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PROJECTIVE MODULES OVER NON-COMMUTATIVE SEMILOCAL RINGS

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An associative ring R with identity is called a *semilocal* ring in case it is semisimple (artinian) modulo its radical $J = J(R)$. A *semiperfect* ring is a semilocal ring in which idempotents lift modulo J. The structure of projective modules over semiperfect rings has been well determined (see [15], [18], [21]). However, though semilocal rings that are not semiperfect occur naturally enough (see Proposition 4 and Example 5) and have enjoyed recent attention in the literature (e.g., in [1], [4], [8], [10]), except when they are commutative (see [11]), the structure of their projective modules is largely unknown. Here we show that if all nonzero projective left modules over a semilocal ring are generators (e.g., if R/J is simple artinian) then they are direct sums of a fixed idempotent-generated principal left ideal (Theorem 1); that if a finite direct sum of copies of each simple module has a projective cover then every indecomposable projective module is finitely generated (Theorem 6 and Corollary 8); and that semilocal rings have only finitely many finitely generated indecomposable projective modules (Theorem 9).

We use notation and results on projective covers contained in [2].

A ring is called (left) p-connected by Bass [3] in case each of its projective left modules is a generator. (We use the notion $M^{(c)}$ to denote, for a cardinal number c and a module M, a direct sum of c copies of M; so ${}_RG$ is a generator in case there is an epimorphism $G^{(c)} \to {}_RR \to 0$.) Hinohara [11] proved that a commutative semilocal ring in which 1 is a primitive idempotent has all its projective modules free. Akasaki [1] generalized this and Kaplansky's well-known theorem [14] on local rings by proving that projective left R-modules are free if R is a p-connected semilocal ring in which each maximal left ideal is two-sided. Our first result eliminates this vestige of commutativity.

1. THEOREM. *If R is a p-connected semilocal ring then there exists a primitive idempotent $e \in R$ such that every projective left R-module is isomorphic to a direct sum of copies of Re.*

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PROOF. Since ${}_R R$ clearly has a.c.c. on direct summands, R contains a primitive idempotent e. Let $T_1, \cdots, T_n$ represent the isomorphism classes of simple left R-modules. Then

$$Re/Je \cong T_1^{(k_1)} \oplus \cdots \oplus T_n^{(k_n)}$$

where, since Re is by hypothesis a generator, each $k_i \neq 0$. Let ${}_R P \neq 0$ be projective. Then, since projective modules are direct sums of countably generated modules [14] we may assume that

$$P/JP \cong T_1^{(c_1)} \oplus \cdots \oplus T_n^{(c_n)}$$

where $0 < c_i \leqq \aleph_0$. If each c_i is infinite then, according to Bass [3] or Beck [5], P is free. If some c_i is finite, renumber $T_1, \cdots, T_n$ so that c_1/k_1 is minimal among $c_1/k_1, \cdots, c_n/k_n$. Then

$$P^{(k_1)}/JP^{(k_1)} \cong T_1^{(c_1 k_1)} \oplus \cdots \oplus T_n^{(c_n k_1)}$$

and

$$Re^{(c_1)}/Je^{(c_1)} \cong T_1^{(k_1 c_1)} \oplus \cdots \oplus T_n^{(k_n c_1)}$$

where $c_i k_1 \geqq k_i c_1$ $(i = 1, \cdots, n)$, so $Re^{(c_1)}/Je^{(c_1)}$ is an epimorph of $P^{(k_1)}$. But, since c_1 is finite, $Re^{(c_1)}$ is a projective cover of $Re^{(c_1)}/Je^{(c_1)}$, and by Uniqueness of Projective Covers [2, Lemma 2.3], $P^{(k_1)} \cong Re^{(c_1)} \oplus Q$. But the T_1-homogeneous components of $P^{(k_1)}/JP^{(k_1)}$ and $Re^{(c_1)}/Je^{(c_1)}$ both have length $c_1 k_1$, so Q cannot map onto T_1. Thus, since R is p-connected, $Q = 0$ and we have found integers $k = k_1$, $c = c_1$ such that

$$P^{(k)} = Re^{(c)} ,$$

if P is not free. In any event, P is a direct sum of indecomposable finitely generated modules, so we may now assume that P is indecomposable and finitely generated. Considering homogeneous components again, we see that $c_i k = k_i c$ $(i = 1, \cdots, n)$. This yields, if $k < c$, a proper (split) epimorphism $P \rightarrow Re \rightarrow 0$ and, if $k > c$, a proper (split) epimorphism $Re \rightarrow P \rightarrow 0$. Thus, by indecomposability, $k = c$, $c_i = k_i$ $(i = 1, \cdots, n)$, and

$$P \cong Re ,$$

which completes the proof.

Noting that ${}_R R \cong Re^{(k)}$ we have at once from the Morita Theorems [17]

2. COROLLARY. *A semilocal ring is p-connected if and only if it is isomorphic to a full matrix ring over a semilocal ring whose projective left modules are all free.*

According to Hinohara's [11, Lemma 4], a commutative semilocal ring is p-connected if and only if its identity is a primitive idempotent. In fact the identity of any p-connected semilocal ring in which each maximal left ideal is two-sided (i.e., R/J is a finite product of division rings) must be a primitive idempotent. Thus the following corollary emphasizes that Theorem 1 generalizes Hinohara's [11] and Akasaki's [1] extensions of Kaplansky's [14] theorem that projective modules over local rings are free.

3. COROLLARY. *Let R be a semilocal ring in which 1 is a primitive idempotent. If R is p-connected then every projective left R-module is free.*

A semilocal ring with only one simple module (i.e., a ring with R/J simple artinian) satisfies the hypothesis of Theorem 1. (See, for example, [2, Proposition 2.7] and [9, page 213].) Such rings occur in the papers [10], [16], [19]. If one of them satisfies Akasaki's hypothesis [1] then it is automatically a matrix ring over a local ring. However, Probert [19] has pointed out that an example of McConnell [16] is a non-commutative domain that is a simple artinian ring of length 2 modulo its radical, so our result properly generalizes Akasaki's. Since McConnell's example is rather complicated, we offer here a method of constructing semilocal domains that is similar to the usual commutative one (e.g., $\boldsymbol{Z}_{\{2,3\}} = \{a/b \mid a, b \in \boldsymbol{Z} \text{ and } b \notin 2\boldsymbol{Z} \cup 3\boldsymbol{Z}\}$).

4. PROPOSITION. *Let D be a principal left and right ideal domain. Let $I \neq 0$ be a proper two-sided ideal in D and let $U = \{u \in D \mid u + I$ is invertible in $D/I\}$. Then D has a classical quotient ring R with respect to U and RI is an ideal of R such that*

$$RI \leqq J(R) \quad \text{and} \quad R/RI \cong D/I .$$

PROOF. For the needed results on quotient rings we refer to Cohn [7] or Jategoankar [13]. Let $I = Da = aD$. Then one can either show that U satisfies the hypotheses of [7, Prop. 3.5] or check that for $u \in U$ and $d \in D$, if $Du_1 = \{x \in D \mid xd \in Du\}$ then u_1 is not a zero divisor modulo I, and hence is invertible in the artinian (see [6, Remark, page 40]) ring D/I. Thus U is a left (and similarly, a right) denominator set, so D has a classical quotient ring R with respect to U. It is straightforward to check that $R/RI \cong D/I$. If $u \in U$ and $b \in I$ then

$$1 - u^{-1}b = u^{-1}(u - b)$$

is invertible in R because $u - b \in U$. So since $RI = \{u^{-1}b \mid u \in U, b \in I\}$, we also have $RI \subseteqq J(R)$.

5. EXAMPLE. *A semilocal principal left and right ideal domain R with $R/J \cong M_2(\boldsymbol{R})$, the 2×2 matrix ring over the real numbers.*

DEMONSTRATION. Let $D = \boldsymbol{C}(x, {}^{-})$, the ring of complex polynomials with multiplication twisted by conjugation, i.e., $xc = \bar{c}x$. (See [7, page 36].) Let $I = \langle x^2 - 1 \rangle$, the ideal generated by $x^2 - 1$. Then $D/I \cong M_2(\boldsymbol{R})$ and we can construct R as in the proposition. (Note: If we take $I = \langle x^4 - 1 \rangle$ we get $R/J \cong M_2(\boldsymbol{R}) \times$ {real quaternions}).

From their conception, it was known that semiperfect rings have all their indecomposable projective modules isomorphic to so-called *primitive one-sided ideals*—those of the form Re (or eR) with e a primitive idempotent in R. However, it was some time later that Klatt [14], Mueller [18], and Warfield [21] independently extended Kaplansky's theorem to semiperfect rings by proving that every projective module over a semiperfect ring is isomorphic to a direct sum of primitive one-sided ideals. Although we do not know how to extend this latter result beyond p-connected semilocal rings, the former extends to a class of semilocal rings that are, in a sense, the antithesis of p-connected ones.

6. THEOREM. *Let R be a semilocal ring with complete orthogonal set of idempotents $e_1 + \cdots + e_n = 1$ such that each Re_i/Je_i is homogeneous. Then every indecomposable projective R-module is isomorphic to a primitive one-sided ideal in R.*

PROOF. First observe that a ring satisfies the hypothesis if and only if it has a complete orthogonal set of primitive idempotents $e_1 + \cdots + e_m = 1$ with each ring e_iRe_i simple artinian modulo its radical. (Cf., [18, Theorem 1].) (Necessity is obvious, and to prove the semilocal part of sufficiency one applies [12, Proposition 1, page 65].) Thus the hypothesis is left-right symmetric.

Given R and $e_1, \cdots, e_n$ as stated, we may assume that $e_1 \cdots, e_n$ are primitive and that

$$Re_i/Je_i \cong T_i^{(k_i)} \qquad (i = 1, \cdots, n)$$

where $T_1, \cdots, T_n$ is a complete irredundant set of representatives of the simple left R-modules. (Note: The assumed $e_1, \cdots, e_n$ may be proper components of the given ones.) Then, since Re_i/Je_i is a projective cover of $T_i^{(k_i)}$ the length of the T_i-homogeneous component of an indecomposable projective left module P must be $\leq k_i$. Thus if ${}_RP$ is indecomposable projective we have a commutative diagram with exact row and column,

$$\begin{array}{ccccc} & & R & & \\ & {}^{f}\swarrow & \downarrow & & \\ P & \xrightarrow{\text{nat.}} & P/JP & \longrightarrow & 0\ , \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

so that $P = \operatorname{Im} f + JP$. Thus, by Nakayama's Lemma, we need only show that indecomposable projectives are finitely generated to insure that they are isomorphic to direct summands of R. So suppose there exist non-finitely generated indecomposable projective left R-modules. Then there exists one P_0 with a minimal number of non-zero homogeneous components in P_0/JP_0, which we may assume correspond to $T_1, \cdots, T_l$, in which the length of the T_1-homogeneous component is minimal. Say

$$P_0/JP_0 = T_1^{(u_1)} \oplus \cdots \oplus T_l^{(u_l)} \qquad (1 \leqq u_i < k_i)\ ,$$

then there is a least positive integer w such that $P_0^{(w)}$ maps onto Re_1, i.e.,

$$P_0^{(w)} \cong Re_1 \oplus Q = Re_1 \oplus Q_1 \oplus \cdots \oplus Q_t$$

with each Q_i indecomposable projective. By minimality of w we must have

$$u_1(w-1) < k_1 \leqq u_1 w$$

and the T_1-homogeneous component of each JQ_i has length

$$u_1 w - k_1 < u_1 w - u_1(w-1) = u_1\ .$$

But then by the minimality assumptions on P_0, each Q_i (and hence P_0) is finitely generated. This contradiction completes the proof.

Observe that we did not show that the indecomposable projective modules over the ring R of Theorem 6 are necessarily isomorphic to any of the Re_i with Re_i/Je_i homogeneous. They need not be.

7. EXAMPLE. *A semilocal ring R with primitive simple modules $S \not\cong T$ and primitive idempotents e, $1-e$, f and $1-f$ such that*

$$Re/Je \cong S^{(2)}\ , \quad R(1-e)/J(1-e) \cong T^{(2)}$$

and

$$Rf/Jf \cong S \oplus T \cong R(1-f)/J(1-f)\ .$$

DEMONSTRATION. Let A be the ring of Example 4. Let $N = J(A)$. Let R be the ring of 2×2 matrices over A whose entries are as indicated:

$$R = \begin{bmatrix} A & A \\ N & A \end{bmatrix}.$$

Then

$$J = \begin{bmatrix} N & A \\ N & N \end{bmatrix}$$

and

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

satisfies $Re/Je \cong S^{(2)}$ and $R(1-e)/J(1-e) \cong T^{(2)}$. To find f, let g be an element of A that is a primitive idempotent modulo N (i.e., $A/N = (A/N)(g+N) \oplus (A/N)(1-g+N)$ where the two terms are isomorphic simples), let

$$g - g^2 = a \in N$$

and let

$$f = \begin{bmatrix} g & 1 \\ a & 1-g \end{bmatrix}.$$

Then, noting that $ag = ga$, we see that f is an idempotent in R, and that neither f nor $1-f$ annihilates $Re/Je \cong S^{(2)}$ or $R(1-e)/J(1-e) \cong T^{(2)}$. Thus Rf and $R(1-f)$ both map onto S and T; so, since $R/J \cong S^{(2)} \oplus T^{(2)}$, we see that $Rf \cong R(1-f)$ is the projective cover of $S \oplus T$ as desired.

It is easy to see that Re, $R(1-e)$ and Rf represent all the isomorphism classes of indecomposable projective left modules over the ring of the preceding example. The ring fRf obtained from this example is also interesting. Modulo its radical it is a product of two copies of the reals, its identity f is a primitive idempotent and, unlike the commutative case [11, Lemma 4], it is not p-connected. Indeed it is Morita equivalent to R, a fact which motivates

8. COROLLARY. *If some finite direct sum of copies of each simple left R-module has a projective cover then R is semilocal and every indecomposable projective R-module is finitely generated.*

PROOF. Sandomiersky [20] observed that a ring is semiperfect if and only if each of its simple modules has a projective cover. A similar argument shows that a ring is semilocal if and only if each of its simple modules embeds in a semisimple module that has a projective cover. Thus the hypothesis implies that R is semilocal.

If the simple R-modules are $T_1, \cdots, T_n$ and

$$P_1 \to T_1^{(k_1)}, \cdots, P_n \to T_n^{(k_n)}$$

are projective covers then $\mathrm{End}_R(P_1 \oplus \cdots \oplus P_n)$ satisfies the hypothesis of Theorem 6 and is Morita equivalent to R.

Over a semiperfect ring the number of isomorphism classes of (finitely generated) indecomposable left projective modules is the same as the number of isomorphism classes of simple left modules, but for semilocal rings either number can be strictly larger than the other (e.g., Example 7 and $\boldsymbol{Z}_{\{2,3\}}$). However, we do have

9. THEOREM. *A semilocal ring has only finitely many isomorphism classes of finitely generated indecomposable projective modules.*

PROOF. Suppose that $T_1, \cdots, T_n$ are the simple modules for a semilocal ring R. Let P and Q be finitely generated indecomposable projective modules with

$$P/JP \cong T_1^{(x_1)} \oplus \cdots \oplus T_n^{(x_n)} \quad \text{and} \quad Q/JQ \cong T_1^{(y_1)} \oplus \cdots \oplus T_n^{(y_n)}.$$

If $x_i \geqq y_i$ $(i = 1, \cdots, n)$ then P maps epimorphically onto Q. Thus

$$(*) \qquad P \cong Q \text{ iff } x_i = y_i \ (i = 1, \cdots, n) \text{ iff } x_i \geqq y_i \ (i = 1, \cdots, n).$$

Let X denote the set of n-tuples of non-negative integers that correspond to the finitely generated indecomposable projective left R-modules in the above manner, and suppose that X is not finite. Then X must be unbounded in at least one coordinate. Renumbering $T_1, \cdots, T_n$, we may assume X is unbounded in the first coordinate to obtain an infinite sequence in X

$$(x_{1i}, x_{2i}, \cdots, x_{ni}), \qquad i = 1, 2, \cdots$$

with

$$x_{11} < x_{12} < \cdots.$$

But then by $(*)$ the $n-1$, tuples $(x_{2i}, \cdots, x_{ni})$ must all be distinct, and renumbering $T_2, \cdots, T_n$ we can find a subsequence

$$(x_{1i_j}, x_{2i_j}, \cdots, x_{ni_j}), \qquad j = 1, 2, \cdots$$

with

$$x_{1i_1} < x_{1i_2} < \cdots \quad \text{and} \quad x_{2i_1} < x_{2i_2} < \cdots.$$

Continuing this process n steps we would obtain $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ with $x_i > y_i$ $(i = 1, \cdots, n)$. But this contradicts $(*)$, so we are done.

10. COROLLARY. *Every semilocal ring is Morita equivalent to a*

semilocal ring over which every finitely generated projective module is isomorphic to a direct sum of primitive one-sided ideals.

11. REMARKS. (a) We do not know whether "left" p-connected implies right—even for semilocal rings. It would be nice to find ideal theoretic conditions that characterize "p-connected." The sufficient ones that we can think of for semilocal rings (e.g., "All proper idempotent ideals contained in $J(R)$") are right-left symmetric.

(b) If every finitely generated projective left R-module is a generator is R p-connected? The answer is yes for commutative semilocal rings because for them the hypothesis implies that 1 is a primitive idempotent, and [11, Lemma 4] applies. If the answer is yes for all (semilocal) rings, then p-connected (semilocal) is left-right symmetric.

(c) Although the indecomposable projective modules over $Z_{[R,3]}$ and over the ring $\{R\}$ mentioned after Example 7 are finitely generated, at this point we see nothing to prevent, a ring R with $R/J \cong S \oplus T$ and $R^{(\aleph_0)} \cong P \oplus Q$ with P and Q indecomposable.

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