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ON COMPACT MINIMAL SURFACES WITH NON-NEGATIVE GAUSSIAN CURVATURE IN A SPACE OF CONSTANT CURVATURE: II

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5. $N_{(b)}$ and $\Omega_{(b)}$ of compact flat surface. As an application of the formulae obtained in the §4, we shall study $N_{(b)}$ and $\Omega_{(b)}$ of a compact flat surface. Let $\bar{M}$ be a space of constant curvature, $c \neq 0$. By the Gauss equation, we have $K_{(2)} = c$ and so $K_{(2)}$ is a positive constant and $c > 0$. Since $f_{(2)}$ is a globally defined non-negative smooth function on M, by $(4.26)_2$, we have $f_{(2)} =$ constant and $A_{(2)} = 0$ on M. By $4N_{(2)} = K_{(2)}^2 - f_{(2)}$, $N_{(2)}$ is also constant on M. By $K_{(2)} > 0$ on M and (3.11), we have $1 \leqq p_1(x) \leqq 2$ at any point of M. Since $N_{(2)}$ is constant, $p_1(x)$ is constant on M. Then the third fundamental forms are defined on a neighborhood of any point of M, i.e., we have $M = \Omega_{(2)}$. If $N_{(2)} = 0$, equivalently, $p_1(x) = 1$ on M, by Lemma 2, there is a 3-dimensional totally geodesic submanifold of $\bar{M}$ such that M is contained in the submanifold as a minimal surface. If $N_{(2)} \neq 0$, then $N_{(2)}$ is a positive constant on M and $p_1(x) = 2$ on M. As $f_{(3)}$ is globally defined on M, by $(4.26)_3$, we have $f_{(3)} =$ constant and $A_{(3)} = 0$. Then we can prove $K_{(3)} =$ constant by virtue of the following Lemma 4 and (4.27).

LEMMA 4. *Let M be a minimal surface in $\bar{M}$. Suppose that*

$$p_a(x) = 2,\ 0 \leqq a \leqq b - 2 \text{ and } p_{b-1}(x) = \text{constant on } \Omega_{(b)}\ ; \tag{5.1}$$

$$\bar{A}_{(b)} = 0 \text{ on } \Omega_{(b-1)}\ ; \tag{5.2}$$

$$K_{(b)} = \text{constant on } \Omega_{(b-1)}\ . \tag{5.3}$$

Then we have

$$N_{(b)} H^{(b)}_{\lambda_{b-1},1} = 0 \text{ on } \Omega_{(b)}\ . \tag{5.4}$$

PROOF. By (5.1), we have $H_\alpha^{(b)} = 0$ for $\alpha \geqq 2b + 1$. Then from (4.18) and (5.2), we obtain

$$H^{(b)}_{(2b-1)} H^{(b)}_{(2b-1),1} + H^{(b)}_{(2b)} H^{(b)}_{(2b),1} = 0\ . \tag{5.5}$$

Since $K_{(b)} =$ constant and (4.24), we get

(5.6) $$\bar{H}^{(b)}_{(2b-1)}H^{(b)}_{(2b-1),1} + \bar{H}^{(b)}_{(2b)}H^{(b)}_{(2b),1} = 0 .$$

It follows from (5.5) and (5.6) that we have

(5.7) $$\{H^{(b)}_{(2b-1)}\bar{H}^{(b)}_{(2b)} - \bar{H}^{(b)}_{(2b-1)}H^{(b)}_{(2b)}\}H^{(b)}_{\lambda_{b-1},1} = 0 .$$

By (4.21) and (5.7), we get (5.4). q.e.d.

From Lemma 4 and $N_{(2)} > 0$ on M, we have

(5.8) $$H^{(2)}_{\lambda_1,1} = 0 \quad \text{on} \quad M(=\Omega_{(2)}) .$$

By $(4.27)_2$ and (5.8), we can see

(5.9) $$K_{(3)} = \frac{K_{(1)}}{N_{(1)}}N_{(2)} = \text{positive constant} .$$

It follows from the $f_{(3)}$'s constancy that $N_{(3)}$ is also constant on $M = \Omega_{(2)}$. Continuing in this way, we can show the following lemma.

LEMMA 5. *Let M be a compact oriented flat minimal surface in $\bar{M}$. If $M = \Omega_{(s-1)}$ and $K_{(b)}$, $N_{(b)}$ are constant on M with $N_{(b)} > 0$, for $2 \leqq b \leqq s$, then we have $M = \Omega_{(s)}$ and $K_{(s+1)}$, $N_{(s+1)}$ are also constant on M with $K_{(s+1)} > 0$.*

PROOF. Since $M = \Omega_{(s-1)}$, by $(4.26)_b$, for $2 \leqq b \leqq s$, we have

(5.10) $$f_{(b)} = \text{constant and } A_{(b)} = 0 \text{ on } M, 2 \leqq b \leqq s .$$

The $N_{(s)}$ being a (positive) constant on M, we have $M = \Omega_{(s)}$. It follows that $f_{(s+1)} = \text{constant}$ and $A_{(s+1)} = 0$ on M by $(4.26)_{s+1}$. Then by $(4.27)_s$, (5.10) and Lemma 4, we get

(5.11) $$K_{(s+1)} = \frac{K_{(s-1)}}{N_{(s-1)}}N_{(s)}(>0 \text{ on } M) .$$

Since $K_{(s+1)}$ and $f_{(s+1)}$ are constant, $N_{(s+1)}$ is also constant on M. q.e.d.

Since dim. $T^{(b)}_x \leqq N$, the Lemma 5 says that there exist some integer q such that $K_{(q)} > 0$ on M but $N_{(q)} = 0$ on M. Thus by the Lemma 2 we have

THEOREM 2. *Let $\bar{M}$ be an N-dimensional Riemannian manifold of constant curvature $c \neq 0$ and $x: M \to \bar{M}$ be an isometric minimal immersion of a compact connected oriented Riemannian 2-manifold into $\bar{M}$ and $x(M)$ is not contained in any totally geodesic submanifold of $\bar{M}$. If the Gaussian curvature of M is identically zero, then $M = \Omega_{(b)}$, $b = 1, \cdots, q-1$, $c > 0$ and N is an odd integer $(=2q-1)$.*

6. Frenet-Borůvka's formula of a flat minimal surface. In this section we study the rigidity problem for a class of flat minimal surfaces.

From a result of §5, we have $f_{(b)} =$ constant for $2 \leqq b \leqq q$ and $f_{(q)} > 0$ on M. Let m be a first integer such that $f_{(m+1)} > 0$ on M and $f_{(b)} = 0$ for $b \leqq m$ $(\leqq q-1)$. In general it is probably $N_{(m+1)} \neq 0$, but we have interested in surfaces with $N_{(m+1)} = 0$ on M. Since $f_{(b)} = 0$ on M, for $2 \leqq b \leqq m$, we have

$$(6.1) \qquad \sum_\alpha h^2_{\alpha 1\cdots 1} = \sum_\alpha h^2_{\alpha 1 \cdots 12}\left(=\frac{1}{2}K_{(b)}\right) > 0 \text{ and } \sum_\alpha h_{\alpha 1\cdots 1} h_{\alpha 1\cdots 12} = 0 \,.$$

Let

$$(6.2) \qquad \begin{aligned} \tilde{e}_{2b-1} &= \frac{\sum h_{\alpha 1\cdots 1} e_\alpha}{\sqrt{\sum h^2_{\alpha 1 \cdots 1}}} \,; \\ \tilde{e}_{2b} &= \frac{\sum h_{\alpha 1\cdots 12} e_\alpha}{\sqrt{\sum h^2_{\alpha 1 \cdots 12}}} \,; \\ E_b &= \tilde{e}_{2b-1} + i\tilde{e}_{2b},\ 2 \leqq b \leqq m \,. \end{aligned}$$

Then for the above vector fields we have

$$(6.3) \qquad H^{(b)}_{(2b-1)} = -iH^{(b)}_{(2b)} = \sqrt{\sum h^2_{\alpha 1\cdots 1}} \,.$$

It follows from $(3.15)_b$ that we have (cf. [7])

$$(6.4) \qquad DE_b = -k_{b-1}\phi E_{b-1} - iw_{2b-1,2b}E_b + k_b\bar{\phi}E_{b+1},\ 1 \leqq b \leqq m-1 \,,$$

where

$$(6.5) \qquad k_1 k_2 \cdots k_{b-1} = \sqrt{\sum h^2_{\underbrace{\alpha 1 \cdots 1}_{b}}} \quad \text{and} \quad E_0 = 0,\ E_1 = e_1 + ie_2 \,.$$

By virtue of the Gauss equation, $K_{(2)} = c$, and (6.3), we have $k_1^2 = c/2$. Since $K^2_{(b)} = 4N_{(b)}$, $2 \leqq b \leqq m$, are positive constant on M, by (6.3) and (5.11), we have $K_{(b)}K_{(b-2)} = K^2_{(b-1)}$, and so $k_1^2 = k_2^2 = \cdots = k^2_{m-1} = c/2$. As $k_b > 0$, we get

$$(6.6) \qquad k_1 = \cdots = k_{m-1} = \sqrt{\frac{c}{2}} \,.$$

Since we supposed $N_{(m+1)} = 0$ on M, we may assume $N = 2m+1$, where N is the dimension of the ambiant space. Then we can put

$$(6.7) \qquad DE_m = -k_{m-1}\phi E_{m-1} - iw_{2m-1,2m}E_m + \Phi_{(m)} \,,$$

where $w_{2m-1,\alpha}$, $\alpha \geqq 2m$, are the differential forms for frames constructed in (6.2) and $\Phi_{(m)} = (w_{2m-1,2m+1} + iw_{2m,2m+1})e_{2m+1}$. By $(3.15)_{m+1}$, (6.3) and (6.5), we can set

$$w_{2m-1,\,2m+1} + iw_{2m,2m+1} = k_m\bar{\phi} \,,$$

where $k_1k_2\cdots k_{m-1}k_m = H^{(m+1)}_{(2m+1)}$. Note that $k_1, \cdots, k_{m-1}$ are real constant but k_m is a complex valued function. From these results, Lemma 4 and $(4.27)_m$, we obtain

$$k_m\bar{k}_m = 2k_{m-1}^2 = c\ . \tag{6.8}$$

The vector $E_1 = e_1 + ie_2$ is defined up to the transformation $E_1 \to E_1^0 = e^{i\tau}E_1$, where τ is real. Under such a change, we have, by (6.2) and (3.17),

$$\phi^0 = e^{i\tau}\phi\ , \tag{6.9}$$
$$E_b^0 = e^{bi\tau}E_b\ ,$$

and $k_1, \cdots, k_{m-1}$ are invariants,

$$k_m^0 = e^{(m+1)i\tau}k_m\ . \tag{6.10}$$

Therefore we may assume $k_m = \sqrt{c}$. By $(4.11)_b$, we have $w_{2b-1,2b} = bw_{12}$, $2 \leqq b \leqq m$, and, by $(4.11)_{m+1}$,

$$w_{12} = 0\ . \tag{6.11}$$

Thus the Frenet-Borůvka's formula for the surface is as follows:

$$\begin{aligned} DE_1 &= \sqrt{\frac{c}{2}}\bar{\phi}E_2\ , \\ DE_b &= -\sqrt{\frac{c}{2}}\phi E_{b-1} + \sqrt{\frac{c}{2}}\bar{\phi}E_{b+1},\ b = 2, \cdots, m-1\ , \\ DE_m &= -\sqrt{\frac{c}{2}}\phi E_{m-1} + \sqrt{c}\,\bar{\phi}E_{m+1}\ , \\ DE_{m+1} &= -\frac{\sqrt{c}}{2}\phi E_m - \frac{\sqrt{c}}{2}\overline{\phi E_m}\ , \end{aligned} \tag{6.12}$$

where $E_{m+1} = e_{2m+1}$. It follows that the minimal surface in consideration is locally uniquely determined up to isometries of $\bar{M}$, if $\bar{M}$ is connected and simply connected, M connected. On the other hand, by $(4.27)_{m+1}$, $N_{(m+1)} = 0$ on M is equivalent to $H_{\alpha,k}^{(m+1)} = 0$ on M. We summarize our results in the following theorem.

THEOREM 3. *Under the same assumption as in Theorem 2, if $K \equiv 0$, there is a first integer m such that $f_{(b)} = 0$ on M, for $b \leqq m$ and $f_{(m+1)} > 0$ on M. If $H_{\alpha,k}^{(m+1)} = 0$ on M, then the Frenet-Borůvka's formula is given by* (6.12). *Furthermore, if $\bar{M}$ is connected and simply connected then such a surface is uniquely determined up to isometries of $\bar{M}$.*

7. Generalized Clifford surface on S^{2m+1}. Let us consider the special case of an isometric minimal immersion $x: M \to S^N(1)$ of the flat surface with $f_{(b)} = 0$ on M, for $b \leqq m$ and $N_{(m+1)} = 0$. Theorem 2 and Theorem 3 have the consequence that the surface must lie on an odd dimensional great sphere $S^{2m+1}(1) \subset S^N(1)$. Thus we may assume $N = 2m + 1$. If e_A is an orthonormal frame of tangent vectors to $S^{2m+1}(1)$ such that e_i is tangent to M at $x \in M$, then $\{x, e_A\}$ is an orthonormal frame in R^{2m+2}, satisfying $(x, x) = 1$, $(x, e_A) = 0$ and $(e_A, e_B) = \delta_{AB}$, where the scalar product is defined for vectors in R^{2m+2}. From these formulae, we have $dE_1 = DE_1 - \phi x$ and $dE_b = DE_b$, $b > 1$. By (6.12) we have

$$
\begin{aligned}
dx &= \frac{1}{2}\bar{\phi}E_1 + \frac{1}{2}\phi\bar{E}_1 , \\
dE_1 &= -\phi x + \frac{1}{\sqrt{2}}\,\bar{\phi}E_2, \\
dE_a &= -\frac{1}{\sqrt{2}}\phi E_{a-1} + \frac{1}{\sqrt{2}}\bar{\phi}E_{a+1},\ a = 2, \cdots, m-1 , \\
dE_m &= -\frac{1}{\sqrt{2}}\phi E_{m-1} + \bar{\phi}E_{m+1} , \\
dE_{m+1} &= -\frac{1}{2}\phi E_m - \frac{1}{2}\overline{\phi E_m} .
\end{aligned} \tag{7.1}
$$

We put

$$X = (X_a, X_{a^*}) \in C^{2m+2},\ x = (x_a, x_{a^*}) \in R^{2m+2} , \tag{7.2}$$

where $X_a = x_a + ix_{a^*}$, $X_{a^*} = x_a - ix_{a^*}$, $a = 1, \cdots, m+1$, $a^* = a + m + 1$. Since the (local) vector field e_A will be considered as a R^{2m+2}-valued function, E_a is the C^{2m+2}-valued function. We can put

$$E_a = (E_{a(1)}, \cdots, E_{a(m+1)}, E_{a(1^*)}, \cdots, E_{a((m+1)^*)}) \in C^{2m+2} . \tag{7.3}$$

Using (7.3), we define a complex vector $F_A \in C^{2m+2}$ as follows:

$$F_A = (F_{A(1)}, \cdots, F_{A(m+1)}, F_{A(1^*)}, \cdots, F_{A((m+1)^*)}) \in C^{2m+2} , \tag{7.4}$$

where $1 \leqq A \leqq 2m + 2$,

$$
\begin{aligned}
&F_{a(b)} = E_{a(b)} + iE_{a(b^*)},\ F_{a(b^*)} = E_{a(b)} - iE_{a(b^*)} , \\
&F_{a^*(b)} = \bar{F}_{(m+2-a)(b^*)},\ F_{a^*(b^*)} = \bar{F}_{(m+2-a)(b)} ,
\end{aligned} \tag{7.5}
$$

and $\bar{F}_{(m+2-a)(b^*)}$ is the b^*-th component of the vector $\bar{F}_{m+2-a}$.
Note that $\bar{F}_{a(b)} \neq F_{a(b^*)}$, $1 \leqq a \leqq m$, since $\bar{E}_{a(b)} \neq E_{a(b^*)}$, but $\bar{F}_{m+1(b)} = F_{m+1(b^*)}$ and $F_{m+1} = F_{m+2}$.

By (6.11) we may take local coordinates $z = x + iy$ such that $ds^2 = dx^2 + dy^2 = dzd\bar{z}$. Then the system of differential equations (7.1) turns as follows:

$$
(7.6)\quad
\begin{aligned}
dX &= \frac{1}{2}\bar{\phi}F_1 + \frac{1}{2}\phi F_{2m+2}\,,\\
dF_1 &= -\phi X + \frac{1}{\sqrt{2}}\bar{\phi}F_2\,,\\
dF_a &= -\frac{1}{\sqrt{2}}\phi F_{a-1} + \frac{1}{\sqrt{2}}\bar{\phi}F_{a+1},\ a = 2, \cdots, m-1\,,\\
dF_m &= -\frac{1}{\sqrt{2}}\phi F_{m-1} + \bar{\phi}F_{m+1}\,,\\
dF_{m+1} &= -\frac{1}{2}\phi F_m - \frac{1}{2}\bar{\phi}F_{m+3}\,,\\
dF_{m+3} &= -\frac{1}{\sqrt{2}}\bar{\phi}F_{m+4} + \phi F_{m+2}\,,\\
dF_{m+p} &= -\frac{1}{\sqrt{2}}\bar{\phi}F_{m+p+1} + \frac{1}{\sqrt{2}}\phi F_{m+p-1},\ p = 4, \cdots, m+1\,,\\
dF_{2m+2} &= -\bar{\phi}X + \frac{1}{\sqrt{2}}\phi F_{2m+1}\,.
\end{aligned}
$$

Since $\phi = dz$, we see immediately that

$$
(7.7)\quad
\begin{aligned}
&\frac{\partial X}{\partial z} = \frac{1}{2}F_{2m+2}\,, && \frac{\partial X}{\partial \bar{z}} = \frac{1}{2}F_1\,,\\
&\frac{\partial F_1}{\partial z} = -X\,, && \frac{\partial F_1}{\partial \bar{z}} = \frac{1}{\sqrt{2}}F_2\,,\\
&\frac{\partial F_a}{\partial z} = -\frac{1}{\sqrt{2}}F_{a-1}\,, && \frac{\partial F_a}{\partial \bar{z}} = \frac{1}{\sqrt{2}}F_{a+1},\ a = 2, \cdots, m-1\,,\\
&\frac{\partial F_m}{\partial z} = -\frac{1}{\sqrt{2}}F_{m-1}\,, && \frac{\partial F_m}{\partial \bar{z}} = F_{m+1}\,,\\
&\frac{\partial F_{m+1}}{\partial z} = -\frac{1}{2}F_m\,, && \frac{\partial F_{m+1}}{\partial \bar{z}} = -\frac{1}{2}F_{m+3}\,,\\
&\frac{\partial F_{m+3}}{\partial z} = F_{m+2} = F_{m+1}\,, && \frac{\partial F_{m+p}}{\partial \bar{z}} = -\frac{1}{\sqrt{2}}F_{m+p+1},\ p = 3, \cdots, m+1\,,\\
&\frac{\partial F_{m+p}}{\partial z} = \frac{1}{\sqrt{2}}F_{m+p-1},\ p = 4, \cdots, m+2\,, && \frac{\partial F_{2m+2}}{\partial \bar{z}} = -X\,.
\end{aligned}
$$

Let ε_b, $b = 1, 2, \cdots, m+1$, be roots of an equation $\varepsilon^{m+1} = \sqrt{-1}$, if m is an even integer and let ε be a non trivial root of an equation $\varepsilon^{2m+2} = 1$ and we set $\varepsilon_b = \varepsilon^b$ if m is an odd integer. The solution of (7.7) is given by

$$(7.8)\quad \begin{aligned} X_b &= \frac{1}{\sqrt{m+1}} \exp \frac{1}{\sqrt{2}}\{\varepsilon_b z - \overline{\varepsilon_b z}\},\ b = 1, \cdots, m+1\ , \\ F_{a(b)} &= (-1)^a \sqrt{2}(\bar{\varepsilon}_b)^a X_b,\ a = 1, \cdots, m\ , \\ F_{m+1(b)} &= F_{m+2(b)} = (-1)^{m+1}(\bar{\varepsilon}_b)^{m+1} X_b\ , \\ F_{a^*(b)} &= \begin{cases} (-1)^{m+1}(\bar{\varepsilon}_b)^m F_{a(b)}\ , & \text{if } a \text{ is even ;} \\ (-1)^m (\bar{\varepsilon}_b)^m F_{a(b)}\ , & \text{if } a(\geqq 2) \text{ is odd .} \end{cases} \end{aligned}$$

Note that $F_{2m+2(b)} = \sqrt{2}\,\varepsilon_b X_b$. We call the above surface on $S^{2m+1}(1)$ the generalized Clifford surface of index m which is the image of a minimal immersion of the Euclidean plane into $S^{2m+1}(1)$. We give the explicit representation of (7.8).

THEOREM 4. *The generalized Clifford surface of index m on $S^{2m+1}(1)$ is given by $(X_1, X_2, \cdots, X_{m+1}) \in C^{m+1}$, where*

$$(7.9)\quad \begin{aligned} X_1 &= \frac{1}{\sqrt{m+1}} e^{i\theta}\ , \quad X_2 = \frac{1}{\sqrt{m+1}} e^{i\tau}\ , \\ X_b &= \frac{1}{\sqrt{m+1}} e^{i((a_b/a_2)\tau - (a_{b-1}/a_2)\theta)}\ , \quad b = 3, 4, \cdots, m+1\ , \\ a_b &= \begin{cases} \sin \dfrac{2(b-1)}{m+1}\pi\ , & \textit{if } m \textit{ is even}\ , \\ \sin \dfrac{(b-1)}{m+1}\pi\ , & \textit{if } m \textit{ is odd}\ . \end{cases} \qquad b = 1, 2, \cdots, m+1\ , \end{aligned}$$

PROOF. (I) If m is even, we have

$$\varepsilon_b = \cos \frac{(4b-3)}{2(m+1)}\pi + i \sin \frac{(4b-3)}{2(m+1)}\pi\ .$$

Then we get

$$(7.10)\quad \sqrt{m+1}\,X_b = \exp\left\{i\sqrt{2}\left(x \sin \frac{(4b-3)}{2(m+1)}\pi + y \cos \frac{(4b-3)}{2(m+1)}\pi\right)\right\}\ .$$

(II) When m is odd, we set $\varepsilon = \cos \pi/(m+1) + i \sin \pi/(m+1)$. Then we have

$$(7.11)\quad \sqrt{m+1}\,X_b = \exp\left\{i\sqrt{2}\left(x \sin \frac{b\pi}{m+1} + y \cos \frac{b\pi}{m+1}\right)\right\}\ .$$

(7.9) follows from (7.10), (7.11) and

$$\theta = \begin{cases} \sqrt{2}\left(x \sin \dfrac{\pi}{2(m+1)} + y \cos \dfrac{\pi}{2(m+1)}\right), & \text{if } m \text{ is even ,} \\ \sqrt{2}\left(x \sin \dfrac{\pi}{m+1} + y \cos \dfrac{\pi}{m+1}\right), & \text{if } m \text{ is odd ,} \end{cases}$$

$$\tau = \begin{cases} \sqrt{2}\left(x \sin \dfrac{5\pi}{2(m+1)} + y \cos \dfrac{5\pi}{2(m+1)}\right), & \text{if } m \text{ is even}, \\ \sqrt{2}\left(x \sin \dfrac{2\pi}{m+1} + y \cos \dfrac{2\pi}{m+1}\right), & \text{if } m \text{ is odd}. \end{cases} \qquad \text{q.e.d.}$$

Let $m = 1$. Then we have

$$(X_1, X_2) = \frac{1}{\sqrt{2}}(e^{i\theta}, e^{i\tau}) \in C^2 .$$

This is the classical Clifford minimal surface on S^3 which is also the minimal immersion of a flat torus.

Let $m = 2$. Then we have

$$(X_1, X_2, X_3) = \frac{1}{\sqrt{3}}(e^{i\theta}, e^{i\tau}, e^{-i(\theta+\tau)}) \in C^3 .$$

This is the generalized Clifford surface on $S^5(1)$. Although the above two mappings induce minimal immersions of a flat torus into the sphere, we can not expect the same results for $m \geqq 3$. For instance, the generalized Clifford surface with index 3:

$$(X_1, \cdots, X_4) = \frac{1}{2}(e^{i\theta}, e^{i\tau}, e^{i(\sqrt{2}\tau-\theta)}, e^{i(\tau-\sqrt{2}\theta)}) \in C^4$$

does not induce a minimal immersion of a flat torus. (We shall remark that a statement of §7 in the Introduction of this paper is incomplete.)

T. Ōtsuki ([15], p. 119) gives a different representation of the solution of (7.6).

8. $f_{(b)}$ and $\Omega_{(b)}$ of compact surfaces with $K \geqq 0$ and $K \not\equiv 0$. In this section, we assume that M is compact, oriented, connected minimal surface on $S^N(1) \subset R^{N+1}$ with

$$K \geqq 0 \quad \text{and} \quad K \not\equiv 0 . \tag{8.1}$$

Then we claim that

$$f_{(b)} = A_{(b)} = 0 \quad \text{on} \quad \Omega_{(b-1)} , \quad \text{for each possible} \quad b . \tag{8.2$_b$}$$

The $(8.2)_b$ follows from a Chern's discussion in [7], but there is a gap in his paper, especially p. 36 in [7]. Therefore we shall give a proof of $(8.2)_b$. We need the following results.

LEMMA 6. ([13]). *Let $x: M \to S^N(1) \subset R^{N+1}$ be an isometric minimal immersion and (u, v) are local isothermal coordinates for M, then $x(u, v)$ is real analytic.*

LEMMA 7. ([7], [18]). *Let $w_\alpha(z)$ be complex-valued functions which satisfy the differential system*

$$\frac{\partial w_\alpha}{\partial \bar{z}} = \sum a_{\alpha\beta} w_\beta\ , \quad 1 \leqq \alpha, \beta \leqq p\ , \tag{8.3}$$

in a neighborhood of $z = o$, where $a_{\alpha\beta}$ are complex valued C^1-functions. Suppose the w_α do not all vanish identically in a neighborhood of $z = 0$:
(1) *Let $w_\alpha = o(|z|^{r-1})$ at $z = 0$, $r \geqq 1$. Then $\lim_{z\to 0} w_\alpha(z)z^{-r}$ exists.*
(2) *Suppose $w_\alpha = o(|z|^{r-1})$, all r. Then $w_\alpha \equiv 0$ in a neighborhood of $z = 0$.*

Since $f_{(2)}$ is a globally defined non-negative smooth function on M, by $(4.26)_2$ and (8.1), we have $(8.2)_2$. If $K_{(2)} \neq 0$ on M, then we have $N_{(2)} = 1/4K_{(2)}^2$ is not identically zero on M and

$$\Omega_{(2)} = \{x \in M\colon N_{(2)} \neq 0 \quad \text{at} \quad x\}\ . \tag{8.4}$$

Let $y \in M - \Omega_{(2)}$. Since $y \in \Omega_{(1)}$, we have $K_{(2)}^2 = 4N_{(2)}$ at y. As $y \bar{\in} \Omega_{(2)}$, by (8.4), we have $K_{(2)} = 0$ at y. By (2.13) and Lemma 7, we can show that the set $M - \Omega_{(2)}$ must be at most finite (cf. [7], [8]). Let z be an isothermal coordinate on a neighborhood U of y in M such that $z = 0$ corresponds to y and $\phi = \lambda dz$ on U. We define a complex valued function $\Lambda_{(3)}$ on an open set $V \subset U$ as follows:

$$\Lambda_{(3)} = \begin{cases} \lambda^6 \sum_{\mu\geqq 5} (\overline{H_\mu^{(3)}})^2 & \text{on} \quad V - \{0\}\ , \\ 0 & \text{on} \quad \{0\}\ , \end{cases} \tag{8.5}$$

where $N_{(2)} \neq 0$ on $V - \{o\}$. We prove that $\Lambda_{(3)}$ is a holomorphic function on V, and thus $\tilde{f}_{(3)} = \lambda^{-12}\Lambda_{(3)}\bar{\Lambda}_{(3)}$ is a smooth function on V, since $\lambda \neq 0$ on V: By (2.15) with $\phi = \lambda dz$ and (4.17), $\lambda^6 \sum_{\mu\geqq 5} (\overline{H_\mu^{(3)}})^2$ is holomorphic on $V - \{0\}$, (cf. [7], p. 36). If we can show that $\Lambda_{(3)}(z)$ is a continuous function on V, then, by the Rado's theorem ([14], p. 53), $\Lambda_{(3)}(z)$ is holomorphic on V.

The continuity of $\Lambda_{(3)}(z)$: Let $\{z_n\} \subset V - \{0\}$ be a sequence such that $z_n \longrightarrow 0$ $(n \longrightarrow \infty)$. Since $x(u, v)$ is real analytic by Lemma 6, we can see that $h_{\lambda_1 i_1 i_2}$'s are also real analytic. In Lemma 7, if $w_\alpha(z)$, $1 \leqq \alpha \leqq p$, are real analytic, we can write $w_\alpha(z) = z^m w'_\alpha(z)$, where $w'_\alpha(z)$ are also real analytic and for some α, $w'_\alpha(0) \neq 0$. It follows that the function defined by

$$\frac{w_\alpha}{\sqrt{\sum w_\beta \bar{w}_\beta}}$$

is meaningful at $z = 0$ and smooth at $z = 0$. Therefore by the above observation and $(4.11)_2$, the (local) vector field E_2 is smooth on a neighborhood of $z = 0$. At the neighborhood of $z = 0$, we have obtained a smooth decomposition $\{e_{\lambda_0}, e_{\lambda_1}, e_{\lambda_2}\}$. By virtue of these vector fields, $w_{\lambda_1\lambda_2}$ and w_{i_3} defined on $\Omega_{(2)}$ tend to bounded forms at $z=0$. Therefore by (3.6), we have $A_{(3)}(z) \to 0$ $(n \to \infty)$. That is, $A_{(3)}(z)$ is a continuous function on V. Since $\lambda \not\equiv 0$, $\tilde{f}_{(3)}$ is smooth on M. As M is compact, $\tilde{f}_{(3)}$ attains a maximum at $p_0 \in M$. If $\tilde{f}_{(3)}(p_0) = 0$, then $f_{(3)}$ is identically zero and thus we have $(8.2)_3$. If $\tilde{f}_{(3)}(p_0) > 0$, we have $p_0 \in \Omega_{(2)}$ and $f_{(3)}$ attains the maximum at p_0. Since $f_{(3)}$ is subharmonic on $\Omega_{(2)}$, by the maximum principle, $f_{(3)} = \text{constant}$, $f_{(3)}K = 0$ and $A_{(3)} = 0$ on $\Omega_{(2)}$, and so $f_{(3)} = 0$ on $\Omega_{(2)}$ by (8.1). Continuing in this way, we can define a smooth decomposition $\{e_{\lambda_0}, e_{\lambda_1}, \cdots, e_{\lambda_{b-1}}\}$ of a (local) frame field e_A at any point of M. Therefore, for the possible b, if we define $\tilde{f}_{(b)}$ as follows:

$$\tilde{f}_{(b)} = \begin{cases} f_{(b)} & \text{on} \quad \Omega_{(b-1)} \\ 0 & \text{on} \quad M - \Omega_{(b-1)}\,, \end{cases} \tag{8.6}$$

then $\tilde{f}_{(b)}$ is a smooth function on M and we have $(8.2)_b$. Summarizing up these results, we get

PROPOSITION. *Let $x: M \to S^N(1) \subset R^{N+1}$ be an isometric minimal immersion of a compact oriented 2-Riemannian manifold with $K \geqq 0$ and $K \not\equiv 0$. Then we have*
(1) $f_{(b)} = 0$ on $\Omega_{(b-1)}$;
(2) $M - \Omega_{(b-1)}$ are at most finite, for the possible b.

Appendix

9. An extrinsic rigidity theorem. Let $x: M \to S^{n+p}(1)$ be an isometric minimal immersion of a compact oriented Riemannian n-manifold M^n into $S^{n+p}(1)$. As an extrinsic rigidity theorem of x, the following DeGiorgi-Simons-Reilly's Theorem is known: Let N be the smooth field of oriented unit normal p-planes of M^n in $S^{n+p}(1)$ and let $A_{n+1}, A_{n+2}, \cdots, A_{n+p}$ be an orthonormal set of vectors in R^{n+p+1}. We put $A = A_{n+1} \wedge A_{n+2} \wedge \cdots \wedge A_{n+p}$ and $U = (N, A)$, where (N, A) means the standard inner product of N and A in exterior algebra. If $U > \sqrt{(2p-2)/(3p-2)}$, x is totally geodesic. In particular if $U > \sqrt{2/3}$, x is so.

S. S. Chern conjectured [6] that if there exists a constant decomposable p-vector $A = A_{n+1} \wedge A_{n+2} \wedge \cdots \wedge A_{n+p}$ such that $(N, A) > 0$, M is totally geodesic.

In the case when $n = 2$ and $p > 2$, we can answer affirmatively to the conjecture of a little generalized form as follows:

THEOREM 5. *Let x be an isometric minimal immersion of a compact oriented Riemannian 2-manifold into $S^N(1)$. If $U > \sqrt{1/2}$, x is totally geodesic.*

PROOF. Reilly's integral formula [16] is, in this case,

$$\int_M \{-K_{(2)}U + Q\}dM = 0 ,$$

where

$$Q = \sum_{i<j,\alpha<\beta,k}\sum\sum (h_{\alpha ik}h_{\beta jk} - h_{\alpha jk}h_{\beta ik})h_{\alpha\beta ij}$$

and

$$h_{\alpha\beta ij} = (e_{n+1} \wedge \cdots \wedge e_{\alpha-1} \wedge e_i \wedge e_{\alpha+1} \wedge \cdots \wedge e_{\beta-1} \wedge e_j \wedge e_{\beta+1} \wedge \cdots \wedge e_{n+p}, A) .$$

The Cauchy-Schwartz inequality implies that

$$Q^2 \leqq \left\{\sum_{\alpha<\beta,i<j}\sum\left(\sum_k (h_{\alpha ik}h_{\beta jk} - h_{\alpha jk}h_{\beta ik})\right)^2\right\}\left\{\sum_{\alpha<\beta,i<k}\sum h^2_{\alpha\beta ik}\right\}$$
$$\leqq 4N_{(2)}(1-U^2) ,$$

because of (3.23) and [16, p. 493], i.e.,

$$Q \leqq 2\sqrt{N_{(2)}(1-U^2)} \leqq K_{(2)}\sqrt{(1-U^2)} ,$$

because of $f_{(2)} \geqq 0$. Thus if $U > \sqrt{1/2}$ we have

$$-K_{(2)}U + Q \leqq -K_{(2)}U + K_{(2)}\sqrt{1-U^2} = K_{(2)}\{\sqrt{1-U^2} - U\} \leqq 0 .$$

This implies that $K_{(2)} = 0$ on M, i.e., x is totally geodesic. q.e.d.

ADDED IN PROOF (May, 1973): (1) The inequality $f_{(2)} \geqq 0$ was used firstly in [11], but the local version of the main theorem in [11] has been proved by Y. C. Wong ([17], Th. 4.9).

(2) We wish to acknowledge that a closely related treatment was announced by T. Itoh in Tokyo, April, 1973, based on the work of T. Ōtsuki.

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