) = \int_{R^n} \frac{\delta^\lambda(y) f(y)}{|x-y|^{n+\lambda} + \delta^{n+\lambda}(x)} dy \tag{1.4}$$

and

$$H'_\lambda(x) = \int_{R^n} \frac{\delta^\lambda(y) f(y)}{|x-y|^{n+\lambda} + \delta^{n+\lambda}(y)} dy . \tag{1.5}$$

In view of the relation $|\delta(x) - \delta(y)| \leqq |x-y|$ we have by Jensen's inequality

$$|x-y|^{n+\lambda} + \delta^{n+\lambda}(x) \approx |x-y|^{n+\lambda} + \delta^{n+\lambda}(y) ,$$

[1] $\complement E$ is the complement of the set E and $|E|$ denotes the Lebesgue measure of E.

hence $H_\lambda(x) \approx H'_\lambda(x)$, so that inequalities for H'_λ immediately lead to inequalities for H_λ. Also, if $x \in P$ then $\delta(x) = 0$, so that

$$H_\lambda(x) = J_\lambda(x) \qquad (x \in P) \tag{1.6}$$

and informations for $H'_\lambda(x)$ on P give informations for $J_\lambda(x)$ on P.

For H_λ and H'_λ, following results are known (see above cited references):

If $f \in L^p(R^n)$, $1 < p < \infty$, *then*[2]

$$\| H_\lambda \|_p \leqq A_p \| f \|_p ; \tag{1.7}$$

if $f \in L^\infty(R^n)$ *and* f *is supported in a (finite) cube* $Q \supset P$, *then*

$$\int_Q e^{c|H_\lambda(x)|/\|f\|_\infty} dx \leqq A |Q| . \tag{1.8}$$

If $|\complement P| < \infty$, *then for any (finite) cube* Q,

$$\int_Q e^{c|H_\lambda(x)|/\|f\|_\infty} dx < \infty . \tag{1.9}$$

On the other hand, John and Nirenberg [5] introduced the notion of functions of bounded mean oscillation (BMO). A function Φ locally integrable on R^n is said to be of *bounded mean oscillation* if

$$\| \Phi \|_* = \sup_Q \frac{1}{|Q|} \int_Q | \Phi(x) - \Phi_Q | \, dx < \infty ,$$

where the supremum ranges over all (finite) cubes in R^n and Φ_Q denotes the mean value of Φ on Q, $\Phi_Q = (1/|Q|) \int_Q \Phi(x) dx$.

They proved that if Φ is of BMO, then

$$\int_Q e^{c|\Phi(x) - \Phi_Q|/\|\Phi\|_*} dx \leqq A |Q| , \tag{1.10}$$

from this we obtain immediately the integrability of $e^{c|\Phi|/\|\Phi\|_*}$ over any cube. This observation and the inequalities (1.8) and (1.9) suggest that the Marcinkiewicz integral of a bounded function would be of BMO. In Section 2 we shall prove that this is true for the Marcinkiewicz integral of the type (1.5), and in Section 3 show an application of this result for an estimate of singular integral of Calderon-Zygmund type, which is an extension of a result due to Hunt [4] for the conjugate function.

2. Marcinkiewicz integrals of bounded functions. In this section

[2] Here and below, A, c may vary from inequalities to inequalities. A and c are always independent of the function f, the set P, the cube, etc., but may depend on the dimension n, the exponent p and the parameter λ or other explicitly indicated parameters.

we shall prove that the Marcinkiewicz integral of a bounded function of the type (1.5) is of BMO. Here we slightly change the notations.

THEOREM 1. *Let P be a closed set in R^n, $\delta(x)$ denote the distance of x from P, and λ be any positive number.*

1°. *If $|\complement P| < \infty$, then for any $\varphi \in L^\infty(R^n)$,*

2°. *If P is arbitrary, then for $\varphi \in L^\infty(R^n)$ supported in a finite cube,*

we define the Marcinkiewicz integral of φ by

$$\Phi(x) = \int_{R^n} \frac{\delta^\lambda(y)\varphi(y)}{|x-y|^{n+\lambda} + \delta^{n+\lambda}(y)}\, dy \, . \tag{2.1}$$

Then Φ is integrable and of BMO on R^n, and

$$\|\Phi\|_* \leqq A \|\varphi\|_\infty \, . \tag{2.2}$$

REMARK. If neither the conditions 1° nor 2° are satisfied, then the integral (2.1) may diverges on a set of positive measure, so that the conditions 1° or 2° is necessary for the validity of the theorem.

PROOF. We begin with the following observation. For any cube Q

$$\int_Q |\Phi(x)|\, dx \leqq \int_{R^n} |\Phi(x)|\, dx = \int_{R^n} \left| \int_{R^n} \frac{\delta^\lambda(y)\varphi(y)}{|x-y|^{n+\lambda} + \delta^{n+\lambda}(y)}\, dy \right| dx$$
$$\leqq \int_{S_\varphi \cap \complement P} |\varphi(y)|\, \delta^\lambda(y) \left\{ \int_{R^n} \frac{dx}{|x-y|^{n+\lambda} + \delta^{n+\lambda}(y)} \right\} dy \, ,$$

where $S_\varphi = \{x \in R^n : \varphi(x) \neq 0\}$. Since the inner integral of the last expression is equal to

$$\int_{R^n} \frac{dx}{|x|^{n+\lambda} + \delta^{n+\lambda}(y)} = \delta^{-\lambda}(y) \int_{R^n} \frac{dx}{|x|^{n+\lambda} + 1} = A\delta^{-\lambda}(y) \, ,$$

we obtain

$$\int_Q |\Phi(x)|\, dx \leqq A \int_{S_\varphi \cap \complement P} |\varphi(y)|\, dy \, ; \tag{2.3}$$

this shows that under the condition 1° or 2°, Φ is integrable on R^n.

Next, to prove that Φ is of BMO, we follow the idea of Fefferman and Stein [3, p. 152]. Let $Q = Q_h$ be a cube with side length h and center x°, and Q_{2h} be the cube with the same center as Q whose sides have length $2h$. We shall estimate Φ in Q writing $\Phi = \Phi_1 + \Phi_2$, where Φ_j arises from φ_j, $\varphi = \varphi_1 + \varphi_2$, $\varphi_1 = \varphi\chi_{Q_{2h}}$, $\varphi_2 = \varphi \cdot (1 - \chi_{Q_{2h}})$ and $\chi_{Q_{2h}}$ is the characteristic function of Q_{2h}. Then in view of (2.3)

$$\int_Q |\Phi_1(x)|\, dx \leqq A \int_{Q_{2h}} |\varphi(x)|\, dx \leqq A \|\varphi\|_\infty |Q| \, . \tag{2.4}$$

To estimate Φ_2, write

$$a_Q = \int_{\complement Q_{2h}} \frac{\varphi(y)\delta^\lambda(y)}{|x^\circ - y|^{n+\lambda} + \delta^{n+\lambda}(y)} dy .$$

Then

$$\Phi_2(x) - a_Q$$
$$= \int_{\complement Q_{2h}} \varphi(y)\delta^\lambda(y)\left[\frac{1}{|x - y|^{n+\lambda} + \delta^{n+\lambda}(y)} - \frac{1}{|x^\circ - y|^{n+\lambda} + \delta^{n+\lambda}(y)}\right]dy .$$

The modulus of the quantity in the brackets of the right side of the last expression does not exceed

$$\frac{A\,|x - x^\circ|\,|\bar{x} - y|^{n+\lambda-1}}{[|\bar{x} - y|^{n+\lambda} + \delta^{n+\lambda}(y)][|x^\circ - y|^{n+\lambda} + \delta^{n+\lambda}(y)]} ,$$

where $\bar{x}$ is a point on the segment joining the points x° and x. Now, if $x \in Q$ and $y \in \complement Q_{2h}$, then

$$\text{(2.5)} \qquad \begin{aligned} &|x - x^\circ| \leqq Ah , \quad |x^\circ - y| \geqq ch \\ &|\bar{x} - y| \approx |x^\circ - y| \approx |x - y| , \end{aligned}$$

so that it follows that for $x \in Q$

$$\text{(2.6)} \qquad |\Phi_2(x) - a_Q| \leqq Ah \int_{\complement Q_{2h}} \frac{|\varphi(y)|\,\delta^\lambda(y)\,|x^\circ - y|^{n+\lambda-1}}{[|x^\circ - y|^{n+\lambda} + \delta^{n+\lambda}(y)]^2} dy .$$

To estimate the last integral, we split the range $\complement Q_{2h}$ into the union of E_1 and E_2, where

$$E_1 = \complement Q_{2h} \cap \{y \in R^n : \delta(y) \leqq |y - x^\circ|\}$$

and

$$E_2 = \complement Q_{2h} \cap \{y \in R^n : \delta(y) > |y - x^\circ|\} .$$

Since $E_1 \subset \{y \in R^n : |y - x^\circ| \geqq ch\}$ in view of (2.5), we obtain

$$\text{(2.7)} \qquad \int_{E_1} \leqq \|\varphi\|_\infty \int_{|y - x^\circ| \geqq ch} \frac{dy}{|x^\circ - y|^{n+1}} \leqq A\,\|\varphi\|_\infty\, h^{-1} .$$

Quite similarly

$$\text{(2.8)} \qquad \begin{aligned} \int_{E_2} &\leqq \|\varphi\|_\infty \int_{ch \leqq |y - x^\circ| < \delta(y)} \frac{dy}{\delta^{n+1}(y)} \\ &\leqq \|\varphi\|_\infty \int_{|y - x^\circ| \geqq ch} \frac{dy}{|x^\circ - y|^{n+1}} \leqq A\,\|\varphi\|_\infty\, h^{-1} . \end{aligned}$$

From (2.4) and (2.9), the relation (2.2) follows immediately, and the proof of Theorem 1 is completed.

John and Nirenberg [5] proved that if Φ is of BMO and integrable and $\|\Phi\|_* \leqq \kappa$, then

$$\int_{R^n} (e^{c|\Phi(x)|/\kappa} - 1)dx \leqq \frac{A}{\kappa}\int_{R^n} |\Phi(x)|\, dx\,. \tag{2.9}$$

Combining this inequality and Theorem 1, we get the following corollary.

COROLLARY. *Under the notations and assumptions of Theorem* 1, *we have for any* $\alpha > 0$

1° *if* $|\complement P| < \infty$, *then*

$$|\{x \in R^n : |\Phi(x)| > \alpha\}| \leqq A(e^{c\alpha/\|\varphi\|_\infty} - 1)^{-1} |\complement P|\,;$$

2° *if* $|S_\varphi| < \infty$ *where* $S_\varphi = \{x \in R^n ; \varphi(x) \neq 0\}$, *then*

$$|\{x \in R^n : |\Phi(x)| > \alpha\}| \leqq A(e^{c\alpha/\|\varphi\|_\infty} - 1)^{-1} |S_\varphi|\,.$$

As the proof shows, it is not necessary to assume in Theorem 1 that δ is the "distance function", and we can extend Theorem 1 as follows:

THEOREM 1′. *Let* δ *be any non negative (finite valued) measurable function in* R^n, *and* λ *be any positive number.*

1°. *If* δ *is supported in a set of finite measure, then for any* $\varphi \in L^\infty(R^n)$,

2°. *If* δ *is arbitrary, then for* $\varphi \in L^\infty(R^n)$ *supported in a set of finite measure,*

3°. *If* δ *is bounded, then for any* $\varphi \in L^\infty(R^n)$,

the "generalised" Marcinkiewicz integral

$$\Phi(x) = \int_{R^n} \frac{\delta^\lambda(y)\varphi(y)}{|x-y|^{n+\lambda} + \delta^{n+\lambda}(y)}\, dy$$

is of BMO on R^n, *and* $\|\Phi\|_* \leqq A\|\varphi\|_\infty$. *In case of* 1° *or* 2°, Φ *is integrable in* R^n.

3. An application. R. Hunt obtained an interesting estimate of the conjugate function: *Let* $f \in L^1(-\pi, \pi)$ *and define its conjugate function* f *by*

$$\tilde{f}(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{2\tan\frac{1}{2}(t-x)}\, dt\,, \tag{3.1}$$

then for any α *and* $\beta > 0$, *we have*

$$|\{x \in (-\pi, \pi) : Mf(x) \leqq \alpha,\ |\tilde{f}(x)| > \alpha\beta\}| \leqq Ae^{-c\beta}\,, \tag{3.2}$$

where Mf is the Hardy-Littlewood maximal function of f.

To prove this result, Hunt used a lemma of Carleson [1] on an estimate of a function of the form

$$\sum_j \int_{I_j} \frac{|I_j|}{|x-y|^2+|I_j|^2} dy \ ;$$

this lemma, however, can be derived from a result concerning the Marcinkiewicz integral, as pointed out by Zygmund [7]. Moreover, Hunt's result can be extended to n-dimensional case.

THEOREM 2. *Let K be a kernel of Calderón-Zygmund type on* R^n*; specifically suppose*

$$K(x) = \Omega(x)/|x|^n \ , \tag{3.3}$$

Ω *is homogeneous of degree zero, and*

$$\int_{S^{n-1}} \Omega(x')dx' = 0 \ ,$$

and

$$\Omega \in \operatorname{Lip} \lambda \ , \qquad \lambda > 0 \ . \tag{3.4}$$

For $f \in L^1(R^n)$*, define*

$$\tilde{f}(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} f(x-y)K(y)dy \ . \tag{3.5}$$

Then for any number $\alpha, \beta > 0$ *and for any cube Q which satisfies*

$$|Q| \geqq \frac{A}{\alpha} \| f \|_1 \tag{3.6}$$

we have

$$|\{x \in Q: |\tilde{f}(x)| > \alpha\beta, Mf(x) \leqq \alpha\}| \leqq Ae^{-c\beta}|Q| \ , \tag{3.7}$$

where Mf is the Hardy-Littlewood maximal function, and A, c are constants depending only on K (more precisely on the bound of Ω*, and the exponent* λ *and the bound of Lipschitz condition for* Ω*) and the dimension n.*

PROOF. Let $P = \{x \in R^n: Mf(x) \leqq \alpha\}$, then P is closed. Combining the Calderón-Zygmund decomposition for the pair f, α, and the Whiteney decomposition of open set into the union of cubes, we obtain the following decompositions of $\complement P$ and f (for a proof see Stein [6, p. 32] or Fefferman [2]):

There exists a sequence of non-overlapping cubes $\{Q_j\}$ such that

$\complement P = \bigcup_j Q_j$

$$|\complement P| = \sum_j |Q_j| \leqq \frac{A}{\alpha} \|f\|_1 , \tag{3.8}$$

$$\frac{1}{|Q_j|} \int_{Q_j} |f(y)| dy \leqq A\alpha , \tag{3.9}$$

$$|f(x)| \leqq \alpha \quad \text{a.e. in } P , \tag{3.10}$$

$$c \operatorname{diam} Q_j \leqq \text{distance } (P, Q_j) \leqq A \operatorname{diam} Q_j \quad \text{where } 1 < c < A . \tag{3.11}$$

Now define g by

$$g(x) = \begin{cases} f(x) & (x \in P) \\ \dfrac{1}{|Q_j|} \displaystyle\int_{Q_j} f(y) dy & (x \in Q_j;\ j = 1, 2, \cdots) \end{cases}$$

and write $f = g + b$. Then

$$|g(x)| \leqq A\alpha \quad \text{a.e.}, \quad \|g\|_1 = \|f\|_1 , \tag{3.12}$$

$$b(x) = 0 \text{ in } P , \quad \int_{Q_j} b(y) dy = 0, \|b\|_1 \leqq A \|f\|_1 . \tag{3.13}$$

Since $\tilde{f} = \tilde{g} + \tilde{b}$, we have by definition of P $|\{x \in Q: |\tilde{f}(x)| > \alpha\beta, Mf(x) \leqq \alpha\}| \leqq |\{x \in Q: |\tilde{g}(x)| > \alpha\beta/2\} \cap P| + |\{x \in Q: |\tilde{b}(x)| > \alpha\beta/2\} \cap P|$. Thus it suffices to prove

$$|\{x \in Q: |\tilde{g}(x)| > \alpha\beta\}| \leqq Ae^{-c\beta} |Q| \tag{3.14}$$

and

$$|\{x \in Q: |\tilde{b}(x)| > \alpha\beta\} \cap P| \leqq Ae^{-c\beta} |Q| \tag{3.15}$$

for any cube Q with (3.6).

Since g is bounded and integrable by (3.12), a result of Fefferman and Stein [2; p. 144] shows that g is of BMO and

$$\|\tilde{g}\|_* \leqq A \|g\|_\infty \leqq A\alpha . \tag{3.16}$$

Therefore by (3.12) and Schwarz inequality we obtain

$$|(\tilde{g})_Q| \leqq \frac{1}{Q} \int_Q |\tilde{g}(y)| dy \leqq \frac{1}{|Q|^{1/2}} \|\tilde{g}\|_2 \leqq \frac{A}{|Q|^{1/2}} \|g\|_2 = A\left(\frac{\|f\|_1 \alpha}{|Q|}\right)^{1/2} .$$

From this and (1.10), we get

$$|\{x \in Q: |\tilde{g}(x)| > \alpha\beta\}| \leqq Ae^{A(\|f\|_1/\alpha|Q|)^{1/2}} e^{-c\beta} |Q| , \tag{3.17}$$

and this reduces to (3.14) for cube Q with $|Q| \geqq (A/\alpha) \|f\|_1$.

Next, let δ be the distance function with respect to P, then Theorem

1, part 1° can be applied to the Marcinkiewicz integral involving this δ. Now it is known (see e.g. Zygmund [8] and Stein [6; Chapter II]) that for $x \in P$

$$|\tilde{b}(x)| \leqq A\alpha \int_{R^n} \frac{\delta^\lambda(y)}{|x-y|^{n+\lambda}} dy \tag{3.18}$$

and as is mentioned in Section 1, the integral on the right hand side of (3.18) is of the same size as the integral

$$\Phi(x) = \int_{R^n} \frac{\delta^\lambda(y)}{|x-y|^{n+\lambda} + \delta^{n+\lambda}(y)} dy$$

for $x \in P$. Thus remembering (3.8) we obtain by Corollary 1, 1°

$$\begin{aligned} &|\{x \in Q: |\tilde{b}(x)| > \alpha\beta\} \cap P| \\ &\leqq |\{x \in Q: \Phi(x) > A\beta\}| \leqq Ae^{-c\beta} |Q| \end{aligned}$$

for Q with $|Q| \geqq A\alpha^{-1} \|f\|_1$, and this proves (3.15). The proof is completed.

References

[1] L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math., 116 (1966), 135-157.
[2] C. Fefferman, Inequalities for strongly singular convolution operators, Acta Math., 124 (1970), 9-36.
[3] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math., 129 (1972), 137-193.
[4] R. Hunt, An estimate of the conjugate function, Studia Math., 44 (1972), 371-377.
[5] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure and Appl. Math., 14 (1961), 415-426.
[6] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton, 1970.
[7] A. Zygmund, On certain lemmas of Marcinkiewicz and Carleson, J. Approximation Theory, 2 (1969), 249-257.
[8] A. Zygmund, On singular integrals, Rend. di Mat., 16 (1957), 468-505.

Tokyo Metropolitan University
Tokyo