THE GENERALIZED PERRON INTEGRALS.*)

By

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Introduction. The notion of differentiation was generalized in many directions. Among them we consider the approximate derivative and Cesàro derivative. As the inverse of those derivatives, approximately continuous Perron integral and Cesàro-Perron integral are defined by J. C. Burkill [1], [2], using the Perron method.

In the definitions of these integrals, he assumed the contintity property of upper and lower functions. Recently S. Saks [3] defined the ordinary Perron integral without using any continuity property of the upper and lower functions, and proved the continuity of the indefinite integral. Hence it arises the problem, whether the notion of continuity of upper and lower functions are superfluous in the Burkill integrals or not.

We will, in this paper, answer this problem affirmatively. By this, the definition of the integrals becomes simply in some way. In §1 we define the approximately continuous Perron integral, or simply (AP)-integral and prove the approximate continuity of the indefinite (AP)-integral. In §2 we define the Cesàro-Perron integral, or simply (CP)-integral and prove the Cesàro continuity of the indefinite (CP)-integral.

1. The approximately continuous Perron integral or (AP)-integral.

Theorem 1.1. If measursble function $f(x)$ has a non-negative lower approximate derivate at each point of $[a, b]$, then $f(a) \leqq f(b)$.

Proof. Since the lower approximate derivate of $f(x)$ is non-negative at $x=a$, $AD\, f(a) \geqq 0$, and then, for any small $\varepsilon (>0)$, the set

(1) $$S \equiv \{x \mid f(x) - f(x) \mid \geqq -\varepsilon (x-a)\}$$

has the point a as a point of density. For a given k $(0<k<1)$, we can find x_1 sufficiently near a, such that

$$f(x_1) - f(a) \geqq -\varepsilon (x_1 - a)$$

and that the set S has average density in (a, x_1) greater than k. Again starting from x_1 we can find x_2 such that

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$$f(x_2)-f(x_1)\geqq-\varepsilon(x_2-x_1)$$

and that the set $\{x|f(x)-f(x_1)\geqq-\varepsilon(x-x_1)\}$ has average density in (x_1, x_2) greater than k. So that $|S\wedge(x_1, x_2)|>k|x_2-x_1|$. Thus proceeding, we get a sequence $\{x_n\}$ which tends to ξ. If $\xi=b$, the proof is completed. Let $\xi<b$. Since the set S has average density greater than k in every intervals (x_n, x_{n+1}), if $(x_m-x_n)/(\xi-x_n)\geqq 1-\delta$, then

$$\frac{|S\wedge(x_n,\xi)|}{\xi-x_n}\geqq\frac{|S\wedge(x_n, x_m)|}{\xi-x_n}\geqq\frac{|S\wedge(x_n, x_m)|}{x_m-x_n}(1-\delta)$$

$$\geqq k(1-\delta), \text{ that is } \lim_{n\to\infty}\frac{|S\wedge(x_n, \xi)|}{\xi-x_n}\geqq k(1-\delta).$$

Hence the set S has not ξ as a point of dispersion. If we assume

$$f(\xi)-f(a)<-\varepsilon(\xi-a),$$

then ξ is not the point of dispersion of the set

$$\{x|f(\xi)-f(x)<-\varepsilon(\xi-x),\ \xi>x\}.$$

So
$$\underline{AD}\, f(\xi)\leqq-\varepsilon,$$

which contradicts the hypothesis. Consequently

$$f(\xi)-f(a)\geqq-\varepsilon(\xi-a),$$

that is

$$f(b)-f(a)\geqq-\varepsilon(b-a).$$

Since ε is arbitrary, we have $f(b)\geqq f(a)$.

Definition 1.1. $U(x)$ $[L(x)]$ is termed upper [lower] function of a measurable $f(x)$ in $[a, b]$, provided that

(1) $U(a)=0$ $[L(a)=0]$,

(2) $\underline{AD}\ U(x)>-\infty$ $[\overline{AD}\ L(x)<+\infty]$ at each point x,

(3) $\underline{AD}\ U(x)\geqq f(x)$ $[\overline{AD}\ L(x)\leqq f(x)]$ at each point x.

Theorem 1.2. The function $U(x)-L(x)$ is increasing and non-negative.

Proof. This follows immediately from Theorem 1.1.

Definition 1.2. If

(1) $f(x)$ has upper and lower functions in $[a, b]$,

(2) $\text{l.u.b.}_L\ L(b)=\text{g.l.b.}_U\ U(b)$,

then $f(x)$ is termed integrable in the approximate Denjoy-Perron sence or (AP)-integrable. The common value (2) of the two bounds is called the definite (AP)-integral of $f(x)$ and denoted by $(AP)\int_a^b f(x)dx$.

Theorem 1.3. If $f(x)$ is (AP)-integrable in $[a, b]$, then $f(x)$ is also in every interval $[a, x]$, $a\leqq x\leqq b$.

Proof. This is evident by Theorem 1.2.

Theorem 1.4. The indefinite integral $F(x)\equiv(AP)\int^{x} f(t)dt$ is approximately continuous.

Proof. For any $\varepsilon>0$, there exist $U(x)$ and $L(x)$ such that

(1) $0\leqq U(x)-F(x)\leqq\varepsilon,\quad 0\leqq F(x)-L(x)\leqq\varepsilon,$

(2) $\underline{AD}\ U(x)>-\infty,\qquad \overline{AD}\ L(x)<+\infty.$

Hence the theorem is immediate.

Theorem 1.5. $F(x)$ is approximately differentiable almost eveywhere and $AD\ F(x)=f(x)$, *a. e.*

Proof. For any $\varepsilon>0$, we can take an $U(x)$ such as

$$U(b)-L(b)<\varepsilon^2.$$

Since $H(x)=U(x)-F(x)$ is increasing, $F'(x)$ exists almost everywhere and

$$|\{x\,|\,F'(x)\geqq\varepsilon\}|<\varepsilon.$$

So that

$$\underline{AD}\ U(x)=F'(x)+\underline{AD}F(x),\ a.e.$$

and

$$\underline{AD}\ F(x)>-\infty,\quad a.e.$$

Since

$$\underline{AD}\ F(x)\geqq\underline{AD}\ U(x)-\varepsilon\geqq f(x)-\varepsilon,$$

except a set of ε-measure, we have

$$\underline{AD}\ F(x)\geqq f(x),\quad a.e.$$

Similarly, using lower functions, we get

$$\overline{AD}\ F(x)\leqq f(x),\quad a.e.$$

Thus we have $\qquad AD\ F(x)=f(x),\quad a.e.$

2. The Cesaro-Perron integral or (*CP*)-integral.

Definition 2.2. We put

$$C(f;a,b)=\frac{1}{b-a}\int_a^b f(x)dx,$$

where the integral is taken in the restricted Denjoy sense.

If $\lim_{h\to0} C(f;x_0,x_0+h)=f(x_0)$, tnen $f(x)$ is termed Cesàro-continuous at x_0.

Definition 2.2. If $\overline{CD}\ f(x_0)=\underline{CD}\ f(x_0)$, where

$$\overline{\lim_{h\to0}}\ \{C(f;x_0,x_0+h)-f(x_0)\}/\frac{1}{2}h=\overline{CD}\ f(x_0),$$

then $f(x)$ is called Ceeàro differentiable at x_0 and we denote the common value by $CD\ f(x_0)$.

Theorem 2.1. If $\underline{CD}\,f(x)\geqq0$ at each point in $[a,b]$, then we have $f(a)\leqq f(b)$.

Proof. Since $\underline{CD}\ f(x)\geqq0$,

(1) $$\{C(f;a,x)-f(a)\}/\frac{1}{2}(x-a)\geqq-\varepsilon,$$

for all $a \leqq x \leqq x_0$, where x_0 is sufficiently near a. If we denote the line $y-f(x)=-\varepsilon(x-a)$, by l_x, then, for all x in $a \leqq x \leqq x_0$, there exists an x_1 such that the point $\{x_1, f(x_1)\}$ lies above the line l_x. For, if not so,

$$\frac{1}{h^2}\int_a^{a+h} [f(t)-f(a)]dt < -\frac{\varepsilon}{h^2}\int_a^{a+h} (t-a)\ dt = -\varepsilon,\ (a<a+h<x_0).$$

That is,

$$\{C(f; a, x)-f(a)\}/\frac{1}{2}(x-a) < -\varepsilon,$$

which contradicts (1).

Thus we can find an x_1 sufficiently near a, such as

$$(2) \qquad f(x_1)-f(a) \geqq -\varepsilon(x_1-a),$$

$$(2') \qquad C(f;\ a,\ x_1)-f(a) \geqq -\frac{1}{2}\varepsilon(x_1-a).$$

Similarly we can find an x_2 near x_1, such that

$$(3) \qquad f(x_2)-f(x_1) \geqq -\varepsilon(x_2-x_1),$$

$$(3') \qquad C(f;\ x_1, x_2)-f(x_1) \geqq -\frac{1}{2}\varepsilon(x_2-x_1).$$

Thus proceeding, we find a sequence $\{x_n\}$ which tends to ξ. If $\xi=b$, the proof is completed. We suppose that $\xi<b$. Then by (2), (3), $\cdots$, we have

$$(4) \qquad C(f; x_m, x_n)-f(a) \geqq C(f; x_m, x_n)-f(x_m)-\varepsilon(x_m-a)$$

and by (2′), (3′), $\cdots$, we have

$$(5) \qquad \int_{x_m}^{x_n} f(t)dt - \sum_{i=m}^{n-1} f(x_i)(x_{i+1}-x_i) \geqq -\frac{1}{2}\varepsilon\sum_{i=m}^{n-1}(x_{i+1}-x_i)^2$$

Using the inequality

$$f(x_i) \geqq f(x_m)-\varepsilon(x_i-x_m) \qquad (i \geqq m)$$

and (5), we have

$$(6) \qquad \int_{x_m}^{x_n} f(t)dt - f(x_m)(x_n-x_m) \geqq -\frac{1}{2}\varepsilon(x_n-x_m)^2.$$

From (4) and (6), we have

$$C(f; x_m, x_n)-f(a) \geqq -\varepsilon(x_m-a)-\frac{1}{2}\varepsilon(x_n-x_m).$$

That is

$$(7) \qquad C(f; x_m, \xi)-f(a) \geqq -\varepsilon(x_m-a)-\frac{1}{2}\varepsilon(\xi-x_m).$$

Since $\underline{CD}\ f(\xi) \geqq 0$, we have

$$(8) \qquad C(f; x_m, \xi)-f(\xi) > \frac{\varepsilon}{2}(\xi-x_n)$$

for sufficiently large m. By (7) and (8)

$$f(\xi)-f(a) \geqq -\varepsilon(x_n-a)-\varepsilon(\xi-x_n)$$

that is

$$f(\xi)-f(a)\geqq-\varepsilon(\xi-a).$$

Consequently

$$f(b)-f(a)\geqq-\varepsilon(b-a),$$

that is

$$f(b)\geqq f(a).$$

Definition 2.3. $U(x)$ [$L(x)$] is termed upper [lower] function of a measurable $f(x)$, provided that if

(1) $U(a)=0$ [$L(a)=0$],

(2) $\underline{CD}\ U(x)>-\infty$ [$\overline{CD}\ L(x)<+\infty$] at each point x,

(3) $\underline{CD}\ U(x)\geqq f(x)$ [$\overline{CD}\ L(x)\leqq f(x)$] at each point x.

Theorem 2.2. $U(x)-L(x)$ is increasing and non-negative.

Proof. This is evident from Theorem 2.1.

Thus we can develope Cesàro-Perron scale of integration by the usual method. This (CP)-integral has the following properties. The proof is done analogously as in §1.

Theorem 2.3. If $f(x)$ is (CP)-integrable in $[a,b]$, then $f(x)$ is so also in any subinterval.

Theorem 2.4. The indefinite integral $F(x)\equiv(CD)\int_a^x f(t)\,dt$ is Cesàro-continuous.

Theorem 2.5. $F(x)$ is Cesàro-derivable almost everywhere and

$$CD\ F(x)=f(x),\ a.e.$$

Literature

[1] Burkill, J. C., The approximately contnuous Perron integral, Math. Zeit., 34(1931), 270-278.

[2] Burkill, J. C., The Cesàro-Perron integral, Proc. London Math. Soe., 34 (1932), 314-322.

[3] Saks, S., Theory of the integral, Warsawa (1937), Chapter VI.