Tôhoku Math. Journ.
27 (1975), 337-354.

ON THE HANKEL TRANSFORM OF DISTRIBUTIONS

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(Received November 21, 1973)

1. Introduction. Let an integral transform $F(y)$ of a complex-valued function $f(x)$ defined over the interval $(-\infty, \infty)$, with respect to the kernel $k(x, y)$ for real parameter y be defined as

$$\text{(i)} \qquad F(y) = \int_{-\infty}^{\infty} f(x)k(x, y)dx \equiv Tf .$$

Let us assume that there exists a function $h(x, y)$ defined for real x, y such that under certain restrictions on $f(x)$ the transform $F(y)$ is inverted by

$$\text{(ii)} \qquad f(x) = \int_{-\infty}^{\infty} F(y)h(x, y)dy .$$

There are mainly two approaches to extend the classical transform (i) to generalized functions. In the first approach a testing function space H is constructed over $(-\infty, \infty)$ which is closed with respect to the classical transform (i) and then the corresponding transform of the generalized function f of the dual space is defined through the generalization of the Parseval's equation as follows

$$\text{(iii)} \qquad \langle Tf, \phi\rangle = \langle f, T\phi\rangle \quad \text{for all} \quad \phi \in H .$$

This approach has been followed by L. Schwartz [7] to extend Fourier transform to distributions of slow growth. Zemanian [9] has also followed this approach to extend Hankel transform to generalized functions.

The second approach consists in defining a testing function space over $(-\infty, \infty)$ containing the Kernel function $k(x, y)$ for each real y and then defining the transform $F(y)$ of the generalized function f by the relation

$$\text{(iv)} \qquad F(y) = \langle f(x), k(x, y)\rangle .$$

This approach has been followed in the extensions of Laplace [12], [13] Mellin [12], Stieltjes [4] transforms to generalized functions. The inversion formula (ii) for generalized functions is then extended by establishing:

$$\lim_{N\to\infty} \left\langle \int_0^N \langle f(x), k(x, y)\rangle h(x, y)dy, \phi\right\rangle = \langle f, \phi\rangle$$

*) This work was supported by the National Research Council Grant No. A5298.
[#] This work was supported by National Research Council Grant No. 240-123.

for each $\phi \in \mathscr{D}(I)$ (generally).

Our goal is to extend the classical Hankel inversion formula established by A. Schwartz [6] to a certain class of generalized functions, following the second approach, which is more natural and explicit and is very well suited for specific computations. Following the first approach Zemanian [9] extended the classical Hankel transform to a certain class of generalized functions of slow growth and proved the inversion theorem for distributions of compact support only. Later Koh and Zemanian [3], following the second approach, gave an extension of the Hankel transform; they also proved an inversion theorem for a larger class of generalized functions. In this paper the following Hankel inversion formula as proved by A. Schwartz [6] is extended to a certain class of generalized functions interpreting convergence in the weak distributional sense, which has distinct advantages over the works done by Zemanian [9] and Koh and Zemanian [3].

Let $\nu > -1/2$ and L consist of all measurable functions defined on $0 < x < \infty$ such that

$$\| f \| = \int_0^\infty | f(x) | \, dm(x) < \infty$$

where

$$dm(x) = [2^\nu \Gamma(\nu + 1)]^{-1} x^{2\nu+1} dx \, .$$

Also let $\mathscr{J}(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu} J_\nu(x)$ for all $x > 0$, where $J_\nu(x)$ is the Bessel function of the first kind of order ν.

THEOREM (A. L. Schwartz). *Suppose $f \in L$ and $\int_0^1 | f(y) y^{\nu+1/2} | \, dy < \infty$. If f is of bounded variation in a neighborhood of x, then,*

$$\lim_{\lambda \to \infty} \int_0^\lambda \mathscr{J}(xu) dm(u) \int_0^\infty f(y) \mathscr{J}(uy) dm(y) = \frac{1}{2} \{ f(x + 0) + f(x - 0) \} \, .$$

We extend the above result to a class of generalized functions and prove some related results. Specifically, we define a generalized Hankel transform following the approach used in [3], and prove the corresponding inversion theorem. In Section 4 we give an example of a class of regular generalized functions which are Hankel transformable in our sense but not in the sense of Koh and Zemanian [3]. Moreover, the inversion formula (Theorem 3) is used to find the distributional solution of a differential equation which cannot be solved by the technique used in [3] (see Remark 1 in Section 5).

At first it would appear that the generalized function space $\mathscr{H}_\mu'$

considered by Zemanian [9] is larger than the space $H'_{\alpha,\delta}(I)$ of ours, but this is not the case. We give examples in Section 4, showing that our testing function space and that used by Zemanian overlap and neither is contained in the other properly. Consequently neither of the generalized function spaces is properly contained in the other.

2. The testing function space $H_{\alpha,\delta}(I)$ and its dual. Let $I = (0, \infty)$, $x \in I$ and α, δ, ν be fixed real numbers satisfying $\nu > -1/2$, $0 < \alpha \leqq \nu + 1/2$ and $\delta \geqq 0$. Let $\xi(x)$ be an infinitely differentiable function defined over I, satisfying $\xi(x) > 0$ for all $x > 0$ and such that

$$\xi(x) = \begin{cases} x^{\nu+1/2+\delta}, & 0 < x < 1 \\ x^{\alpha-2}, & x \geqq 2 . \end{cases}$$

Now define $H_{\alpha,\delta}(I)$ to be the collection of all infinitely differentiable complex-valued functions $\phi(x)$ on I with the property

$$\gamma_k(\phi(x)) = \sup_{0<x<\infty} \left| \xi(x) \Delta_x^k \left(\frac{\phi(x)}{m'(x)} \right) \right| < \infty$$

for each $k = 0, 1, \cdots$, where $\Delta_k = D_x^2 + ((2\nu + 1)/x)D_x$; $D_x = d/dx$ and $m'(x) = [2^\nu \Gamma(\nu + 1)]^{-1} x^{2\nu+1}$.

The sequence $\{\gamma_k\}_{k=0}^\infty$ is a separating collection of seminorms [11, p. 7] which generates the topology of $H_{\alpha,\delta}(I)$. It can be readily seen that $H_{\alpha,\delta}(I)$ is a locally convex, sequentially complete, Hausdorff topological vector space. The dual space of $H_{\alpha,\delta}(I)$ is denoted by $H'_{\alpha,\delta}(I)$.

NOTE 1. Let $\mathscr{D}(I)$ denote the space of infinitely differentiable functions with compact support on I, equipped with the usual topology. The dual space $\mathscr{D}'(I)$ is the space of Schwartz distributions on I [11, p. 33]. It is easy to check that the space $\mathscr{D}(I) \subset H_{\alpha,\delta}(I)$, and that the topology of $\mathscr{D}(I)$ is stronger than that induced on it by $H_{\alpha,\delta}(I)$. Hence the restriction of any $f \in H'_{\alpha,\delta}(I)$ to $\mathscr{D}(I)$ is in $\mathscr{D}'(I)$.

NOTE 2. We point out that the space $H_{\alpha,\delta}(I)$ is not in general closed with respect to differentiation. For example take a function on I defined as $\phi(x) = x$, where $\nu > -1/2$. Obviously $\phi(x) \in H_{\alpha,\delta}(I)$ for $\nu - 1/2 < \delta < \nu + 1/2$ but the derivatives of $\phi(x)$ do not belong to $H_{\alpha,\delta}(I)$. Hence we cannot define distributional differentiation in $H'_{\alpha,\delta}(I)$ in the way it was defined in $\mathscr{D}'(I)$ [10, p. 47].

NOTE 3. Let $f(x)$ be a locally integrable function defined for $x > 0$ and satisfying $\int_0^\infty (|f(x)|)/(\xi(x))\, dm(x) < \infty$. Then $f(x)$ generates a regular generalized function in $H'_{\alpha,\delta}(I)$ defined by

$$(1) \qquad \langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)dx \ .$$

In fact,

$$(2) \qquad \begin{aligned} |\langle f, \phi \rangle| &= \left| \int_0^\infty \frac{f(x)}{\xi(x)} \xi(x) \frac{\phi(x)}{m'(x)} dm(x) \right| \\ &\leqq \beta_0(\phi(x)) \int_0^\infty \frac{|f(x)|}{\xi(x)} dm(x) < \infty \ , \end{aligned}$$

which shows that (1) defines a functional f on $H'_{\alpha,\delta}(I)$. The linearity and continuity of f follow from (1) and (2) respectively.

NOTE 4. For $\nu > -1/2$ the regular generalized functions determined in [3, (ix)] may be easily shown to be contained in $H'_{\alpha,\delta}(I)$.

NOTE 5. Let $\nu + 1/2 > \alpha > 2$. Then the regular generalized function space generated by the elements of the function space for which the inversion formula of Schwartz [6] is valid is contained in $H'_{\alpha,0}(I)$.

3. The generalized Hankel transform. For $f \in H'_{\alpha,\delta}(I)$, define its generalized Hankel transform by the relation

$$(3) \qquad F(y) = (\mathscr{H}_\nu f)(y) = \langle f(x), m'(x)\mathscr{J}(xy) \rangle \ , \quad y > 0 \ .$$

Notice that (3) is well defined since using $\Delta_x^k \mathscr{J}(xy) = (-1)^k y^{2k} \mathscr{J}(xy)$, for $k = 0, 1, 2, \cdots$, it follows easily that $m'(x)\mathscr{J}(xy) \in H_{\alpha,\delta}(I)$ for fixed $y > 0$.

It can also be verified that $m'(x)(\partial^k/\partial y^k)\mathscr{J}(xy) \in H_{\alpha,\delta}(I)$ for each $y > 0$ and $k = 1, 2$.

THEOREM 1. *For $y > 0$, let $F(y)$ be the generalized Hankel transform of f; then $F(y)$ is differentiable and*

$$\frac{d}{dy} F(y) = \left\langle f(x), m'(x) \frac{\partial}{\partial y} \mathscr{J}(xy) \right\rangle \ .$$

PROOF. Let h be an arbitrary increment in y. Without any loss of generality assume $0 < |h| < y/2$. Now

$$\frac{F(y+h) - F(y)}{h} = \left\langle f(x), m'(x) \frac{\mathscr{J}\{x(y+h)\} - \mathscr{J}(xy)}{h} \right\rangle \ .$$

Let $\theta_h(x)$ denote the expression

$$\frac{\mathscr{J}\{x(y+h)\} - \mathscr{J}(xy)}{h} - \frac{\partial}{\partial y} \mathscr{J}(xy) \ .$$

We will show that $m'(x)\theta_h(x)$ converges to zero in $H_{\alpha,\delta}(I)$ as $h \to 0$. Our result will then follow from the continuity of $f(x)$. Now, for any non-negative integer k

$$\xi(x)\Delta_x^k(\theta_h(x)) = \xi(x)(-1)^k\Big[\frac{(y+h)^{2k}\mathscr{J}\{x(y+h)\}-y^{2k}\mathscr{J}(xy)}{h}-\frac{\partial}{\partial y}(y^{2k}\mathscr{J}(xy)\Big]$$
$$= (-1)^k\xi(x)\Big[\frac{1}{h}\int_y^{y+h}\int_y^u\frac{\partial^2}{\partial t^2}(t^{2k}\mathscr{J}(xt))dt\,du\Big]$$
$$= (-1)^k2^\nu\Gamma(\nu+1)\xi(x)x^{-\nu}\frac{1}{h}\Big[\int_y^{y+h}\int_y^u\{(2k-\nu)(2k-\nu-1)t^{2k-\nu-2}J_\nu(xt)$$
$$+ 2(2k-\nu)xt^{2k-\nu-1}J_\nu^{(1)}(xt) + x^2t^{2k-\nu}J_\nu^{(2)}(xt)\}dt\,du\Big]$$
$$= I_1 + I_2 + I_3 ,$$

where

$$I_1 = (-1)^k2^\nu\Gamma(\nu+1)(2k-\nu-1)\frac{1}{h}\int_y^{y+h}\int_y^u\{\xi(x)(xt)^{-\nu}J_\nu(xt)\}t^{2k-2}dt\,du ,$$

and I_2, I_3 are defined similarly.

Using the asymptotic orders of Bessel functions and their derivatives, it can be shown that for each $m = 0, 1$ and 2, the expression $|\xi(x)x^{-\nu+m}J_\nu^{(m)}(xt)|$ is uniformly bounded for all $x > 0$ and $y/2 \leqq t \leqq 3y/2$. Let B_m, $m = 0, 1, 2$, be the corresponding bounds. Then for fixed $y > 0$,

$$|I_1| \leqq 2^\nu\Gamma(\nu+1)\,|(2k-\nu)(2k-\nu-1)|\,B_0\frac{1}{|h|}\Big|\int_y^{y+h}\int_y^u t^{2k-2}dt\,du\Big|$$
$$\leqq 2^\nu\Gamma(\nu+1)\,|(2k-\nu)(2k-\nu-1)|\,B_0\frac{\left(\frac{3y}{2}\right)^{2k}}{\left(\frac{y}{2}\right)^2}|h| \longrightarrow 0 \text{ as } h\longrightarrow 0 .$$

Similarly it can be shown that for fixed $y > 0$, I_2 and I_3 both converge to zero as $h \longrightarrow 0$ uniformly for all $x > 0$. Hence for each $k = 0, 1, 2, \cdots$, $\xi(x)\Delta_x^k(\theta_h(x)) \longrightarrow 0$ as $h \longrightarrow 0$, uniformly for all $x > 0$.

This completes the proof of the theorem.

REMARK. We do not know whether the second order derivative of $F(y)$ exists in general.

THEOREM 2. *Let $F(y)$ be the generalized Hankel transform of $f \in H'_{\alpha,\delta}(I)$. Then*

$$|F(y)| = O(y^{\mathrm{Min}(2-\alpha,0)}) , \quad y \longrightarrow 0+$$

and

$$|F(y)| = O(y^{2r-\nu-1/2}) , \quad y \longrightarrow \infty ,$$

where r is a non-negative integer.

PROOF. In view of a general result [11, Th. 1.8.1], there exists a constant $C > 0$ and a non-negative integer r such that

$$\begin{aligned} |\langle f(x), m'(x)\mathscr{J}(xy)\rangle| = |F(y)| &\leqq C \max_{0\leqq k\leqq r} \gamma_k(\mathscr{J}(xy)m'(x)) \\ &= C \max_{0\leqq k\leqq r} \sup_{0<x<\infty} |\xi(x)\Delta_x^k \mathscr{J}(xy)| \\ &= C2^\nu\Gamma(\nu+1) \max_{0\leqq k\leqq r} \sup_{0<x<\infty} |\xi(x)y^{2k}(xy)^{-\nu}J_\nu(xy)| . \end{aligned}$$

We now evaluate the quantity $\xi(x)y^{2k}(xy)^{-\nu}J_\nu(xy)$ (for $k = 0$ or r according as $y \to 0+$ or $y \to \infty$) by dividing the x-line into three parts $0 < x \leqq 1$, $1 < x \leqq 1/y$ and $1/y < x < \infty$ for $y \to 0+$ and $0 < x \leqq 1/y$, $1/y < x \leqq 1$ and $1 < x < \infty$ for $y \to \infty$ and thus conclude that

$$|F(y)| = \begin{cases} O(y^{\mathrm{Min}(2-\alpha,0)}) , & y \to 0+ \\ O(y^{2r-\nu-1/2}) , & y \to \infty . \end{cases}$$

This completes the proof of Theorem 2.

Before giving the inversion theorem we prove the following required lemmas.

LEMMA 1. *Let $0 < \alpha \leqq \nu + 1/2$. Then for fixed $x > 0$,*

$$m'(t)\int_0^\eta \mathscr{J}(yt)\mathscr{J}(xy)dm(y) \to 0 \ \ in\ H_{\alpha,\delta}(I)\ \ as\ \eta \to 0+ .$$

PROOF. For any non-negative integer k we have

$$\begin{aligned} &\left|\xi(t)\Delta_t^k \int_0^\eta \mathscr{J}(yt)\mathscr{J}(xy)dm(y)\right| \\ &\leqq \xi(t)\int_0^\eta |\mathscr{J}(yt)\mathscr{J}(xy)y^{2k}|\, dm(y) \\ &\leqq [2^\nu\Gamma(\nu+1)]^{-1}M\int_0^\eta |(ty)^\alpha\mathscr{J}(ty)|\,|\mathscr{J}(xy)|\, y^{2\nu+1-\alpha+2k}dy , \end{aligned} \tag{4}$$

where $\xi(t) \leqq Mt^\alpha$ for all $t > 0$, and an appropriate constant $M > 0$.

Since $0 < \alpha \leqq \nu + 1/2$, $|(ty)^\alpha\mathscr{J}(ty)|$ is bounded for all $t, y > 0$. Therefore, in view of the fact that $|\mathscr{J}(xy)| \leqq 1$, there is a constant K (depending on ν and α) such that the left-hand side of (4) is bounded by

$$K\int_0^\eta y^{2\nu+2k+1-\alpha}dy ,$$

which clearly approaches zero as $\eta \to 0+$, independently of t.

LEMMA 2. *If $f \in H'_{\alpha,\delta}(I)$ then for fixed $x > 0$ and any positive number N we have*

$$\begin{aligned}(5)\qquad &\int_0^N \langle f(t), m'(t)\mathscr{J}(yt)\rangle \mathscr{J}(xy)dm(y) \\ &= \left\langle f(t), m'(t)\int_0^N \mathscr{J}(yt)\mathscr{J}(xy)dm(y)\right\rangle .\end{aligned}$$

PROOF. In view of Theorems 1 and 2, the integral on the left-hand side of (5) exists. It can be shown readily that for fixed $x > 0$, $m'(t)$ times the integral on the right-hand side of (5) belongs to $H_{\alpha,\delta}(I)$, and therefore the right-hand side of (5) is meaningful.

To prove (5), we need to establish that for $\eta > 0$,

$$\begin{aligned}(6)\qquad &\int_\eta^N \langle f(t), m'(t)\mathscr{J}(yt)\rangle \mathscr{J}(xy)dm(y) \\ &= \left\langle f(t), m'(t)\int_\eta^N \mathscr{J}(yt)\mathscr{J}(xy)dm(y)\right\rangle ,\end{aligned}$$

which can be proved by using the technique of Riemann sums. The proof is very similar to that of Theorem 2 in [5], and therefore is omitted. The result (5) now follows by letting $\eta \to 0+$ in (6) and applying Lemma 1.

At this point, let us denote $\int_0^N \mathscr{J}(yt)\mathscr{J}(xy)dm(y)$ by the symbol $G_N(t, x)$.

LEMMA 3. *For positive numbers a and b, one has*

$$\lim_{N\to\infty}\int_a^b G_N(t, x)dm(x) = \begin{cases} 1 , & t \in (a, b) \\ \dfrac{1}{2} , & t = a, t = b \\ 0 , & t \notin [a, b] .\end{cases}$$

PROOF. The result follows quite readily from Hankel's inversion formula [2, p. 96].

LEMMA 4. *Let $\phi(x) \in \mathscr{D}(I)$ with support contained in $[a, b]$, $0 < a < b$. Then*

$$(7)\qquad m'(t)\int_a^b G_N(t, x)\phi(x)dx \to \phi(t) \text{ in } H_{\alpha,\delta}(I) \text{ as } N \to \infty .$$

PROOF. It can be easily seen that $\Delta_t G_N(t, x) = \Delta_x G_N(t, x)$. Therefore

$$\begin{aligned}\Delta_t\int_a^b G_N(t, x)\varphi(x)dx &= \int_a^b \Delta_x G_N(t, x)\varphi(x)dx \\ &= \int_a^b G_N(t, x)\Delta_x\left(\frac{\phi(x)}{m'(x)}\right)dm(x)\end{aligned}$$

(by integration by parts) .

Operating with Δ_t successively and applying Lemma 3, we get

$$\lim_{N\to\infty}\left[\Delta_t^k\int_a^b G_N(t, x)\phi(x)dx - \phi(t)\right]$$
$$= \lim_{N\to\infty}\int_a^b G_N(t, x)[\phi_k(x) - \phi_k(t)]dm(x) ,$$

where $\phi_k(x)$ denotes $\Delta_x^k\left(\frac{\phi(x)}{m'(x)}\right) \in \mathscr{D}(I)$ for $k = 0, 1, 2, \cdots$. Our problem is now reduced to proving that

$$\lim_{N\to\infty}\xi(t)\int_a^b G_N(t, x)[\psi(x) - \psi(t)]dm(x) = 0 , \tag{8}$$

uniformly for all $t > 0$ where $\psi(x) \in \mathscr{D}(I)$ with support contained in $[a, b]$.

Substitute

$$G_N(t, x) = 2^\nu\Gamma(\nu + 1)\frac{(xt)^{-\nu}N[xJ_{\nu+1}(Nx)J_\nu(Nt) - tJ_{\nu+1}(Nt)J_\nu(Nx)]}{(x^2 - t^2)}$$

as given in [8, p. 134] into (8), and express the left-hand side of (8) as $I = I_1 - I_2$ where

$$I_1 = N\xi(t)t^{-\nu}J_\nu(Nt)\int_a^b x^{\nu+2}\frac{\psi(x) - \psi(t)}{x^2 - t^2}J_{\nu+1}(Nx)dx$$

and

$$I_2 = N\xi(t)t^{-\nu+1}J_{\nu+1}(Nt)\int_a^b x^{\nu+1}\frac{\psi(x) - \psi(t)}{x^2 - t^2}J_\nu(Nx)dx .$$

As before, one can find a constant K' such that $\xi(t) \leqq K't^{\alpha-2}$, for all $t > 0$. Moreover for $t > b$, $\psi(t) = 0$. Then using the asymptotic orders of Bessel functions, the fact that $\psi(x)$ is bounded, and the inequality $|x^2 - t^2| > t^2 - b^2$ for $t > b$, we can find a constant B (depending on ν) such that

$$|I_2| \leqq B\frac{t^{\alpha-(\nu+3/2)}}{t^2 - b^2}\int_a^b x^{\nu+1/2}dx , \quad \text{as} \quad N \to \infty .$$

Therefore for an arbitrary $\varepsilon > 0$, there exists a number $L > b$ sufficiently large that

$$\lim_{N\to\infty}|I_2| < \varepsilon \quad \text{for all} \quad t > L . \tag{9}$$

For $a \leqq t \leqq b$, using the inequality $\xi(t) \leqq K't^{\alpha-2}$ and an analogue of the Riemann-Lebesgue lemma [8, p. 457] we get

$$
\begin{aligned}
|I_2| &\leqq K' t^{\alpha-(\nu+1/2)-1}\sqrt{N}\, o\left[\frac{1}{\sqrt{N}}\right] \quad \text{as} \quad N \to \infty \\
&\leqq K' a^{\alpha-(\nu+1/2)-1}\sqrt{N}\, o\left[\frac{1}{\sqrt{N}}\right] \quad \text{as} \quad N \to \infty\,.
\end{aligned} \tag{10}
$$

In the above asymptotic order, uniformity with respect to t is implied. For further details see [1, p. 41].

Similarly for $b < t \leqq L$ we have

$$
|I_2| \leqq K' b^{\alpha-(\nu+1/2)-1}\sqrt{N}\, o\left[\frac{1}{\sqrt{N}}\right] \quad \text{as} \quad N \to \infty\,. \tag{11}
$$

Lastly, when $0 < t < a$, find a constant K'' such that $\xi(t) \leqq K'' t^{\nu+1/2+\delta}$ and hence arrive at

$$
|I_2| \leqq K'' a^{1+\delta}\sqrt{N}\, o\left[\frac{1}{\sqrt{N}}\right] \quad \text{as} \quad N \to \infty\,. \tag{12}
$$

Note that the asymptotic orders in (11) and (12) are also uniform with respect to t.

Combining (9), (10), (11) and (12) we conclude that

$$
\overline{\lim_{N\to\infty}}\, |I_2| = 0 \quad \text{uniformly for all} \quad t > 0\,.
$$

Therefore,

$$
\lim_{N\to\infty} I_2 = 0 \qquad \text{uniformly for all} \quad t > 0\,. \tag{13}
$$

Similar techniques can be used to show that

$$
\lim_{N\to\infty} I_1 = 0 \qquad \text{uniformly for all} \quad t > 0\,,
$$

which combined with (13) proves the required result.

Now we prove the following inversion theorem.

THEOREM 3. *Let $f \in H'_{\alpha,\delta}(I)$ and let $F(y)$ be the generalized Hankel transform of f. Then for each $\phi \in \mathscr{D}(I)$,*

$$
\left\langle \int_0^N F(y)\mathscr{J}(xy)dm(y),\, \phi(x)\right\rangle \to \langle f, \phi\rangle, \text{ as } N \to \infty\,. \tag{14}
$$

PROOF. Suppose that the support of $\phi(x)$ is contained in (a, b), $0 < a < b$. We prove (14) by justifying the steps in the following manipulations.

$$\left\langle \int_0^N F(y)\mathscr{J}(xy)\,dm(y),\ \phi(x)\right\rangle$$

$$= \int_a^b \int_0^N F(y)\mathscr{J}(xy)dm(y)\phi(x)d(x) \tag{15}$$

$$= \int_a^b \phi(x)d(x) \int_0^N \langle f(t),\ m'(t)\mathscr{J}(yt)\rangle \mathscr{J}(xy)dm(y)$$

$$= \int_a^b \left\langle f(t),\ m'(t)\int_0^N \mathscr{J}(yt)\mathscr{J}(xy)\,dm(y)\right\rangle \phi(x)d(x) \tag{16}$$

$$= \left\langle f(t),\ m'(t)\int_a^b G_N(t,\ x)\Big(\frac{\phi(x)}{m'(x)}\Big)dm(x)\right\rangle \to \langle f(t),\ \phi(t)\rangle \text{ as } N\to\infty\ . \tag{17}$$

Step (15) is obvious in view of Theorems 1 and 2. Step (16) follows from Lemma 2. Step (17) is obtained by applying techniques similar to those used in proving Lemma 2. The final step comes from Lemma 4.

This completes the proof of Theorem 3.

Next, we give a structure formula for the restriction of an element $f \in H'_{\alpha,\delta}(I)$ to $\mathscr{D}(I)$.

THEOREM 4. *Let f be an arbitrary element of $H'_{\alpha,\delta}(I)$. Then there exist bounded measurable functions $g_i(x)$ defined for $x > 0$ for $i = 0, 1, 2, \cdots, r$ where r is some non-negative integer depending upon f such that for an arbitrary $\phi \in \mathscr{D}(I)$ we have*

$$\langle f,\ \phi\rangle = \left\langle \sum_{i=0}^{r} \Delta_x^i \left\{\frac{\xi(x)}{m'(x)}\Big(-\Delta_x \int_0^x g_i(t)dt + \frac{2\nu+1}{x} g_i(x)\Big)\right\},\ \phi(x)\right\rangle .$$

PROOF. In view of the boundedness property of generalized functions there exists a constant $C > 0$ and a non-negative integer r depending upon f such that all $\phi \in \mathscr{D}(I)$

$$|\langle f,\ \phi\rangle| \leqq C \operatorname*{Max}_{0\leqq k\leqq r} \gamma_k(\phi)$$

$$\leqq C \operatorname*{Max}_{0\leqq k\leqq r}\ \sup_{0<x<\infty} \left|\xi(x)\Delta_x^k\Big(\frac{\phi(x)}{m'(x)}\Big)\right|$$

$$= C \operatorname*{Max}_{0\leqq k\leqq r}\ \sup_{0<x<\infty} \left|\frac{\xi(x)}{m'(x)}\Big(D_x^2 - \frac{2\nu+1}{x}D_x + \frac{2\nu+1}{x^2}\Big)^k \phi(x)\right|$$

$$= C \operatorname*{Max}_{0\leqq k\leqq r}\ \sup_{0<x<\infty} \int_0^x \left| D_t\Big[\frac{\xi(t)}{m'(t)}\Big(D_t^2 - \frac{2\nu+1}{t}D_t + \frac{2\nu+1}{t^2}\Big)^k \phi(t)\Big]\right| dt$$

$$\leqq C \sum_{k=0}^{r}\ \sup_{0<x<\infty} \int_0^\infty \left| D_t\Big\{\frac{\xi(t)}{m'(x)}\Big(D_t^2 - \frac{2\nu+1}{t}D_t + \frac{2\nu+1}{t^2}\Big)^k \phi(t)\Big\}\right| dt\ .$$

Therefore, in view of the Riesz representation theorem and Hahn-Banach theorem there exist bounded measurable functions $g_i(x)$ defined over $I=(0,\infty)$, $i=0,1,\cdots,r$ satisfying

$$\langle f,\phi\rangle=\sum_{i=0}^{r}\left\langle g_i(x),\ D_x\left[\frac{\xi(x)}{m'(x)}\left(D_x^2-\frac{2\nu+1}{x}D_x+\frac{2\nu+1}{x^2}\right)^i\phi(x)\right]\right\rangle$$

$$=\left\langle\sum_{i=0}^{r}\Delta_x^i\left\{\frac{\xi(x)}{m'(x)}(-D^2)\int_0^x g_i(t)\,dt\right\},\ \phi(x)\right\rangle .$$

[By integration by parts]

$$=\left\langle\sum_{i=0}^{r}\Delta_x^i\frac{\xi(x)}{m'(x)}\left(-\Delta_x+\frac{2\nu+1}{x}D_x\right)\int_0^x g_i(t)dt,\ \phi(x)\right\rangle$$

$$=\left\langle\Delta_x^i\left\{\frac{\xi(x)}{m'(x)}\left((-\Delta_x)\int_0^x g_i(t)\,dt\right)+\frac{2\nu+1}{x}g_i(x)\right\},\ \phi(x)\right\rangle .$$

This completes the proof of Theorem 4.

4. In this section we give a few interesting examples comparing the space $H'_{\alpha,\delta}(I)$ with the generalized function spaces considered by Koh and Zemanian [3] and Zemanian [9].

EXAMPLE 1. Let $f(x)$ be a locally integrable function defined for $x>0$ such that

$$f(x)=\begin{cases}O[x^{-\nu-3/2+\eta+\delta}]\ , & x\longrightarrow 0+\\ O[x^{\alpha-2\nu-4-\varepsilon}]\ , & x\longrightarrow\infty\ ,\end{cases}$$

where ε and η are some positive numbers. Clearly $f(x)\in H'_{\alpha,\delta}(I)$ as a regular generalized function in view of Note 3 in Section 2.

Now, let $\phi(x)$ be an infinitely differentiable function defined over I satisfying

$$\phi(x)=\begin{cases}0\ , & 0<x<1\\ e^{ax}\ , & x\geqq 2;\ a>0\ .\end{cases}$$

It is easy to show that $\phi(x)$ belongs to the testing function space considered by Koh and Zemanian [3]. As $\int_0^\infty f(x)\phi(x)dx$ does not exist, it follows that $f(x)$ does not belong to the corresponding dual space.

EXAMPLE 2. We have seen that for each fixed $y>0$, $m'(x)\mathscr{J}(xy)\in H_{\alpha,\delta}(I)$. However $m'(x)\mathscr{J}(xy)\notin\mathscr{H}_\nu$ (the testing function space considered by Zemanian [9]), because $m'(x)\mathscr{J}(xy)$ is not of rapid descent.

EXAMPLE 3. Let $\phi(x)$ be an infinitely differentiable function defined over I such that

$$\phi(x)=\begin{cases}x^{\nu+1/2}\ , & 0<x<1,\ \nu\geqq-\dfrac{1}{2}\\ e^{-x}\ , & x\geqq 2\ .\end{cases}$$

It is easy to verify that $\phi(x) \in \mathscr{H}_\nu$ [9], but $\phi(x) \notin H_{\alpha,\delta}(I)$.

From Examples 2 and 3 and the fact that the space $\mathscr{D}(I)$ is contained in both the spaces $\mathscr{H}_\nu$ and $H_{\alpha,\delta}(I)$, it follows that the spaces $\mathscr{H}_\nu$ and $H_{\alpha,\delta}(I)$ overlap, and neither is contained in the other. Therefore the generalized function spaces $\mathscr{H}_\nu'$ and $H'_{\alpha,\delta}(I)$ also overlap and neither is contained in the other.

5. Now we will apply our inversion formula to the solution of certain differential equations.

We define an operator $\Delta_x^*: H'_{\alpha,\delta}(I) \to H'_{\alpha,\delta}(I)$ given by the relation

$$\langle \Delta_x^* f(x), \phi(x)\rangle = \left\langle f(x), m'(x)\, \Delta_x\left(\frac{\phi(x)}{m'(x)}\right)\right\rangle$$

for all $f \in H'_{\alpha,\delta}(I)$ and $\phi(x) \in H_{\alpha,\delta}(I)$, $0 < \alpha \leqq \nu + 1/2$; $\delta \geqq 0$. Let us call the operator Δ_x^* the adjoint of the operator $\Delta_x = D_x^2 + ((2\nu + 1)/x)D_x$. It can also be shown that for all $k = 1, 2, 3, \cdots$ and $\phi(x) \in H_{\alpha,\delta}(I)$ one will have

$$\langle (\Delta_x^*)^k f(x), \phi(x)\rangle = \left\langle f(x), m'(x)\Delta_x^k\left(\frac{\phi(x)}{m'(x)}\right)\right\rangle .$$

It can be readily seen that if f is a regular distribution in $H'_{\alpha,\delta}(I)$ generated by a member of $\mathscr{D}(I)$, then

$$\Delta_x^* f \equiv \Delta_x f .$$

For each $k = 1, 2, 3, \cdots$ and $y > 0$ one can show that

$$\langle (\Delta_x^*)^k f(x), \mathscr{J}(xy)m'(x)\rangle = (-1)^k y^{2k} \langle f(x), \mathscr{J}(xy)m'(x)\rangle .$$

That is,

$$\mathscr{H}_\nu[(\Delta_x^*)^k f(x)] = (-1)^k y^{2k} \mathscr{H}_\nu[f(x)] . \tag{18}$$

Now consider the operator equation

$$P(\Delta_x^*)u = g , \tag{19}$$

where $g \in H'_{\alpha,\delta}(I)$ and P is any polynomial whose zeros do not lie on the negative real axis. Our object is to find a generalized function $u \in H'_{\alpha,\delta}(I)$ satisfying the operator equation (18).

Taking the generalized Hankel transform of both sides of (19) and using (18) we get

$$P(-y^2)U(y) = G(y) , \tag{20}$$

where U and G are the generalized Hankel transform of u and g respectively.

We now wish to find $u \in H'_{\alpha,\delta}(I)$ such that $\mathscr{H}u = G(y)/P(-y^2)$. We claim that for each $\phi \in \mathscr{D}(I)$

$$\langle u, \phi\rangle = \lim_{N\to\infty}\left\langle \int_0^N \frac{G(y)}{P(-y^2)}\mathscr{J}(xy)\,dm(y),\ \phi(x)\right\rangle . \tag{21}$$

We know that for every $g \in H'_{\alpha,\delta}(I)$ there exists a non-negative integer r satisfying

$$\mathscr{H}g = G(y) = O[y^{2r-\nu-1/2}]\ ,\quad y \longrightarrow \infty\ .$$

Let $Q(x)$ be a polynomial of degree $r+1$ defined by

$$Q(x) = x^{r+1} + 1 \text{ if } r \text{ is odd and}$$
$$Q(x) = x^{r+1} - 1 \text{ if } r \text{ is even}\ .$$

The fact that the right hand side integral in (21) converges in the distributional sense as $N \longrightarrow \infty$ can now be proved as follows

$$\begin{aligned}
&\left\langle \int_0^N \frac{G(y)}{P(-y^2)}\mathscr{J}(xy)dm(y),\ \phi(x)\right\rangle\\
&= \left\langle Q(\varDelta_x)\int_0^N \frac{G(y)\mathscr{J}(xy)dm(y)}{P(-y^2)Q(-y^2)},\ \phi(x)\right\rangle\\
&= \left\langle \int_0^N \frac{G(y)}{P(-y^2)Q(-y^2)}\mathscr{J}(xy)dm(y),\ m'(x)Q(\varDelta_x)\left\{\frac{\phi(x)}{m'(x)}\right\}\right\rangle
\end{aligned} \tag{22}$$

[by integration by parts] .

A careful computation now shows that using (22) we can find $K, M > 0$ such that for all $N_1, N_2 > K$ we have

$$\left|\left\langle \int_{N_1}^{N_2} \frac{G(y)}{P(-y^2)}\mathscr{J}(xy)dm(y),\ \phi(x)\right\rangle\right| \leqq M\int_{N_1}^{N_2}\frac{y^{2r}dy}{|P(-y^2)Q(-y^2)|}$$
$$\longrightarrow 0 \text{ as } N_1, N_2 \longrightarrow \infty\ .$$

Therefore,

$$\lim_{N\to\infty}\left\langle \int_0^N \frac{G(y)}{P(-y^2)}\mathscr{J}(xy)dm(y),\ \phi(x)\right\rangle \text{ exists}$$

and in view of the completeness of $\mathscr{D}'(I)$ there exists $f \in \mathscr{D}'(I)$ such that

$$\lim_{N\to\infty}\left\langle \int_0^N \frac{G(y)}{P(-y^2)}\mathscr{J}(xy)dm(y),\ \phi(x)\right\rangle = \langle f, \phi\rangle\ . \tag{23}$$

The function f as determined in (23) is the restriction of $u \in H'_{\alpha,\delta}(I)$ to $\mathscr{D}(I)$. We now prove that f satisfies the differential equation

$$P(\Delta_x)u = g \text{ on } \mathscr{D}(I) . \tag{24}$$

In view of the continuity of the operation of differentiation and multiplication by $1/x$ in $\mathscr{D}'(I)$ one can show that

$$\lim_{N\to\infty}\left\langle P(\Delta_x)\int_0^N \frac{G(y)}{P(-y^2)}\mathscr{J}(xy)dm(y),\, \phi(x)\right\rangle = \langle P(\Delta_x)f,\, \phi\rangle \tag{25}$$

$$\text{for all } \phi \in \mathscr{D}(I) .$$

Therefore, using Theorem 3, in (25) we get

$$\langle g, \phi\rangle = \langle P(\Delta_x)f, \phi\rangle .$$

Thus f determined in (23), which belongs to $\mathscr{D}'(I)$ and is the restriction of $u \in H'_{\alpha,\delta}(I)$ to $\mathscr{D}(I)$, satisfies the distributional differential equation

$$P(\Delta_x)f = g . \tag{26}$$

Now observe that $\mathscr{J}(axi) = 2^\nu\Gamma(\nu+1)(axi)^{-\nu}J_\nu(axi)$ satisfies the distributional differential equation

$$\left(D^2 + \frac{2\nu+1}{x}D - a^2\right)u = 0 . \tag{27}$$

Using the method of variation of parameters one can show that the general solution of (27) in $\mathscr{D}'(I)$ is given by

$$u(x) = \mathscr{J}(axi)\left[c\int_1^x [t^{2\nu+1}\mathscr{J}^2(ati)]^{-1}dt + d\right]$$

where c and d are arbitrary constants.

Hence for a polynomial $P(x) = (x - a_1^2)(x - a_2^2)\cdots(x - a_n^2)$, where the a_i are distinct real numbers, the general solution of the distributional differential equation (24) in $\mathscr{D}'(I)$ is given by

$$u(x) = f + \sum_{k=1}^n \mathscr{J}(a_kxi)\left[c_k\int_1^x [t^{2\nu+1}\mathscr{J}^2(a_kti)]^{-1}dt + d_k\right], \tag{28}$$

where c_k and d_k are arbitrary constants and f is the distribution in $\mathscr{D}'(I)$ as determined in (23).

REMARK. In equation (24), let $g(x)$ be a regular distribution as given in Example 1, Section 4. With such a choice of g, the differential equation (24) cannot be solved by using the transform technique of Koh and Zemanian as used in [3], whereas it is solvable by our transform technique.

A Dirichlet problem in cylindrical co-ordinates:

We will now find the conventional function $u(r, z)$ on the domain

$\{(r, z): 0 < r < \infty, 0 < z < \infty\}$ which satisfies the differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2\nu + 1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0\,, \quad r, z > 0\,, \quad \nu > -\frac{1}{2} \tag{29}$$

and the following boundary conditions,

(a) $\dfrac{\partial u}{\partial r}(r, z) = o\left[\dfrac{1}{r^{\nu+1/2}}\right]$, as $r \to \infty$ for a fixed $z > 0$.

(b) $u(r, z) = o\left[\dfrac{1}{r^{\nu+1/2}}\right]$, as $r \to \infty$ for a fixed $z > 0$.

(c) $\dfrac{\partial u}{\partial r}(r, z) = o\left[\dfrac{1}{r^{2\nu+1}}\right]$, $r \to 0+$ for a fixed $z > 0$.

(d) $\lim_{z\to\infty} u(r, z) = 0$ in $\mathscr{D}'(I)$.

(e) $\lim_{z\to 0+} u(r, z) = f$ in $\mathscr{D}'(I)$ where $f \in H'_{\alpha,\delta}(I)$.

We will find the solution in the space $H'_{\alpha,\delta}(I)$ with $0 < \alpha \leqq \nu + 1/2$ and $\delta \geqq 0$. Taking the Hankel transform of both sides of (31) (for fixed $z > 0$) with respect to the kernel $\mathscr{J}(ry)m'(r)$ and using integration by parts we get

$$-y^2 U(y, z) + \frac{\partial^2 U(y, z)}{\partial z^2} = 0\,. \tag{30}$$

In view of the condition (c) and the fact that $\nu + 1/2 > 0$ it follows that $u(r, z) = o[1/r^{2\nu+1}]$, $r \to 0+$. Therefore, the limit terms in the integration by parts vanish in the light of conditions (a), (b) and (c). We have also assumed that $\mathscr{H}(\partial^2 u(r, z)/\partial z^2) = (\partial^2/\partial z^2)U(y, z)$. Solving (30) we get,

$$U(y, z) = A(y)e^{yz} + B(y)e^{-yz}\,.$$

In view of the conditions (d) and (e) it is reasonable to assume (though we do not care to justify it) that

$$\lim_{z\to\infty} U(y, z) = 0 \text{ and } \lim_{z\to 0+} U(y, z) = \mathscr{H}(f) = F(y)\,.$$

Therefore we get,

$$U(y, z) = F(y)e^{-yz}\,.$$

Using the inversion formula stated in Theorem 3 we get

$$u(r, z) = \lim_{N\to\infty}\int_0^N F(y)e^{-yz}\mathscr{J}(yr)dm(y) \text{ in } \mathscr{D}'(I)\,. \tag{31}$$

For each $\phi \in \mathscr{D}(I)$ one can show that

$$\langle u(r, z), \phi(r)\rangle = \int_0^\infty \phi(r)dr \int_0^\infty F(y)e^{-yz}\mathscr{J}(yr)dm(y) . \tag{32}$$

Now one can observe from (32) that

$$u(r, z) = \int_0^\infty F(y)e^{-yz}\mathscr{J}(yr)dm(y) , \quad r, z > 0 . \tag{33}$$

Using the asymptotic orders of $F(y)$ as established in Section 2 we can justify that

$$\left(\nabla_r + \frac{\partial^2}{\partial z^2}\right)u = \int_0^\infty F(y)\left\{\nabla_r + \frac{\partial^2}{\partial z^2}\right\}e^{-yz}\mathscr{J}(yr)dm(y) .$$

Therefore $u(r, z)$ as defined in (31) satisfies the differential equation (29).

The boundary conditions (a) and (b) can be verified easily in view of the analogue of the Riemann-Lebesgue Lemma [8; p. 457]. Again, for $\nu > -1/2$ there exists $N > 0$ satisfying, $|\mathscr{J}'(x)x^{\nu+1/2}| \leqq N$ uniformly for all $x > 0$. Therefore,

$$\left| r^{2\nu+1}\frac{\partial u}{\partial r}\right| \leqq Nr^{\nu+1/2}\int_0^\infty |F(y)|\, e^{-yz}y^{\nu+3/2}dy \longrightarrow 0 \text{ as } r \longrightarrow 0+ .$$

This verifies (c). The verification of (d) is trivial. To verify (e) we choose a polynomial $Q(x)$ as defined in Section 4 and then we have

$$\begin{aligned}\langle u(r, z), \phi(r)\rangle &= \int_0^\infty \phi(r)drQ(\Delta_r)\int_0^\infty \frac{F(y)e^{-yz}}{Q(-y^2)}\mathscr{J}(yr)dm(y)\\ &= \int_0^\infty m'(r)Q(\Delta_r)\left\{\frac{\phi(r)}{m'(r)}\right\}dr\int_0^\infty \frac{F(y)e^{-yz}\mathscr{J}(yr)dm(y)}{Q(-y^2)}\end{aligned}$$

[by integration by parts] .

That is,

$$\begin{aligned}&\langle u(r, z), \phi(r)\rangle\\ &\quad = \lim_{N\to\infty}\int_a^b m'(r)Q(\Delta_r)\left\{\frac{\phi(r)}{m'(r)}\right\}dr\int_0^N \frac{F(y)e^{-yz}\mathscr{J}(yr)dm(y)}{Q(-y^2)} .\end{aligned} \tag{34}$$

We assume that the support of $\phi(x)$ is contained in (a, b), $b > a > 0$. The right hand side expression in (34) converges uniformly for all $z > 0$ as $N \longrightarrow \infty$. Therefore, letting $z \longrightarrow 0+$ in (34) and switching the limit operation with respect to N and z in the right hand side of (34) we get

$$\begin{aligned}&\lim_{z\to0+}\langle u(r, z), \phi(r)\rangle\\ &\quad = \lim_{N\to\infty}\int_a^b Q(\Delta_r)\left\{\frac{\phi(r)}{m'(r)}\right\}dm(r)\int_0^N \frac{F(y)\mathscr{J}(yr)dm(y)}{Q(-y^2)}\end{aligned}$$

or

$$\lim_{z\to 0+}\langle u(r, z), \phi(r)\rangle = \lim_{N\to\infty}\int_a^b \phi(r)dr \int_0^N F(y)\mathscr{J}(yr)dm(y)$$

[By integration by parts]

$$= \langle f, \phi\rangle$$

[In view of Theorem 3] .

Thus the condition (e) is also verified. The solution obtained is unique in the sense of equality over $\mathscr{D}(I)$ in view of the following uniqueness theorem.

THEOREM 5. *For $f, g \in H'_{\alpha,\delta}(I)$ let us define $\mathscr{H}f = F(y)$ and $\mathscr{H}g = G(y)$ for all $y > 0$. If $F(y) = G(y)$ for all $y > 0$ then $f = g$ in the sense of equality over $\mathscr{D}(I)$.*

PROOF. By Theorem 3

$$f - g = \lim_{N\to\infty}\int_0^N [F(y) - G(y)]\mathscr{J}(xy)dm(y) = 0 .$$

ACKNOWLEDGEMENT. The authors express their gratefulness to a referee for his constructive criticisms. Thanks are also due to Professor K. Hardy of the Department of Mathematics, Carleton University, Ottawa who was kind enough to improve the presentation of the manuscript.

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