

**NOTES ON BANACH SPACE (X):  
VITALI-HAHN-SAKS' THEOREM AND K-SPACES.\*)**

By

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Since H. Freudenthal [4] established the spectral representation for the vector-lattices, the closed analogy between the theory of the vector-lattices and that of the additive set functions are pointed out by several authors. As its consequence the well-known theorem of Radon-Nikodym are abstractly handled.

This note<sup>1)</sup> lies in this direction, and firstly we prove the Hahn-decomposition theorem and the Vitali-Hahn-Saks theorem of the additive set functions for the Banach-lattices. The former, as proved by G. Birkhoff [2], is already known in strictly monotone Banach lattices. But, as will be seen in the following, if we restrict the linear functional suitably, then Birkhoff's proof is applicable in some more general cases. The later is obtained by T. Ogasawara [12] and H. Nakano [10] independently. Ogasawara's proof depends on his representation theory and concrete case of the Vitali-Hahn-Saks theorem. On the other hand, Nakano's proof is fine but does not contain the classical theory of the additive set functions. In this note, we prove it containing both cases, using the method of S. Saks [13]. Hence it may be observed with some interest.

As an application of the above theorems, we prove in §3 some structure theorems due to T. Ogasawara [11] on K-spaces. In §2, we will prove some lemmas, which are due to T. Ogasawara [11], [12]. We gave a sketch of proof, which does not depend on the concrete representation theory, and then seems to be somewhat simpler than those of T. Ogasawara.

Throughout this note, we use the terminologies of the text books of G. Birkhoff [2] and S. Banach [1] without any explanation. But there is one different point which is the notion of the "ideal" of the vector lattices. We use it here as "closed admissible l-ideal" in [3] or "complemented normal

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1) A part of this note is contained in Japanese [9]. The author expresses his hearty thanks to Prof. S. Kakutani who gave the author valuable remarks.

subspace" in [2].

1. We concern, in this article, with complete Banach lattices. We will begin by the following definitions.

**Definition 1.** A set  $S = \{x_\alpha\}$  in a complete Banach lattice is said to have the Moore-Smith property provided that  $x_\alpha, x_\beta \in S$  imply  $x_\alpha \vee x_\beta \in S$  (or dually). And a set  $S$  of positive elements  $\{x_\alpha\}$  with the Moore-Smith property is said to be order-convergent to zero, symbolically  $x_\alpha \downarrow 0$ , if the greatest lower bound of  $S$  is zero.

**Definition 2.** A linear functional  $f$  defined on a complete Banach lattice  $E$  is said to be order-continuous provided that  $f(x_\alpha)$  converges to zero in the Moore-Smith sense whenever  $x_\alpha$  order-converges to zero, that is, for any  $\varepsilon > 0$  and any  $x_\alpha \downarrow 0$  there exists  $\alpha$  such that  $x_\beta < x_\alpha$  implies  $|f(x_\beta)| < \varepsilon$ .

This notion of order-continuity, due to T. Ogasawara [5], seems to play an essential rôle in the theory of complete Banach lattices, for it resembles the notion of the absolute continuity in the theory of the additive set functions. Many theorems on absolutely continuous set functions are generalized to the concept of the order-continuous linear functionals on the complete Banach lattice. Firstly, the so-called Jordan-Hahn decomposition theorem of the additive set functions is proved in the complete Banach lattice as following:

**Theorem 1.** By mean of an order-continuous linear functional, every complete Banach lattice is decomposed into direct sum of positive, negative and null ideals.

By positive, negative and null ideals, we mean complemented normal subspaces with elements  $x$  such that  $0 < y \leq |x|$  implies  $f(y) > 0$ ,  $f(y) < 0$ ,  $f(y) = 0$  respectively, as defined in [2] § 151. This theorem is proved already by G. Birkhoff [2] for the strictly monotone normed Banach lattices. But examining his proof, we see that the strict monotonicity of the norm is not essentially used and it can be replaced by the order-continuity of the given linear functional. Therefore, proof of the theorem is done parallel as that of G. Birkhoff. Hence we omit the details.

By Theorem 1, we see, that for any order-continuous linear functional  $f$  the order-continuity of  $|f|$  and  $g$  follows for any  $|g| \leq |f|$ . For,  $f^+(x) = f(x)$  where  $x$  is the component of  $x$  in the positive ideal, and  $0 \leq g \leq f$  implies  $0 \leq g(x) \leq f(x)$  for any  $x \geq 0$ . Thus, we have immediately:

**Corollary 1.** All order-continuous linear functionals on a complete

Banach lattice form a metrically closed normal subspace in the conjugate space.

Nextly, we show that the Vitali-Hahn-Saks theorem can be proved in the complete Banach lattice, using the method of the proof due to S. Saks [13].

**Theorem 2.** If an enumerable sequence of the order-continuous linear functionals converges weakly on a complete Banach lattice, then the limit is also order-continuous.

**Proof.** Let  $\{f_n\}$  be a sequence converging weakly to  $f_0$ , and  $\{x_\alpha\}$  be a Moore-Smith set of positive elements order-converging to zero. Without loss of generality, we can assume  $0 \leq x_\alpha \leq 1$ . By  $I$  we denote the interval from 0 to 1, and put

$$f = \sum_i \frac{1}{2^i} |f_i|.$$

By Corollary 1,  $f$  exists and is a positive order-continuous linear functional.

Since, by Theorem 1, it suffices to show the theorem for the positive ideal of  $f$ , we may assume without loss of generality, that 1 belongs to the positive ideal of  $f$  and  $f(1)=1$ . Then, by a theorem due to G. Birkhoff [2; Theorem 3.13],  $f$  defines a continuous metric on the interval  $I$  considered as a distributive lattice. Since its metric topology coincides with star convergence,  $f_i$  becomes continuous on the metric space  $I$ .

Let  $\varepsilon$  be any positive number and put

$$H_n = \{x \mid \varepsilon \geq |f_n(x) - f_m(x)|, m \geq n, x \in I\},$$

then  $H_n$  is closed in  $I$  and  $\bigcap_n H_n = I$ . Since, by the above used theorem due to G. Birkhoff, the lattice completeness of  $I$  implies the metric completeness,  $I$  becomes a set of the second category. Hence, by the well-known Baire's category theorem, some  $H_n$  contains a sphere  $S$  with center  $a$  and radius  $\lambda$ .

Now, if we put for any  $x$  such that  $f(x) < \lambda$ ,

$$y_1 = a \wedge (1-x) \quad \text{and} \quad y_2 = x + y_1,$$

then  $y_1$  and  $y_2$  belong to the sphere  $S$ . Hence, by the definition of  $H_n$ ,

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(y_2 - y_1) - f_m(y_2 - y_1)| \\ &\leq |f_n(y_1) - f_m(y_1)| + |f_n(y_2) - f_m(y_2)| \leq 2\varepsilon \end{aligned}$$

for all  $m \geq n$ . Therefore, for any  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $f(x) < \delta$  implies  $|f_m(x)| < 3\varepsilon$  for all  $m$ . Thus we have  $|f_0(x)| < 3\varepsilon$ . This proves the theorem.

Since the essential point of the proof is the positive ideal of  $f$  includes all positive and negative ideals of  $f_i$ , and since in the case of the absolutely

continuous set functions this is satisfied by the existence of the given measure, the later half of the proof gives the following well-known Vitali-Hahn-Saks theorem:

**Theorem 3.** If a sequence of additive and absolutely continuous set functions, defined on a space with finite completely additive measure, converges on every set, then the limit is also absolutely continuous and moreover the functions are equi-absolutely continuous.

It is also remarked, that Theorem 2 is true for the sequentially order-continuous case, that is, if  $\lambda_m \downarrow 0$  implies  $f_n(x_m) \rightarrow 0$  for all  $m$  and  $f_m$  converges weakly to  $f_0$ , then  $f_0(x_m) \rightarrow 0$ . In this case, the proof requires some modification, since the Hahn-decomposition theorem does not hold to the sequentially order-continuous case.

2. We will give some applications of Theorem 1. We begin to state the following theorem due to Ogasawara [11]:

**Theorem 4.** For a complete Banach lattice, the following three conditions are equivalent:

Condition F1: Every linear functional is order-continuous,

Condition F2:  $x_n \downarrow 0$  implies  $\|x_n\| \rightarrow 0$ ,

Condition F3:  $x_\alpha \downarrow 0$  implies  $\|x_\alpha\| \rightarrow 0$ .

F2 follows from F1 by virtue of Dini's theorem for monotone sequence of continuous functions, F3 follows from F2 in same manner as Theorem 8 of the next article, and evidently F3 implies F1.

In this article we assume that the given Banach lattice  $E$  is complete and satisfies one of the above conditions. Thus Theorem 1 is applicable for any linear functional  $f$  on  $E$ . Therefore, if we denote by  $P_f, N_f$  the positive and null ideals of  $f > 0$  respectively, then by the definition of join of two positive functionals  $f$  and  $g$  we have

$$f \vee g(x) = \sup \{f(y) + g(z) \mid x = y + z \geq 0, y, z \geq 0\}.$$

It is easily verified that  $P_f \vee P_g = P_f \cup P_g$  and  $N_f \vee N_g = N_f \cap N_g$ . And dually it holds for the meet,  $P_f \wedge P_g = P_f \cap P_g$  and  $N_f \wedge N_g = N_f \cup N_g$ .

Therefore if we put

$$I^b = \bigvee \{P_f \mid f \in I\}$$

for an ideal  $I$  in  $E^*$ , the conjugate space of  $E$ , then we see by above equalities, that the correspondence  $I \rightarrow I^b$  determines a lattice homomorphism from the structure lattice  $L^*$ , the lattice of ideals in  $E^*$ , to the structure lattice  $L$  of  $E$ .

Conversely, as was proved essentially by Kantorovitch [8],  $f_\alpha \downarrow 0$  implies

$x(f_\alpha) = f_\alpha(x) \rightarrow 0$ . Hence, we can apply Theorem 1 taking elements of the given space as linear functionals on the conjugate space. And the following lemma credits the above arguments in this case.

**Lemma 1.** Every Banach lattice is closed sub-lattice in the second conjugate space.

Since the linearity and the closedness follow from the general theory of the Banach spaces, it is sufficient to show that the order relation and the lattice operations are preserved. Concerning the former, if  $x$  is positive in  $E$  and not positive in  $E^{**}$ , the second conjugate of  $E$ , then, applying Theorem 1 to  $x$  as a linear functional on  $E^*$ , we have a non-zero negative ideal of  $x$  in  $E^*$ , that is,  $f(x) < 0$  for some  $f > 0$ , which is a contradiction. Concerning the later, it is sufficient to prove that  $f(x)$  attains its supremum  $f(x^+)$  in the interval  $(0, f)$ . Since the existence of a functional  $g$  taking supremum value in the interval follows from Theorem 1, it needs to prove that  $f(x)$  takes the value  $f(x^+)$  in the interval. This can be proved similarly as the proof of the Hahn-Banach extension theorem appealing with Zorn's lemma, for the positiveness and boundedness of linear functionals are of finite restrictions.

We will now put

$$I^\# = \vee \{ P | x | | x \in I \}$$

for an ideal  $I$  in  $E$ , and we call the correspondences  $I \rightarrow I^b$ ,  $I \rightarrow I^\#$  as derived correspondences. Then we have the following duality theorem due to T. Ogasawara:

**Theorem 5.** The structure lattices of the conjugate space and the complete Banach lattice which satisfies the Condition F are isomorphic under the derived correspondences.

To prove this we need a lemma:

**Lemma 2.** If a positive element  $x$  belongs to  $I^b$ , then there exists at least one positive linear functional  $f$  in  $I$  with  $f(x) > 0$ .

**Proof.** By the hypothesis  $x \in I^b$ , it is evidently excluded by some null ideal  $Nf$  for  $f \in I$ . If we assume that  $f$  is positive, it is  $f(x) > 0$  by the definition of the positive ideal.

**Proof of Theorem 6.** Since it is obvious  $I^{b\#} \geq I$ , we can assume that there exists a linear functional  $f$  in  $I^{b\#}$ , orthogonal to all  $g \in I$ . Hence by Lemma 1  $f > 0$  implies the existence of a positive element  $x$  in  $I^b$  with  $f(x) > 0$ . On the other hand,  $f \wedge g = 0$  for all  $g \in I$  implies  $Pf \cap Pg = 0$ , from which it may be seen  $Pf \cap I^b = 0$  easily. This is a contradiction.

In order to give an alternative form of Theorem 6, we introduce a notion due to H. Freudenthal [4]:

**Definition 3.** An element  $1$  is a principal unit of  $E$  if  $1 \wedge x > 0$  for any  $x > 0$ . If a principal unit exists and  $E$  is complete lattice, then the set of all  $e$  with  $e \wedge (1 - e) = 0$  is called unit-lattice of  $E$ .

If a principal unit exists in a complete Banach lattice, then the structure lattice is isomorphic to the unit-lattice. In this case Theorem 6 may be stated as follows:

**Theorem 7.** If the principal units exist for the conjugate space and the complete Banach lattice satisfying Condition C, then the unit-lattices are isomorphic.

Following H. Freudenthal [4], every complete Banach lattice  $E$  with a principal unit is a metrically closed hull of linear combinations of the unit-lattice  $B$ . Hence if we term this fact by " $E$  is constructed on a complete Boolean algebra  $B$ ", then Theorem 6 can be restated in such a way that  $E$  and  $E^*$  can be constructed on the same Boolean algebra up to isomorphism. This point gives a possibility to construct a representation theory of  $E$  and  $E^*$  in the same time, as similar manner as the abstract  $(L)$ -space in S, Kakutani [6]. But since this problem do not concern to the below considerations, we do not go to further.

For the later use, we prove the following two lemmas.

**Lemma 3.** If an element  $\ddot{x}$  of the second conjugate space of a complete Banach lattice, not belonging to the original lattice, is dominated by an element of the original space, then the unit lattice with respect to  $\ddot{x}$  contains at least one element which is not contained in the given space.

**Proof.** By the spectral theorem due to H. Freudenthal [4], every element in  $E^{**}$ , dominated by  $1 \in E$ , can be approximated uniformly by finite linear combinations of the unit-lattice of  $E^{**}$  with respect to  $1$ . Therefore, the lemma follows from Theorem 7. 21 of [2] and from the metrically closedness of  $E$  as a subspace of  $E^{**}$ .

**Lemma 4.** If a complete Banach lattice satisfies the conditions of Theorem 5, then it is closed normal subspace in the second conjugate space.

**Proof.** By Lemma 3, it suffices to prove, that every element  $e$  of the unit lattice of  $E^{**}$  with respect to any element  $1$  in  $E$  belongs to  $E$ . Since  $\ddot{e}$  is order-continuous on  $E^*$  by Corollary 1, it has non-zero positive ideal  $P$  in  $E^*$ . Hence, by Theorem 6, there exists non-zero ideal  $P^b$ , and  $P^b$  is contained

in the principal ideal of 1 in  $E$ . Since  $E$  is complete,  $P^b$  is a principal ideal by some element  $e$  of the unit lattice of  $E$  with  $\bar{e} \wedge (1-e) = 0$ , and we have  $\bar{e} - e < 0$  in  $E^{**}$ . Since  $e - \bar{e}$  is order-continuous, it has positive ideal different from zero, and by  $(e - \bar{e}) \wedge \bar{e} = 0$  it also differs from  $P^b$ . This is a contradiction.

As a consequence of Lemma 4, we get a theorem due to T. Ogasawara [11]:

**Corollary 2.** If a complete Banach lattice satisfies Condition F, then its intervals are weakly compact.

For, since every interval in  $E^{**}$  is closed with respect to the weak topology as functionals<sup>2)</sup> on  $E^*$  and the unit sphere of  $E^{**}$  is weakly compact with respect to that topology, Lemma 4 gives the above statement.

3. We conclude this paper by proving some results due to T. Ogasawara [11] without use of the representation theory. We will now introduce the following definitions:

**Definition 4.** A (complete) Banach lattice with Condition F is called a K-space if it satisfies the following condition:

Condition L:  $0 \leq x_n \leq x_{n+1}$ , and the norm boundedness of  $x_n$  imply the existence of  $\bigvee_n x_n$ .

As T. Ogasawara [11] pointed out, Conditions F and L are equivalent to

Condition K:  $0 \leq x_n \leq x_{n+1}$  and norm boundedness imply the strong convergence of  $x_n$  to its supremum  $\bigvee_n x_n$ .

Moreover, we can prove the following theorem similarly as Theorem 7.23 of G. Birkhoff [2], which he proved for the strictly monotone normed Banach lattices:

**Theorem 8.** A Banach lattice is a K-space if and only if it satisfies the following

Condition B: Every metrically bounded Moore-Smith set converges strongly to its bound.

**Proof.** Since the sufficiency is obvious, it remains to prove the necessity. Let  $S$  be a Moore-Smith set in the K-space  $E$ , and suppose  $\sup \{ \|x_\alpha\| \mid x_\alpha \in S \} = 1$  and  $x_\alpha \geq 0$ . Let  $S'$  be the set of all supremums of enumerable set of  $S$ , then  $S'$  has also the Moore-Smith property. If  $A$  is a linearly ordered subset of  $S'$ , then  $A$  converges its supremum in  $S'$ , for, if otherwise, there exists enumerable infinite sequence  $x_{\alpha(i)} \in A$  with  $|x_{\alpha(i+1)} - x_{\alpha(i)}| \geq \varepsilon$

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2) On the weak topologies of the Banach spaces, cf. S. Kakutani [5].

which leads to a contradiction. Therefore,  $S'$  is inductively ordered and then by Zorn's lemma it has maximal element  $a$ . But since  $S'$  has the Moore-Smith property,  $a$  becomes its maximum. The strong convergence follows from Condition K.

From Theorem 8 and Lemma 4 it follows as a generalization of the well-known theorem of Radon-Nikodym.

**Theorem 9.** K-space is an ideal in the second conjugate space.

**Proof.** By the last footnote in G. Birkhoff's book Chap. VII, it suffices to show that every order-bounded set of  $E$  has a least upper bound in  $E$ . But, this is an obvious consequence of Theorem 8, since the set  $S$  with bound  $x \in E^{**}$  converges its supremum and  $E$  is metrically closed.

From this and Theorem 2 we can prove a sort of duality theorem due to T. Ogasawara:

**Theorem 10.** A Banach lattice is a K-space, if and only if, it coincides with all order-continuous linear functionals on the conjugate space.

**Proof.** If the theorem does not hold, then by Theorem 8 there is an order-continuous linear functional  $\check{x}$  with  $\check{x} \wedge x = 0$  in  $E^{**}$  for all positive  $x$  of  $E$ . Hence by Theorem 1  $\check{x}$  has a non-zero ideal  $P$  in  $E^*$ , and  $f \in I$  implies  $f(x) = 0$  for all positive  $x$  in  $E$ . This is a contradiction.

Conversely, if the condition is satisfied by a Banach lattice, then Condition F holds by Theorem 5. If  $0 \leq x_n \leq x_{n+1}$  and its norm is bounded, then  $x_n$  converges weakly on  $E^*$  to a functional  $\check{x}$  in  $E^{**}$ . But  $\check{x}$ , as a consequence of Theorem 2, is order-continuous, and by hypothesis it belongs to  $E$ . This proves Condition L and then the proof is completed.

In the similar manner, Theorem 2 is also applicable to prove the following theorem due to T. Ogasawara.

**Theorem 11.** If the conjugate space of a Banach lattice is separable, then it is a K-space.

**Proof.** By a theorem due to S. Banach [1],  $E^{**}$  is weakly separable by the assumption, and then it is possible to approximate every element of  $E^{**}$  by an enumerable sequence of element of  $E$ . Hence, by a theorem due to Kantorovitch [8] and Theorem 2, every element of  $E^{**}$  is order-continuous on  $E^*$ . Therefore, by Theorem 5,  $E^*$  satisfies Condition F. On the other hand, Condition L holds for any Banach lattice which is conjugate to a some Banach lattice. Hence the lemma is proved.

**Theorem 12.** A Banach lattice is a K-space if and only if it is weakly complete.



**Proof.** Since the sufficiency is obvious, we prove only the necessity. If a sequence of a  $K$ -space converges weakly, then its limit belongs to the second conjugate space in general. But it is order-continuous by Theorem 2, and then it belongs to the given lattice by Theorem 10. This completes the proof.

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