ON A SPACE WITH AFFINE CONNECTION WHICH HAS NO CLOSED PATH.*)

By

Tominosuke Ôtsuki.

In the geometry of a system of curves which are solutions of the differential equation such as

$$\frac{d^2v}{du^2} = A(u,v)\left(\frac{dv}{du}\right)^3 + 3B(u,v)\left(\frac{dv}{du}\right)^2 + 3C(u,v)\frac{dv}{du} + D(u, v),$$

that is, projective or affine geodesics which we shall call here paths, when we wish to study the properties in the large after the model of those in Riemannian spaces, it will come into question what problems to treat.

We shall now consider a space R_2 which is a connected and closed differentiable manifold of class C^r $(r\geqq3)$ and in which the coefficients Γ^i_{jk} of an affine connexion of class C^{r-2} are given in admissible coordinates. Moreover let us suppose that R_2 is complete in the sense of H. Hopf and W. Rinow, and consider its universal covering space R^*. In addition, we shall suppose that there is one and only one path through any two points in R^*.

For the sake of simplicity, suppose that R_2 is orientable and its genus is not zero, then for each point of R_2 a closed arc of path will be determined, whose initial and end point are the given points and which belongs to a given homotopic class of closed sensed curves; owing to this fact we can define a continuous mapping of R_2 into the space of sensed curves Ω, in which we define the metric by the method of H. Seifert and W. Threlfall in their work: Variationsrechnung im Grossen (Theorie von Marston Morse). Thus the existence of this mapping shows that a continuous vector-field will be determined on R_2, the direction of whose vector is the tangent of the closed arc. Hence according to H. Hopf the genus of R_2 must be 1, that is, R_2 must be homeomorphic to the torus. It will be evident that on a sphere we can not give such system of paths. On the other hand, any Riemannian space homeomorphic to the torus has at least one closed geodesic corresponding to any homotopic class as is well known, and this is based on the fact that in the

*) Received November 1st, 1947.

space Ω we can define such a functional that its critical points in the abstract sense of Morse's theory correspond to closed geodesics. Is it a characteristic property for geodesics of Riemannian spaces ? In short, is it generally affirmed that there is at least one closed path in such affinely connected spaces that are homeomorphic to the torus ? But this will be denied by the following example.

Two independent variables x_1, x_2 may be thought of as the coordinates of R^* and, if $x_1' \equiv x_1$, $x_2' \equiv x_2$ (mod 1), (x_1', x_2') and (x_1, x_2) may define the same point in R_2. Let us suppose that the coefficients of the affine connexion are given by

$$(1) \qquad \Gamma^1_{11}=\lambda,\ \Gamma^1_{12}=0,\ \Gamma^1_{22}=-\lambda\mu^2,\ \Gamma^2_{11}=1,\ \Gamma^2_{12}=0,\ \Gamma^2_{22}=-\mu^2,$$

where λ and μ are irrational numbers such that $\lambda^2 \neq \mu^2$.

Then the equations of path are

$$(2) \qquad \begin{cases} \dfrac{d^2x_1}{dt^2}+\lambda\left(\dfrac{dx_1}{dt}\dfrac{dx_1}{dt}-\mu^2\dfrac{dx_2}{dt}\dfrac{dx_2}{dt}\right)=0, \\[2mm] \dfrac{d^2x_2}{dt^2}+\dfrac{dx_1}{dt}\dfrac{dx_1}{dt}-\mu^2\dfrac{dx_2}{dt}\dfrac{dx_2}{dt}=0, \end{cases}$$

where t is an affine parameter. Integrating these, we shall obtain

$$(3) \qquad ae^{-2\mu(x_1-\lambda x_2)}+be^{(\lambda-\mu)(x_1+\mu x_2)}=c,\ (a^2+b^2\neq 0).$$

Accordingly, it can be observed that one and only one path passes through any two given points in R^*. For any two integers n_1, n_2 $(n_1^2+n_2^2\neq 0)$, the condition that the tangent vectors at the both end points of the path which joins two points (x_1, x_2) and (x_1+n_1, x_2+n_2) have the same direction is, as obtained by easy calculation,

$$(4) \qquad ab\{e^{-2\mu(n_1-\lambda n_2)}-e^{(\lambda-\mu)(n_1+\mu n_2)}\}=0.$$

But this does not happen in our case, because for $ab=0$ (3) becomes $x_1+\mu x_2=\text{const.}$ or $x_1-\lambda x_2=\text{const.}$, while these paths will not pass through the two points, hence $ab\neq 0$. The other factor of the left hand side of (4) is not zero too. Thus we see that this space has no closed path.

Next, calculating the angle θ between the above two directions, we get

$$(5) \qquad \tan\theta=\frac{(\lambda+\mu)\,k\,(h-1)}{(\lambda^2+1)k^2h+(\lambda\mu-1)k(h+1)+\mu^2+1},$$

where we have put

$$k=\frac{-2\mu\{e^{(\lambda-\mu)(n_1+\mu n_2)}-1\}}{\{(\lambda-\mu)e^{-2\mu(n_1-\lambda n_2)}-1\}},\quad h=e^{-(\lambda+\mu)(n_1-\mu n_2)}.$$

Let us now take the m-th convergent $\dfrac{P_m}{Q_m}$ of the continued fraction of μ. It is well known that

$$Q_m > 0, \quad Q_m \to \infty \quad \text{as } m \to \infty,$$

and

$$| P_m - \mu Q_m | < \frac{1}{Q_m}.$$

Hence, if we put $n_1 = P_m$, $n_2 = Q_m$ and denote the corresponding k, h, θ by k_m, h_m, θ_m respectively. Then we have, by virtue of (5)

$$\lim k_m = \frac{-2\mu}{\lambda - \mu}, \quad \lim h_m = 1$$

and

$$\lim \theta_m = 0.$$

Thus it is shown that in the phase space of this space there are arcs of paths returning so near to its initial points (line-elements) as possible.

From this example we have seen that it will be out of the question that whether there exist always closed paths or not. But two questions may arise what under conditions there exist closed paths and whether the above stated fact of approximation can be generally affirmed or not.

Kyûshû University, Kurume.